MATH 281
Introduction to Real Analysis

Winter 2009

Instructor:	Navin Kashyap Office: Jeffery Hall 410 Phone: 533-6640, Email: `nkashyap@mast.queensu.ca`
Office hours:	Thursdays 3:30–5:00, or by appointment
Course Web Site:	`http://www.mast.queensu.ca/~math281` All assignments and important announcements will be posted here.
Lectures:	Slot 15 (Tuesday 12:30, Thursday 11:30, Friday 1:30), Jeffery 128
Tutorials:	Section A (Arts & Science): Wednesday 2:30 in Jeffery 101 Section B (Applied Science): Thursday 2:30 in Jeffery 101
Prerequisite:	MATH 120 or MATH 280, or other previous exposure to limits and continuity.
Assignments:	There will be 10 homework assignments, due on Fridays, in class. Late homeworks will NOT be accepted. Students are allowed to work on and hand in assignments in teams of up to three members. Members of the same team will get the same marks.
Midterm Test	Scheduled for Friday, Feb. 27, in class.
Evaluation:	Each homework assignment will be worth 2%. The lowest homework mark will be dropped, meaning that only 9 homeworks, accounting for a total of 18%, will count towards the final course mark. The mid-term will be worth 22%, and the final exam 60%. If a student misses the midterm without making prior arrangement with the instructor, then the final exam will account for 82% of the student's course mark.
Textbook:	MATH 281 Course Reader, by O. Nielsen and D. Norman (available at the Queen's campus bookstore)
Additional references:	The following books have been placed on course reserve at the Douglas Engineering & Science library: T. Bromwich, *An Introduction to the Theory of Infinite Series* G.H. Hardy, *A Course of Pure Mathematics*, 10th ed. K. Knopp, *Infinite Sequences and Series* W. Rudin, *Principles of Mathematical Analysis* T. Tao, *Analysis* (vols. 1 & 2)
Course Description:	See next page

Course Description (courtesy of Prof. J. Mingo)

A subtitle for the course might be 'the analysis of the infinite' as this title was used by Newton (*De Analysi per Aequationes Numero Terminorum Infinitas*) and Euler (*Introductio in analysi infinitorum*) probably to distinguish their work from algebra where one has only a finite number of terms.

The use of series enables us to extend the calculations done in calculus to functions which have a series representation. For example the Bessel functions occur in the analysis of heat transfer. We may write the 0^{th} order Bessel function of the first kind $J_0(x) = 1/(2\pi)\int_0^{2\pi}\cos(t - x\sin t)\,dt$ as

$$J_0(2x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(n!)^2}$$

In order to do calculus on such functions we need to be able to differentiate and integrate a series term by term.

If we wish to find an expression for the integral

$$G(z) = \frac{1}{2\pi}\int_{-2}^{2} \frac{1}{z-t}\sqrt{4-t^2}\,dt$$

we can write $1/(z-t)$ as a series:

$$\frac{1}{z-t} = \frac{1}{z}\frac{1}{1-t/z} = \frac{1}{z}\sum_{n=0}^{\infty}\frac{t^n}{z^n}$$

Then we integrate term by term:

$$G(z) = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{2\pi}\int_{-2}^{2} t^n\sqrt{4-t^2}\,dt\right)\frac{1}{z^n}$$

By symmetry. the integrals with odd powers of t are zero, and by making the substitution $t = 2\cos(\theta)$ we find

$$\frac{1}{2\pi}\int_{-2}^{2} t^{2n}\sqrt{4-t^2}\,dt = \frac{1}{n+1}\binom{2n}{n}$$

Thus

$$\frac{1}{z}G\Big(\frac{1}{z}\Big) = \sum_{n=0}^{\infty}\frac{1}{n+1}\binom{2n}{n}z^{2n}$$

To sum this series we note that

$$\frac{d}{dt}\sum_{n=0}^{\infty}\frac{1}{n+1}\binom{2n}{n}t^{n+1} = \sum_{n=0}^{\infty}\binom{2n}{n}t^n$$

and we recall that this is the Taylor series of $1/\sqrt{1-4t}$. Thus

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{1}{n+1}\binom{2n}{n}t^n &= \frac{1}{t}\int\frac{1}{\sqrt{1-4t}}\,dt \\ &= \frac{1}{t}\left(c + \frac{-\sqrt{1-4t}}{2}\right)\end{aligned}$$

where c is chosen to equal 1 to make the last expression (after an invocation of L'Hôpital's rule) have the value 0 when $t = 0$.

Thus

$$\sum_{n=0}^{\infty}\frac{1}{n+1}\binom{2n}{n}t^n = \frac{t-\sqrt{1-4t}}{2t}$$

Hence

$$\frac{1}{z}G\Big(\frac{1}{z}\Big) = \frac{z^2-\sqrt{1-4z^2}}{2z^2}$$

So

$$G(z) = \frac{z-\sqrt{z^2-4}}{2z}$$

There are a few steps in the calculation above that require further justification; by the end of the course you should be able to identify and justify these steps.

Thus the goal of the course is to be able to understand and calculate with sequences and series of real numbers and real-valued functions. The basic problem is to expand a function into a series and then to perform operations on this series: addition, multiplication, differentiation, and integration. Along the way we shall learn how to write a mathematical proof, develop some topology of $\mathbb{R}^n$ and sufficient tools for dealing with the convergence of infinite series.