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WINNING ON A PROGRESSIVE SLOT MACHINE

by

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(Lee-Jeff Bell did his undergraduate work at Queen's and Carleton and is now a Master's student in the Mathematics and Statistics Department at Queen's. One of his major interests is in the mathematical theory of gambling).

A progressive slot machine is a slot machine with a jackpot that increases (slowly) until it is won. When the jackpot is large enough, a person playing the machine will have a positive expected value. In this article a particular type of 3 reel nickel machine with two progressive jackpots is analyzed.

The slot machine has a lever on its side. A player inserts from 1 to 5 nickels into the coin slot and pulls the lever. This starts 3 reels inside the machine spinning. Each reel has 22 symbols on it; the symbols are pieces of fruit or the number '7'. A window on the front of the machine allows the player to see a portion of each reel. After a short time the reels stop spinning, one at a time. The stopping of each reel is a random process. At this point, three symbols are visible in the window; one from each reel. For each pattern of 3 symbols, there is a payoff. The payoffs range from nothing up to the size of a jackpot, which can reasonably be expected to be a few thousand dollars. A jackpot can be won only if 5 nickels have been inserted into the machine before the lever is pulled.

The machine has 2 progressive jackpots, both of which start at $100.00. The current values of both jackpots are displayed on the front of the machine. At any given time, one of the jackpot values is lighted up. To win a jackpot, a player must spin a '7-7-7' combination in the window. If the '7-7-7' combination does occur, and the player has played 5 nickels, he/she wins the amount of the jackpot which is currently lighted up. For every 2 nickels that are inserted into the machine, each jackpot is increased by $0.005, the lighted jackpot becomes unlighted, and the unlighted jackpot becomes lighted.

It is usually the case that one jackpot is much larger than the other. We will call the former 'jackpot 1' and the latter 'jackpot 2'. The first thing that we shall compute is the expected cost of winning jackpot 1, if we insert 5 nickels into the machine every time that we pull the lever.

There is one '7' on each reel. Thus the probability of spinning a '7-7-7' on a pull of the lever is 1 in 22^3. But we want to know how many pulls of the lever it will take (on average) to win jackpot 1 .

Suppose that at the beginning of play jackpot 2 is lighted, but that after inserting 1 nickel into the machine jackpot 1 is lighted up. Then after inserting 3 nickles into the machine, jackpot 2 is lighted up. After 5 nickles, jackpot 1 is lighted up and we pull the lever. If we keep careful track of the lights, we see that after inserting 5 more nickels into the machine, jackpot 1 is again lighted up. We pull the lever. Another 5 nickels, and jackpot 2 is lighted up. We pull the lever. After the next 5 nickels, jackpot 2 is again lighted up. We pull the lever. This 20 nickel cycle repeats.

Thus the expected number of times that we must pull the lever in order to win jackpot 1 is 2×22^3. Since it costs 25¢ = $\$\frac{1}{4}$ each time we pull the lever, the expected amount of money that we must put in the machine to win jackpot 1 is $\$2 \times 22^3/4 = \$5324.00$. This is not the expected cost of winning jackpot 1.

We need to know one more fact. If we put $\$x$ into the machine and do not win a jackpot, what is the expected return? On our machine it is $\$\frac{3}{4}x$. Thus if we start with $\$x$ and put it all into the machine (5 nickels at a time) and also put all the winnings into the machine and we do not win a jackpot, the total amount of money that we have put into the machine is $\$x + \frac{3}{4}x + (\frac{3}{4})^2 x + \ldots = \$4x$.

Thus the expected cost of winning jackpot 1 is $\frac{\$5324.00}{4} = \$1331.00$. Note that in the 2×22^3 pulls of the lever that we make, we not only expect to win jackpot 1 exactly once, but also jackpot 2 exactly once. So, to see if we should play a given machine, we need only check and see whether or not the sum of the two jackpots that we could win is over \$1331.00. The expected value of the jackpots that we could win is not the value of the jackpots which is currently displayed.

