Let p be an odd prime and ~~let~~ x, y, z be relatively prime integers such ~~that~~ $x^p+y^p+z^p=0$.

Form $u_1 = \frac{x^p y^p}{z^{2p}}$, $u_2 = \frac{y^p z^p}{x^{2p}}$, $u_3 = \frac{z^p x^p}{y^{2p}}$.

$u = u_1, u_2, u_3$ are roots of

$$0 = (u-u_1)(u-u_2)(u-u_3)$$

$$0 = u^3 - (u_1+u_2+u_3)u^2 + (u_1u_2+u_2u_3+u_3u_1)u - u_1u_2u_3$$

$$u_1u_2u_3 = {}^{+}1 \text{ ——— } (1).$$

~~$(u_1u_2+u_2u_3+u_3u_1)$~~

$$(u_1+u_2+u_3) = \frac{(x^{3p}y^{3p}+y^{3p}z^{3p}+z^{3p}x^{3p})}{x^{2p}y^{2p}z^{2p}} \text{ ——— } (2).$$

$$(x^{3p}y^{3p}+y^{3p}z^{3p}+z^{3p}x^{3p})$$

$$= (x^py^p+y^pz^p+z^px^p)^3 - 6x^{2p}y^{2p}z^{2p} - 3\left(x^{2p}y^{3p}z^p + y^{2p}z^{3p}x^p + z^{2p}x^{3p}y^p + x^{2p}y^pz^{3p} + y^{2p}z^px^{3p} + z^{2p}x^py^{3p}\right) \text{ ——— } (3)$$

$$= (x^py^p+y^pz^p+z^px^p)^3 - 6x^{2p}y^{2p}z^{2p} - 3x^py^pz^p\left(x^py^{2p}+y^pz^{2p}+z^px^{2p} + x^pz^{2p}+y^px^{2p}+z^py^{2p}\right). \text{ ——— } (4).$$

~~$(u_1+u_2+u_3) =$~~

$$(u_1u_2+u_2u_3+u_3u_1) = \left(\frac{x^py^py^pz^py^{2p}}{x^{2p}y^{2p}z^{2p}} + \cdots + \cdots\right)$$

$$= \frac{1}{x^py^pz^p}(y^{3p}+z^{3p}+x^{3p}) \text{ ——— } (5)$$

But, if $x^p+y^p+z^p=0$, then

$$x^{3p}+y^{3p}+z^{3p}=(x^p+y^p+z^p)^3-6x^py^pz^p-3\begin{pmatrix}x^{2p}y^p+y^{2p}z^p+z^{2p}x^p\\+x^{2p}z^p+y^{2p}x^p+z^{2p}y^p\end{pmatrix}$$

$$=0^3-6x^py^pz^p-3\begin{bmatrix}x^py^p(x^p+y^p)\\+y^pz^p(y^p+z^p)\\+z^px^p(z^p+x^p)\end{bmatrix}$$

$$=-6x^py^pz^p-3\begin{bmatrix}x^py^p(-z^p)+y^pz^p(-x^p)\\+z^px^p(-y^p)\end{bmatrix}$$

$$=-6x^py^pz^p+9x^py^pz^p=3x^py^pz^p. \quad (6)$$

By (5) and (6),

$$(u_1u_2+u_2u_3+u_3u_1)=+3 \quad (7)$$

In (4),

$$\begin{pmatrix}x^py^{2p}+y^pz^{2p}+z^px^{2p}\\+x^pz^{2p}+y^px^{2p}+z^py^{2p}\end{pmatrix}$$

$$=x^py^p(y^p+x^p)+y^pz^p(z^p+y^p)+z^px^p(x^p+z^p)$$

$$=x^py^p(-z^p)+y^pz^p(-x^p)+z^px^p(-y^p)$$

$$=(-3x^py^pz^p) \quad (8)$$

By (4) and (8),

$$(x^{3p}y^{3p}+y^{3p}z^{3p}+z^{3p}x^{3p})=$$

$$(x^py^p+y^pz^p+z^px^p)^3-6x^{2p}y^{2p}z^{2p}-3x^py^pz^p(-3x^py^pz^p)$$

$$=(x^py^p+y^pz^p+z^px^p)^3+3x^{2p}y^{2p}z^{2p}. \quad (9)$$

Now $(x^p y^p + y^p z^p + z^p x^p)$

$= \frac{1}{2}\left[(x^p y^p + x^p z^p) + (y^p z^p + y^p x^p) + (z^p x^p + z^p y^p)\right]$

