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Modeling stochastic anomalies in an SIS system

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ABSTRACT
We propose a stochastic SIS model to include both a Gaussian and Poissonian perturbation to account for noise and anomalies in the transmission rate. Conditions are given for stability to the disease free equilibrium and for positive Harris recurrence with a unique invariant measure for the endemic.

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1. Introduction

For many infectious diseases, the rate at which susceptible individuals contract the infection undergoes large stochastic perturbations. For example, large-scale public gatherings at social events can result in a sudden influx of new infections beyond what would be expected through typical transmission routes. Likewise, large-scale social avoidance (as might occur if media reports of disease spread instill panic) can result in a sudden and temporary decease in the rate of infection. Such events are not well modeled by Brownian motion. Instead, events of this type are captured by a Poisson term. The epidemiological model analyzed in this paper is stochastically perturbed with an Itô and Poisson intergral, where the Poisson integral captures the effects of "anomalies," rare events that have an impact on the transmission rate.

Many authors have considered continuous time stochastic epidemiological models [1–5], however, none of these models have considered anomalies that affect the system. Kuske et al. [5] investigated an SIR model perturbed by multiple Itô integrals. Using a linear approximation, the authors give conditions for an epidemic to occur, where this event is mathematically described by the existence of a unique stationary measure, and this measure is able to be given explicitly. Roa et al. [1] consider a stochastic SIR model with a nonlinear incident rate, and after analyzing the deterministic model, they perturb the system with a two Itô integrals. The authors then show that these systems are well defined and give conditions for almost sure convergence to the endemic and the disease free equilibrium. Yu et al. [3] analyze a White noise two-group SIR model with a stationary deterministic equilibrium, and determine sufficient conditions for the endemic equilibrium to be stochastically stable. The models examined in

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[2, 5] are very close to the model considered in this article. This model is explored in detail below.

2. Well-posed model

We consider the following SIS model:

$$\begin{aligned} S'(t) &= -\beta S(t)I(t) - \mu S(t) + \mu + \lambda I(t) \\ I'(t) &= \beta S(t)I(t) - (\lambda + \mu)I(t), \end{aligned} \tag{1}$$

where, $S(t)$ and $I(t)$ denote the frequencies of the susceptible and infected at time t, and $S(t) + I(t) = 1$. The constant μ represents the birth and death rate (newborns are assumed to be susceptible), λ represents the recovery rate for the individuals that are infected, and we take β as the average number of contacts per day.

For the SIS model, if $\beta > \mu + \lambda$, then an epidemic will occur, and if $\beta \leq \mu + \lambda$ then the process will converge to the disease-free equilibrium. In other words, if $\beta \leq \mu + \lambda$ then the point $(1, 0)$ is globally asymptotically stable, and if $\beta > \mu + \lambda$ then the point $(\frac{\mu+\lambda}{\beta}, 1 - \frac{\mu+\lambda}{\beta})$ (the endemic equilibrium) is globally asymptotically stable.

For the event of an anomaly affecting the transmission rate, we momentarily denote this quantity as h. However, this value may not always be the same. To account for this, we take $h(x)$ as a function that determines the impact of the anomaly on the population when the impact has "strength" $x \in \mathbb{R}$, that is, how much the anomaly affects the population. We assume this anomaly happens with a Poisson distribution, say N, with intensity measure $\nu(\cdot)$. The intensity for the Poisson process is the value $\nu(\mathbb{R})$. For any interval of time, the total impact to the population is $\int_0^t \int_{\mathbb{R}} h(x)N(ds, dx)$. To make sense of this integral, for any $B \in \mathcal{B}(\mathbb{R}^d \backslash \{0\})$, (the Borel σ-algebra) $N(t, B)$ is Poisson process with intensity $\nu(B)$, where the anomaly has "strength" x, where x is strictly in B. The integral accounts for all possible anomalies that may affect the dynamic. The net effect over this time interval is $\int_0^t S(s-)I(s-) \int_{\mathbb{R}} h(x)N(ds, dx)$, where $I(s-)$ denotes the left limit.

For the triple $(\Omega, \mathcal{F}, P)$, we assume that $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$ is a right-continuous filtration, and that $\mathcal{F}_0$ contains all of the null sets of $\mathcal{F}$. Define $N(dt, dy)$ as a Poisson measure with $\nu(\cdot)$ as it's intensity measure, and take $W(t)$ as standard Brownian motion independent of the Poisson measure. We assume that $\nu(\mathbb{R}) < \infty$, (which implies that ν is a Lévy measure), and denote $\tilde{N}(dt, dy) := N(dt, dy) - \nu(dy)dt$. In our setting, "dt" represents the Lebesgue integral. See [6–8] for further information.

For the stochastic perturbations, define σ^2 as the variance of the Brownian motion, and $h(y)$ as the affect of random jumps in the population. We assume that $h(y)$ is continuously differentiable on $\mathbb{R}$, and that $\min h(y) > -1$, and $\max h(y) < 1$. For the SIS model, equation (1) becomes the right-continuous stochastic differential equation

$$\begin{aligned} dS(t) &= (-\beta S(t)I(t) - \mu S(t) + \mu + \lambda I(t))dt - \sigma S(t)I(t)dW(t) \\ &\quad + \int_{\mathbb{R}} h(y)S(t-)I(t-)N(dt, dy), \\ dI(t) &= (\beta S(t)I(t) - (\lambda + \mu)I(t))dt + \sigma S(t)I(t)dW(t) \\ &\quad - \int_{\mathbb{R}} h(y)S(t-)I(t-)N(dt, dy). \end{aligned} \tag{2}$$

Gray et al. [4] analyzed an SIS model very similar to the one we are considering, but only considered an Itô integral for their stochastic forcing term. While both underlying deterministic models assume a population of expected size, say N, Gray et al. [4] assume that the number of new infections in $[t, t+dt)$ is $\beta S(t)I(t)dt$, while equation (2) assumes this number is $\beta S(t)\frac{I(t)}{N}dt$. The authors found that for $R_0^S := \frac{\beta N}{\mu+\lambda} - \frac{\sigma^2 N^2}{2(\mu+\lambda)}$, if $R_0^S < 1$ then the process will converge almost surely to the disease free equilibrium, and if $R_0^S > 1$, the process is recurrent (and hence an epidemic occurs), and admits a unique invariant measure. Due to the slight differences in assumptions, we found very similar results, where the continuous stochastic version of equation (2) (i.e., $h \equiv 0$) yields $R_0^S = \frac{\beta}{\mu+\lambda} - \frac{\sigma^2}{2(\mu+\lambda)}$. However, these results are only supplementary to our analysis of the right-continuous process.

