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Smooth Hilbert schemes: Their classification and geometry

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Abstract. Closed subschemes in projective space with a fixed Hilbert polynomial are parametrized by a Hilbert scheme. We classify the smooth ones. We identify numerical conditions on a polynomial that completely determine when the Hilbert scheme is smooth. We also reinterpret these smooth Hilbert schemes as generalized partial flag varieties and describe the subschemes being parametrized.

Overview

Hilbert schemes are crucial for compactifying families of subschemes and constructing moduli spaces. Among these parameter spaces, the Hilbert schemes of points on a projective surface are exceptional. Being smooth, they have a wider range of applications including deep results in algebraic geometry, combinatorics, and representation theory; see [1, 11, 13, 21]. In contrast, little is known about geometric properties of other Hilbert schemes. Even the geometry of $\operatorname{Hilb}^p(\mathbb{P}^m)$, the Hilbert scheme parametrizing closed subschemes in projective m-space $\mathbb{P}^m$ with Hilbert polynomial p, is poorly understood when $m \geqslant 3$. Although Hartshorne [14] shows that each $\operatorname{Hilb}^p(\mathbb{P}^m)$ is path-connected, celebrated insights into these Hilbert schemes typically highlight pathologies. For example, Mumford [20] exhibits an irreducible component in $\operatorname{Hilb}^{14t-23}(\mathbb{P}^3)$ that is generically non-reduced, Ellia, Hirschowitz, and Mezzetti [5] show that the number of irreducible components in $\operatorname{Hilb}^{dt+c}(\mathbb{P}^3)$ is not bounded by a polynomial in $\mathbb{Q}[c, d]$, and Vakil [29] proves that every singularity type appears in some $\operatorname{Hilb}^p(\mathbb{P}^4)$. As a counterpoint, this article classifies the smooth $\operatorname{Hilb}^p(\mathbb{P}^m)$ and describes their geometry.

Our primary theorem uses integer partitions to characterize smooth Hilbert schemes. A partition λ is an r-tuple $\lambda := (\lambda_1, \lambda_2, \dots, \lambda_r)$ of integers satisfying $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_r \geqslant 1$.

Theorem A. *For any positive integer m and any polynomial p in $\mathbb{Q}[t]$, the Hilbert scheme* $\operatorname{Hilb}^p(\mathbb{P}^m)$ *is a smooth irreducible variety if and only if there exists an integer partition*

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$\lambda := (\lambda_1, \lambda_2, \dots, \lambda_r)$ *such that*

$$p(t) = \sum_{i=1}^{r} \binom{t + \lambda_i - i}{\lambda_i - 1}$$

and one of the following seven conditions holds:

(1) $m = 2 \geqslant \lambda_1$,

(2) $m \geqslant \lambda_1$ *and* $\lambda_r \geqslant 2$,

(3) $\lambda = (1)$ *or* $\lambda = (m^{r-2}, \lambda_{r-1}, 1) = (\overbrace{m, \dots, m}^{(r-2)\text{-times}}, \lambda_{r-1}, 1)$, *where* $r \geqslant 2$ *and* $m \geqslant \lambda_{r-1} \geqslant 1$,

(4) $\lambda = (m^{r-s-3}, \lambda_{r-s-2}^{s+2}, 1)$, *where* $r - 3 \geqslant s \geqslant 0$ *and* $m - 1 \geqslant \lambda_{r-s-2} \geqslant 3$,

(5) $\lambda = (m^{r-s-5}, 2^{s+4}, 1)$, *where* $r - 5 \geqslant s \geqslant 0$,

(6) $\lambda = (m^{r-3}, 1^3)$, *where* $r \geqslant 3$,

(7) $\lambda = (m + 1)$ *or* $r = 0$.

These combinatorial conditions encode the underlying geometry. Rewriting the polynomial p in terms of an integer partition λ is equivalent to $\mathrm{Hilb}^p(\mathbb{P}^m)$ being nonempty; see Remark 2.4. The partition $\lambda = (\lambda_1)$ corresponds to the Grassmannian of $(\lambda_1 - 1)$-dimensional planes in $\mathbb{P}^m$ and $\lambda = (m^r) = (m, m, \dots, m)$ corresponds to the Hilbert scheme parametrizing hypersurfaces of degree r in $\mathbb{P}^m$; both well-known families are covered by Condition (2) and the case $\lambda = (1)$ in Condition (3). More generally, every point on a Hilbert scheme satisfying Condition (3) where $\lambda = (m^d, \ell, 1)$ corresponds to the scheme-theoretic union of a hypersurface of degree d, a linear subspace of dimension $\ell - 1$, and a point. Similarly, general points on a Hilbert scheme satisfying Condition (5) with $\lambda = (m^d, 2^c, 1)$ correspond to the union of a hypersurface of degree d, a plane curve of degree c, and a point, whereas those satisfying Condition (4) with $\lambda = (m^d, \ell^c, 1)$ correspond to the union of a hypersurface of degree d, a hypersurface of degree c contained in an ℓ-dimensional linear subspace, and a point. The minor discrepancies in Conditions (4) and (5), arising from the integer partitions $\lambda = (2^2, 1)$ and $\lambda = (2^3, 1)$, are required because the Hilbert schemes with points corresponding to two skew lines and a twisted cubic curve are singular; see Example 4.4. For Condition (6), a general point on the Hilbert scheme with $\lambda = (m^d, 1^3)$ corresponds to the union of a hypersurface of degree d and 3 reduced points. For completeness, observe that the unique point on a Hilbert scheme satisfying Condition (7) corresponds to either $\mathbb{P}^m$ or the empty scheme.

The list of conditions in Theorem A is new and answers Lin's question [17]. However, the challenge lies in proving that this list is exhaustive. Understanding the geometry of Condition (2) is, unexpectedly, the key to overcoming this challenge. Our geometric interpretation in this condition relies on expanding the traditional notion of a residual scheme. To be more precise, consider a hypersurface D in $\mathbb{P}^m$. The residual scheme of a closed immersion $D \subseteq X$ in $\mathbb{P}^m$ is the unique closed subscheme $Y \subset X$ such that their defining ideal sheaves on $\mathbb{P}^m$ satisfy $\mathcal{I}_X = \mathcal{I}_Y \cdot \mathcal{I}_D$. Geometrically, the scheme X is the union of Y and D. Building on this concept, a closed immersion $Y \subset X$ in $\mathbb{P}^m$ is a *residual inclusion* if there exists a linear subspace Λ in $\mathbb{P}^m$ containing X and a hypersurface D in Λ such that Y is the residual scheme of $D \subseteq X$ in Λ. We define a *residual flag* in $\mathbb{P}^m$ to be a chain

$$\varnothing = X_{e+1} \subset X_e \subset \cdots \subset X_1$$

such that, for all $1 \leqslant i \leqslant e$, the closed immersion $X_{i+1} \subset X_i$ is a residual inclusion; see Definition 1.4. Unlike other flags, the scheme X_i routinely fails to be equidimensional. Informally, a residual flag extends a partial flag like a multiset extends a set: the degree of each hypersurface in a residual flag is analogous to the multiplicity of each element in a multiset. Proposition 1.11 demonstrates that the parameter spaces representing residual flags are projective bundles over partial flag varieties.

Beyond the classification in Theorem A, our second major contribution proves that a general point on the smooth Hilbert schemes satisfying Conditions (2)–(7) corresponds to either a residual flag or the union of a residual flag and a point. For all integers m greater than 2, this describes the closed subschemes parametrized by a smooth $\mathrm{Hilb}^p(\mathbb{P}^m)$. In the first case, we deduce that these smooth Hilbert schemes are projective bundles over partial flag varieties; see Theorem 3.2. In particular, every point on a Hilbert scheme satisfying Conditions (2)–(3) corresponds to a residual flag. In the second case, the smooth Hilbert schemes are birational to the product of $\mathbb{P}^m$ and a projective bundle over a partial flag variety; see Proposition 3.8 and Example 3.10. In other words, we realize the smooth Hilbert schemes $\mathrm{Hilb}^p(\mathbb{P}^m)$ as suitable generalizations of partial flag varieties.

The success in classifying these smooth Hilbert schemes suggests new questions that may be tractable. What conditions on the partition λ imply that $\mathrm{Hilb}^p(\mathbb{P}^m)$ is irreducible? How does one extend this result to Quot schemes or nested Hilbert schemes? What is the analogue if $\mathbb{P}^m$ is replaced with a smooth toric variety, a complete intersection, or a Grassmannian?

Strategy of proof. We analyze $\mathrm{Hilb}^p(\mathbb{P}^m)$ via the induced action of the general linear group. A point in this Hilbert scheme is Borel-fixed if the stabilizer of the corresponding closed subscheme contains all lower triangular matrices. Every nonempty $\mathrm{Hilb}^p(\mathbb{P}^m)$ has a distinguished Borel-fixed point called the lexicographic point. This point shows that rewriting the polynomial p in terms of the integer partition λ is equivalent to $\mathrm{Hilb}^p(\mathbb{P}^m)$ being nonempty; see [18] or [14, Corollary 5.7]. Reeves and Stillman [25, Theorems 1.4 and 4.1] establish that the lexicographic point is always smooth and determine the dimension of the unique irreducible component containing it. Proving smoothness reduces, in principle, to computing the dimension of the tangent space at the other Borel-fixed points. Unfortunately, the rapid growth in the number of these points and the complexity of individual points overwhelm a brute-force attack.

To circumvent these complications, we identify a new family of subschemes in $\mathbb{P}^m$ that correspond to singular points on $\mathrm{Hilb}^p(\mathbb{P}^m)$. Residual flags are used to describe these points and to prove that they are singular. By exploiting the geometry of residual flags, our analysis reduces to the Hilbert schemes with an integer partition $\lambda = ((m-1)^d, \ell^c, 1)$, where $m-2 \geqslant \ell \geqslant 1$; see Proposition 4.3. In this situation, we construct an explicit monomial ideal and exhibit a number of linearly independent deformations; see Lemma 4.2. Since this number exceeds the dimension of the lexicographic component, this ideal corresponds to a singular point on $\mathrm{Hilb}^p(\mathbb{P}^m)$.

In hindsight, the complete classification of smooth Hilbert schemes is obtained from just a few families of Borel-fixed points. Only three singular families, in addition to our new family, are required; see Examples 4.4–4.5. For smoothness, only one family other than the lexicographic points is needed; see Theorem 3.2 and Proposition 3.8. This reduction may be the most surprising development. The relevant Borel-fixed points also, fortuitously, avoid technicalities arising in positive characteristic, thereby producing uniform results over the integers.

All seven conditions in Theorem A correspond to Hilbert schemes that are known to be smooth. Fogarty's article [6, Theorems 1.4 and 2.4] shows that Conditions (1) and (6) guarantee smoothness. Serving as our initial inspiration, Staal's thesis [27, Theorem 1.1] establishes that Conditions (2)–(3) correspond to smooth Hilbert schemes. Likewise, Ramkumar's preprint [24, Theorem A] proves that Conditions (4)–(5) are associated to smooth Hilbert schemes. Under Condition (7), the Hilbert scheme is just one point. Despite recognizing each separate condition, the consolidated list does not already appear in the literature and is not obviously complete.

Computational experience. Although independent of our proofs, calculations using the software *Macaulay2* [10] were indispensable in the discovery of our results. Recoding the Hilbert polynomial as an integer partition gives a novel method of sampling nonempty Hilbert schemes. Using *Macaulay2*, we made a systematic search of the Borel-fixed points, for all $3 \leqslant m \leqslant 7$, exposing Conditions (2)–(6). We learn, a posteriori, that Conditions (2)–(5) imply that Hilbert schemes have at most two Borel-fixed points. Our computational experiments suggest that the number of parts in the integer partition λ equal to 1 governs the size of the intersection graph for the irreducible components in the Hilbert scheme.