$= \frac{1}{2}\left[x^p(y^p + z^p) + y^p(z^p + x^p) + z^p(x^p + y^p)\right]$

$= \frac{1}{2}\left[x^p(-x^p) + y^p(-y^p) + z^p(-z^p)\right]$

$= -\frac{1}{2}(x^{2p} + y^{2p} + z^{2p})$. So what?

Now $(x^p y^p + y^p z^p + z^p x^p)$

$= (x^p y^p - z^{2p})$

$= (y^p z^p - x^{2p})$

$= (z^p x^p - y^{2p})$

So $(x^p y^p + y^p z^p + z^p x^p)^3$

$= (x^p y^p - z^{2p})(y^p z^p - x^{2p})(z^p x^p - y^{2p})$

$= \Big[x^{2p} y^{2p} z^{2p} - x^{2p} y^{2p} z^{2p}$
$- x^p y^p y^p z^p y^{2p} - x^p y^p z^p x^p x^{2p}$
$- y^p z^p z^p x^p z^{2p}$
$+ x^p y^p x^{2p} y^{2p} + y^p z^p y^{2p} z^{2p}$
$+ z^p x^p z^{2p} x^{2p} \Big]$

$= -x^p y^p z^p (x^{3p} + y^{3p} + z^{3p})$
$+ (x^{3p} y^{3p} + y^{3p} z^{3p} + z^{3p} x^{3p})$ —— (10)

By (6) and (10),

$$(x^p y^p + y^p z^p + z^p x^p)^3 = (-x^p y^p z^p)(3x^p y^p z^p) + (x^{3p} y^{3p} + y^{3p} z^{3p} + z^{3p} x^{3p}) \qquad (11)$$

But (11) is equivalent to (9). No progress!

But by (2), (4), (8) and (9),

$$(u_1 + u_2 + u_3) = \frac{(x^p y^p + y^p z^p + z^p x^p)^3}{x^{2p} y^{2p} z^{2p}} + 3, \qquad (12)$$

So $$0 = u^3 - u^2\left[3 + \frac{(x^p y^p + y^p z^p + z^p x^p)^3}{(x^{2p} y^{2p} z^{2p})}\right] + 3u - 1 \qquad (13)$$

$$0 = (u-1)^3 - u^2\left[\frac{(x^p y^p + y^p z^p + z^p x^p)^3}{x^{2p} y^{2p} z^{2p}}\right]$$

$$0 = (u-1)^3 - u^2\left[\frac{(x^p y^p - z^{2p})(y^p z^p - x^{2p})(z^p x^p - y^{2p})}{x^{2p} y^{2p} z^{2p}}\right] \qquad (14)$$

$$0 = (u-1)^3 - u^2\left[(u_1 - 1)(u_2 - 1)(u_3 - 1)\right] \qquad (16)$$

(16) is satisfied by $u = u_1$, $u = u_2$, $u = u_3$.

$$0=(u_1-1)^3-u_1^2(u_1-1)(u_2-1)(u_3-1) \quad \text{(17)}$$

$$(u_1-1)(u_2-1)(u_3-1)=\frac{(u_1-1)^3}{u_1^2} \quad \text{(18)}$$

Similarly,

$$(u_1-1)(u_2-1)(u_3-1)=\frac{(u_2-1)^3}{u_2^2} \quad \text{(19)}$$

and $$(u_1-1)(u_2-1)(u_3-1)=\frac{(u_3-1)^3}{u_3^2} \quad \text{(20)}$$

By (1), $u_3=\frac{1}{u_1u_2}$. (21)