In this article, we will show that equation (2) is well defined, and we give conditions for stability of the disease free equilibrium, as well as giving conditions for a *strong* epidemic. The term strong refers to the type of recurrence. Since the recurrent property tells us that the process will return to a neighborhood of the deterministic endemic equilibrium in finite time, this represents a more natural dynamic of an epidemic.

Given that the initial condition is in the interior of the simplex, we will show that for all finite time, equation (2) is almost surely in the simplex. Define $K(t) = (S(t), I(t))$, $\Delta_2 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 > 0 \text{ and } x_1 + x_2 = 1\}$, and $[\cdot,\cdot]$ as the standard quadratic variation. Throughout this article, we take the bold $\mathbf{x} := (x_1, x_2)$ as a vector, take x as a one-dimensional dummy variable, and define x_0 as the one-dimensional initial condition. For simplicity, we define $S_c(t)$ and $I_c(t)$ as the continuous part of the process.

Proposition 2.1. *For all finite t, given that* $\mathbf{x} \in \Delta_2$, $P_{\mathbf{x}}(K(t) \in \Delta_2) = 1$.

Proof. Consider the mapping on the simplex $G((x_1, x_2)) = x_1 + x_2$, and define $U(t) = G(K(t))$. Itô's lemma yields

$$\begin{aligned}
dU(t) &= \frac{\partial G}{\partial y_1}(K(t))dS_c(t) + \frac{1}{2}\frac{\partial G}{\partial y_2}(K(t))dI_c(t) + \frac{1}{2}\frac{\partial^2 G}{\partial y_1^2}(K(t))[dS_c(t), dS_c(t)] \\
&\quad + \frac{1}{2}\frac{\partial^2 G}{\partial y_2^2}(K(t))[dI_c(t), dI_c(t)] + \frac{1}{2}\frac{\partial^2 G}{\partial y_1 \partial y_2}(K(t))[dS_c(t), dI_c(t)] \\
&\quad + \frac{1}{2}\frac{\partial^2 G}{\partial y_2 \partial y_1}(K(t))[dI_c(t), dS_c(t)] + \int_{\mathbb{R}} [G(K(t) + (h(y)S(t-)I(t-), \\
&\quad - h(y)S(t-)I(t-))) - G(K(t))]N(dt, dy) \\
&= (-\beta S(t)I(t) - \mu S(t) + \mu + \lambda I(t))dt - \sigma S(t)I(t)dW(t) + (\beta S(t)I(t) \\
&\quad - (\lambda + \mu)I(t))dt + \sigma S(t)I(t)dW(t) + \int_{\mathbb{R}} [(S(t) + h(y)S(t-)I(t-) + I(t) \\
&\quad - h(y)S(t-)I(t-)) - (S(t) + I(t))]N(dt, dy) \\
&= (-\mu(S(t) + I(t)) + \mu)dt.
\end{aligned}$$

Thus, if the process is in Δ_2, then $dU(t) = (-\mu(S(t) + I(t)) + \mu)dt = (-\mu + \mu)dt = 0$. Therefore, if $K(t) \in \Delta_2$ then $U(t) = 1$.

Finally, we show that $K(t)$ does not hit or jump over the boundary in finite time. Define $\Psi((x_1, x_2)) = \log(x_2/x_1)$ for $(x_1, x_2) \in \Delta_2$ (and notice that Ψ is a homeomorphic mapping from $\Delta_2 \to \mathbb{R}$), take τ as the first time $K(t)$ leaves the open simplex (i.e., such that $I(\tau) \leq 0$ or $S(\tau) \leq 0$), and $Z(t) := \Psi(K(t))$ for $t < \tau$. We will apply [9, theorem 2.1] to the process $Z(t)$ in order to show this $Z(t)$ does not explode in finite time, that is, $P_{\mathbf{x}}(\tau = \infty) = 1$. By

Itô's lemma we have

$$
\begin{aligned}
dZ(t) = {} & \frac{\partial \Psi}{\partial x_1}(K(t))dS_c(t) + \frac{\partial \Psi}{\partial x_2}(K(t))dI_c(t) + \frac{1}{2}\frac{\partial^2 \Psi}{\partial x_1^2}(R(t))[dS_c(t), dS_c(t)] \\
& + \frac{1}{2}\frac{\partial^2 \Psi}{\partial x_2^2}(K(t))[dI_c(t), dI_c(t)] + \frac{1}{2}\frac{\partial^2 \Psi}{\partial x_1 \partial x_2}(K(t))[dS_c(t), dI_c(t)] \\
& + \frac{1}{2}\frac{\partial^2 \Psi}{\partial x_2 \partial x_1}(K(t))[dI_c(t), dS_c(t)] + \int_{\mathbb{R}} [\Psi(K(t) + (h(y)S(t-)I(t-), \\
& - h(y)S(t-)I(t-))) - \Psi(K(t))]N(dt, dy) \\
= {} & \frac{-1}{S(t)}((-\beta S(t)I(t) - \mu S(t) + \mu + \lambda I(t))dt - \sigma S(t)I(t)dW(t)) + \frac{1}{I(t)}((\beta S(t)I(t) \\
& - (\lambda+\mu)I(t))dt + \sigma S(t)I(t)dW(t)) + \frac{\sigma^2}{2S^2(t)}S^2(t)I^2(t)dt + \frac{-\sigma^2}{2I^2(t)}S^2(t)I^2(t)dt \\
& + \int_{\mathbb{R}} \left[\log\left(\frac{I(t) - h(y)S(t-)I(t-)}{S(t) + h(y)S(t-)I(t-)}\right) - \log(I(t)/S(t))\right] N(dt, dy) \\
= {} & \left(\beta I(t) + \mu + \frac{-\mu}{S(t)} - \lambda\frac{I(t)}{S(t)}\right) dt + \sigma I(t)dW(t) + (\beta S(t) - (\lambda+\mu))dt + \sigma S(t)dW(t) \\
& + \frac{\sigma^2}{2}I^2(t)dt + \frac{-\sigma^2}{2}S^2(t)dt + \int_{\mathbb{R}} \log\left(\frac{1 - h(y)S(t-)}{1 + h(y)I(t-)}\right) N(dt, dy) \\
= {} & \left(\beta\Psi_1^{-1}(Z(t)) + \mu + \frac{-\mu}{\Psi_2^{-1}(Z(t))} - \lambda e^{Z(t)} + \beta\Psi_2^{-1}(Z(t)) - (\lambda+\mu) + \frac{\sigma^2}{2}\Psi_1^{-1}(Z(t))^2\right. \\
& \left. + \frac{-\sigma^2}{2}\Psi_2^{-1}(Z(t))^2\right) dt + \sigma(\Psi_1^{-1}(Z(t)) + \Psi_2^{-1}(Z(t)))dW(t) \\
& + \int_{\mathbb{R}} \log\left(\frac{1 - h(y)\Psi_2^{-1}(Z(t))}{1 + h(y)\Psi_1^{-1}(Z(t))}\right) N(dt, dy),
\end{aligned}
$$