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1. Residual flags

In this section, we introduce the notion of a residual flag and demonstrate that the scheme parametrizing these objects is smooth and projective. Throughout, we work over a locally noetherian base scheme S and E denotes a coherent $\mathcal{O}_S$-module. The projectivization of the graded symmetric algebra $\operatorname{Sym}(E)$ is denoted by $\mathbb{P}(E) := \operatorname{Proj}(\operatorname{Sym}(E))$.

Grassmannians. For any S-scheme T, let E_T denote the pull-back of E to T. A surjection of coherent T-modules $E_T \to F$ gives a closed immersion

$$\mathbb{P}(F) \subseteq \mathbb{P}(E_T) = \mathbb{P}(E) \times_S T.$$

If F is locally free of constant rank $n+1$, then the subscheme $\mathbb{P}(F)$ is an n-plane in $\mathbb{P}(E_T)$. The set of T-valued points of the functor $\operatorname{Gr}(n, \mathbb{P}(E))$ is the set of the n-planes in $\mathbb{P}(E_T)$. The S-scheme representing this functor is projective; see [12, Proposition 9.8.4]. When E is locally free of constant rank $m+1$, the map $\operatorname{Gr}(n, \mathbb{P}(E)) \to S$ is smooth of relative dimension $(n+1)(m-n)$.

Flag varieties. Consider an e-tuple $n := (n_1, n_2, \dots, n_e)$ of nonnegative integers such that $n_1 > n_2 > \cdots > n_e \geqslant 0$. For any S-scheme T, a flag of type n in $\mathbb{P}(E_T)$ is a chain of closed immersions

$$\mathbb{P}(F_e) \subset \mathbb{P}(F_{e-1}) \subset \cdots \subset \mathbb{P}(F_1)$$

where each $\mathbb{P}(F_i)$ is an n_i-plane in $\mathbb{P}(E_T)$ for all $1 \leqslant i \leqslant e$. The set of T-valued points of the functor $\operatorname{Flag}(n, \mathbb{P}(E))$ is the set of flags of type n in $\mathbb{P}(E_T)$. The S-scheme representing this functor is projective; see [12, Proposition 9.9.3]. A flag is a succession of Grassmannians. Hence, when E is locally free of constant rank $n_0 + 1$, it follows that the map $\mathrm{Flag}(n, \mathbb{P}(E)) \to S$ is smooth of relative dimension $\sum_{i=1}^{e} (n_i + 1)(n_{i-1} - n_i)$.

Relative divisors. A closed subscheme $D \subset \mathbb{P}(E)$ is a relative effective Cartier divisor if it is flat over S and its ideal sheaf is invertible. The divisor D has degree d if, for each geometric point $\mathrm{Spec}(k) \to S$, the fibre $D \times_S \mathrm{Spec}(k)$ is a hypersurface of degree d in $\mathbb{P}^m \cong \mathbb{P}(E) \times_S \mathrm{Spec}(k)$; compare with [15, Corollary 1.1.5.2]. Using the dual sheaf

$$E^* := \mathcal{H}om(E, \mathcal{O}_S),$$

we may parametrize these divisors in $\mathbb{P}(E)$; see [6, Proposition 1.2] and [16, Exercise 1.4.1.4].

Lemma 1.1. *Assume that E is a locally free sheaf and let E^* be its dual. For all nonnegative integers d, the S-scheme $\mathbb{P}(\mathrm{Sym}^d(E^*))$ represents the functor of relative effective Cartier divisors in $\mathbb{P}(E)$ having degree d.*

Proof. Let T be an S-scheme and set

$$F_T := (\mathrm{Sym}^d(E^*))_T = \mathrm{Sym}^d((E^*)_T).$$

For a line bundle L on T and a surjection $F_T \to L$, we see that L^* is an invertible subsheaf of F_T^*. The ideal sheaf generated by L^* in $\mathrm{Sym}(E^*)_T$ is invertible. It determines a hypersurface of degree d fibrewise in $\mathbb{P}(E_T)$ and a relative effective Cartier divisor in $\mathbb{P}(E_T)$ of degree d. □

Residual scheme. Consider the closed immersion $D \subseteq X$ in $\mathbb{P}(E)$, where D is a relative effective Cartier divisor. Let $\mathcal{I}_D$ and $\mathcal{I}_X$ denote the ideal sheaves of the closed subschemes D and X in $\mathbb{P}(E)$. The residual scheme to D in X is the closed subscheme Y in $\mathbb{P}(E)$ defined by the colon ideal sheaf

$$\mathcal{I}_Y := (\mathcal{I}_X : \mathcal{I}_D) = \mathcal{I}_X \cdot \mathcal{I}_D^{-1}.$$

It follows that $\mathcal{I}_X = \mathcal{I}_Y \cdot \mathcal{I}_D$ and X is the union of the subschemes D and Y; see [7, Definition 9.2.1] and [6, pp. 512–513].

Definition 1.2. For any positive integer d, a closed immersion $Y \subset X$ in $\mathbb{P}(E)$ is a *d-residual inclusion* if there exists a relative effective Cartier divisor D in $\mathbb{P}(E)$ of degree d such that the closed subscheme Y is the residual scheme to D in X with respect to $\mathbb{P}(E)$.

Lemma 1.3. *Let d be a positive integer and let $Y \subset X$ in $\mathbb{P}(E)$ be a d-residual inclusion. The map $Y \to S$ is flat if and only if the map $X \to S$ is flat.*

Proof. The existence of a relative effective Cartier divisor D in $\mathbb{P}(E)$ such that Y is the residual scheme to D in X yields the short exact sequence

$$0 \to \mathcal{O}_Y(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0,$$

where multiplication by a local equation for D defines the injective map. Since the sheaf $\mathcal{O}_D$ is flat over S, we deduce that $\mathcal{O}_Y$ and $\mathcal{O}_Y(-D)$ are flat over S if and only if $\mathcal{O}_X$ is flat over S. □

Definition 1.4. Let $(n, d) := (n_1, d_1), (n_2, d_2), \dots, (n_e, d_e)$ be a sequence of pairs of positive integers such that $n_1 > n_2 > \dots > n_e > 0$. For any S-scheme T, a *residual flag* of type (n, d) in $\mathbb{P}(E_T)$ is a chain of closed immersions $\varnothing = X_{e+1} \subset X_e \subset X_{e-1} \subset \dots \subset X_1$ in $\mathbb{P}(E_T)$ such that, for all $1 \leqslant i \leqslant e$, the following properties are satisfied:

(i) the scheme X_i is flat over T,

(ii) the scheme X_i is contained in some n_i-plane $\mathbb{P}(F_i) \subseteq \mathbb{P}(E_T)$, and

(iii) the closed immersion $X_{i+1} \subset X_i$ is a d_i-residual inclusion in $\mathbb{P}(F_i)$.

Remark 1.5. For any residual flag, the third property for $i = e$ asserts that closed immersion $\varnothing = X_{e+1} \subset X_e$ is a d_e-residual inclusion. In other words, the closed subscheme X_e is a relative effective Cartier divisor of degree d_e in some n_e-plane $\mathbb{P}(F_e) \subseteq \mathbb{P}(E_T)$.

Remark 1.6. Residual flags generalize flags of linear subspaces. To be more explicit, assume E is locally free of constant rank $m + 1$ and let T be an S-scheme. Given a flag $\Lambda_e \subset \Lambda_{e-1} \subset \dots \subset \Lambda_1$ of type n in $\mathbb{P}(E_T)$, where $m > n_1$, there exists a flag

$$\mathbb{P}(F_e) \subset \mathbb{P}(F_{e-1}) \subset \dots \subset \mathbb{P}(F_1)$$

of type $(n_1 + 1, n_2 + 1, \dots, n_e + 1)$ in $\mathbb{P}(E_T)$ such that the n_i-plane Λ_i is a hyperplane in $\mathbb{P}(F_i)$ for all $1 \leqslant i \leqslant e$. Setting $X_{e+1} := \varnothing$, we define $X_i := \Lambda_i \cup X_{i+1}$ by a descending induction. It follows that $X_{i+1} \subset X_i$ is a 1-residual inclusion for all $1 \leqslant i \leqslant e$. Thus, the chain of closed immersions $\varnothing \subset X_e \subset X_{e-1} \subset \dots \subset X_1$ is a residual flag of type $(n_1 + 1, 1)$, $(n_2 + 2, 1), \dots, (n_e + 1, 1)$ in $\mathbb{P}(E_T)$.

Example 1.7. We illustrate how to recursively construct the defining ideal for the closed subschemes in a residual flag. To this end, let $(n, d) := (3, 2), (2, 4)$ and set $S = \operatorname{Spec}(k)$, where k is a field. The residual flags of type (n, d) in $\mathbb{P}^3 := \operatorname{Proj}(k[x_0, x_1, x_2, x_3])$ are nested pairs $X_2 \subset X_1$ in $\mathbb{P}^3$ such that X_2 is a planar curve of degree 4 and X_2 is a 2-residual scheme in X_1. The defining ideal of X_2 has the form

$$I_{X_2} := \langle f_1, f_2 \rangle,$$

where f_1 is a homogeneous polynomial in $k[x_0, x_1, x_2, x_3]$ of degree 1 and f_2 is a homogeneous polynomial in $k[x_0, x_1, x_2, x_3]$ of degree $d_2 = 4$ that is not divisible by f_1. The 2-plane $\mathbb{P}(F_2)$ containing X_2 is given by the vanishing of the linear form f_1. The defining ideal of the closed subscheme X_1 in $\mathbb{P}^3$ has the form

$$I_{X_1} := g \cdot I_{X_2} = \langle g f_1, g f_2 \rangle,$$

where $g \in k[x_0, x_1, x_2, x_3]$ is a homogeneous polynomial of degree $d_1 = 2$. Geometrically, the scheme X_1 is the union of the quadratic hypersurface defined by the vanishing of g and the planar quartic curve X_2. For the special configuration in which $g = f_1^2$, the defining ideal of the closed subscheme X_1 is $I_{X_1} = \langle f_1^3, f_1^2 f_2 \rangle = \langle f_1^2 \rangle \cap \langle f_1^3, f_2 \rangle$. $\diamond$

Functor of residual flags. The pullback of a residual scheme is again a residual scheme; see [6, Lemma 1.3]. It follows that residual flags define a contravariant functor from the category of S-schemes to the category of sets. Let $(n, d) := (n_1, d_1), (n_2, d_2), \dots, (n_e, d_e)$ be a sequence of pairs of positive integers such that $n_1 > n_2 > \cdots > n_e$ and let E be coherent $\mathcal{O}_S$-module. For any S-scheme T, the set of T-valued points of the functor $\operatorname{Flag}(n, d, \mathbb{P}(E))$ is defined to be the set of residual flags of type (n, d) in $\mathbb{P}(E_T)$.