By (20) and (21),

$$(u_1-1)(u_2-1)(u_3-1)=u_1^2u_2^2\left(\frac{1}{u_1u_2}-1\right)^3$$

$$=-\frac{1}{u_1u_2}(u_1u_2-1)^3 \quad \text{(22)}$$

By (18) and (22),

$$\frac{(u_1-1)^3}{u_1^2}+\frac{1}{u_1u_2}(u_1u_2-1)^3=0 \quad \text{(23)}$$

$$\frac{(u_1-1)^3}{(u_1u_2-1)^3}=-\frac{u_1}{u_2} \quad \text{(24)}$$

$$=-\frac{x^{3p}}{z^{3p}} \quad \text{(25)}$$

(25) gives by cube root

$$\frac{(u_1-1)}{(u_1u_2-1)} = -\frac{x^p}{z^p} \qquad (26),$$

$$\frac{\left(\frac{x^p y^p}{z^{2p}}-1\right)}{\left(\frac{y^{2p}}{z^p x^p}-1\right)} = -\frac{x^p}{z^p} \qquad (27),$$

(27) is

$$\frac{(x^p y^p - z^{2p})}{(y^{2p} - z^p x^p)} = -1 \quad \text{known1}$$

Let (16) be

$$(u-1)^3 - 3ku^2 = 0 \qquad (28),$$

Put $v = (u-1-k)$. Then

$$(v+k)^3 = 3k(v+k+1)^2$$

$$v^3 + 3kv^2 + 3k^2v + k^3 = 3kv^2 + 6k(k+1)v + 3k(k+1)^2$$

~~$v^3 = 3k^2v + 2k^3$.~~

$$(a-b)^3 = (a^3-b^3) + 3ab(a-b). \qquad (29)$$

$$v = (a-b). \qquad (30)$$

$$ab = k^2 \qquad (31)$$

$$a^3 - b^3 = 2k^3 \qquad (32)$$

(29) yields $a^3b^3 = k^6$ —— (33)

(30) yields $(a^6 - 2a^3b^3 + b^6) = 4k^6$ —— (34)

(32) + 4 x (31) yields

$(a^6 + 2a^3b^3 + b^6) = 8k^6$

$(a^3+b^3)^2 = 8k^6$ —— (35)

(33) yields

$(a^3+b^3) = \pm 2\sqrt{2}k^3$ —— (36)

(30) and (34) give

$2a^3 = (2k^3 \pm 2\sqrt{2}k^3)$

$a^3 = k^3(1 \pm \sqrt{2})$

$a = (k)\sqrt[3]{(1 \pm \sqrt{2})}$ —— (37)

(30) and (34) also give

$2b^3 = \pm 2\sqrt{2}k^3 - 2k^3$

$b^3 = (\pm\sqrt{2} - 1)k^3$

$b = (k)\sqrt[3]{(\pm\sqrt{2} - 1)}$ —— (38)

By (37) and (38) and (30)

$V = (a-b) = (k)(\sqrt[3]{1 \pm \sqrt{2}}) - (k)(\sqrt[3]{\pm\sqrt{2} - 1})$ —— (39)

$$V^3 + 3kV^2 + 3k^2V + k^3 = 3kV^2 + 6(k+1)kV + 3k(k+1)^2 \quad \text{(29)}$$

$$V^3 = V\left[6k^2 + 6k - 3k^2\right] + \left(3k(k+1)^2 - k^3\right) \quad \text{(30)}$$

$$(a-b)^3 = 3ab(a-b) + (a^3 - b^3) \quad \text{(31)}$$

$$V = a - b \quad \text{(32)}$$

$$ab = (k^2 + 2k) \quad \text{(33)}$$

$$(a^3 - b^3) = \left(3k(k+1)^2 - k^3\right) \quad \text{(34)}$$

$$a^3b^3 = k^3(k+2)^3 \quad \text{(35)}$$

$$(a^6 - 2a^3b^3 + b^6) = \left[3k(k+1)^2 - k^3\right]^2 \quad \text{(36)}$$

$$(a^6 + 2a^3b^3 + b^6) = \left[3k(k+1)^2 - k^3\right]^2 + 4k^3(k+2)^3 = (a^3 + b^3)^2 \quad \text{(37)}$$

$$a^3 = \frac{1}{2}\left[3k(k+1)^2 - k^3 \pm \sqrt{\left[3k(k+1)^2 - k^3\right]^2 + 4k^3(k+2)^3}\right] \quad \text{(38)}$$

$$b^3 = \frac{1}{2}\left[\pm \sqrt{\left[3k(k+1)^2 - k^3\right]^2 + 4k^3(k+2)^3} - \left(3k(k+1)^2 - k^3\right)\right] \quad \text{(39)}$$

$$V = (a-b),\ (a\omega - b\omega^2),\ (a\omega^2 - b\omega), \quad \text{(40)}$$

where $\omega = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$. (41)