where, for $x \in \mathbb{R}$, $\Psi^{-1}(x) = \frac{1}{1+e^x}(1, e^x) := (\Psi_1^{-1}(x), \Psi_2^{-1}(x))$.

Now, for $\mathfrak{B}$ defined as the infinitesimal generator of $Z(t)$ and the function $V(x) = 1 + x^2$ (where $x \in \mathbb{R}$), we derive the following inequality:

$$
\begin{aligned}
\mathfrak{B}V(x) = {} & \left(\beta\Psi_1^{-1}(x) + \mu + \frac{-\mu}{\Psi_2^{-1}(x)} - \lambda e^x + \beta\Psi_2^{-1}(x) - (\lambda+\mu) + \frac{\sigma^2}{2}\Psi_1^{-1}(x)^2\right. \\
& \left. + \frac{-\sigma^2}{2}\Psi_2^{-1}(x)^2\right) 2x + \sigma^2(\Psi_1^{-1}(x) + \Psi_2^{-1}(x))^2 \\
& + \int_{\mathbb{R}} \left[\left(x + \log\left(\frac{1 - h(y)\Psi_2^{-1}(x)}{1 + h(y)\Psi_1^{-1}(x)}\right)\right)^2 - x^2\right] \nu(dy) \\
\leq {} & \left[\sigma^2(\Psi_1^{-1}(x) + \Psi_2^{-1}(x))^2 + \int_{\mathbb{R}} \log\left(\frac{1 - h(y)\Psi_2^{-1}(x)}{1 + h(y)\Psi_1^{-1}(x)}\right)^2 \nu(dy)\right] \\
& + (2\beta\Psi_1^{-1}(x) + 2\beta\Psi_2^{-1}(x) + \sigma^2\Psi_1^{-1}(x)^2 + 2\int_{\mathbb{R}} \log\left(\frac{1 - h(y)\Psi_2^{-1}(x)}{1 + h(y)\Psi_1^{-1}(x)}\right) \nu(dy)) \cdot x \\
:= {} & \varphi_1(x) + \varphi_2(x) \cdot x.
\end{aligned}
$$

Noting that $0 \leq \Psi_1^{-1}(x), \Psi_2^{-1}(x) \leq 1$, and $\log(\frac{1-h(y)\Psi_2^{-1}(x)}{1+h(y)\Psi_1^{-1}(x)}) \leq \log(\frac{1-\min h(y)}{1+\min h(y)})$, one can see that there exists a constant K such that $3 \cdot \max_{x\in\mathbb{R}}\{\varphi_1(x), \varphi_2(x)\} < K$. Thus, for $-1 \leq x \leq 1$, $\mathfrak{B}V(x) \leq K \leq KV(x)$. Also for $|x| > 1$, $\mathfrak{B}V(x) \leq \varphi_1(x) + \varphi_2(x) \cdot x \leq \varphi_1(x) + \varphi_2(x) \cdot x^2 \leq KV(x)$. Therefore, invoking [9, theorem 2.1], we may conclude that $P_{\mathbf{x}}(\tau = \infty) = 1$. □

3. Analysis of the disease-free equilibrium

Since $I(t) = 1 - S(t)$, we are able to focus the analysis on $S(t)$. We may rewrite $S(t)$ as

$$dS(t) = (-\beta S(t)(1 - S(t)) - \mu S(t) + \mu + \lambda(1 - S(t)))dt - \sigma S(t)(1 - S(t))dW(t) \\ + \int_A h(y)S(t-)(1 - S(t-))N(dt, dy). \quad (3)$$

Since our analysis is simplified to a one-dimensional process, we set $x \in (0, 1)$ for the rest of the article, and define L as the infinitesimal generator for $S(t)$.

For the lemma below, we define $\tau_\epsilon = \inf\{t \geq 0 : S(t) \geq 1 - \epsilon\}$. We will show that the hitting time τ_ϵ has finite expectation, and then apply this result to the analysis of the disease free equilibrium

Lemma 3.1. *For any initial condition* $0 < x_0 < 1$, *we have* $E_{x_0}[\tau_\epsilon] < \infty$.

Proof. We follow the proof of [10, theorem 4.2]. Define $f(x) = e^\gamma - e^{\gamma x}$ for $\gamma > 0$ and $x \in (0, 1)$, and fix an arbitrarily small $\epsilon > 0$. Dynkin's formula yields

$$E_{x_0}[f(S(\tau_\epsilon \wedge T))] = f(x_0) + E_{x_0}\left[\int_0^{\tau_\epsilon \wedge T} Lf(S(t))dt\right].$$

We now determine an appropriate upper bound for $Lf(S(t))$. To adjust for the possibility of the process jumping out of the interval, we define $\epsilon_0 = \sup\{S(\tau_\epsilon)\}$. Since the process does not hit the boundary in finite time, $\epsilon_0 < 1$ a.s. We see that

$$\begin{aligned} Lf(x) &= -\gamma(-\beta x(1 - x) - \mu x + \mu + \lambda(1 - x))e^{\gamma x} - \frac{\gamma^2\sigma^2 x^2(1 - x)^2}{2} e^{\gamma x} \\ &\quad + \int_{\mathbb{R}} \left[e^{\gamma x} - e^{\gamma(x + h(y)x(1-x))}\right] \nu(dy) \\ &= \gamma\left(\beta x - \mu - \lambda - \frac{\gamma\sigma^2 x^2(1 - x)}{2}\right)(1 - x)e^{\gamma x} + \int_{\mathbb{R}} [e^{\gamma x} - e^{\gamma(x + h(y)x(1-x))}]\nu(dy) \end{aligned}$$