Suppose we look around and find a machine with jackpots of \$160.00 and \$200.00 respectively. What is our expected gain playing this machine for 2×22^3 pulls, and what is our expected rate of hourly earnings?

As noted above, 1/20th of the money put into the machine goes into each jackpot. Thus if we put our \$5324.00 into the machine. each jackpot increases by \$5324.00/20 = \$266.20. Since we have the same likelyhood of winning the jackpot on each pull of the lever, each jackpot will have increased by an average of \$266.20/2 = \$133.10 at the time that it is won. Then our expected win on the two jackpots is \$1600.00 + 200.00 + 2 × 133.10 = \$2066.20, so our expected gain is \$2066.20 - 1331.00 = \$735.20.

The next thing we want to know is the number of pulls per hour that we can average on this machine. My (conservative) estimate of this, based on practical experience, is about 300 pulls per hour. Since we have to pull the lever 2×22^3 times, it will take us $2 \times 22^3/300 = 71.0$ hours. Thus, our expected gain is \$10.35/hour. This is pretty good, but it can be improved upon.

Let us return to contemplating throwing nickels into the machine and watching to see which jackpot is lighted. Suppose we try a little variation on our original pattern of playing 5 nickels per pull. Instead, we play 5 nickels, 5 nickels, then 2 nickels, and repeat this pattern. We end up playing 5 nickels against jackpot 1, again 5 nickels against jackpot 1, then 2 nickels against jackpot 2. Note that we can no longer win jackpot 2 - a jackpot can only be won if 5 nickels have been played. If we spin a '7-7-7' and have only played 2 nickels, we win the paltry sum of \$40.00, an amount which we shall ignore. How much better is this method than the original?

What is the expected amount of money that we have to put into the machine using the '5/5/2' method in order to win jackpot 1? We make 22^3 pulls at 25¢ each and $\frac{1}{2} \times 22^3$ pulls at 10¢ each, so we put \$2662.00 + 532.40 = \$3194.40 into the machine. Proceeding as before, we have that the expected cost of winning jackpot 1 is \$3194.40/4 = \$798.60. Our expected win on jackpot 1 is \$1600.00 + 3194.40/2.20 = \$1679.86 so our expected gain is \$1679.86 - 798.60 = \$881.26.

Our $22^3 + \frac{1}{2} \times 22^3$ pulls take us $\frac{22^3 + \frac{1}{2} \times 22^3}{300} = 53.24$ hours. Thus our expected gain playing the '5/5/2' method is \$16.55/hour, which is a significant improvement over the \$10.35/hour of the '5/5/5/5' method.

Finally, we address a problem which many of you may have thought of by now. If we start out with \$x and put it, and all resulting profits except wins against jackpot 1, into the machine, what guarantee is there that we will win jackpot 1? What about the possibility that we could win jackpot 1 more than once? (Of course, if we win jackpot 1 before our original \$x is gone, we find a new machine with a large jackpot 1 and continue play on the new machine). We shall use the more profitable '5/5/2' method of play.

Since the expected cost of winning jackpot 1 is \$798.60, the expected number of times we win jackpot 1 is x/798.60. Call this number λ. The number of times that we win jackpot 1 will be in the Poisson distribution. That is, the probability that we win exactly k jackpots, k an integer, is

$$e^{-\lambda} \frac{(\lambda)^k}{k!} .$$

Suppose that we start out with \$798.60, so that $\lambda = 1$. Then $e^{-1} = 36.8\%$ of the time we do not win jackpot 1 at all. Another $e^{-1} = 36.8\%$ of the time, we win jackpot 1 once. $\frac{1}{2} e^{-1} = 18.4\%$ of the time, we win jackpot 1 twice. $\frac{1}{3!} e^{-1} = 6.1\%$ of the time, we win jackpot 1 three times. The remaining 1.9% of the time, we win jackpot 1 at least four times.

Our final word on progressive machines is this: Provided we can find a machine with a large jackpot, progressive slot machines can be played with an excellent expected value. Unfortunately, a large amount of capital is required and there is a large chance of loosing it all. As compensation for this, there is a large chance of winning close to the expected value, and a moderate chance of winning many times the expected value. It is up to the individual to decide whether or not he/she is willing to accept the risk of playing and losing.