The ~~roots~~ V_1, V_2, V_3 in (41) are all real if
$\sqrt{[3k(k+1)^2 - k^3]^2 + 4k^3(k+2)^3}$ is imaginary,
ie if $\left([3k(k+1)^2 - k^3]^2 + 4k^3(k+2)^3\right) < 0.$
ie if $\left[\begin{array}{l}[2k^3+6k^2+3k]^2 \\ +4k^3(k^3+6k^2+12k+8)\end{array}\right] < 0.$
ie if $\left\{\begin{array}{l}\left[\begin{array}{l}4k^6+24k^5+12k^4+36k^4 \\ +36k^3+9k^2\end{array}\right] \\ +[4k^6+24k^5+48k^4+32k^3]\end{array}\right\} < 0$
ie if $(8k^6+48k^5+96k^4+68k^3+9k^2) < 0.$ —— (42)

For (42) to be satisfied, we certainly need $k<0$,
(42) is taken with (28), to show that $(u-1)^3 < 0$, ie $u<1$ (& maybe $u<0$) (& maybe $u<-1$). —— (43)

(42) is
$$\left[\begin{array}{r}8k^2(k^2+3k+\tfrac{3}{2})^2 - 72k^3 - 18k^2 \\ +68k^3+9k^2\end{array}\right] < 0.$$

$$8k^2(k^2+3k+\tfrac{3}{2})^2 < (4k^3+9k^2) \quad \text{—— (44)}$$

$k \neq 0$, so (44) is

$$\left(k^2+3k+\frac{3}{2}\right)^2 < \left(\frac{k}{2}+\frac{9}{8}\right) \quad (45)$$

$$\left[\left(k+\frac{3}{2}\right)^2-\frac{3}{4}\right]^2 < \left(\frac{k}{2}+\frac{9}{8}\right) \quad (46)$$

(46) cannot hold if RHS < 0, ie if $k < -\frac{9}{4}$. So $k > -\frac{9}{4}$ ——— (47)

$$f(\nu) = \left[\left(k+\frac{3}{2}\right)^2-\frac{3}{4}\right]^2-\left(\frac{k}{2}+\frac{9}{8}\right) \quad (48)$$

$$f(-2) = \left[\left(-\frac{1}{2}\right)^2-\frac{3}{4}\right]^2-\left(-1+\frac{9}{8}\right)$$

$$= \left(1+1-\frac{9}{8}\right) \not< 0.$$

So $f(-2) \not< 0$. $k=-2$ is excluded. ——— (49)

$$f\left(\frac{3}{2}\right) = \left[3^2-\frac{3}{4}\right]^2-\left(\frac{3}{4}+\frac{9}{8}\right)$$

$$= \left[\left(\frac{9}{4}\right)^2-\frac{15}{8}\right] = \left(\frac{81}{16} - \frac{30}{16}\right) \not< 0.$$

So $k=\frac{3}{2}$ is excluded ——— (50).

But, by (42), $k < 0$ ——— (51).

The k's making $\left[\left(k+\frac{3}{2}\right)^2-\frac{3}{4}\right]^2$ minimum are $k=\left(-\frac{3}{2}-\frac{\sqrt{3}}{2}\right)$ and $k=\left(-\frac{3}{2}+\frac{\sqrt{3}}{2}\right)$.

$\left(-\frac{3}{2}-\frac{\sqrt{3}}{2}\right) \not> -\frac{9}{4}$. So $k=\left(-\frac{3}{2}-\frac{\sqrt{3}}{2}\right)$ is excluded ——— (52).