Lemma 1.8. *Assume that (n_1, d_1) is a pair of positive integers and E is a coherent sheaf on S. Let F be the universal quotient sheaf on the Grassmannian $\operatorname{Gr}(n_1, \mathbb{P}(E))$ and let F^* denote its dual.*

(i) *When $d_1 > 1$, the S-scheme $\mathbb{P}(\operatorname{Sym}^{d_1}(F^*))$ represents the functor of residual flags of type (n_1, d_1) in $\mathbb{P}(E)$. The structure map of this S-scheme is the composition of the canonical maps $\mathbb{P}(\operatorname{Sym}^{d_1}(F^*)) \to \operatorname{Gr}(n_1, \mathbb{P}(E))$ and $\operatorname{Gr}(n_1, \mathbb{P}(E)) \to S$.*

(ii) *When $d_1 = 1$, the Grassmannian $\operatorname{Gr}(n_1 - 1, \mathbb{P}(E))$ represents the functor of residual flags of type $(n_1, 1)$ in $\mathbb{P}(E)$.*

Proof. Let T be an S-scheme. Remark 1.5 shows that residual flags of type $(n_1, 1)$ in $\mathbb{P}(E_T)$ are $(n-1)$-planes, so part (ii) follows. Assume that $d_1 > 1$. A T-valued point of $\mathbb{P}(\operatorname{Sym}^{d_1}(F^*))$ consists of a line bundle L_T on T and a surjection $\operatorname{Sym}^{d_1}((F_T)^*) \to L_T$ together with a T-valued point of $\operatorname{Gr}(n_1, \mathbb{P}(E))$. Since

$$(\operatorname{Sym}^{d_1}(F^*))_T = \operatorname{Sym}^{d_1}((F_T)^*),$$

Lemma 1.1 demonstrates that L_T corresponds to a relative effective Cartier divisors of degree d_1 in the n_1-plane $\mathbb{P}(F_T)$. The T-valued point of $\operatorname{Gr}(n_1, \mathbb{P}(E))$ corresponds to the n_1-plane $\mathbb{P}(F_T) \subseteq \mathbb{P}(E_T)$. Thus, the S-scheme $\mathbb{P}(\operatorname{Sym}^{d_1}(F^*))$ represents the residual flags of type (n_1, d_1) in $\mathbb{P}(E)$. □

Remark 1.9. When the base scheme S is the spectrum of a field k and $d_1 > 1$, the parameter space for the residual flags of type (n_1, d_1) in $\mathbb{P}^m := \operatorname{Proj}(k[x_0, x_1, \dots, x_m])$ is the variety of degree d_1 hypersurfaces in n_1-planes in $\mathbb{P}^m$; see [7, Example 14.7.12].

Latent planes. The definition of a residual flag $\varnothing \subset X_e \subset X_{e-1} \subset \cdots \subset X_1$ of type (n, d) in $\mathbb{P}(E_T)$ includes the existence of flag of linear subspaces. For all $1 \leqslant i \leqslant e$, the scheme X_i lies in some n_i-plane $\mathbb{P}(F_i) \subseteq \mathbb{P}(E_T)$. When $d_e > 1$, we will refer to the set $\{\mathbb{P}(F_i) : 1 \leqslant i \leqslant e\}$ as the latent planes of the residual flag. In the special case $d_e = 1$, the scheme X_e is itself a $(n_e - 1)$-plane and the latent planes are $\{X_e\} \cup \{\mathbb{P}(F_i) : 1 \leqslant i \leqslant e-1\}$.

Lemma 1.10. *Let (n, d) be the type of a residual flag.*

(i) *When $d_e > 1$, there exists a morphism $\operatorname{Flag}(n, d, \mathbb{P}(E)) \to \operatorname{Flag}(n, \mathbb{P}(E))$ sending a residual flag to its flag of latent planes.*

(ii) *When $d_e = 1$, there exists a morphism $\operatorname{Flag}(n, d, \mathbb{P}(E)) \to \operatorname{Flag}(n^\circ, \mathbb{P}(E))$ sending a residual flag to its flag of latent planes, where $n^\circ := (n_1, n_2, \dots, n_{e-1}, n_e - 1)$.*

Proof. We need to show that the latent planes are unique and form a flag. Let T be an S-scheme and let $\varnothing \subset X_e \subset X_{e-1} \subset \cdots \subset X_1$ be a residual flag of type (n, d) in $\mathbb{P}(E_T)$. Suppose that, for some $1 \leqslant i \leqslant e$, the scheme X_i is contained in two distinct n_i-planes $\mathbb{P}(F_i)$ and $\mathbb{P}(F_i{}')$. The closed immersion $X_{i+1} \subset X_i$ is a d_i-residual inclusion in $\mathbb{P}(F_i)$, so there exists a relative effective Cartier divisor D_i in $\mathbb{P}(F_i)$ such that X_i is the union of D_i and X_{i+1}. Since the codimension of X_i in $\mathbb{P}(F_i)$ equals 1, we deduce that $X_i = D_i = \mathbb{P}(F_i) \cap \mathbb{P}({F_i}')$. It follows that either the n_i-plane containing X_i is unique or $i = 1$, $D_1 = X_1$, and $d_1 = 1$. Thus, each scheme X_i is contained in a unique plane having the dimension of its corresponding latent plane, so both assertions follow. □

Representability. The pivotal result in this section shows that the functor of residual flags is representable. Moreover, it realizes this parameter space as a generalization of a partial flag variety.

Proposition 1.11. *Assume that $(n, d) := (n_1, d_1), (n_2, d_2), \dots, (n_e, d_e)$ is the type of a residual flag and E is a coherent sheaf on S. For all $1 \leqslant i \leqslant e$, let F_i^* denote the dual of the universal quotient sheaf on the Grassmannian $\mathrm{Gr}(n_i, \mathbb{P}(E))$.*

(i) *When $d_e > 1$, we have the Cartesian square*

$$\begin{array}{ccc} \mathrm{Flag}(n, d, \mathbb{P}(E)) & \longrightarrow & P \\ \downarrow & & \downarrow \\ \mathrm{Flag}(n, \mathbb{P}(E)) & \longrightarrow & G, \end{array}$$

where

$$P := \mathbb{P}(\mathrm{Sym}^{d_1}(F_1^*)) \times_S \mathbb{P}(\mathrm{Sym}^{d_2}(F_2^*)) \times_S \cdots \times_S \mathbb{P}(\mathrm{Sym}^{d_e}(F_e^*)),$$
$$G := \mathrm{Gr}(n_1, \mathbb{P}(E)) \times_S \mathrm{Gr}(n_2, \mathbb{P}(E)) \times_S \cdots \times_S \mathrm{Gr}(n_e, \mathbb{P}(E))$$

(ii) *When $d_e = 1$, setting $n^\circ := (n_1, n_2, \dots, n_{e-1}, n_e - 1)$ gives the Cartesian square*

$$\begin{array}{ccc} \mathrm{Flag}(n, d, \mathbb{P}(E)) & \longrightarrow & P' \\ \downarrow & & \downarrow \\ \mathrm{Flag}(n^\circ, \mathbb{P}(E)) & \longrightarrow & G', \end{array}$$

where

$$\begin{aligned} P' := {} & \mathbb{P}(\mathrm{Sym}^{d_1}(F_1^*)) \times_S \mathbb{P}(\mathrm{Sym}^{d_2}(F_2^*)) \times_S \cdots \\ & \times_S \mathbb{P}(\mathrm{Sym}^{d_{e-1}}(F_{e-1}^*)) \times_S \mathrm{Gr}(n_e - 1, \mathbb{P}(E)), \\ G' := {} & \mathrm{Gr}(n_1, \mathbb{P}(E)) \times_S \mathrm{Gr}(n_2, \mathbb{P}(E)) \times_S \cdots \\ & \times_S \mathrm{Gr}(n_{e-1}, \mathbb{P}(E)) \times_S \mathrm{Gr}(n_e - 1, \mathbb{P}(E)). \end{aligned}$$

In both cases, the functor $\mathrm{Flag}(n, d, \mathbb{P}(E))$ *is represented by a projective S-scheme.*

Proof. The bottom horizontal arrows in the squares are closed immersions; see for instance [12, Proposition 9.9.3]. To prove the projectivity of $\mathrm{Flag}(n, d, \mathbb{P}(E))$, it is enough to show that these squares are Cartesian.

Assume that $d_e > 1$. Let T be an S-scheme and let $\varnothing \subset X_e \subset X_{e-1} \subset \cdots \subset X_1$ be a residual flag of type (n, d) in $\mathbb{P}(E_T)$. For all $1 \leqslant i \leqslant e$, there exists a relative effective Cartier divisor D_i in $\mathbb{P}(F_i)$ such that $X_i = D_i \cup X_{i+1}$. Lemma 1.8 implies that the product of projective bundles P over the product of Grassmannians G parametrizes e-tuples of relative effective Cartier divisors of degree d_i contained in some n_i-plane in $\mathbb{P}(E)$ for all $1 \leqslant i \leqslant e$. Hence, we have a morphism from $\operatorname{Flag}(n, d, \mathbb{P}(E))$ to P sending the residual flag $\varnothing \subset X_e \subset X_{e-1} \subset \cdots \subset X_1$ to the e-tuple $(D_1 \subset \mathbb{P}(F_1), D_2 \subset \mathbb{P}(F_2), \ldots, D_e \subset \mathbb{P}(F_e))$. Combined with the morphism in Lemma 1.10, we obtain a morphism from $\operatorname{Flag}(n, d, \mathbb{P}(E))$ to the fibre product $\operatorname{Flag}(n, \mathbb{P}(E)) \times_G P$.

We want to exhibit the inverse of this morphism. A T-valued point of this fibre product is a flag $\mathbb{P}(F_e) \subset \mathbb{P}(F_{e-1}) \subset \cdots \subset \mathbb{P}(F_1)$ of type n in $\mathbb{P}(E_T)$ together with an e-tuple of relative effective Cartier divisors D_i in $\mathbb{P}(F_i)$ for all $1 \leqslant i \leqslant e$. Setting $X_{e+1} := \varnothing$, we define $X_i := D_i \cup X_{i+1}$ for all $1 \leqslant i \leqslant e$ by descending induction. By construction, we have a chain $\varnothing \subset X_e \subset X_{e-1} \subset \cdots \subset X_1$ of closed immersions in $\mathbb{P}(E_T)$. Since Lemma 1.3 shows that, for all $1 \leqslant i \leqslant e$, the scheme X_i is flat over T, this chain is a residual flag of type (n, d) in $\mathbb{P}(E_T)$. As the pull-back of a residual scheme is again a residual scheme, this construction is also functorial. We conclude that the scheme $\operatorname{Flag}(n, d, \mathbb{P}(E))$ and the fibre product $\operatorname{Flag}(n, \mathbb{P}(E)) \times_G P$ are isomorphic.

The proof for the case $d_e = 1$ is very similar. Since the scheme X_e is a $(n_e - 1)$-plane in $\mathbb{P}(E_T)$, we have a natural morphism from $\operatorname{Flag}(n, d, \mathbb{P}(E))$ to the product P' sending the residual flag to $(D_1 \subset \mathbb{P}(F_1), D_2 \subset \mathbb{P}(F_2), \ldots, D_{e-1} \subset \mathbb{P}(F_{e-1}), X_e \subset \mathbb{P}(E_T))$. Using Lemma 1.10, we obtain a morphism from $\operatorname{Flag}(n, d, \mathbb{P}(E))$ to the appropriate fibre product. As above, one exhibits the inverse morphism by defining $X_i := D_i \cup X_{i+1}$ for all $1 \leqslant i \leqslant e-1$. □

Corollary 1.12. *Let $(n, d) := (n_1, d_1), (n_2, d_2), \ldots, (n_e, d_e)$ be the type of a residual flag and let E be a locally free sheaf on S of constant rank $n_0 + 1$ where $n_0 \geqslant n_1$.*

(i) *When $d_e > 1$, the structure map* $\operatorname{Flag}(n, d, \mathbb{P}(E)) \to S$ *is smooth of relative dimension*

$$\sum_{i=1}^{e} \left[\binom{n_i + d_i}{d_i} - 1 + (n_i + 1)(n_{i-1} - n_i) \right].$$

(ii) *When $d_e = 1$, the structure map* $\operatorname{Flag}(n, d, \mathbb{P}(E)) \to S$ *is smooth of relative dimension*

$$-(n_{e-1} - n_e) + \sum_{i=1}^{e} \left[\binom{n_i + d_i}{d_i} - 1 + (n_i + 1)(n_{i-1} - n_i) \right].$$

Proof. For all $1 \leqslant i \leqslant e$, the sheaf $\operatorname{Sym}^{d_i}(F_i^*)$ on the Grassmannian $\operatorname{Gr}(n_i, \mathbb{P}(E))$ is locally free of constant rank $\binom{n_i+d_i}{d_i}$. Using the relative dimensions of flag varieties and Grassmannians, Proposition 1.11 shows that the map $\operatorname{Flag}(n, d, \mathbb{P}(E)) \to S$ is smooth of the claimed relative dimensions. □

Question 1.13. When E is locally free, the fibre product interpretation leads to a presentation for the Chow ring of $\operatorname{Flag}(n, d, \mathbb{P}(E))$. Specifically, one combines the formula for the Chow ring of a projective bundle with the description of the Chow ring of a partial flag variety; see [7, Examples 8.3.4 and 14.7.16]. Moreover, the cycle map on $\operatorname{Flag}(n, d, \mathbb{P}(E))$ from its Chow ring to its integral cohomology ring is an isomorphism; see [7, Example 19.1.11]. Which aspects of Schubert calculus on partial flag varieties extend to the parameter space of residual flags?