JOHN COLEMAN DAY

Former and present students, colleagues and associates of John Coleman gathered on November 8, 1980 in order to honour him on his retirement as Head of the Department.

Two lectures were presented in the afternoon:

G. de B. Robinson, University of Toronto - "John Coleman in the Mathematical World"

and Nathan Mendelsohn, University of Manitoba - "Some Reflections of 35 Years".

In the evening over 125 people gathered in the Faculty Club for a banquet. We were honoured by the presence of many Queen's officials, distinguished visitors and members of the Coleman family. Among those present were President James Ham of the University of Toronto and Principal Ronald Watts of Queen's, both of whom spoke briefly after dinner. John Coleman reminisced about many of the people who had influenced him in his early years and recalled many memories of his days in the Department and of his life at Queen's.

The evening concluded with musical entertainment provided by members of the Department: pianist Valery Lloyd-Watts, the Collegium Mathematicum choral group, soloists Norm Rice, Angela Broekhoven, Judith Taylor and Terry Smith, and The Stochastics modern dance company. The entire performance was inspired and directed by Terry Smith.

THE MATHEMATICS AND ENGINEERING PROGRAM AT QUEEN'S

The program in Mathematics and Engineering was developed in 1964 at Queen's in response to a need for applied mathematicians who can work on engineering problems in research and development laboratories. It is the only such course offered in Canada and one of the few in North America. Traditionally, the applied mathematician has worked in the fields of theoretical physics and statistics. However, in recent years there has been a fruitful interplay between modern mathematics and engineering. The increasingly complex systems in use and under development today require correspondingly sophisticated mathematical descriptions and analyses. Because of this, all engineering students are studying much more mathematics than before. However, there is a need for people with special ability and training in mathematics to work with other engineers on problems which require a combination of engineering insight and rigorous mathematics.

The first two years of the course are devoted to studies in the fundamentals of mathematics, science and engineering. In the third and fourth years the student proceeds to more advanced studies in mathematics. He or she also chooses an area of engineering and takes the same advanced courses in this area as do the engineering students of other departments. Thus, Mathematics and Engineering students should attain the same competence in at least one area of engineering as do students in more traditional branches of applied science. At present, the areas in which options are offered are:

A. Theremosciences
B. Control and Communication
C. Computer Science
D. Applied Mechanics
E. Structures
F. Process Control

The student in Mathematics and Engineering attends many classes with his fellows in Applied Science and many other classes with students in the Honours Mathematics course in Arts and Science. Thus, when he leaves Queen's he will be able to serve as a bridge between the engineers on the one hand and the pure mathematicians on the other.

Graduates of the program find employment is a wide variety of fields. The following data apply to the graduates of the program from its inception up to and including the class of 1979. On graduation

- 46% were employed by engineering companies (Imperial Oil, AMOCO, Ontario Hydro, IBM and other computing companies, Bell Northern, engineering consultants, etc.)
- 26% went on to graduate work (in electrical engineering, mathematics and statistics, business (MBA), mechanical engineering, computing science, etc.)
- 12% continued their training (in actuarial work, education, law, accounting, etc.)
- 11% were employed by government agencies (National Research Council, Atomic Energy of Canada, etc.)
- 5% no information on the remainder.

Some of the advantages of the Queen's Mathematics and Engineering Program are:

(i) the program is unique in Canada

(ii) small classes; at present there are 24 students in 2nd year

(iii) wide choice of options available within the program

(iv) wide choice of graduate programs available; on completion of the program, a student may be qualified for graduate work in mathematics, statistics, computer science, many of the standard engineering disciplines, operations research or business administration.

PEOPLE IN THE NEWS

Terry Smith attended the ASQC/ASA thirty-sixth annual Conference on Applied Statistics in Newark, New Jersey on December 4,5, 1980.

Grace and Morris Orzech's book on "Plane Algebraic Curves" has just been published by Marcel Dekker; it is a textbook for advanced undergraduate or beginning graduate students which lays a foundation for further study in algebraic geometry, commutative algebra and algebraic functions fields.