For $k=\left(-\frac{3}{2}+\frac{\sqrt{3}}{2}\right)$,

$$f\left(-\frac{3}{2}+\frac{\sqrt{3}}{2}\right) = 0^2-\left(\frac{9}{8}+\frac{1}{2}\left[-\frac{3}{2}+\frac{\sqrt{3}}{2}\right]\right) < 0.$$

So $k=\left(-\frac{3}{2}+\frac{\sqrt{3}}{2}\right)$ is included ——— (53).

By (17),

$$0 = (u_1 - 1)^3 - u_1^2(u_1 - 1)(u_2 - 1)(u_3 - 1) \quad \text{——— (17)}.$$

So, $u_1 \neq 0$ by x, y, z relatively prime, so

$$(u_1 - 1)^2 = u_1^2(u_2 - 1)(u_3 - 1) \quad \text{——— (54)}.$$

$$u_1^2[1 - (u_2 - 1)(u_3 - 1)] - 2u_1 + 1 = 0.$$

$$u_1 = \frac{+2 \pm \sqrt{4 - 4(1)[1 - (u_2 - 1)(u_3 - 1)]}}{2[1 - (u_2 - 1)(u_3 - 1)]} \quad \text{——— (55)}.$$

Therefore, u_1 being rational,

$\{4 - 4(1)[1 - (u_2 - 1)(u_3 - 1)]\}$ is a perfect square.

$(u_2 - 1)(u_3 - 1)$ is a perfect square ——— (56).

So is $(u_3 - 1)(u_1 - 1)$ ——— (57).

So is $(u_1 - 1)(u_2 - 1)$ ——— (58).

Is $(u_1 - 1)(u_2 - 1)(u_3 - 1)$ a perfect square? So then, by (17), $(u_1 - 1)$ is a perfect square. And so is $(u_2 - 1)$, $(u_3 - 1)$. Logical?

$$\frac{(x^p y^p - z^{2p})}{z^{2p}} = \lambda^2, \ \lambda \text{ rational. Maybe?}$$

$$(x^p y^p - z^{2p}) = A^2, \ A \text{ integral. Maybe?} \quad \text{——— (59)}$$

But this (59) is impossible if $0 < |x| < |y| < |z|$.

Proved result? No! see over.

By (17),

$$(u_1-1)^3 = \frac{u_1^2}{x^{2p}y^{2p}z^{2p}}(xy - z^{2p})^3$$

$\frac{u_1^2}{x^{2p}y^{2p}z^{2p}}$ is a perfect cube of rational.

$\frac{u_1}{x^p y^p z^p}$ is a perfect cube of a rational.

$\frac{1}{z^{3p}}$ is a perfect cube of a rational.
Known!

We get $|z|^{2p} - x^p y^p = A^2$. $A > 0$.

$$(|z|^p - A)(|z|^p + A) = x^p y^p.$$

$x^p = |z|^p - A$ maybe and $|z|^p + A = y^p$ maybe.

So $x^p + y^p = 2|z|^p$. } maybe, so only
But $x^p + y^p = |z|^p$. } zero solutions.

Probably not a proof

From (55),

$$\{u_1 - u_1(u_2-1)(u_3-1)\} = 1 \pm \sqrt{(u_2-1)(u_3-1)}.$$
$$= 1 \pm \frac{(x^p y^p - z^{2p})}{x^p y^p}.$$
$$= \frac{z^{2p}}{x^p y^p} \text{ or } \left(2 - \frac{z^{2p}}{x^p y^p}\right). \quad (60)$$

$$= [u_1 - u_1 u_2 u_3 + (u_2+u_3)u_1 - u_1].$$

$$= [-1 + \ldots (u_1u_2 + u_1u_3 + u_2u_3) - u_2u_3]$$

$$= [-1 + 3 - u_2u_3] = (2 - u_2u_3) \quad (61)$$

$$(2 - u_2u_3) = \left(2 - \frac{z^{2p}}{x^p y^p}\right) \text{ or } - \frac{z^{2p}}{x^p y^p}. \quad (62), (63)$$

(62) is clearly true by $u_1u_2u_3 = 1$ and

$$u_1 = \frac{x^p y^p}{z^{2p}}.$$

The other root is $(2 - u_2u_3) = \frac{1}{u_1}$.

$$(2u_1 - u_1u_2u_3) = -1.$$

$$2u_1 = -2.$$

$u_1 = -1$. Impossible because x, y, z are relatively prime.