Question 1.14. For line bundles on $\mathbb{P}(E)$, the higher-direct images under the structure map to the base scheme S are well-understood. Similarly, the Borel–Weil–Bott theorem describes the higher-direct images of line bundles on a flag variety under the structure map to the base scheme. What is the common refinement for line bundles on the parameter space $\operatorname{Flag}(n, d, \mathbb{P}(E))$?

2. Hilbert polynomials and residual flags

Using the geometry of residual flags, we explain the combinatorial formula for the Hilbert polynomials. We show that the type of a residual flag encodes the Hilbert polynomial of its largest subscheme. We also prove that every lexicographic ideal (other than the zero ideal and unit ideal) determines a residual flag. Let $R := k[x_0, x_1, \dots, x_m]$ denote the standard graded polynomial ring over a field k and set $\mathbb{P}^m := \operatorname{Proj}(R)$.

Integer partitions. We repackage the type of a residual flag as an integer partition. Given a sequence $(n_1, d_1), (n_2, d_2), \dots, (n_e, d_e)$ of pairs of positive integers such that

$$n_1 > n_2 > \cdots > n_e > 0,$$

the partition $\lambda := (\lambda_1, \lambda_2, \dots, \lambda_r)$ satisfies $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_r \geqslant 1$ and

$$\lambda = (\underbrace{n_1, n_1, \dots, n_1}_{d_1\text{-times}}, \underbrace{n_2, n_2, \dots, n_2}_{d_2\text{-times}}, \dots, \underbrace{n_e, n_e, \dots, n_e}_{d_e\text{-times}}).$$

The length of λ is $r := d_1 + d_2 + \cdots + d_e$. It can be convenient to use a notation for integer partitions that indicates the number of times each integer occurs as a part; see [19, Subsection 1.1]. The expression $\lambda = (\dots, i^{a_i}, \dots, 2^{a_2}, 1^{a_1})$ means that, for all positive integers i, exactly a_i of the parts in λ are equal to i. For instance, we have $\lambda = (n_1^{d_1}, n_2^{d_2}, \dots, n_e^{d_e})$.

To describe Hilbert polynomials, we treat a binomial coefficient with a variable in its numerator as a polynomial. Specifically, for all integers c, we set

$$\binom{t}{c} := \begin{cases} \dfrac{1}{c!}(t)(t-1)\cdots(t-c+1) & \text{if } c \geqslant 0, \\ 0 & \text{if } c < 0. \end{cases}$$

The polynomial $\binom{t}{c}$ has rational coefficients and degree c.

Lemma 2.1. *Let (n, d) be the type of a residual flag and let $\lambda := (\lambda_1, \lambda_2, \dots, \lambda_r)$ be its associated integer partition. For any residual flag $\varnothing \subset X_e \subset X_{e-1} \subset \cdots \subset X_1$ of type (n, d) in $\mathbb{P}^m$, the Hilbert polynomial of the closed scheme X_1 in $\mathbb{P}^m$ is*

$$p(t) = \sum_{i=1}^{r} \binom{t + \lambda_i - i}{\lambda_i - 1}.$$

Proof. We proceed by induction on e. The closed immersion $X_2 \subset X_1$ is a d_1-residual inclusion in some n_1-plane contained in $\mathbb{P}^m$. Hence, there exists a relative effective Cartier divisor D_1 in this n_1-plane and a short exact sequence of sheaves

$$0 \to \mathscr{O}_{X_2}(-D_1) \to \mathscr{O}_{X_1} \to \mathscr{O}_{D_1} \to 0. \tag{2.1}$$

When $e = 1$, we have $X_2 = \varnothing$. The closed scheme X_1 is the divisor D_1, so its Hilbert polynomial is

$$p(t) = \binom{t+n_1}{n_1} - \binom{t+n_1-d_1}{n_1}$$
$$= \sum_{i=1}^{d_1}\left[\binom{t+n_1-i+1}{n_1} - \binom{t+n_1-i}{n_1}\right] = \sum_{i=1}^{d_1}\binom{t+n_1-i}{n_1-1}.$$

The partition associated to the residual flag $\varnothing \subset X_1$ is $\lambda = (n_1^{d_1})$, so the base case holds.

Suppose that $e > 1$. The residual flag

$$\varnothing \subset X_e \subset X_{e-1} \subset \cdots \subset X_2$$

has $e-1$ subschemes and its integer partition is $(n_2^{d_2}, n_3^{d_3}, \dots, n_e^{d_e})$. The induction hypothesis implies that the Hilbert polynomial of the closed scheme X_2 is

$$\sum_{i=d_1+1}^{r}\binom{t+\lambda_i-i+d_1}{\lambda_i-1}.$$

From the short exact sequence (2.1), we deduce that the Hilbert polynomial of the closed scheme X_1 is

$$p(t) = \sum_{i=d_1+1}^{r}\binom{t+\lambda_i-i+d_1}{\lambda_i-1} + \sum_{i=1}^{d_1}\binom{t+n_j-i}{n_j-1} = \sum_{i=1}^{r}\binom{t+\lambda_i-i}{\lambda_i-1}. \qquad \square$$

Remark 2.2. Let X be a closed subscheme in $\mathbb{P}^m$ having Hilbert polynomial

$$p(t) = \sum_{i=1}^{r}\binom{t+\lambda_i-i}{\lambda_i-1}.$$

The dimension of X is $\lambda_1 - 1$ and the degree of X is the number d_1 of parts in λ equal to λ_1. One verifies that the arithmetic genus of X is

$$(-1)^{\lambda_1-1}\sum_{i=2}^{r}\binom{\lambda_i-i}{\lambda_i-1} = \sum_{i=2}^{r}(-1)^{\lambda_1-\lambda_i}\binom{i-2}{\lambda_i-1}.$$

Remarkably, the Gotzmann Regularity Theorem shows that the length r of λ is an upper bound on the Castelnuovo–Mumford regularity of the saturated ideal defining the closed subscheme X in $\mathbb{P}^m$; see [8, Lemma 2.9] or [2, Theorem 4.3.2].

Lexicographic ideals. We identify a special residual flag from the geometric properties of a distinguished monomial ideal. The lexicographic order on the monomials in the polynomial ring $R = k[x_0, x_1, \dots, x_m]$ is defined by declaring $x_0^{b_0}x_1^{b_1}\cdots x_m^{b_m} > x_0^{c_0}x_1^{c_1}\cdots x_m^{c_m}$ whenever the first nonzero entry in the integer sequence $(b_0 - c_0, b_1 - c_1, \dots, b_m - c_m)$ is positive. A lexicographic ideal I is a monomial ideal in R such that, for all integers j, the homogeneous component I_j is the k-vector space spanned by the largest $\dim_k I_j$ monomials in lexicographic order.

As the cornerstone of our approach, we recount a variant of the Macaulay characterization [18] of the Hilbert functions for homogeneous ideals in R. Recall that the Hilbert function $h_{R/I}\colon \mathbb{Z} \to \mathbb{N}$ of a homogeneous ideal I in R is defined, for all integers j, by

$$h_{R/I}(j) := \dim_k (R/I)_j .$$

Lemma 2.3. *Let p be a numerical polynomial having degree less than m. The following statements are equivalent.*

(a) *There exists a closed subscheme X in $\mathbb{P}^m$ whose Hilbert polynomial is p.*
(b) *There exists an integer partition $\lambda := (\lambda_1, \lambda_2, \dots, \lambda_r)$ such that $p(t) = \sum_{i=1}^{r} \binom{t+\lambda_i - i}{\lambda_i - 1}$.*
(c) *There exists an integer partition $\lambda := (\lambda_1, \lambda_2, \dots, \lambda_r)$ such that $p(t) = \sum_{i=1}^{r} \binom{t+\lambda_i - i}{\lambda_i - 1}$ and a lexicographic ideal $L(\lambda)$ in R such that, for all integers j, we have*

$$h_{R/L(\lambda)}(j) = \sum_{i=1}^{r} \binom{j+\lambda_i - i}{j - i + 1}.$$

Outline of proof. We sketch the details because a proof may be derived from other accounts of Macaulay's work via identities for binomial coefficients; see [9, Table 174].

- (a) $\Rightarrow$ (b): Let ℓ be a fixed sufficiently large integer. One uses the ℓ-th Macaulay representation for the integer $p(\ell)$ to obtain an integer partition $\lambda := (\lambda_1, \lambda_2, \dots, \lambda_r)$ such that
$$p(t) = \sum_{i=1}^{r} \binom{t+\lambda_i - i}{\lambda_i - 1};$$
see [2, Lemma 4.2.6 and Corollary 4.2.14] or [27, Proposition 2.3].
- (b) $\Rightarrow$ (c): One verifies that the function $h\colon \mathbb{Z} \to \mathbb{N}$ defined by
$$h(j) := \sum_{i=1}^{r} \binom{j+\lambda_i - i}{j - i + 1},$$
for all integers j, satisfies the inequality $(h(j))^{\langle j \rangle} \geqslant h(j)$; see [2, Theorem 4.2.10].
- (c) $\Rightarrow$ (a): One takes X to be the closed subscheme of $\mathbb{P}^n$ defined by the ideal $L(\lambda)$. $\square$

Remark 2.4. By using the conjugate integer partition, [18] and [14, Corollary 5.7] give an alternative condition via sums of differences of binomial coefficients that is equivalent to Lemma 2.3 (b); see [27, Lemma 2.4]. Our expression as a sum of binomial coefficients is a slight modification of the Gotzmann decomposition [8, Section 2].

We specify monomial generators and a primary decomposition of the lexicographic ideal $L(\lambda)$; these generators are also listed in [25].

