Math 281
An Introduction to Real Analysis
Winter 2007

Instructor:

James A. Mingo, `mingo@post`, 533-2444

Office Hours: Jeffery 404

Tuesday: 2:30 - 3:30
Thursday: 1:30 - 2:30

Grading Scheme:

eleven assignments (best ten count) 20%
midterm 20%
final examination 60%.

Lectures & Tutorials:

lectures - slot 11, Jeffery 128
tutorials - Jeffery 225
Thursdays 2:30 & 4:30

Teaching Assistants:

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Textbook:

Math 281, Course Notes by O. Nielsen and D. Norman; available at the Queen's bookstore.

Website:

`www.mast.queensu.ca/~math281`

Assignments:

are due each Thursday in class or in my office mailbox by 4:30 pm. I will use the best ten scores attained. You may if you wish do the assignments in teams. A team may consist of up to three people; all team members receive the same grade.

Midterm:

will be in class on Thursday March 1 (week 7), the midterm will cover (approximately) the material up to the end of week 5. If you will be unable to write because of extra-curricular activities please notify J. Mingo in writing by January 12. All special needs students must submit their requests for accommodations by January 12. Exceptions can be made for emergencies but they must be beyond the control of the applicant and well documented.

Course Description:

A subtitle for the course might be 'the analysis of the infinite' as this title was used by Newton (*De Analysi per Aequationes Numero Terminorum Infinitas*) and Euler (*Introductio in analysi infinitorum*) probably to distinguish their work from algebra where one has only a finite number of terms.

The use of series enables us to extend the calculations done in calculus to functions which have a series representation. For example the Bessel functions occur in the analysis of heat transfer. We may write the 0^{th} order Bessel function of the first kind $J_0(x) = 1/(2\pi)\int_0^{2\pi} \cos(t - x\sin t)\,dt$ as

$$J_0(2x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(n!)^2}$$

In order to do calculus on such functions we need to be able to differentiate and integrate a series term by term.

If we wish to find an expression for the integral

$$G(z) = \frac{1}{2\pi}\int_{-2}^{2} \frac{1}{z-t}\sqrt{4-t^2}\,dt$$

we can write $1/(z-t)$ as a series:

$$\frac{1}{z-t} = \frac{1}{z}\frac{1}{1-t/z} = \frac{1}{z}\sum_{n=0}^{\infty}\frac{t^n}{z^n}$$

Then we integrate term by term:

$$G(z) = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{2\pi}\int_{-2}^{2} t^n\sqrt{4-t^2}\,dt\right)\frac{1}{z^n}$$

By symmetry. the integrals with odd powers of t are zero, and by making the substitution $t = 2\cos(\theta)$ we find

$$\frac{1}{2\pi}\int_{-2}^{2} t^{2n}\sqrt{4-t^2}\,dt = \frac{1}{n+1}\binom{2n}{n}$$

Thus

$$\frac{1}{z}G\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty}\frac{1}{n+1}\binom{2n}{n}z^{2n}$$

To sum this series we note that

$$\frac{d}{dt}\sum_{n=0}^{\infty}\frac{1}{n+1}\binom{2n}{n}t^{n+1} = \sum_{n=0}^{\infty}\binom{2n}{n}t^{n}$$

and we recall that this is the Taylor series of $1/\sqrt{1-4t}$ Thus

$$\sum_{n=0}^{\infty}\frac{1}{n+1}\binom{2n}{n}t^{n} = \frac{1}{t}\int\frac{1}{\sqrt{1-4t}}\,dt$$

$$= \frac{1}{t}\left(c + \frac{-\sqrt{1-4t}}{2}\right)$$

where c is chosen to equal 1 to make the last expression (after an invocation of L'Hôpital's rule) have the value 0 when $t = 0$.

Thus

$$\sum_{n=0}^{\infty}\frac{1}{n+1}\binom{2n}{n}t^{n} = \frac{t-\sqrt{1-4t}}{2t}$$

Hence

$$\frac{1}{z}G\left(\frac{1}{z}\right) = \frac{z^2-\sqrt{1-4z^2}}{2z^2}$$

So

$$G(z) = \frac{z-\sqrt{z^2-4}}{2z}$$

There a few steps in the calculation above that require further justification; by the end of the course you should be able to identify and justify these steps.

Thus the goal of the course is to be able to understand and calculate with sequences and series of real numbers and real valued functions. The basic problem is to expand a function into a series and then to perform operations on this series: addition, multiplication, differentiation, and integration. Along the way we shall learn how to write a mathematical proof, develop some topology of $\mathbb{R}^n$ and sufficient tools for dealing with the convergence of infinite series.

Assignment 1: (*Due January 18*)

In questions 1, 2, and 3 decide if the given statement is true or false. If it is true give a proof; if false give a counter-example.

1) *a*) $\forall x \in \mathbb{R}\ \exists y \in \mathbb{R}$ s.t. $x > y$.
   b) $\exists y \in \mathbb{R}$ s.t. $\forall x \in \mathbb{R}\ x > y$.

2) $\exists y \in \mathbb{R}$ s.t. $\forall x \in \mathbb{R}\ xy \geq 0$.

3) $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N\ \sqrt{n+1} - \sqrt{n} < 10^{-6}$.

4) *a*) For $0 \leq a < b$ show that

$$(n+1)a^n < \frac{b^{n+1}-a^{n+1}}{b-a} < (n+1)b^n$$

b) Let $a_n = (1+n^{-1})^n$ and $b_n = (1+n^{-1})^{n+1}$. Show that for each $n \geq 1$

$$a_n < a_{n+1} < b_{n+1} < b_n$$

c) Show that $\sup\{a_m \mid m \geq 1\}$ and $\inf\{b_n \mid n \geq 1\}$ exist and

$$\sup\{a_m \mid m \geq 1\} \leq \inf\{b_n \mid n \geq 1\}$$

Practice Questions:

1) Let $A = \{x \in \mathbb{R} \mid x^2 < 2\}$
   a) Show that $A \subset [-2, 2]$.
   b) Suppose that y is the least upper bound of A. Show that

   i) $y \geq 1$;
   ii) $y^2 \geq 2$;
   iii) $y^2 \leq 2$;
   iv) $y^2 = 2$.

2) Exercise 2 in the Appendix on Logic.

3) Exercise 3 in the Appendix on Logic.

Books on reserve in Douglas Library:

P. Bhatnagar, The Theory of Infinite Series, QA 295 B555

T. Bromwich, Infinite Series, QA 295 B85

K. Davidson & A. Donsig, Real Analysis with Real Applications

L. Euler, Foundations of Differential Calculus, QA E8813

R. Gordon, Real Analysis, A First Course

G. Hardy, A Course of Pure mathematics, QA 303 H26

G. Klambauer, Aspects of Calculus, QA 303 K654

K. Knopp, Infinite Sequences and Series, QA 295 K72

M. Stoll, Introduction to Real Analysis

W. Trench, Introduction to Real Analysis