Norman Pullman presented a paper and chaired a section at the Third Caribbean Conference on Combinatorics and Computing, January 5-8, 1981 at the University of the West Indies, Cave Hill, Barbados. He also gave an invited lecture at Penn State University.

Paulo Ribenboim has given invited lectures at Carleton and the University of New Brunswick, as well as "The Jeffery Lecture" at Acadia. He has attended the AMS meeting in San Francisco and the Western Number Theory Conference in Tuscon, Arizona.

L.L. Campbell, S.R. Caradus, K.K. Oberai and P. Ribenboim attended the Annual Winter Meeting of the American Mathematical Society in San Francisco in January 1981.

T.W.F. Stroud presented a talk "Empirical Bayes versions of Stein-type estimators" at the NBER-NSF Bayesian Seminar in Econometrics at the University of Chicago, October 31, 1980.

Tony Geramita was invited last May by the National Research Council of Italy to give a sequence of lectures at the University of Catania in Sicily. He spoke on "Multiplicity of Intersection - an Algebraic Approach" and on "Algebraic Properties of the Singular Point formed by s intersecting lines in M-space". This latter work reported on research done jointly with Ferruccio Orecchia of the University of Genoa while he was visiting Queen's.

Dan Norman spoke to a session at the fall meeting of the Quinte - St. Lawrence Mathematics Association, on the subject of preparing students for university-level mathematics courses.

Peter Taylor led a workshop on the teaching of calculus at the June 1980 meeting of the Canadian Math Education Study Group at Laval University. In July 1980 he gave an invited talk at the Conference on Animal Conflict at the University of Sheffield, U.K. In February 1981 he was invited to Acadia University to speak on public key cryptography and on curriculum design. On his return journey he visited McGill University and addressed the Biology Colloquium on the lek paradox - a problem in sexual selection.

John Ursell attended the Second Victoria Symposium on Non-Standard Analysis held at University of Victoria, British Columbia from June 23 to 28, 1980 and presented a research paper entitled "Non-Standard Analysis and an integral on a semigroup".

Rick Mollin has joined the Department as an Assistant Professor. Rick received his B.A. and M.A. in mathematics from the University of Western Ontario and his Ph.D. from Queen's in 1975. His thesis was entitled "The Group of Algebras With Uniformly Distributed Invariants" and was done under the supervision of Ian Hughes. Since getting his degree he has been at McMaster and the University of Lethbridge.

The Bureaucracy of Omnitopia

by

T. H. Merrett

(Tim Merrett, Queen's 1964, wrote the article on "Databases, Relations and Functional Dependences" which appeared in the October 1979 issue of the Communicator.)

Once upon a time in the near Kingdom of Omnitopia, King Ruler was playing a game of Mediates with Shufflesmug, the Royal Secretary, and listening to his complaints.

"And Farmer Sunney was asking again today to graze his cow in the Olympic Training Field. He even brought hay into my office to show me how much better it was there than in his own field," fumed Shufflesmug, concluding a long list of administrative grievances. "Oh, I wish I had an assistant to stop people from coming to see me!"

King Ruler promised to find out how other countries handled these problems during his forthcoming Royal Tour.

He soon learned that every neighbouring Kingdom was using the modern techniques of Bureaucracy. Being an outsider and somewhat of a mathematician King Ruler was easily able to identify the three Principles of Bureaucracy.

(1) Principle of Mediation:
Access to any bureaucrat must be mediated by another bureaucrat.

(2) Principle of Buck-passing:
Every bureaucrat must be mediator for some other bureacrat.

(3) Principle of Hierarchy:
Mediation and Buck-passing must not be circular.

King Ruler perceived that the ideal bureaucracy grounded in these principles must be infinite in size. Since he felt it would be unwise to commit more than half the population of the Kingdom of Omnitopia (which was 27,182,818 souls) to a bureaucracy, he decided to weaken the Principle of Hierarchy.

(3') No cycle of mediation or buck-passing must be smaller than a given size.