Proposition 2.5. *Let $\lambda := (\lambda_1, \lambda_2, \dots, \lambda_r)$ be an integer partition and let a_j be the number of parts in λ equal to j, for all positive integers j. When $n \geqslant \lambda_1$, the corresponding lexicographic ideal is*

$$L(\lambda) = \langle x_0^{a_m+1}, x_0^{a_m} x_1^{a_{m-1}+1}, \dots, x_0^{a_m} x_1^{a_{m-1}} \cdots x_{m-3}^{a_3} x_{m-2}^{a_2+1}, x_0^{a_m} x_1^{a_{m-1}} \cdots x_{m-2}^{a_2} x_{m-1}^{a_1} \rangle.$$

Moreover, the unique irredundant irreducible decomposition of this monomial ideal is

$$L(\lambda) = \bigcap_{\substack{1 \leqslant i \leqslant m \\ a_i \neq 0}} \langle x_0^{a_m+1}, x_1^{a_{m-1}+1}, \dots, x_{m-i-1}^{a_{i+1}+1}, x_{m-i}^{a_i} \rangle.$$

Proof. We first establish the decomposition. As each intersectand is generated by powers of the variables and no two have the same dimension, this intersection is the irredundant irreducible decomposition of some monomial ideal. It remains to show that this ideal is $L(\lambda)$. Since $a_j = 0$ for all $j > \lambda_1$, each irreducible ideal contains $\langle x_0, x_1, \dots, x_{m-\lambda_1-1} \rangle$ and we may assume that $\lambda_1 = m$.

We proceed by induction on the number e of positive entries in $(a_1, a_2, \dots, a_m)$. When $e = 1$, the integer partition is (m^{a_m}) and the ideal is $\langle x_0^{a_m} \rangle$. The ideal $\langle x_0^{a_m} \rangle$ is lexicographic. Since the monomials $\{1, x_0, \dots, x_0^{a_m-1}\}$ form a basis as free $k[x_1, x_2, \dots, x_m]$-module for the quotient $R/\langle x_0^{a_n} \rangle$, it follows that

$$h_{R/\langle x_0^{a_m} \rangle}(j) = \sum_{i=1}^{a_m} \binom{j+m-i}{j-i+1}.$$

Lemma 2.3 (c) implies that $L(m^{a_m}) = \langle x_0^{a_m} \rangle$ and the base case holds.

Assume that $e > 1$. Set $I := L(m^{a_m}) = \langle x_0^{a_m} \rangle$. The induction hypothesis implies that

$$J := \bigcap_{\substack{1 \leqslant i \leqslant m-1 \\ a_i \neq 0}} \langle x_0^{a_m+1}, x_1^{a_{m-1}+1}, \dots, x_{m-i-1}^{a_{i+1}+1}, x_{m-i}^{a_i} \rangle$$

is the lexicographic ideal associated to the integer partition $\nu := (\lambda_{a_m+1}, \lambda_{a_m+2}, \dots, \lambda_r)$. Since the intersection of lexicographic ideals is again lexicographic, it is enough to prove that the ideals $I \cap J$ and $L(\lambda)$ have the same Hilbert function. From the short exact sequences of graded R-modules

$$0 \to \frac{R}{I \cap J} \to \frac{R}{I} \oplus \frac{R}{J} \to \frac{R}{I+J} \to 0$$

and

$$0 \to \frac{R(-a_n)}{J} \to \frac{R}{J} \to \frac{R}{I+J} \to 0,$$

we see that $h_{R/I \cap J}(j) = h_{R/I}(j) + h_{R/J}(j) - h_{R/(I+J)}(j) = h_{R/I}(j) + h_{R/J}(j - a_n)$ for all integer j. The equality $J = L(\nu)$ implies that

$$h_{R/J}(j) = \sum_{i=a_m+1}^{r} \binom{j+\lambda_i-i+a_m}{j-i+a_m+1},$$

so we deduce that

$$\begin{aligned} h_{R/I \cap J}(j) &= \sum_{i=1}^{a_m} \binom{j+\lambda_i-i}{j-i+1} + \sum_{i=a_m+1}^{r} \binom{j+\lambda_i-i}{j-i+1} \\ &= \sum_{i=1}^{r} \binom{j+\lambda_i-i}{j-i+1}, \end{aligned}$$

which by Lemma 2.3 (c) completes the induction.

Lastly, we establish that the given set of monomials generate $L(\lambda)$. The intersection of monomial ideals is generated by the least common multiples of their monomial generators, so we observe

$$\begin{aligned} x_0^{a_m} x_1^{a_{m-1}} \cdots x_{m-i-1}^{a_{i+1}} x_{m-i}^{a_i+1} &= \operatorname{lcm}\big(x_{m-i}^{a_i+1}, x_{m-i}^{a_i+1}, \dots, x_{m-i}^{a_i+1}, x_{m-i}^{a_i}, x_{m-i-1}^{a_{i+1}}, \dots, x_0^{a_m}, \\ x_0^{a_m} x_1^{a_{m-1}} \cdots x_{m-2}^{a_2} x_{m-1}^{a_1} &= \operatorname{lcm}\big(x_{m-1}^{a_1}, x_{m-2}^{a_2}, \dots, x_0^{a_m}\big) \end{aligned}$$

for all $2 \leqslant i \leqslant m$. Hence, each of the given monomials is a least common multiple of a generator from the irreducible components. Since each variable x_i appears as a minimal generator in an irreducible component with exponent either a_{m-i} or $a_{m-1}+1$, we see that the least common multiple of any subset of generators for the irreducible components is divisible by at least one of the given monomials, so the opposite inclusion also holds. □

We complete our converse to Lemma 2.1 by relating lexicographic ideals to residual flags. This geometric interpretation for the lexicographic ideal $L(\lambda)$ appears to be new.

Corollary 2.6. *Let (n, d) be the type of a residual flag in $\mathbb{P}^m$. For all $1 \leqslant i \leqslant e$, let X_i be the closed subscheme in $\mathbb{P}^m$ defined by the lexicographic ideal $L(n_i^{d_i}, n_{i+1}^{d_{i+1}}, \dots, n_e^{d_e})$. The chain $\varnothing \subset X_e \subset X_{e-1} \subset \cdots \subset X_1$ of closed immersions forms a residual flag of type (n, d) in $\mathbb{P}^m$.*

Proof. For all i with $1 \leqslant i \leqslant e$, the irreducible decomposition in Proposition 2.5 gives $X_{i+1} \subset X_i$ and the monomial generators in Proposition 2.5 establish that the closed subscheme X_i is contained in the n_i-plane in $\mathbb{P}^m$ defined by the monomial ideal $\langle x_0, x_1, \dots, x_{m-n_i-1}\rangle$. For all positive integers j, let a_j be the number of parts in the partition $(n_1^{d_1}, n_2^{d_2}, \dots, n_e^{d_e})$ equal to j. Restricting to the linear subspace

$$\mathbb{P}^{n_i} := \operatorname{Proj}(k[x_{m-n_i}, x_{m-n_i+1}, \dots, x_m])$$

in $\mathbb{P}^m$, Proposition 2.5 also shows that the closed subscheme X_i is defined by the monomial ideal

$$\begin{aligned} I_i := \big\langle x_{m-n_i}^{a_{n_i}+1},\ & x_{m-n_i}^{a_{n_i}} x_{m-n_i-1}^{a_{n_i+1}+1}, \dots, x_{m-n_i}^{a_{n_i}} x_{m-n_i-1}^{a_{n_i+1}+1} \cdots x_{m-3}^{a_3} x_{m-2}^{a_2+1}, \\ & x_{m-n_i}^{a_{n_i}} x_{m-n_i-1}^{a_{n_i+1}+1} \cdots x_{m-2}^{a_2} x_{m-1}^{a_1}\big\rangle. \end{aligned}$$

It follows that $I_i = x_{m-n_i}^{a_{n_i}} \cdot J$ where

$$\begin{aligned} J &:= \big\langle x_{m-n_i}, x_{m-n_i-1}^{a_{n_i+1}+1}, \dots, x_{m-n_i-1}^{a_{n_i+1}+1} \cdots x_{m-3}^{a_3} x_{m-2}^{a_2+1}, x_{m-n_i-1}^{a_{n_i+1}+1} \cdots x_{m-2}^{a_2} x_{m-1}^{a_1}\big\rangle \\ &= \big\langle x_{n-n_i}, x_{n-n_i-1}, \dots, x_{n-n_{i+1}-1}\big\rangle + I_{i+1}. \end{aligned}$$

As $a_{n_i} = d_i$, the closed immersion $X_{i+1} \subset X_i$ is a d_i-residual inclusion in $\mathbb{P}^{n_i}$. Thus, the chain $\varnothing \subset X_e \subset X_{e-1} \subset \cdots \subset X_1$ is a residual flag of type (n, d) in $\mathbb{P}^m$. □

Remark 2.7. By definition, the lexicographic point in a Hilbert scheme corresponds to the closed subscheme in $\mathbb{P}^m$ defined by the lexicographic ideal. Note that [25, Theorem 1.4] proves that the lexicographic point is smooth, so this point lies on a unique irreducible component called the lexicographic component; and [25, Theorem 4.1] computes the dimension of this component. When $(n, d) := (n_1, d_1), (n_2, d_2), \dots, (n_e, d_e)$ is the type of a residual flag,

$n_0 := m$, and λ is its associated integer partition, the lexicographic component determined by $L(\lambda)$ in the polynomial ring $R := k[x_0, x_1, \ldots, x_m]$ is

$$\begin{cases} \displaystyle\sum_{i=1}^{e} N_i & \text{if } n_e > 1 \text{ and } d_e > 1, \\ \displaystyle -(n_{e-1} - n_e) + \sum_{i=1}^{e} N_i & \text{if } n_e > 1 \text{ and } d_e = 1, \\ \displaystyle n_0\, d_e + \sum_{i=1}^{e-1} N_i & \text{if } n_e = 1 \text{ and } d_{e-1} > 1, \\ \displaystyle n_0\, d_e - (n_{e-2} - n_{e-1}) + \sum_{i=1}^{e-1} N_i & \text{if } n_e = 1 \text{ and } d_{e-1} = 1, \\ n_0\, d_e & \text{if } n_e = 1 \text{ and } e = 1, \end{cases} \tag{2.2}$$

where

$$N_i := \binom{n_i + d_i}{d_i} - 1 + (n_i + 1)(n_{i-1} - n_i).$$

Question 2.8. The saturated monomial ideals defining residual flags of type (n, d) in $\mathbb{P}^m$ determine the torus-fixed points in the parameter space $\operatorname{Flag}(n, d, \mathbb{P}(E))$. Following Proposition 2.5, the irredundant irreducible decomposition for each such monomial ideal has a combinatorial description. Can one use this perspective to count these monomial ideals and, thereby, compute the Euler characteristic of the projective scheme $\operatorname{Flag}(n, d, \mathbb{P}(E))$?

3. Geometry of smooth Hilbert schemes

In this section, we identify the Hilbert schemes isomorphic to a parameter space of residual flags. Exploiting this identification, we describe the closed subscheme corresponding to a general point on any smooth Hilbert scheme. Thus, we obtain a birational description of all smooth Hilbert schemes.

Geometry. Let E be a locally free sheaf on a locally noetherian scheme S. The projective bundle $\mathbb{P}(E)$ carries a tautological invertible sheaf that is relatively ample over S. We compute the Hilbert polynomials for closed subschemes in $\mathbb{P}(E)$ relative to this tautological bundle. For any numerical polynomial p, the set of T-valued points of the functor $\operatorname{Hilb}^p(\mathbb{P}(E))$ is the set of closed subschemes $X \subseteq \mathbb{P}(E_T)$ that are flat over the S-scheme T and have Hilbert polynomial p. The S-scheme representing this functor is projective; see [16, Theorem 1.4].