Recalling the inequality $-e^x \leq -1 - x$, we have the inequality

$$\begin{aligned} \int_{\mathbb{R}} [e^{\gamma x} - e^{\gamma(x+h(y)x(1-x))}]\nu(dy) &= e^{\gamma x}\int_{\mathbb{R}} [1 - e^{\gamma h(y)x(1-x)}]\nu(dy) \\ &\leq e^{\gamma x}\int_{\mathbb{R}} [1 - 1 - \gamma h(y)x(1 - x)]\nu(dy) \\ &= -\gamma e^{\gamma x}x(1 - x)\int_{\mathbb{R}} h(y)\nu(dy). \end{aligned}$$

Thus,

$$Lf(x) \leq \gamma\left(\beta x - x\int_{\mathbb{R}} h(y)\nu(dy) - \frac{\gamma\sigma^2 x^2(1 - x)}{2} - \mu - \lambda\right)(1 - x)e^{\gamma x}$$

Now, take γ large enough so that $\beta x - x\int_{\mathbb{R}} h(y)\nu(dy) - \frac{\gamma\sigma^2x^2(1-x)}{2} - \mu - \lambda < 0$ for all $x \in [0, 1 - \epsilon_0]$, and define

$$\alpha := \min_{x\in[0,1-\epsilon_0]} \left|\beta x - x\int_{\mathbb{R}} h(y)\nu(dy) - \frac{\gamma\sigma^2 x^2(1 - x)}{2} - \mu - \lambda\right|.$$

Thus, $Lf(x) \leq -\gamma\varphi(1 - x)e^{\gamma x}$. Noting that $-e^{\gamma x} \leq -1$, we conclude that $0 \leq E_{x_0}[S(\tau_\epsilon \wedge T)] \leq f(x_0) - \epsilon_0\gamma\alpha E_{x_0}[\tau_\epsilon \wedge T]$. Therefore, taking $T \to \infty$, the bounded convergence theorem yields $E_{x_0}[\tau_\epsilon] < \infty$. □

Theorem 3.1. *Suppose that* $\int_{\mathbb{R}} h(y)\nu(dy) < 0$ *and* $\frac{\sigma^2}{2} \leq \beta$. *If* $\beta < \mu + \lambda + \int_{\mathbb{R}} h(y)\nu(dy)$, *then for initial condition* $\mathbf{x} \in \Delta_2$,

$$P_{\mathbf{x}}\left(\lim_{t\to\infty} K(t) = (1,0)\right) = 1.$$

Proof. In our proof we employ the stochastic Lyapunov method to the process $S(t)$, defined by equation (3). For $x \in (0,1)$, define $g(x) = 1 - x$ as our Lyapunov function. Thus,

$$\begin{aligned} Lg(x) &= (-\beta x(1-x) + \mu(1-x) + \lambda(1-x))(-1) + \int_{\mathbb{R}} [\{1 - (x + h(y)x(1-x))\} \\ &\quad -(1-x)]\nu(dy) \\ &= -\left(-\beta x + \mu + \lambda + \int_{\mathbb{R}} h(y)\nu(dy)x\right)(1-x) \leq -\left(-\beta + \mu + \lambda + \int_{\mathbb{R}} h(y)\nu(dy)\right)(1-x). \end{aligned}$$

Therefore, [11, theorem 4 and remark 2] tells us that for an $\epsilon > 0$, there exists a neighborhood of $(1,0)$, say U, such that

$$P_{\mathbf{x}}\left(\lim_{t\to\infty} K(t) = (1,0)\right) \geq 1 - \epsilon,$$

for $\mathbf{x} \in U \cap \Delta_2$.

Now, take an arbitrary $\epsilon > 0$ and $\mathbf{x} \in \Delta_2$, and define the set $M = \{\lim_{t\to\infty} K(t) = (1,0)\}$. The strong Markov property yields

$$P_{\mathbf{x}}(M) = E_{\mathbf{x}}[E_{K(\tau_\epsilon)}[\chi_M]] \geq 1 - \epsilon.$$

Since ϵ was arbitrary, the theorem follows. □

Remark 3.1. [11, theorem 4] is stated for a jump-diffusion with a compensated Poisson measure. However, since we assumed that $\nu(\mathbb{R}) < \infty$, we may rewrite equation (3) as

$$\begin{aligned} dS(t) &= \left(-\beta S(t)I(t) - \mu S(t) + \mu + \lambda I(t) + \int_{\mathbb{R}} h(y)S(t-)I(t-)\nu(dy)\right)dt - \sigma S(t)I(t)dW(t) \\ &\quad + \int_{\mathbb{R}} h(y)S(t-)I(t-)\tilde{N}(dt,dy), \end{aligned}$$

and, thus, we may apply this theorem. The infinitesimal generator remains unchanged.

Corollary 3.1. *Suppose that* $\int_{\mathbb{R}} h(y)\nu(dy) > 0$ *and* $\frac{\sigma^2}{2} \leq \beta$, *and there exists a constant* $0 < \varrho < 1$ *such that* $\beta < \mu + \lambda + \varrho \int_{\mathbb{R}} h(y)\nu(dy)$. *Then*

$$P_{\mathbf{x}}\left(\lim_{t\to\infty} K(t) = (1,0)\right) = 1.$$

Proof. Taking a neighborhood $U = \{(x_1, x_2) \in \Delta_2 : x_1 > \varrho\}$ and $g(x)$ defined in theorem 3.1, we have that

$$Lg(x) \leq -\left(-\beta + \mu + \lambda + \varrho \int_{\mathbb{R}} h(y)\nu(dy)\right)(1-x).$$

[11, theorem 4 and remark 2] gives us, for $\mathbf{x} \in U$,

$$P_{\mathbf{x}}\left(\lim_{t\to\infty} K(t) = (1,0)\right) \geq 1 - \epsilon,$$

for some ϵ. The rest of the proof follows as the one given in theorem 3.1. □

The inequalities derived in theorem 3.1 are very natural and intuitive, however, they lack the variance term $\frac{\sigma^2}{2}$. To include the variance, a different method is applied to analyze the process. The inequality is very similar, however, the jump term only includes the values of $h(y)$ that are negative. Under an appropriate large variance and relatively small jump function, this inequality would give a better bound.

Following Gray et al. [4], we derive conditions for $I(t)$ to almost surely flow to zero exponentially. We use equation (2) and the equality $I(t) = 1 - S(t)$ to simplify the analysis.