He reasoned that while a bureaucrat would, like Shufflesmug, not tolerate being his own assistant, nor being his assistant's assistant, he might, if the bureaucracy was sufficiently confused, fail to notice a longer cycle.

Shufflesmug expressed great admiration for the Principle of Mediation and the Principle of Buck-passing and began right away to create a bureaucracy for the Kingdom of Omnitopia. He did this by expropriating Farmer Sunney's land and building an enormous office tower of concrete and glass.

King Ruler compensated Farmer Sunney for the loss of his pasture by giving him the Olympic Training Field. He appointed Shufflesmug to the nominal post of Chief Bureaucrat. Then, when the wheels of the Bureaucracy of Omnitopia were turning smoothly, he went on permanent holiday to Miami and lived happily ever after.

What is the smallest cycle in the Bureaucracy of Omnitopia?

THE GEOREGE L. EDGETT STATISTICAL LABORATORY at Queen's University

by

Louis Broekhoven

Statistics is a subject which finds application in almost every area of research: it is used extensively in agriculture, biology, psychology engineering, the social sciences, medicine and to a lesser extent in many other disciplines. It is natural, therefore, that over the years many departments at Queen's found a need for statistical expertise in both teaching and research and solved their problems in a variety of ways. Some hired statisticians, others encouraged their more mathematically inclined staff members to learn about statistics and still others relied heavily on the Department of Mathematics and Statistics. Although all this activity did produce an increased awareness and use of statistics on campus, the quality of instruction and practise was somewhat uneven.

The University Senate became concerned about the quality of statistical teaching throughout the university and in 1969 set up a committee to investigate the matter. This committee, and a successor, considered the problem for four years. It was realized that it would be undesirable to concentrate all teaching of statistics in one department since much subject-matter expertise in the area of application would be lost. At the same time it was not possible to exercise effective control over the quality of teaching in many departments. It also found a considerable demand for statistical consulting, much of which was not being served. It decided therefore that setting up a centre to provide sound statistical advice might be the best means for solving both problems. In 1973 Queen's STATLAB was formed by the action of senate. Since then it has progressed steadily. In 1979 it was renamed the George L. Edgett Statistical Laboratory in honour of Professor Edgett who for so long was the only teacher of statistics at Queen's (and in Canada.)

The statistical laboratory serves the entire university and provides advice on all statistical matters. All pure and applied science departments have used its services as have many of the social science and medical departments. Not surprisingly some of the purely Arts departments, such as French and English, have needed statistical help. The administration also uses its services, for example, in analysing salary data or in devising fair means for selecting students for the residences. The Arts and Science Undergraduate Society have come for help in processing their student evaluation forms.

The Statistical Laboratory serves two main functions. First of all, it provides statistical advice to the many researchers on campus who require it and thus serves the university. The Mathematics and Statistics department also benefits because the constant stream of practical problems provides stimulus for research and helps to keep the statisticians in touch with reality. The Laboratory helps in the teaching program because many of the problems are suitable as undergraduate projects for statistics students. Students at Queen's thus benefit from both theoretical and practical training which is a com-ination most employers seek.

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The ALGEBRA GROUP of the Department had an active year in 1980. They offered five graduate courses: Structure of Algebras (M. Orzech), Algebraic Topology (L. Roberts), Modular Representations (I. Hughes), Commutative Algebra (G. Orzech), Fermat's Last Theorem (P. Ribenboim). They also held seminars on Number Theory, Representation Theory of Artin Algebras, Algebraic Curves, Commutative Algebra, Group Theory. The Group also welcomed over thirty visitors whose stays ranged from a week to a year and whose interests ranged over the entire field.

A Clever Little Puzzle - The Vicar and the Verger

by

Dave Mason

The vicar and the verger were out for a stroll one Sunday afternoon.

Verger: How was attendance at Church this morning?

Vicar: Not so good. There were just three females in attendance. The product of their ages was 2450 and the sum of their ages was twice yours. Can you tell me their ages?

Verger: (after some thought): No, I'm afraid not.

Vicar: You could if I told you that I was the oldest one in Church this morning.