Lemma 3.1. *Let E be locally free sheaf on S of constant rank $m + 1$ and fix a polynomial p in $\mathbb{Q}[t]$. The Hilbert scheme $\operatorname{Hilb}^p(\mathbb{P}(E))$ is smooth over S if and only if the fibre $\operatorname{Hilb}^p(\mathbb{P}^m)$ is nonsingular over every geometric point of S.*

Proof. Let $X \subseteq \mathbb{P}^m$ be a closed subscheme in the fibre of $\mathbb{P}(E)$ over a geometric point in S. When X corresponds to a smooth point on $\operatorname{Hilb}^p(\mathbb{P}^m)$, [16, Theorem 2.10] proves that the structure map $\mathrm{Hilb}^p(\mathbb{P}(E)) \to S$ is flat at this geometric point. Therefore, this structure map is smooth if and only if its the fibre is nonsingular over every geometric point. $\square$

Theorem 3.2. *Let (n, d) be the type of a residual flag and let $\lambda := (\lambda_1, \lambda_2, \dots, \lambda_r)$ be its associated integer partition. Set $p(t) = \sum_{i=1}^{r} \binom{t+\lambda_i - i}{\lambda_i - 1}$. Assume that E is a locally free sheaf on X of constant rank $m+1$. The natural morphism*

$$\pi\colon \mathrm{Flag}(n, d, \mathbb{P}(E)) \to \mathrm{Hilb}^p(\mathbb{P}(E))$$

sending a residual flag $\varnothing \subset X_e \subset X_{e-1} \subset \cdots \subset X_1$ to the closed subscheme $X_1 \subset \mathbb{P}(E)$, is an isomorphism if and only if one of the two conditions holds:

(2) *$m \geqslant \lambda_1$ and $\lambda_r \geqslant 2$,*

(3) *$\lambda = (1)$ or $\lambda = (m^{r-2}, \lambda_{r-1}, 1)$, where $r \geqslant 2$ and $m \geqslant \lambda_{r-1} \geqslant 1$.*

In both cases, the Hilbert scheme $\mathrm{Hilb}^p(\mathbb{P}(E))$ is smooth over S.

Proof. Suppose that $S = \mathrm{Spec}(k)$ for some algebraically closed field k. We note that [27, Theorem 1.1] shows that Conditions (2) and (3) characterize when a nontrivial Hilbert scheme $\mathrm{Hilb}^p(\mathbb{P}^m)$ has a unique Borel-fixed point. Moreover, [27, Lemma 5.6] proves that the target $\mathrm{Hilb}^p(\mathbb{P}^m)$ is nonsingular and irreducible in this situation. Proposition 1.11 and Corollary 1.12 show that the source $\mathrm{Flag}(\kappa, d, \mathbb{P}(E))$ is a smooth projective variety. Since π is injective, it is enough to certify that the dimensions of the source and target agree. Using Corollary 1.12 and equation (2.2), one verifies that the dimension of the lexicographic component in $\mathrm{Hilb}^p(\mathbb{P}^m)$ equals the dimension of the parameter space of residual flags if and only if Conditions (2) or (3) holds.

Suppose that S is any locally noetherian scheme. Lemma 3.1 implies that the Hilbert scheme is smooth. Hence, the source and target of the morphism π are smooth. Since the induced morphism on fibres over any geometric point is an isomorphism, the result follows. $\square$

Example 3.3. The conditions in Theorem 3.2 cover the well-known cases of hypersurfaces and Grassmannians. Consider a partition $\lambda = (\lambda_1^r)$ and set

$$p(t) := \sum_{i=1}^{r} \binom{t + \lambda_1 - i}{\lambda_1 - 1}.$$

When $\lambda_1 = m$, each point in $\mathrm{Hilb}^p(\mathbb{P}(E))$ corresponds to a hypersurface of degree r in $\mathbb{P}(E)$; see Lemma 1.8. More generally, each point in $\mathrm{Hilb}^p(\mathbb{P}(E))$ corresponds to a hypersurface of degree r lying some λ_1-dimensional linear subspace of $\mathbb{P}(E)$. In the special case $r = 1$, the Hilbert scheme $\mathrm{Hilb}^p(\mathbb{P}(E))$ is the Grassmannian parametrizing $(\lambda_1 - 1)$-dimensional linear subspaces in $\mathbb{P}(E)$. $\diamond$

Example 3.4. For the integer partition $\lambda = (1^2)$, Theorem 3.2 shows that each point in the Hilbert scheme $\mathrm{Hilb}^2(\mathbb{P}(E))$ correspond to a hypersurface of degree 2 lying on some line in $\mathbb{P}(E)$. Alternatively, the Hilbert scheme of two points in $\mathbb{P}(E)$ is also known to be the blow-up of the diagonal of the quotient scheme $\mathbb{P}(E) \times_S \mathbb{P}(E)/\mathfrak{S}_2$, where the symmetric group $\mathfrak{S}_2$ on two elements acts by permuting the factors in the product $\mathbb{P}(E) \times_S \mathbb{P}(E)$. $\diamond$

Remark 3.5. The geometry of residual flags also explains why the morphism π cannot be surjective when Conditions (2) and (3) fail to holds. To avoid these conditions, we may assume that $\lambda_r = 1$, $r \geqslant 3$, and $m > \lambda_{r-2}$. The smallest scheme X_e in the residual flag is a degree d_e hypersurface in a line. Hence, the map π cannot be surjective when $d_e \geqslant 3$. When $d_e \leqslant 2$, there exists a line Λ_e containing X_e. The defining properties of a residual flag require the line Λ_e to be contained in the latent plane Λ_{e-1} which by assumption has dimension less than m. Since a general line is not contained such a plane, the map π also not surjective in this case.

Remark 3.6. The strategy outlined in Question 1.13 also leads to a description of the Chow ring (and integral cohomology ring) of the Hilbert schemes classified in Theorem 3.2.

Example 3.7. Two trivial Hilbert schemes, not covered by Theorem 3.2, are nevertheless particular Grassmannians. When $\lambda = (m + 1)$ and

$$p(t) = \binom{t+m}{m},$$

the Hilbert scheme $\mathrm{Hilb}^p(\mathbb{P}(E))$ is a one point corresponding to closed subscheme $\mathbb{P}(E)$ itself. When $r = 0$, the Hilbert scheme $\mathrm{Hilb}^0(\mathbb{P}(E))$ is a one point corresponding to empty scheme in $\mathbb{P}(E)$. $\diamond$

Birationality. Before examining the birational geometry of the other smooth Hilbert schemes, we remember that some Hilbert schemes factor into a product of Hilbert schemes. Let $\lambda = (m^s, \lambda_{s+1}, \lambda_{s+2}, \dots, \lambda_r)$ be a partition with $m > \lambda_{s+1}$ and set

$$p(t) := \sum_{i=1}^{r} \binom{t+\lambda_i - i}{\lambda_i - 1}.$$

We see that $p(t) = q_1(t) + q_2(t - s)$, where

$$q_1(t) := \sum_{i=1}^{s} \binom{t+m-i}{m-1}$$

is the Hilbert polynomial for a hypersurface of degree s in $\mathbb{P}(E)$ and

$$q_2(t) := \sum_{i=1}^{r-s} \binom{t+\lambda_{i+s} - i}{\lambda_{i+s} - 1}.$$

It follows from Lemma 1.1 that the Hilbert scheme parametrizing of these hypersurfaces is $\mathbb{P}(\mathrm{Sym}^s(E^*))$ and [6, p. 514, Remark 2] yields there is a natural splitting

$$\mathrm{Hilb}^p(\mathbb{P}(E)) \cong \mathbb{P}(\mathrm{Sym}^s(E^*)) \times_S \mathrm{Hilb}^{q_2}(\mathbb{P}(E)). \tag{3.1}$$

Given an integer partition $\lambda := (\lambda_1, \lambda_2, \dots, \lambda_r)$, the new integer partition $\lambda \cup (1)$ is defined to be $(\lambda_1, \lambda_2, \dots, \lambda_r, 1)$ and has length $r + 1$; see [19, Section 1.1].

Proposition 3.8. *Let (n, d) be a residual type and let $\lambda := (\lambda_1, \lambda_2, \dots, \lambda_r)$ be its integer partition. Set*

$$p(t) := \sum_{i=1}^{r} \binom{t+\lambda_i - i}{\lambda_i - 1}.$$

Assume that one of the following conditions holds:

(4) $\lambda \cup (1) = (m^{r-s-3}, \lambda_{r-s-2}^{s+2}, 1)$, *where* $r-3 \geqslant s \geqslant 0$, *and* $m-1 \geqslant \lambda_{r-s-2} \geqslant 3$, *or*

(5) $\lambda \cup (1) = (m^{r-s-5}, 2^{s+4}, 1)$, *where* $r-5 \geqslant s \geqslant 0$.

The Hilbert scheme $\operatorname{Hilb}^{p+1}(\mathbb{P}(E))$ *is smooth over* S*. Moreover, a general point on this Hilbert scheme corresponds to the disjoint union of a residual flag of type* (n,d) *and a point.*

Proof. From the splitting in (3.1), it is enough to consider an integer partition $\lambda = (\lambda_1^r)$ where $r \geqslant 1$ and $\lambda_1 \geqslant 2$, excluding the two integer partitions (2^2) and (2^3). By Lemma 3.1, we many assume that the base scheme is the spectrum of an algebraically closed field. Over a field of characteristic zero, [24, Theorem A] classifies all Hilbert schemes with precisely two Borel-fixed points; [24, Theorem B] and [28, Theorem 1.1] also describe this classification over any algebraically closed field. Conditions (4) and (5) guarantee that the Hilbert scheme $\operatorname{Hilb}^{p+1}(\mathbb{P}^n)$ has two Borel-fixed points. By computing the dimension of the tangent space at the non-lexicographic Borel-fixed point, [24, Theorem A] demonstrates that $\operatorname{Hilb}^{p+1}(\mathbb{P}^n)$ is nonsingular.

To understand a general point, consider the universal flag X_1 of type (λ_1, s) in $\mathbb{P}(E)$ and the universal closed subscheme Z having length one on $\mathbb{P}(E) = \operatorname{Hilb}^1(\mathbb{P}(E))$. As the structure map is proper, their intersection determines a closed subset in $\operatorname{Flag}(\lambda_1, s, \mathbb{P}(E)) \times_S \mathbb{P}(E)$. Let U denote the open complement. There is a morphism $\psi \colon U \to \operatorname{Hilb}^{p+1}(\mathbb{P}(E))$ induced by sending the pair (X_1, Z) to their disjoint union. The source and target of ψ are smooth S-schemes and, using Corollary 1.12 and equation (2.2), one verifies that they have the same relative dimension. Over each geometric point in S, the induced morphism on the fibres is an open immersion. We conclude that

$$\psi \colon \operatorname{Flag}(\lambda_1, s, \mathbb{P}(E)) \times_S \mathbb{P}(E) \dashrightarrow \operatorname{Hilb}^{p+1}(\mathbb{P}(E))$$

is a birational map. □

Remark 3.9. Under the hypothesis of Proposition 3.8, the scheme $\operatorname{Hilb}^{p+1}(\mathbb{P}(E))$ and the product $\operatorname{Flag}(m, d, \mathbb{P}(E)) \times_S \mathbb{P}(E)$ are birational. However, these schemes are not isomorphic. For instance, the existence of two different Borel-fixed points on the Hilbert scheme implies that there is more than one way to embedded a point of multiplicity 1 into the lexicographic ideal; see Proposition 2.5 and Proposition 4.3.

Example 3.10. The integer partition $\lambda = (1^r)$ is associated to the constant Hilbert polynomial r. The Hilbert scheme $\operatorname{Hilb}^r(\mathbb{P}(E))$ is known to be smooth in two cases: [6, Theorem 2.4] applies when $m = 2$ and [3, equation (0.2.1)] applies when $r \leqslant 3$. In either case, this Hilbert scheme is birational to the r-fold symmetric product $\mathbb{P}(E) \times_S \mathbb{P}(E) \times_S \cdots \times_S \mathbb{P}(E)/\mathfrak{S}_r$ where the symmetric group $\mathfrak{S}_r$ on r elements acts by permuting the factors in the product $\mathbb{P}(E) \times_S \mathbb{P}(E) \times_S \cdots \times_S \mathbb{P}(E)$.