Look at $(x^p y^p + y^p z^p + z^p x^p) = u$

$$= (x^p y^p - z^{2p}) = (y^p z^p - x^{2p})$$

$$= (z^p x^p - y^{2p}).$$

$$3u = (x^p y^p + y^p z^p + z^p x^p - x^{2p} - y^{2p} - z^{2p})$$

$$= 3(x^p y^p + y^p z^p + z^p x^p) - (x^p + y^p + z^p)^2$$

$u = u$. Known!

Look at $u = (x^p y^p + y^p z^p + z^p x^p)$

$$= (x^p y^p - z^{2p}) = (y^p z^p - x^{2p}) = (z^p x^p - y^{2p})$$

$$3u = (x^p y^p + y^p z^p + z^p x^p - x^{2p} - y^{2p} - z^{2p})$$

~~$3u \equiv 0 (\mathrm{mod}$~~

$u \equiv 1 \pmod 2$ ——— (65)

$3u \equiv 3 \pmod 6$. ——— (66)

If $3 | xyz$, then $u \equiv 2 \pmod 3$. ——— (67)

So $2u \equiv 4 \pmod 6$ ——— (68)

So, by (66) & (68),

$u \equiv -1 \pmod 6$ ——— (69).

$(x^{2p} + y^{2p} + z^{2p}) \equiv (4+3+1) \pmod 6$ if $2|x$ & $3|y$.

$\equiv 2 \pmod 6$,

$3u \equiv (u-2) \pmod 6$; true if $u \equiv -1 \pmod 6$,

$(x^{2p} + y^{2p} + z^{2p}) \equiv (0+1+1) \pmod 6$ if $6|x$.

$\equiv 2 \pmod 6$. Same.

By (18) and (19)

$$\frac{(u_1-1)^3}{u_1^2} = (u_1-1)(u_2-1)(u_3-1) = \frac{(u_2-1)^3}{u_2^2}.$$

$$\frac{\left(\frac{x^p y^p}{z^{2p}} - 1\right)^3}{\left(\frac{x^p y^p}{z^{2p}}\right)^2} = \frac{\left(\frac{y^p z^p}{x^{2p}} - 1\right)^3}{\left(\frac{y^p z^p}{x^{2p}}\right)^2}$$

By (18) and (19),

$$\frac{(u_1-1)^3}{u_1^2} = \frac{(u_2-1)^3}{u_2^2}. \quad \text{——— (70).}$$

(70) is

$$u_1(u_3^3 u_2^3)(u_1-1)^3 = u_2(u_3^3 u_1^3)(u_2-1)^3$$

$$u_1\left(1-\frac{1}{u_1}\right)^3 = u_2\left(1-\frac{1}{u_2}\right)^3,$$

$$\frac{u_1}{u_2} = \left(\frac{1-\frac{1}{u_2}}{1-\frac{1}{u_1}}\right)^3$$

$\frac{u_1}{u_2}$ is a perfect cube of a rational.

$$\frac{\left(\frac{x^p y^p}{z^{2p}}\right)}{\left(\frac{y^p z^p}{x^{2p}}\right)} \text{ is } \text{-----------}.$$

From (55),

$$u_1 = \frac{1 \pm \sqrt{(u_2-1)(u_3-1)}}{[1-(u_2-1)(u_3-1)]}$$

$$= \frac{1 \pm \frac{(x^p y^p - z^{2p})}{x^p y^p}}{1 - \frac{(x^p y^p - z^{2p})^2}{x^{2p} y^{2p}}}$$

$$= \frac{1}{1 - \frac{(x^p y^p - z^{2p})}{x^p y^p}} \text{ or } \frac{1}{1 + \frac{(x^p y^p - z^{2p})}{x^p y^p}}.$$

$$= \frac{x^p y^p}{z^{2p}} \text{ or } \frac{x^p y^p}{(2x^p y^p - z^{2p})}. \quad \text{——— (71)}$$