With that seemingly useless tidbit of information, the Verger immediately rattled off the ages of the three females.

What were the ages of the Vicar, Verger and the three females?
(Clue: $2450 = 2 \times 5 \times 5 \times 7 \times 7$)

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Among VISITORS TO THE DEPARTMENT this winter are

Paulo Maroscia (Roma)	Commutative Algebra
Michael Katz (Haifa)	Logic
Wally Wallis (Newcastle, NSW)	Combinatorics
Herb Shank (Waterloo)	Combinatorics
S. K. Gupta (Indian Statistical Inst.)	Algebra
Samuel Goldberg (Urbana, Illinois)	Differential Geometry

NEWS FROM GRADUATES

1973

John Redding

After graduation from Queen's in Mathematics and Engineering John completed an M.Sc. degree in Electrical Engineering at MIT, with a thesis in the area of switching power supplies. His first employment was with the Bell Northern subsidiary in California. After several job changes, he accepted the position of Eastern Sales Manager for Analog Devices Inc., a Massachusetts firm engaged in the supply of microprocessor-based process control equipment. At last word, John had left Analog Devices, and was busy forming his own company in the microprocessor field.

1979

Dom de Caen

Received his M.Sc. from Queen's in 1979, and is now working on his Ph.D. in graph theory at the University of Toronto. He has coauthored three papers with Norm Pullman on "Clique Coverings of Graphs" and one with Dave Gregory on "Primes in the Semigroup of Boolean Matrices". He has also published "A Note on Path and Cycle Decomposition of Graphs" and co-authored a paper with Dave Gregory and Norm Pullman on "The Boolean Rank of Zero-one Matrices".

Al Donald

Also received his M.Sc. from Queen's in 1979, and is now working on his Ph.D. in epidemiology at the University of Western Ontario. He has coauthored with Norm Pullman a paper on "Clique Coverings of Graphs" as well as authoring a paper on "An Upper Bound for the Path Number of a Graph".

1980

Frank Dixon

Frank has been in Calgary for the last year working with a small geophysical exploration company doing computer programming. He says that promotional and salary prospects look very good!

1979

Les Gulko

Les is doing graduate work at MIT in nuclear engineering and is planning next to study business.

MASTER'S DEGREES AWARDED RECENTLY BY THE DEPARTMENT

NAME	SUPERVISOR	TITLE
LIZOTTE-VOYER, Danielle	B.J. Kirby	Economic Operation of Hydro-Thermal Electric Power System
McGUGAN, James	N. Pullman and S.G. Akl	Local Neighbourhood Search Heuristics for the Travelling Salesman Problem
HODDER, James	B.J. Kirby	Recent Research into a Minimization Problem

PH.D.'S AWARDED RECENTLY BY THE DEPARTMENT

HAMILTON, David	D.G. Watts	Experimental Design of Non-linear Regression Models
SPINNEY, Sandra	K. Oberai	Spectral Theory of Unbounded Operators on a Banach Space
O'SHEA, Donal	A.J. Coleman	On μ-Equivalent Families of Singularities

PROBLEM SECTION

by

Peter Taylor

The following problem was posed in the last issue.

Problem No. 3

A Four Dimensional Polyhedron

Let K be the set of all 3×3 "doubly stochastic" matrices: all entries ≥ 0 and row and column sums equal to 1. K can be realized as a subset of nine-dimensional space. What does it look like? Give a geometric description.

Solution

The problem is not as formidable as first appears. K itself is not nine-dimensional but really only four; there are four "degrees of freedom" in specifying such a matrix. And K is a regular convex polyhedron with lots of symmetry, like, for example, the icosahedron, but one dimension higher. Still you can't quite visualize it but I gave you a start by asking how you might describe an icosahedron to an inhabitant of a 2-dimensional world. You could describe and count its vertices, edges, and triangular faces, and say something about how they fit together. You could even describe a family of parallel 2-dimensional cross-sections. So try the same thing for K.