Using the splitting (3.1), this analysis extends to the integer partition $\lambda = (m^{r-s}, 1^s)$ where $r \geqslant s \geqslant 0$. Set

$$p(t) := \sum_{i=1}^{r} \binom{t + \lambda_i - i}{\lambda_i - 1}.$$

Assuming $m = 2$ or $r \leqslant 3$, a general point on $\operatorname{Hilb}^p(\mathbb{P}(E))$ corresponds to the disjoint union of a hypersurface of degree $r - s$ and s isolated points. ◇

Remark 3.11. The seven conditions in Theorem A imply that $\mathrm{Hilb}^p(\mathbb{P}(E))$ is smooth over S: Example 3.10 handles Conditions (1) and (6), Theorem 3.2 handles Conditions (2) and (3), Proposition 3.8 handles Conditions (4) and (5), and Example 3.7 handles Condition (7). In particular, we have a birational description for all of these smooth Hilbert schemes.

4. General classification

The final section completes our classification of smooth Hilbert schemes. By identifying enough singular points, we prove that our list of smooth Hilbert schemes is exhaustive.

Nearly lexicographic points. We specify a novel point on a Hilbert scheme by perturbing a lexicographic ideal. Geometrically, this nearly lexicographic point corresponds to a residual flag with an embedded point whose nilpotent elements do not lie in the smallest linear subspace containing the residual flag.

Lemma 4.1. *Let $\lambda := (\lambda_1, \lambda_2, \ldots, \lambda_r)$ be an integer partition and set*

$$p(t) := \sum_{i=1}^{r} \binom{t + \lambda_i - i}{\lambda_i - 1}.$$

Fix $m > \lambda_1$ and consider both the lexicographic ideal $L(\lambda)$ and the monomial ideal

$$J := \langle x_0,\, x_1, \ldots, x_{n-\lambda_1-2},\, x_{m-\lambda_1-1}^2,\, x_{m-\lambda_1},\, x_{n-\lambda_1+1}, \ldots, x_{n-1} \rangle$$

in the polynomial ring $R = k[x_0, x_1, \ldots, x_m]$. The closed subscheme in $\mathbb{P}^m$ defined by the homogeneous ideal $K := L(\lambda) \cap J$ has Hilbert polynomial $p + 1$ and corresponds to a point on the lexicographic component of the Hilbert scheme $\mathrm{Hilb}^{p+1}(\mathbb{P}^m)$.

Proof. By Lemma 2.3, the Hilbert polynomial of the closed subscheme defined by the lexicographic ideal $L(\lambda)$ is

$$p(t) = \sum_{i=1}^{r} \binom{t + \lambda_i - i}{\lambda_i - 1}.$$

Proposition 2.5 implies that

$$L(\lambda) + J = \langle x_0, x_1, \ldots, x_{m-1} \rangle.$$

For all integers j greater than 1, the sets $\{x_{m-\lambda_1-1} x_m^{j-1}, x_n^j\}$ and $\{x_m^j\}$ form bases for the j-th homogeneous components of R/J and $R/(L(\lambda) + J)$ respectively. It follows that their Hilbert polynomials are the constants 2 and 1. From the short exact sequence of graded R-modules

$$0 \to \frac{R}{L(\lambda) \cap J} \to \frac{R}{L(\lambda)} \oplus \frac{R}{J} \to \frac{R}{L(\lambda) + J} \to 0,$$

we deduce that $p + 1$ is the Hilbert polynomial of the closed subscheme in $\mathbb{P}^m$ corresponding to the monomial ideal $K = L(\lambda) \cap J$.

Using Proposition 2.5, we also deduce that the saturation $(L(\lambda) : x_{m-1}^{\infty})$ is equal to the saturation $(K : x_{m-1}^{\infty})$. Since both $L(\lambda)$ and K are Borel-fixed ideals, [26, Theorem 6] shows that the point corresponding to the ideal K lies on the lexicographic component. □

Tangent spaces. We demonstrate that these nearly lexicographic points are singular for a special class of integer partitions. The Zariski tangent space at the point in the Hilbert scheme $\mathrm{Hilb}^{p}(\mathbb{P}^{m})$ corresponding to the closed subscheme X in $\mathbb{P}^{m}$ with ideal sheaf $\mathcal{I}_X$ is naturally isomorphic to $\mathrm{Hom}_{\mathbb{P}^m}(\mathcal{I}_X, \mathcal{O}_X) = \mathrm{Hom}_X(\mathcal{I}_X/\mathcal{I}_X^2, \mathcal{O}_X)$; see [16, Theorem 2.8].

Lemma 4.2. *Suppose that $\lambda := ((m-1)^{r-s-1}, (m-n)^{s+1})$ is an integer partition, where $r-2 \geqslant s \geqslant 0$ and $m-1 \geqslant n \geqslant 2$. Set*

$$p(t) := \sum_{i=1}^{r} \binom{t+\lambda_i - i}{\lambda_i - 1}.$$

The Hilbert scheme $\mathrm{Hilb}^{p+1}(\mathbb{P}^{m})$ is singular at the point corresponding to the saturated monomial ideal

$$\begin{aligned} K &:= x_0 \cdot \langle x_0, x_1, \dots, x_{m-1} \rangle + x_1^{r-s-1} \cdot \langle x_1, x_2, \dots, x_{n-1}, x_n^{s+1} \rangle \\ &= L(\lambda) \cap \langle x_0^2,\ x_1, x_2, \dots, x_{m-1} \rangle \end{aligned}$$

in the polynomial ring $R = k[x_0, x_1, \dots, x_m]$.

Proof. From the monomial generators for the lexicographic ideal $L(\lambda)$ appearing in Proposition 2.5, we see that the given monomials generate the ideal K. It remains to show that the dimension of the Zariski tangent space at the nearly lexicographic point is larger than the dimension of the Zariski tangent space at the lexicographic point. Equation (2.2) establishes that the dimension of later is less than or equal to (with equality holding when $s > 0$)

$$\binom{m+r-s-2}{r-s-1} + \binom{m-n+s+1}{s+1} + (m-n+1)(n-1) + 2m - 2\,. \tag{4.1}$$

To estimate the dimension of the Zariski tangent space at the nearly lexicographic point, we examine the sheaf on $\mathbb{P}^m$ corresponding to the graded R-module $\mathrm{Hom}_R(K, R/K)$. Since the variable x_m does not divide any of the generators of the ideal K, the dimension of this tangent space is greater than or equal to $\dim_k \mathrm{Hom}_R(K, R/K)_0$; see [25, Lemma 3.1]. Because K is a stable monomial ideal, the Eliahou–Kervaire resolution [22, Theorem 2.3] yields a homogeneous free presentation. The minimal syzygies among the generators of the ideal K are given by the block matrix

$$\Theta := \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 & \cdots & \mathbf{A}_{m-1} & \mathbf{B}_1 & \mathbf{B}_2 & \cdots & \mathbf{B}_{n-1} & \mathbf{C} \end{bmatrix},$$

where, for all $1 \leqslant i \leqslant m-1$ and all $1 \leqslant j \leqslant n-1$, we have

$$\mathbf{A}_i^{\mathsf{T}} := \begin{array}{cc} \begin{array}{cccccccccccccc} x_0^2 & x_0x_1 & \cdots & x_0x_{i-2} & x_0x_{i-1} & x_0x_i & x_0x_{i+1} & \cdots & x_0x_{m-1} & x_1^{r-s} & x_1^{r-s-1}x_1 & \cdots & x_1^{r-s-1}x_{n-1} & x_1^{r-s-1}x_n^{s+1} \end{array} & \\ \left[\begin{array}{cccccccccccccc} 0 & 0 & \cdots & 0 & -x_i & x_{i-1} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & -x_{i+1} & 0 & x_{i-1} & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -x_{m-1} & 0 & 0 & \cdots & x_{i-1} & 0 & 0 & \cdots & 0 & 0 \end{array}\right] & \begin{array}{c} i \\ i+1 \\ \vdots \\ m-1 \end{array} \end{array},$$

$$\mathbf{B}_j^{\mathsf{T}} := \begin{array}{cc} \begin{array}{ccccccccccccc} x_0^2 & x_0x_1 & \cdots & x_0x_{m-1} & x_1^{r-s} & \cdots & x_1^{r-s-1}x_{j-2} & x_1^{r-s-1}x_{j-1} & x_1^{r-s-1}x_j & x_1^{r-s-1}x_{j+1} & \cdots & x_1^{r-s-1}x_{n-1} & x_1^{r-s-1}x_n^{s+1} \end{array} & \\ \left[\begin{array}{ccccccccccccc} 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & -x_j & x_{j-1} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & -x_{j+1} & 0 & x_{j-1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & -x_{n-1} & 0 & 0 & \cdots & x_{j-1} & 0 \end{array}\right] & \begin{array}{c} j \\ j+1 \\ \vdots \\ n-1 \end{array} \end{array},$$

and

$$\mathbf{C}^{\mathsf{T}} := \begin{array}{cc} \begin{array}{cccccccccccccc} x_0^2 & x_0x_1 & \cdots & x_0x_{n-1} & x_0x_n & x_0x_{n+1} & \cdots & x_0x_{m-1} & x_1^{r-s} & x_1^{r-s-1}x_2 & \cdots & x_1^{r-s-1}x_{n-1} & x_1^{r-s-1}x_n^{s+1} \end{array} & \\ \begin{bmatrix} 0 & 0 & \cdots & 0 & -x_1^{r-x-1}x_n^s & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & x_0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & -x_n^{s+1} & 0 & \cdots & 0 & x_1 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & -x_n^{s+1} & \cdots & 0 & x_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & -x_n^{s+1} & x_{n-1} \end{bmatrix} & \begin{array}{c} 0 \\ 1 \\ 2 \\ \vdots \\ m-1 \end{array} \end{array}.$$

It follows that

$$\operatorname{Hom}_R(K, R/K) = \operatorname{Ker}(\operatorname{Hom}_R(\Theta, R/K)).$$

The two $(m+n) \times 1$-matrices defined by

$$\mathbf{D}_0^{\mathsf{T}} := \begin{array}{cc} \begin{array}{cccccccccc} x_0^2 & x_0x_1 & x_0x_1 & \cdots & x_0x_{m-1} & x_1^{r-s-1} & x_1^{r-s-1}x_2 & \cdots & x_1^{r-s-1}x_{n-1} & x_1^{r-s-1}x_n^{s+1} \end{array} & \\ \begin{bmatrix} x_0 & x_1 & x_2 & \cdots & x_n & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} & 0 \end{array}$$

and

$$\mathbf{D}_1^{\mathsf{T}} := \begin{array}{cc} \begin{array}{cccccccccc} x_0^2 & x_0x_1 & x_0x_1 & \cdots & x_0x_{m-1} & x_1^{r-s-1} & x_1^{r-s-1}x_2 & \cdots & x_1^{r-s-1}x_{n-1} & x_1^{r-s-1}x_n^{s+1} \end{array} & \\ \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & x_1 & x_2 & \cdots & x_{n-1} & x_n^{s+1} \end{bmatrix} & 1 \end{array}$$

satisfy

$$\Theta^{\mathsf{T}}\, \mathbf{D}_0 = \mathbf{0} \quad \text{and} \quad \Theta^{\mathsf{T}}\, \mathbf{D}_1 = \mathbf{0}.$$