Theorem 3.2. *If $\frac{\sigma^2}{2} \leq \beta$ and $\beta \leq \mu + \lambda + \frac{\sigma^2}{2} + \int_{\{y\in\mathbb{R}:h(y)\leq 0\}} h(y)\nu(dy)$ then*

$$\limsup_{t\to\infty} \log(I(t)) \leq \beta - \mu - \lambda - \frac{\sigma^2}{2} - \int_{\{y\in\mathbb{R}:h(y)\leq 0\}} h(y)\nu(dy) < 0 \ a.s.,$$

hence, $I(t)$ tends to zero exponentially almost surely.

Proof. From the generalized Itô's formula, we see that

$$\begin{aligned}\log(I(t)) = &\int_0^t (\beta(1-I(s)) - \mu - \lambda - \frac{\sigma^2}{2}(1-I(s))^2)ds + \int_0^t \sigma(1-I(s))dB(s) \\ &+ \int_0^t \int_{\mathbb{R}} (\log(I(s) - h(y)(1-I(s))I(s)) - \log(I(s)))N(ds,dy),\end{aligned}$$

recalling that ds is the Lebesgue integral. Considering the integrand in the Lebesgue integral, we have

$$\begin{aligned}&\beta(1-I(s)) - \mu - \lambda - \frac{\sigma^2}{2}(1-I(s))^2 \\ &= \beta - \mu - \lambda - \frac{\sigma^2}{2} + \left(\frac{\sigma^2}{2} - \beta\right) I(s) - \frac{\sigma^2}{2} I(s)^2 \leq \beta - \mu - \lambda - \frac{\sigma^2}{2}.\end{aligned}$$

Furthermore, for the integrand in the Poisson integral we have,

$$\begin{aligned}&\log(I(s) - h(y)(1-I(s))I(s)) - \log(I(s)) \\ &= \log(1 - h(y)(1-I(s))) \leq -h(y)(1-I(s)) \leq -h(y),\end{aligned}$$

where the last inequality holds if $h(y) \leq 0$. Taking the two inequalities above, we see that

$$\begin{aligned}\limsup_{t\to\infty} \frac{1}{t}\log(I(t)) \leq &\beta - \mu - \lambda - \frac{\sigma^2}{2} + \limsup_{t\to\infty} \frac{1}{t}\int_0^t \sigma(1-I(s))dB(s) \\ &+ \limsup_{t\to\infty} \frac{1}{t}\int_0^t \int_{\{y\in\mathbb{R}:h(y)\leq 0\}} -h(y)N(ds,dy).\end{aligned}$$

The Law of Large Numbers for Martingales (see [12]) tells us that $\limsup_{t\to\infty} \frac{1}{t}\int_0^t \sigma(1 - I(s))dB(s) = 0$, and [6, theorem 36.5] tells us that

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t \int_{\{y\in\mathbb{R}:h(y)\leq 0\}} -h(y)N(ds,dy) = -\int_{\{y\in\mathbb{R}:h(y)\leq 0\}} h(y)\nu(dy).$$

Thus,

$$\limsup_{t\to\infty} \frac{1}{t}\log(I(t)) \leq \beta - \mu - \lambda - \frac{\sigma^2}{2} - \int_{\{y\in\mathbb{R}:h(y)\leq 0\}} h(y)\nu(dy) < 0,$$

and, therefore, $I(t)$ almost surely converges to 0 exponentially. □

3.1. Conditions for an epidemic

Define $\hat{S}(t)$ as the continuous stochastic differential equation of (2), hence,

$$d\hat{S}(t) = (-\beta\hat{S}(t)(1-\hat{S}(t)) - \mu\hat{S}(t) + \mu + \lambda(1-\hat{S}(t)))dt - \sigma\hat{S}(t)(1-\hat{S}(t))dW(t).$$

We will show that the process is positive Harris recurrent, which is a very strong property. This is a close representation of the dynamic of an endemic since the recurrent characteristic tells us, for any initial condition, that the process will almost surely return to a neighborhood of the deterministic endemic equilibrium. It will be shown that the majority of the mass of the invariant measure lies in a neighborhood of $\frac{\mu+\lambda}{\beta}$, where the size of the neighborhood is contingent on the size of the stochastic perturbations.

We denote $\frac{\mu+\lambda}{\beta}$ as p for simplicity. For $\delta > 0$, define the neighborhood $U_\delta := (p-\delta, p+\delta) \cap [0,1]$. Throughout this section, we utilize the Kullback–Leibler distance centered at $(p, 1-p)$, which we define as $\varphi(x) = p\log(p/x) + (1-p)\log((1-p)/(1-x))$, for $x \in (0,1)$.

Proposition 3.1. *If $\beta \geq \frac{\sigma^2}{2} + \mu + \lambda$ then $\hat{S}(t)$ is recurrent. Moreover, the invariant measure, denoted as $\pi(\cdot)$, exists, is unique, and for $\delta > 0$, $\pi(U_\delta) \geq 1 - \frac{\sigma^2 p_m}{2\beta\delta^2}$, where $p_m := \max\{p, 1-p\}$ and $\frac{\sigma}{2}\sqrt{\frac{2p_m}{\beta}} < \delta$.*

Proof. By [13, theorem 16], we will analyze when $q(x) := \int_{\hat{S}(0)}^{x} \exp\{-2\int_{z_0}^{y} \frac{\alpha(z)}{\gamma^2(z)}dz\}dy$, where $z_0 \in (0,1)$. First, we notice that

$$\begin{aligned}\exp\left\{-2\int_{z_0}^{y}\frac{\alpha(z)}{\gamma^2(z)}dz\right\} &= \exp\left\{\frac{-2}{\sigma^2}\int_{z_0}^{y}\frac{((\mu+\lambda)-\beta z)(1-z)}{z^2(1-z)^2}dz\right\} \\ &= \exp\left\{\frac{-2}{\sigma^2}\int_{z_0}^{y}\left[\frac{(\mu+\lambda)-\beta}{z} + \frac{(\mu+\lambda)-\beta}{1-z} + \frac{(\mu+\lambda)}{z^2}\right]dz\right\} \\ &= C\exp\left\{\ln\left(z^{\frac{2\beta-2(\mu+\lambda)}{\sigma^2}}(1-z)^{\frac{2(\mu+\lambda)-2\beta}{\sigma^2}}\right) + 2(\mu+\lambda)\sigma^2 z\right\} \\ &= Cz^{\frac{2\beta-2(\mu+\lambda)}{\sigma^2}}(1-z)^{\frac{2\beta-2(\mu+\lambda)}{\sigma^2}}\exp\left\{\frac{2(\mu+\lambda)}{\sigma^2 z}\right\},\end{aligned}$$