But nobody did, at least they didn't report their progress to me. It is very much an exploratory problem, and for anyone who's never done this type of geometric-algebraic thing before, considerable amounts of exploration might be required just to see what to expect. But the results are quite rewarding and can teach you a lot about the relationship between geometry and algebra. By way of solution, let me describe some of the basic structure of this polyhedron.

The vertices are the "permutation" matrices: those with 3 ones (and 6 zeros). There are 6 of these. There are two types of edges (one-dimensional faces). The set of all matrices

$$\begin{bmatrix} 1-x & x & 0 \\ x & 1-x & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ for x between 0 and 1 forms an edge and there are 9 edges of this type, one for every 2×2 submatrix. But also the set of matrices $$\begin{bmatrix} 1-x & x & 0 \\ 0 & 1-x & x \\ x & 0 & 1-x \end{bmatrix}$$ for $0\leq x\leq 1$, forms an edge and there are 6 edges of this type (think of how to choose the three zeros). Call these type 1 and type 2, edges. To verify that these segments are indeed edges it's not enough to simply notice that each is the line segment between two vertices. (Think of the icosohedron; the line segment between two vertices need not be an edge.) We must also observe that if any point of the segment is halfway between two other points of K, then those two other points are also in the segment. This is not so hard to do. Thus there are 15 edges in all. Since there are only 6 vertices and there are 15 ways of choosing 2 objects from 6, we get the surprising result that every pair of vertices is connected by an edge! This happens in 3-space for the 3-simplex (the tetrahedron) and we immediately wonder whether our polyhedron is the 4-simplex. But this could not be since the 4-simplex has 5 vertices, and our polyhedron has 6. Can you think of a polyhedron in 3-space eith 5 vertices that has this property?

Again, by counting, we can verify that every vertex subtends 5 edges, 3 of type 1 and 2 of type 2. Now we must look at the two-dimensional faces. One such is the set of matrices

$$\begin{bmatrix} 1-x-y & x & y \\ x & 1-x & 0 \\ y & 0 & 1-y \end{bmatrix}$$ for $0\leq x\leq 1$, $0\leq y\leq 1$ and $x+y\leq 1$. This is easily seen to be a triangle with two type 1 edges ($x=0$ and $y=0$) and one type 2 edge ($x+y=1$). There are 18 faces of this type (think of choosing the 2×2 submatrix containing the two zeros). It is not hard to satisfy yourself that every 2-face is of this type. Every type 1 edge is a boundary for exactly 4 of these triangles, and every type 2 edge is a boundary for 3 of these triangles.

Finally we look for the 3-dimensional faces. First note that matrices of K with all entries > 0 must be in the interior of K, whereas a matrix with at least one 0 entry will be in at least one 3-face. Indeed for each (i,j) the set of all matrices in K with $a_{ij} = 0$ is a 3-face. These are all the three faces and there are 9 of them. What do they look like? Take the one defined by $a_{11} = 0$. The 2×2 upper left corner looks like $\begin{vmatrix} 0 & x \\ y & z \end{vmatrix}$. The three parameters x,y and z determine a member of K if and only if

$$\begin{aligned} 0\le x,y,z&\le 1 \\ x+z&\le 1 \\ y+z&\le 1 \\ x+y+z&\ge 1 \quad . \end{aligned}$$

From a 3-dimensional sketch it can be seen that these conditions determine a tetrahedron. Thus every 3-face is a tetrahedron bounded by 4 triangles and 6 edges (two type 2 and four type 1). Every triangle bounds two such tetrahedra.

We can go on and try to see how these 9 tetrahedra are put together to make the polyhedron K, but I think I've said enough. For now anyway.

Problem No. 4

Summing The Harmonic Series

This problem will teach you something about one of the standard mathematical constants, not quite so famous as π and e, but worth knowing about all the same. (This is your hint!)

Every series nut knows that the harmonic series $\sum_{n=1}^{\infty} 1/n$ diverges. That is given any $M > 0$ we can always find a sufficiently large N such that

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{N} \ge M \quad .$$

Take $M = 10$ and find the smallest N for which this inequality holds. No computers or programmable hand calculators please. (Though you can use them to check.) I want an easily verifiable analytic solution available 50 years ago.