By (71), unorthodox ~~root~~,

$$u_1 = \frac{x^p y^p}{(2x^p y^p - z^{2p})} = - \frac{x^p y^p}{(x^{2p} + y^{2p})} \quad \text{——— (72)}$$

So $u_2 = - \dfrac{y^p z^p}{(y^{2p} + z^{2p})}$ ——— (73)

and $u_3 = - \dfrac{z^p x^p}{(z^{2p} + x^{2p})}$ ——— (74)

(72), (73) and (74) with $u_1 u_2 u_3 = 1$ ——— (75) and (1)

show ~~that~~

$$(x^{2p}+y^{2p})(y^{2p}+z^{2p})(z^{2p}+x^{2p}) = -x^{2p}y^{2p}z^{2p} \quad \text{——— (76)}$$

This cannot hold for real x, y, z.

Is this a proof ?

Otherwise, $u_1 u_2 u_3 = 1$ gives

$$\left(-\frac{x^p y^p}{(x^{2p}+y^{2p})}\right)\left(\frac{y^p z^p}{x^{2p}}\right)\left(\frac{z^p x^p}{y^{2p}}\right) = 1, \text{ ie } x^{2p}+y^{2p}+z^{2p} = 0.$$

Impossible.

OR $\left(-\dfrac{x^p y^p}{(x^{2p}+y^{2p})}\right)\left(-\dfrac{y^p z^p}{(y^{2p}+z^{2p})}\right)\left(\dfrac{z^p x^p}{y^{2p}}\right) = 1,$

ie $(x^{2p}+y^{2p})(y^{2p}+z^{2p}) = x^{2p}z^{2p},$

ie $y^{2p}(y^{2p}+x^{2p}+z^{2p}) = 0.$ Impossible.

Is this a proof ? of inconsistency ?

By (14), $(u-1)^3 = u^2 \frac{(x^p y^p + y^p z^p + z^p x^p)^3}{x^{2p} y^{2p} z^{2p}}$ —— (14).

But $2(x^p y^p + y^p z^p + z^p x^p) = (x^p + y^p + z^p)^2 - (x^{2p} + y^{2p} + z^{2p})$

$= D^2 - (x^{2p} + y^{2p} + z^{2p})$ —— (77)

$2(x^p y^p - z^{2p}) = -(x^{2p} + y^{2p} + z^{2p})$. —— (78).

~~$(x^p + y^p)^2 = z^{2p}$.~~

Similarly,

$2(y^p z^p - x^{2p}) = -(x^{2p} + y^{2p} + z^{2p})$ —— (79)

and $2(z^p x^p - y^{2p}) = -(x^{2p} + y^{2p} + z^{2p})$ —— (80)

Add (77), (78) & (79),

$\left.\begin{matrix} 2(x^p y^p + y^p z^p + z^p x^p) \\ -2(z^{2p} + x^{2p} + y^{2p}) \end{matrix}\right\} = -3(x^{2p} + y^{2p} + z^{2p})$ —— (81).

This brings us to (7?).

(24) & (25) give $\frac{(u_1-1)^3}{(u_1 u_2 - 1)^3} = -\frac{u_1}{u_2} = -\frac{x^{3p}}{y^{3p} z^{3p}}$ —— (24) & (25)

$\times \frac{1}{u_3^3}$: $\frac{(u_1-1)^3}{(u_1 u_2 u_3 - u_3)^3} = \left(-\frac{x^{3p}}{y^{3p}}\right) \times \left(\frac{y^{2p}}{x^p z^p}\right)^3$

$\frac{(u_1-1)^3}{(u_3-1)^3} = + \frac{y^{3p}}{z^{3p}}$. —— (82)

(82) gives $\frac{(u_1-1)}{(u_3-1)} = \frac{y^p}{z^p}$. ——— (83).

Similarly, $\frac{(u_2-1)}{(u_1-1)} = \frac{z^p}{x^p}$ ——— (84).

and $\frac{(u_3-1)}{(u_2-1)} = \frac{x^p}{y^p}$ ——— (85).