Thus, for all $2 \leqslant i \leqslant m$, the column in the product $x_i\, \mathbf{D}_0$ represents a nonzero element in $\operatorname{Hom}_R(K, R/K)_0$. The column in the product of the matrix $\mathbf{D}_1$ with any monomial of degree $r-s-1$ in the variables $x_1, x_2, \ldots, x_m$ (excluding x_1^{r-s-1}) also represents a nonzero element in $\operatorname{Hom}_R(K, R/K)_0$. There are

$$(m-1) + \binom{m+r-s-2}{r-s-1} - 1$$

columns of these products. Since all entries in the product $x_0\,\Theta$ lie in the ideal K, the $m+n$ columns of the square matrix

$$\begin{array}{cc} \begin{array}{ccccccccc} 1 & 2 & \cdots & m & 1 & 2 & \cdots & n-1 & n \end{array} & \\ \begin{bmatrix} x_0x_m & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & x_0x_m & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & x_0x_m & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & x_0x_n^{r-s-1} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & x_0x_n^{r-s-1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & x_0x_n^{r-s-1} & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & x_0x_m^{r-1} \end{bmatrix} & \begin{array}{c} x_0^2 \\ x_0x_1 \\ \vdots \\ x_0x_{m-1} \\ x_1^{r-s} \\ x_1^{r-s-1}x_2 \\ \vdots \\ x_1^{r-s-1}x_{n-1} \\ x_1^{r-s-1}x_n^{s+1} \end{array} \end{array}$$

represent nonzero elements in $\operatorname{Hom}_R(K, R/K)_0$. Similarly, all of entries in the bottom n rows of the matrix Θ when multiplied by the monomial x_1^{r-s-1} lie in the ideal K. Hence, for all

$n \leqslant j \leqslant m$, the columns of the matrices

$$\begin{array}{c} \begin{array}{cccc} 1 & 2 & \cdots & n-1 \end{array} \\ \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ x_1^{r-s-1}x_j & 0 & \cdots & 0 \\ 0 & x_1^{r-s-1}x_j & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_1^{r-s-1}x_j \\ 0 & 0 & \cdots & 0 \end{bmatrix} \end{array}$$

and

$$\begin{array}{cc} \begin{array}{cccc} 1 & 2 & \cdots & \binom{m-n+s+1}{s+1}-1 \end{array} & \\ \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ x_1^{r-s-1}x_n^s x_{n+1} & x_1^{r-s-1}x_n^s x_{n+2} & \cdots & x_1^{r-s-1}x_m^{s+1} \end{bmatrix} & \begin{array}{c} x_0^2 \\ x_0x_1 \\ \vdots \\ x_0x_{m-1} \\ x_1^{r-s} \\ x_1^{r-s-1}x_2 \\ \vdots \\ x_1^{r-s-1}x_{n-1} \\ x_1^{r-s-1}x_n^{s+1} \end{array} \end{array}$$

represent nonzero elements in $\operatorname{Hom}_R(K, R/K)_0$. Each nonzero entry in the bottom row of the second matrix is the product of x_1^{r-s-1} and a monomial of degree $s+1$ in the variables $x_n, x_{n+1}, \ldots, x_m$ (excluding x_n^{s+1}). Hence, there are $\binom{m-n+s+1}{s+1} - 1$ columns in this matrix.

The number of distinct columns representing nonzero elements in $\operatorname{Hom}_R(K, R/K)_0$ is

$$N := (m-1) + \binom{m+r-s-2}{r-s-1} - 1 + (m+n) \\ + (m-n+1)(n-1) + \binom{m-n+s+1}{s+1} - 1.$$

By comparing their nonzero entries, we see that these N columns are linearly independent. The difference between the number N and equation (4.1) is $n-1$. As $n \geqslant 2$, we conclude that the Hilbert scheme is singular at the point corresponding to the monomial ideal K. □

Proposition 4.3. *Let $\lambda := (\lambda_1, \lambda_2, \ldots, \lambda_r)$ be an integer partition such that $\lambda \cup (1)$ has at least three distinct parts or $\lambda := (\lambda_1^{r-s-1}, 1^{s+1})$, where $r-2 \geqslant s \geqslant 0$ and $\lambda_1 > 1$. Fix $m > \lambda_1$, set*

$$p(t) := \sum_{i=1}^{r} \binom{t+\lambda_i - i}{\lambda_i - 1},$$

and consider the monomial ideal

$$J := \langle x_0,\, x_1, \ldots, x_{m-\lambda_1-2},\, x_{m-\lambda_1-1}^2,\, x_{m-\lambda_1},\, x_{m-\lambda_1+1}, \ldots, x_{m-1} \rangle$$

in the polynomial ring $R = k[x_0, x_1, \ldots, x_m]$*. The Hilbert scheme* $\mathrm{Hilb}^{p+1}(\mathbb{P}^m)$ *is singular at the point corresponding to the saturated monomial ideal* $K := L(\lambda) \cap J$*.*

Proof. Lemma 4.1 shows that the nearly lexicographic point lies on the lexicographic component of the Hilbert scheme $\mathrm{Hilb}^{p+1}(\mathbb{P}^m)$. We reduce the analysis to a special case.

The inclusion $\langle x_0, x_1, \ldots, x_{m-\lambda_1-2} \rangle \subset K$ implies that the nearly lexicographic point is contained in a $(\lambda_1 + 1)$-plane. Hence, we may assume that $m = \lambda_1 + 1$.

Assume that $\lambda \cup (1)$ has at least three distinct parts. The lexicographic ideal $L(\lambda \cup (1))$ is the flat limit of a one-parameter family whose general member is the sum of the lexicographic ideal $L(\lambda)$ and the ideal of a disjoint point. Since the dimension of the Zariski tangent space at a point in family is an upper-semicontinuous function, we may also assume that $\lambda_r > 1$. Applying Corollary 2.6, the lexicographic ideal $L(\lambda \cup (1))$ determines a residual flag $\varnothing \subset X_e \subset X_{e-1} \subset \cdots \subset X_1$ in $\mathbb{P}^m$. The hypotheses ensure that $e \geqslant 3$. The closed subscheme X_{e-2} lies in some linear space $\Lambda \subseteq \mathbb{P}^m$. We may deform the scheme X_{e-2} in the linear space Λ and leave the rest of the residual flag $X_{e-3} \subset X_{e-4} \subset \cdots \subset X_1$ unchanged. If the closed scheme X_{e-2} corresponds to a singular point on the Hilbert scheme in Λ, then it follows that the closed scheme X_1 corresponds to a singular point on the Hilbert scheme. Thus, we may assume that $e = 3$.

With these reductions, it suffices to consider $\lambda = ((m-1)^{r-s-1}, (m-n)^{s+1})$, where $r - 2 \geqslant s \geqslant 0$ and $m - 1 \geqslant n \geqslant 2$. In this special case, Lemma 4.2 proves that the dimension of the Zariski tangent space at the nearly lexicographic point exceeds the dimension of the lexicographic component. Therefore, the Hilbert scheme $\mathrm{Hilb}^{p+1}(\mathbb{P}^m)$ is singular at the point corresponding to the saturated monomial ideal $K := L(\lambda) \cap J$. □

Other singular examples. In addition to our family of singular Hilbert schemes, the classification of smooth Hilbert schemes relies on three other singular families.

Example 4.4. Two familiar Hilbert schemes explain the curious gap between Conditions (4) and (5) in Proposition 3.8. By appealing to the splitting in (3.1), it is enough to understand integer partitions $(2, 1)$, $(2^2, 1)$ and $(2^3, 1)$. The first of these is already covered by both Theorem 3.2 and Example 3.10. In contrast, the Hilbert schemes in the other two cases are singular.

The integer partition $(2^2, 1)$ is associated to the Hilbert polynomial $2t + 2$. The Hilbert scheme $\mathrm{Hilb}^{2t+2}(\mathbb{P}^m)$ is singular; it has two irreducible components. A general point on one component corresponds to a pair of skew lines and a general point on the other corresponds to the union of a plane conic and an isolated point; compare with [4, Theorem 1.1].

The integer partition $(2^3, 1)$ is associated to the Hilbert polynomial $3t + 1$. The Hilbert scheme $\mathrm{Hilb}^{3t+1}(\mathbb{P}^m)$ is again singular because it has two irreducible components. A general point in first component corresponds to a twisted cubic curve and a general point in the other corresponds to the union of a plane cubic and an isolated points; see [23, Theorem]. ◇

Example 4.5. For completeness, we give an explicit description of another well-known singular Hilbert scheme. For any nonnegative integer s, consider the partition $\lambda = (1^{s+4})$

whose associated Hilbert polynomial is the constant $s+4$. For all $m \geqslant 3$, the Hilbert scheme $\mathrm{Hilb}^{s+4}(\mathbb{P}^m)$ is singular at the point corresponding to the saturated homogeneous ideal

$$B(s) := \langle x_0, x_1, \ldots, x_{m-4}, \; x_{m-3}^2, x_{m-3}\,x_{m-2}, x_{m-3}\,x_{m-1}, x_{m-2}^2, \; x_{m-2}\,x_{m-1}, x_{m-1}^{s+2} \rangle$$

in the polynomial ring $R = k[x_0, x_1, \ldots, x_m]$; compare with [3, Lemma 1.4]. ◇

Theorem 4.6. *Let E be a locally free sheaf on a locally noetherian scheme S of constant rank $m+1$ and let p be a numerical polynomial. The Hilbert scheme $\mathrm{Hilb}^p(\mathbb{P}(E))$ is smooth and irreducible over S if and only if there exists an integer partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ such that*

$$p(t) = \sum_{i=1}^{r} \binom{t + \lambda_i - i}{\lambda_i - 1}$$

and one of the following seven conditions holds:

(1) $m = 2 \geqslant \lambda_1$,

(2) $m \geqslant \lambda_1$ *and* $\lambda_r \geqslant 2$,

(3) $\lambda = (1)$ *or* $\lambda = (m^{r-2}, \lambda_{r-1}, 1)$, *where* $r \geqslant 2$ *and* $m \geqslant \lambda_{r-1} \geqslant 1$,

(4) $\lambda = (m^{r-s-3}, \lambda_{r-s-2}^{s+2}, 1)$, *where* $r-3 \geqslant s \geqslant 0$ *and* $m-1 \geqslant \lambda_{r-s-2} \geqslant 3$,

(5) $\lambda = (m^{r-s-5}, 2^{s+4}, 1)$, *where* $r-5 \geqslant s \geqslant 0$,

(6) $\lambda = (m^{r-3}, 1^3)$, *where* $r \geqslant 3$,

(7) $\lambda = (m+1)$ *or* $r = 0$.

Proof. Remark 3.11 already shows that each condition implies that the Hilbert scheme is smooth. Hence, it suffices to prove that the Hilbert scheme has a singular point when the integer partition $\lambda := (\lambda_1, \lambda_2, \ldots, \lambda_r)$ does not satisfy Conditions (1)–(7). To bypass Conditions (1) and (2), we must have $m \geqslant 3$ and $\lambda_r = 1$. By Lemma 3.1, we may assume that S is the spectrum of an algebraically closed field. Using the splitting in (3.1), we may also assume that $m > \lambda_1$. For the remaining integer partitions with one distinct part, Example 4.5 describes a singularity. When the integer partition has two distinct parts, there are three outstanding cases, namely $\lambda = (2^2, 1)$, $\lambda = (2^3, 1)$, or $\lambda = (\lambda_1^{r-s-2}, 1^{s+2})$, where $r-1 \geqslant s \geqslant 0$ and $\lambda_1 > 1$. Example 4.4 and Proposition 4.3 exhibit their singularities. Finally, Proposition 4.3 also identifies a singular point whenever the integer partition has at least three distinct parts. □

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