where $C = \exp\{\ln(z_0^{-\frac{2\beta-2(\mu+\lambda)}{\sigma^2}}(1-z_0)^{-\frac{2(\mu+\lambda)-2\beta}{\sigma^2}}) - \exp\{\frac{2(\mu+\lambda)}{\sigma^2 z_0}\}\}$. Considering the function with respect to z, $\exp\{\frac{2(\mu+\lambda)}{\sigma^2 z}\}$, one can see that $q(0) = -\infty$. Since the functions with respect to z, $z^{\frac{2\beta-2(\mu+\lambda)}{\sigma^2}}$ and $\exp\{\frac{2(\mu+\lambda)}{\sigma^2 z}\}$, are bounded on the interval $[\hat{S}(0), 1]$, we may conclude $q(1) = \infty$ if and only if $\frac{2(\mu+\lambda)-2\beta}{\sigma^2} \leq -1$. But $\frac{2(\mu+\lambda)-2\beta}{\sigma^2} \leq -1$ if and only if $\beta \geq \frac{\sigma^2}{2} + \mu + \lambda$. Thus, $q(1) = \infty$ by our assumption on β.

Now, for $\tilde{\gamma}(x) := q'(q^{-1}(x))\gamma(q^{-1}(x))$, we need to determine that $\int_0^1[\tilde{\gamma}(x)]^{-2}dx < \infty$. Using the equality $q'(q^{-1}(x)) = 1/(q^{-1})'(x)$, and for the function $g(x)$, where $g'(x) = \gamma^{-2}(x)$, we see that

$$\int_0^1[\tilde{\gamma}(x)]^{-2}dx = \int_0^1[\gamma^{-2}(q^{-1}(x))(q^{-1})'(x)](q^{-1})'(x)dx = \int_{q^{-1}(0)}^{q^{-1}(1)} g'(z)dz.$$

Since $0 < q^{-1}(0) < q^{-1}(1) < 1$, we are able to conclude from [13, theorem 1.17] that the process is recurrent, and $\pi(\cdot)$ exists and is unique.

To finish the second statement of the theorem, we apply the infinitesimal generator of $\hat{S}(t)$, which we call $\mathfrak{L}$, to $\varphi(x)$. Thus, for $x \in (0, 1)$, we find that

$$\begin{aligned}\mathfrak{L}\varphi(x) &= ((\mu+\lambda)-\beta x)(1-x)\frac{x-p}{x(1-x)} + \frac{\sigma^2}{2}x^2(1-x)^2\frac{x^2-2px+p}{x^2(1-x)^2}\\ &= \frac{-\beta}{x}(x-p)^2 + \frac{\sigma^2}{2}(x^2-2px+p) \le -\beta(x-p)^2 + \frac{\sigma^2 p_m}{2}.\end{aligned}$$

For an initial condition $x_0 \in (0, 1)$,

$$\begin{aligned}0 \le E_{x_0}[\varphi(\hat{S}(t))] = \varphi(x_0) + E_{x_0}\left[\int_0^t \mathfrak{L}\varphi(\hat{S}(u))du\right] &\le \varphi(x_0)\\ -\beta E_{x_0}\left[\int_0^t (\hat{S}(u)-p)^2 du\right] &+ t\frac{\sigma^2 p_m}{2},\end{aligned}$$

which implies

$$E_{x_0}\left[\int_0^t (\hat{S}(u)-p)^2 du\right] \le \varphi(x_0)/\beta + t\frac{\sigma^2 p_m}{2\beta}.$$

Now for $\frac{\sigma^2 p_m}{2\beta} < \delta^2$, the complement of U_δ, denoted by U_δ^C, we see that

$$\pi(U_\delta^C) = \lim_{t\to\infty}\frac{1}{t}E_{x_0}\left[\int_0^t \mathbf{1}_{U_\delta^C}(\hat{S}(u))du\right] \le \lim_{t\to\infty}\frac{1}{t}E_{x_0}\left[\int_0^t \frac{(\hat{S}(u)-p)^2}{\delta^2}du\right] \le \frac{\sigma^2 p_m}{2\beta\delta^2}.$$

Therefore, $\pi(U_\delta) \ge 1 - \frac{\sigma^2 p_m}{2\beta\delta^2}$. □

Notice that the inequality $\pi(U_\delta) \ge 1 - \frac{\sigma^2 p_m}{2\beta\delta^2}$ is not necessary since we are able to explicitly write out the distribution of the invariant measure (see [13]). However, the inequality is nice comparison to the analogous inequality given in the following theorem, and we are able to see the effect jumps have on the dynamic.

For the main theorem below, we define $H(y) = \max\{\frac{-ph(y)}{1+h(y)}, \frac{(1-p)h(y)}{1-h(y)}\}$, and for $\overline{U}_\delta$ denoting the closure of a neighborhood δ about p, we define $\tau_{\overline{U}_\delta}$ as the first time $S(t)$ enters this space. The proof of this theorem utilizes [14, theorem 5.2].

Theorem 3.3. *Assume that* $\beta \ge \int_{\mathbb{R}} H(y)\nu(dy) + \frac{\sigma^2}{2} + \mu + \lambda$, *and there exists* $\delta > 0$ *where* $\delta > \sqrt{\frac{\sigma^2 p_m + \int_{\mathbb{R}} H(y)\nu(dy)}{2\beta}}$ *and* $U_\delta \subset [0, 1]$. *Then,* $E_{x_0}[\tau_{\overline{U}_\delta}] \le \frac{\varphi(x_0)}{\beta\delta^2 - \frac{\sigma^2 p_m}{2} - \int_{\mathbb{R}} H(y)\nu(dy)}$, $S(t)$ *is recurrent, the invariant measure, denoted as* $\pi_J(\cdot)$, *exists, is unique, and* $\pi_J(U_\delta) \ge 1 - \frac{\sigma^2 p_m + \int_{\mathbb{R}} H(y)\nu(dy)}{2\beta\delta^2}$.