(83) is $\frac{y^p}{z^p} = \frac{\left(\frac{x^p y^p}{z^{2p}} - 1\right)}{\left(\frac{z^p x^p}{y^{2p}} - 1\right)} = \frac{y^{2p}}{z^{2p}}$ ——— (86)

(86) shows ~~that~~ $z^p = y^p$ ——— (87).

~~Impossible~~ by x, y, z relatively prime.

(There was a misprint on side (17)).

$z < 0$; $x, y > 0$. $|z| > |x|$; $|z| > |y|$. ~~$0 < x < y$.~~

$u_1 = \frac{x^p y^p}{z^{2p}} = \frac{x^p y^p}{(x^{2p} + 2x^p y^p + y^{2p})}$

$= \frac{x^p y^p}{(x^p - y^p)^2 + 4x^p y^p} > 0$

$< \frac{1}{4}$.

$u_2 < 0$; $u_2 = \frac{y^p z^p}{x^{2p}} = \frac{y^p z^p}{(z^p - y^p)^2 + 4y^p z^p}$

$< \frac{1}{4}$.

$|u_2| = \frac{y^p |z|^p}{(|z|^p - y^p)^2}$

$= + \frac{y^p |z|^p}{(|z|^p + y^p)^2 - 4y^p |z|^p}$.

$$|u_2| = \frac{y^p|z|^p}{(|z|^p+y^p)^2 - 4y^p|z|^p} > 0 \quad > \frac{y^p|z|^p}{(8y^p|z|^p - 4y^p|z|^p)}$$

We used the hypothesis $= \frac{1}{4}$. So $u_2 < -\frac{1}{4}$.

$(|z|^p+y^p)^2 < 8y^p|z|^p$.

$(|z|^p-y^p)^2 < 4y^p|z|^p$.

$x^{2p} < 4y^p|z|^p$ even if $0 < y < x < |z|$.

$x^{2p} < \{(|z|^p+y^p)^2 - (|z|^p-y^p)^2\}$

$= \{(|z|^p+y^p)^2 - x^{2p}\}$

$2x^{2p} < (|z|^p+y^p)^2$ even if $0 < y < x < |z|$ …

This is hypothetical.

~~Actually $4x^{2p} < (|z|^p+y^p)^2$ if $0 < x < y$.~~

$3x^{2p} < 4y^p|z|^p$ where from? (88).

$3(|z|^p-y^p)^2 < 4y^p|z|^p$ —— (89).

$3(|z|^p+y^p)^2 < (4+3\times 4)\, y^p|z|^p$

$3(|z|^p+y^p)^2 < 16y^p|z|^p$.

$3(2y^p+x^p)^2 < 16y^p(y^p+x^p)$.

$(12y^{2p}+12y^px^p+3x^{2p}) < (16y^{2p}+16xy$ …

$(4y^{2p}+4x^py^p-3x^{2p}) > 0$. REALLY …

$\{(y^p+\frac{1}{2}x^p)^2 - \frac{3}{4}x^{2p} - \frac{1}{4}x^{2p}\} > 0$

$(y^p+\frac{1}{2}x^p) > x^p$, $y^p > \frac{1}{2}x^p$, $0 < y < x$ …

$4|z|^p y^p = \{(|z|^p + y^p)^2 - (|z|^p - y^p)^2\}$
$= \{(2y^p + x^p)^2 - x^{2p}\}$
$= 4y^{2p} + 4y^p x^p = 4y^p|z|^p$. yes!

$4|z|^p y^p = \{(|z|^p + y^p)^2 - (|z|^p - y^p)^2\}$

If $y > x > 0 > z$, ~~then~~ $2y^p > |z|^p$, so

$2|z|^{2p} < 4y^p|z|^p = \{(2y^p + x^p)^2 - x^{2p}\}$

$2(y^p + x^p)^2 < \{4y^{2p} + 4y^p x^p\}$

$\{2y^{2p} + 4y^p x^p + 2x^p\} < \{4y^{2p} + 4y^p x^p\}$

$2x^{2p} < 2y^{2p}$

$x < y$. ~~Contradiction!~~ ~~Inconsistency?~~