Proof. Taking the function φ as above, for $x \in (0, 1)$, we see that

$$\begin{aligned}L\varphi(x) &= ((\mu+\lambda)-\beta x)(1-x)\frac{x-p}{x(1-x)} + \frac{\sigma^2}{2}x^2(1-x)^2\frac{x^2-2px+p}{x^2(1-x)^2}\\ &+ \int_{\mathbb{R}}\left[p\log\left(\frac{p}{x+h(y)x(1-x)}\right) + (1-p)\log\left(\frac{1-p}{1-x-h(y)x(1-x)}\right)\right.\\ &\left. - p\log\left(\frac{p}{x}\right) - (1-p)\log\left(\frac{1-p}{1-x}\right)\right]\nu(dy)\end{aligned}$$

$$= \frac{-\beta}{x}(x-p)^2 + \frac{\sigma^2}{2}(x^2 - 2px + p) + \int_{\mathbb{R}} \left[p \log\left(\frac{1}{1+h(y)(1-x)}\right) + (1-p)\log\left(\frac{1}{1-h(y)x}\right)\right]\nu(dy)$$

$$\leq -\beta(x-p)^2 + \frac{\sigma^2 p_m}{2} + \int_{\mathbb{R}} \left[p \log\left(\frac{1}{1+h(y)(1-x)}\right) + (1-p)\log\left(\frac{1}{1-h(y)x}\right)\right]\nu(dy).$$

Using the inequality $\log(x) \leq x - 1$, we see that

$$p\log\left(\frac{1}{1+h(y)(1-x)}\right) + (1-p)\log\left(\frac{1}{1-h(y)x}\right) \leq p\left(\frac{1}{1+h(y)(1-x)} - 1\right) + (1-p)\left(\frac{1}{1-h(y)x} - 1\right)$$

$$= p\frac{-h(y)(1-x)}{1+h(y)(1-x)} + (1-p)\frac{h(y)}{1-h(y)x} \leq \begin{cases} \dfrac{-ph(y)}{1+h(y)} & \text{if } h(y) \leq 0 \\ \dfrac{(1-p)h(y)}{1-h(y)} & \text{if } h(y) > 0 \end{cases}.$$

Thus, $L\varphi(x) \leq -\beta(x-p)^2 + \frac{\sigma^2 p_m}{2} + \int_{\mathbb{R}} H(y)\nu(dy)$. Now for $\delta > 0$ that holds our assumption, and $x \in [0, 1]\backslash U_\delta$, $L\varphi(x) \leq -\beta\delta^2 + \frac{\sigma^2 p_m}{2} + \int_{\mathbb{R}} H(y)\nu(dy)$. By our assumption on δ, $\varphi(S(t)) - t(-\beta\delta^2 + \frac{\sigma^2 p_m}{2} + \int_{\mathbb{R}} H(y)\nu(dy))$ is a super martingale for $t \in [0, \tau_{\overline{U}_\delta})$, and therefore, $\varphi(x) \geq (\beta\delta^2 - \frac{\sigma^2 p_m}{2} - \int_{\mathbb{R}} H(y)\nu(dy))E_{x_0}[\tau_{\overline{U}_\delta}]$.

We now show that $S(t)$ holds the properties in [14, theorem 5.2]. To show the ψ-irreducible condition (page 1674 [14]), we define the Borel measure $\psi(O) = M(O \cap U_\delta)$, where M is the Lebesgue measure, and $\eta_O := \int_0^\infty \mathbf{1}_{\{S(t)\in O\}}dt$, (which is the occupancy time). Since U_δ is strictly in the simplex, the strong Markov property tells us that $S(t)$ is recurrent in U_δ. Thus, if $\psi(O) > 0$ then $\mathbb{E}_{x_0}[\eta_O] > 0$ for $x_0 \in (0, 1)$.

To show the aperiodic condition ([14, p. 1675]), we use the set U_δ. Before we show this condition holds, we will note that since we assumed the Poisson measure is independent of the Wiener processes, the jumps are only dependent on time. We first note that since $\nu(\mathbb{R}) < \infty$, for every time t, there is a positive (but very small) probability that there has not been a jump. Under this event, the process has the form of $\hat{S}(t)$. Independence of the Brownian and Poisson terms coupled with proposition 3.1 tells us there exists a time T, such that for any $t \geq T$ and $x \in U_\delta$, $P_t(x, U_\delta) > 0$, that is, the transition probability at some time is always positive.

To finish the proof, we give a function V, where $V \geq 1$, is unbounded off of level sets, and there exists $c, b \in \mathbb{R}_+$ such that $LV(x) \leq -cV(x) + b$, (see [14, p. 1679]). Define $V(x) = -\log(x) + 1$. Then

$$LV(x) = ((\mu+\lambda) - \beta x)(1-x)\frac{-1}{x} + \frac{\sigma^2}{2}x^2(1-x)^2\frac{1}{x^2} + \int_{\mathbb{R}} [-\log(x + h(y)x(1-x)) + 1 + \log(x) - 1]\nu(dy)$$

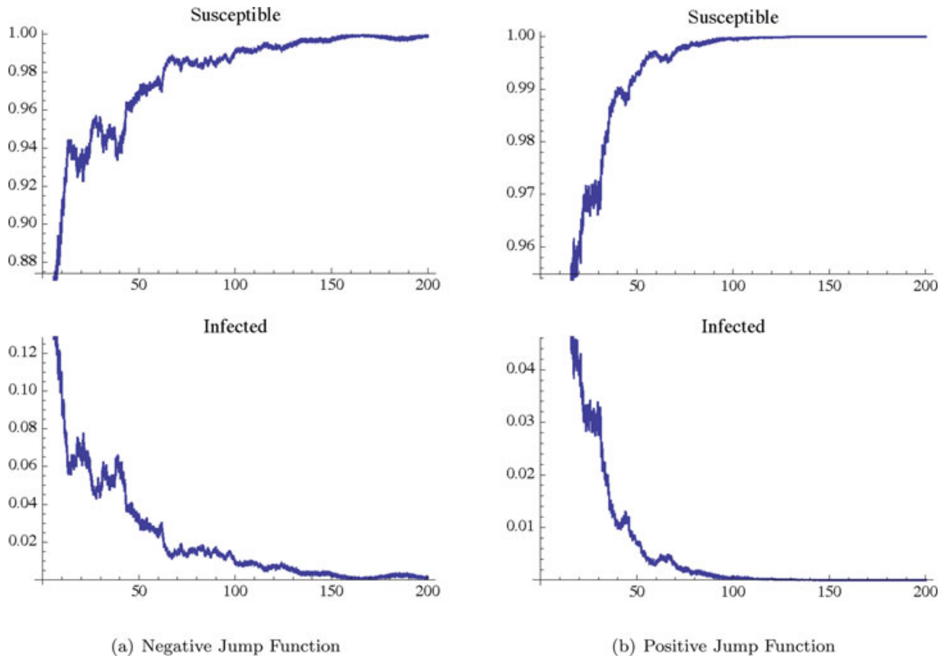

Figure 1. For simulations, the initial condition are $S_0 = .6$ and $I_0 = .4$: parameter values are a) $\beta = .3$, $\sigma = .1$, $\lambda = .3$, $\mu = .1$, $\nu(\mathbb{R}) = 1$, and $h(y) \equiv -.1$; and b) $\beta = .45$, $\sigma = .1$, $\lambda = .3$, $\mu = .1$, $\nu(\mathbb{R}) = 1$, and $h(y) \equiv .1$.

$$= \frac{\beta}{x}(x-p)(1-x) + \frac{\sigma^2}{2}(1-x)^2 + \int_{\mathbb{R}} \log\left(\frac{x}{x+h(y)x(1-x)}\right)\nu(dy)$$

$$= \frac{\beta}{x}(x-p)(1-x) + \frac{\sigma^2}{2}(1-x)^2 - \int_{\mathbb{R}} \log(1+h(y)(1-x))\nu(dy)$$

$$\leq \frac{\beta}{x}(x-p)(1-x) + \frac{\sigma^2}{2} - \log(1+h_{min})\nu(\mathbb{R})$$

$$= \left(\frac{\beta(x-p)(1-x)}{xV(x)} + \frac{\sigma^2}{2V(x)} - \frac{\log(1+h_{min})\nu(\mathbb{R})}{V(x)}\right)V(x)$$

Noting that $\frac{\beta(x-p)(1-x)}{xV(x)} \to -\infty$ and $\frac{\sigma^2}{2V(x)} - \frac{\log(1+h_{min})\nu(\mathbb{R})}{V(x)} \to 0$ as $x \to 0$, and $\frac{\beta(x-p)(1-x)}{xV(x)} + \frac{\sigma^2}{2V(x)} - \frac{\log(1+h_{min})\nu(\mathbb{R})}{V(x)} \to \sigma^2 - \log(1+h_{\min})\nu(\mathbb{R})$ as $x \to 1$, one can see there exists constants $c, b \in \mathbb{R}_+$, such that $LV(x) \leq -cV(x) + b$. Thus, the invariant measure $\pi_J(\cdot)$ exists and is unique.

Following the proof in proposition 3.1 yields $\pi_J(U_\delta) \geq 1 - \frac{\sigma^2 p_m + \int_{\mathbb{R}} H(y)\nu(dy)}{2\beta\delta^2}$. □

The simulations below agree with the conclusion of the theorems. Figure 1 shows for both cases that the disease free equilibrium is globally asymptotically stable. Figure 1a holds the hypotheses of theorem 3.2, where the jump function negative. While Figure 1b has the assumption that the jump function is positive, and holds the hypotheses of corollary 3.1.

Figure 2 shows the recurrent nature of the process, simulating an epidemic. As before, the simulations display the dynamics when the jump function is a negative or positive. Notice the prominence of the process showing the strength of the recurrence.

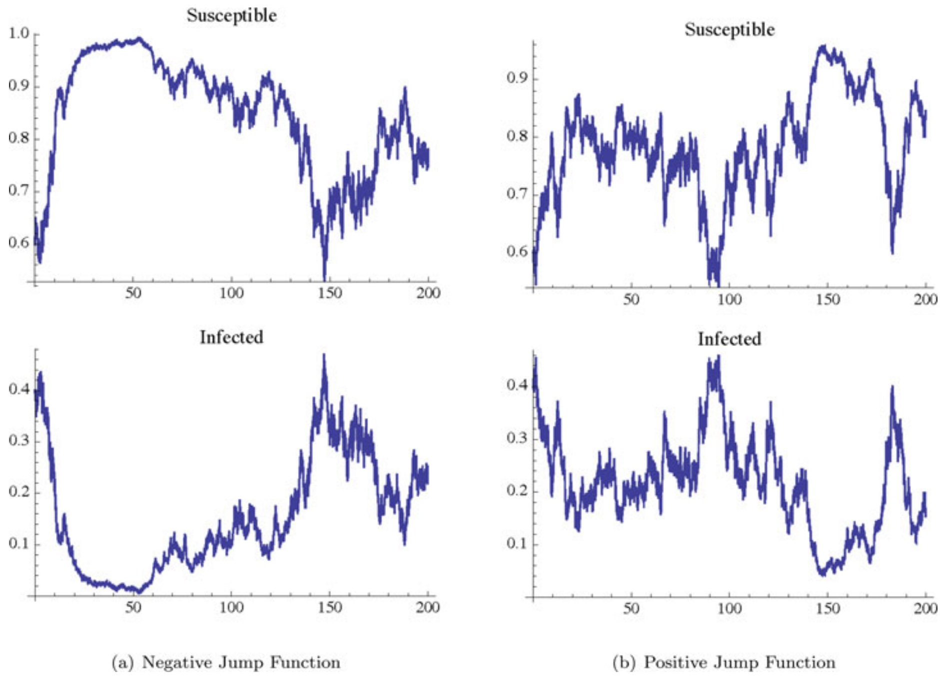

Figure 2. As in Figure 1, the initial condition are $S_0 = .6$ and $I_0 = .4$: parameter values are a) $\beta = .4$, $\sigma = .2$, $\lambda = .2$, $\mu = .15$, $\nu(\mathbb{R}) = .3$, and $h(y) \equiv -.1$, and b) $\beta = .5$, $\sigma = .2$, $\lambda = .2$, $\mu = .15$, $\nu(\mathbb{R}) = .3$, and $h(y) \equiv .1$.

4. Conclusion

In this article, we have shown that anomalies in the transmission rate have a significant influence in the epidemiological dynamics. The condition for the disease free equilibrium in theorem 3.1 are rather intuitive, except the variance does not show up in the equality. With the Lyapunov exponent method, another inequality for the disease free equilibrium is given. These two conditions display quite well the complexity that is added to the dynamics.

For an endemic equilibrium to occur, the inequality in theorem 3.3 gives a condition for the process to be positive recurrence with a unique invariant measure. Since this is the strongest type of recurrence, the endemic is also strong. Although this condition does give an interval for which β has not been characterized, it implies the possibility of different strengths of recurrence, and hence, of the endemic. This is in contrast to the model with just an Itô perturbation. This further displays the complexity that anomalies add to the system, and emphasizes the importance of including them when considering the transmission rate.

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