FLT(1).

~~Let~~ $p$ be an odd prime and ~~let~~ $x, y, z$ be relatively prime integers ~~such that~~ $x^p+y^p+z^p=0$.

~~Put~~ $u_1 = \frac{z^{2p}}{x^p y^p}$, $u_2 = \frac{x^{2p}}{y^p z^p}$, $u_3 = \frac{y^{2p}}{z^p x^p}$.

$u = u_1, u_2, u_3$ are ~~roots~~ of

$$0 = u^3 - (u_1+u_2+u_3)u^2 + (u_1u_2+u_2u_3+u_3u_1)u - u_1u_2u_3.$$

~~But~~ $u_1u_2u_3 = 1$;

$u_1+u_2+u_3 = 3$;

$u_1u_2+u_2u_3+u_3u_1 = 3 + (u_1-1)(u_2-1)(u_3-1)$.

$$0 = (u-1)^3 + (u_1-1)(u_2-1)(u_3-1)u$$

~~Put~~ $u = u_1$, cancel $(u_1-1)$ as $u_1 \neq 1$.

$$\begin{aligned}(u_1-1)^2 &= -u_1(u_2-1)(u_3-1)\\ &= -u_1u_2u_3 + (u_1)(u_2+u_3) - u_1\\ &= -1 + u_1(3) - u_1^2 - u_1.\\ &= -1 + 2u_1 - u_1^2 = -(u_1-1)^2.\end{aligned}$$

~~Is it~~ $u_1u_2+u_2u_3+u_3u_1 = 3 - (u_1-1)(u_2-1)(u_3-1)$.

~~If it~~ is, ~~then~~

$$(u-1)^3 = u(u_1-1)(u_2-1)(u_3-1).$$

~~Put~~ $u = u_1$ and cancel $(u_1-1)$; ~~get~~

$$(u_1-1)^2 = u_1(u_2-1)(u_3-1).$$

$$u_1^2 - \{2 + (u_2-1)(u_3-1)\}u_1 + 1 = 0. \quad \text{------} (1)$$

~~smellone~~.

Having fixed ~~that~~ $u_1 u_2 u_3 = 1$ and ~~that~~ $u_2 = \frac{x^{2p}}{y^p z^p}$ and $u_3 = \frac{y^{2p}}{z^p x^p}$, there is only one possibility for $u_1$, so (1) has a repeated root.

$$\{2 + (u_2 - 1)(u_3 - 1)\}^2 = 4 \quad \text{------------ (2)}$$

$$(u_2 - 1)(u_3 - 1)[4 + (u_2 - 1)(u_3 - 1)] = 0 \quad \text{-------- (3)}$$

$0 = [\ ]$ yields $4 + (1 - (u_2 + u_3) + u_2 u_3) = 0$

$$5 - 3 + u_1 + \frac{1}{u_1} = 0$$

$u_1^2 - 2u_1 + 1 = 0$. So $u_1 = 1$, ie $z^{2p} = x^p y^p$ --- (4)

~~$u_1 = \frac{1}{2} + \frac{\sqrt{3}}{2}$, not rational --- (4).~~

Otherwise, (3) yields $u_2 = 1$, ie $x^{2p} = y^p z^p$ --- (5)

or $u_3 = 1$, ie $y^{2p} = z^p x^p$ --- (6).

In (5) & (6), $x, y, z$ are not relatively prime.

So initial assumptions contradicted.

Is this a proof?

If $z^{2p} = x^p y^p$ and $-z^p = (x^p + y^p)$, then

$$x^p y^p = z^{2p} = (-z^p)^2 = (x^p + y^p)^2$$

$$(x^p + \tfrac{1}{2} y^p)^2 + \tfrac{3}{4} y^{2p} = 0 \quad \text{------------ (7)}$$

So $x = 0$, $y = 0$, $z = 0$ ------------ (8).

Again, in $x^{2p} = y^p z^p$, if $y > x > 0 > z$, then $x^{2p} < 0$.

A propos the repeated root, $u_1 = \frac{z^{2p}}{x^p y^p}$ fits (1) and maybe so does $\frac{x^p y^p}{z^{2p}} = u_1$. Sad that it seems not to be repeated root.

Solve (1):

$$u_1 = \frac{2+(u_2-1)(u_3-1) \pm \sqrt{\{2+(u_2-1)(u_3-1)\}^2 - 4(1)(1)}}{2} \qquad (9)$$

In (9), $\{2+(u_2-1)(u_3-1)\}^2 - 4$

$$= 4(u_2-1)(u_3-1) + (u_2-1)^2(u_3-1)^2$$

$$= \{4+(u_2-1)(u_3-1)\}(u_2-1)(u_3-1)$$

$$= \frac{(x^p y^p + y^p z^p + z^p x^p)^2}{(y^p z^p)^2 (z^p x^p)^2}\left[4(y^p z^p)(z^p x^p) + (x^p y^p + y^p z^p + z^p x^p)^2\right] \qquad (10)$$

$$[\;] = (x^p y^p - z^{2p})^2 + 4(x^p y^p)(z^{2p})$$

$$= (x^p y^p + z^{2p})^2 \qquad (11)$$

So $$\sqrt{\;} = \frac{(x^p y^p + y^p z^p + z^p x^p)}{x^p y^p z^{2p}}(x^p y^p + z^{2p}). \qquad (12)$$

(10), (11) & (12) into (9) to yield

$$u_1 = \frac{2 + \frac{(x^p y^p - z^{2p})^2}{x^p y^p z^{2p}} \pm \frac{(x^p y^p - z^{2p})(x^p y^p + z^{2p})}{x^p y^p z^{2p}}}{2}$$

$$= \frac{(x^{2p} y^{2p} + z^{4p}) \pm (x^{2p} y^{2p} - z^{4p})}{2x^p y^p z^{2p}}$$

FLT (4)

$$= \frac{x^p y^p}{z^{2p}} \text{ or } \frac{z^{2p}}{x^p y^p}.$$

[redacted]

If it is $(u-1)^3 = (u_1-1)(u_2-1)(u_3-1)u$, ~~put~~ $u=(v+1)$,
~~get~~ $v^3 = 3k(v+1)$ ---------- (26)

$v=(a-b)$, $(a-b)^3 = 3(ab)(a-b) + (a^3-b^3)$,

$ab = k$ ---------- (13),

$(a^3-b^3) = 3k$ ---------- (14),

$(a^6 - 2a^3b^3 + b^6) = 9k^2$ ---------- (15),

$4a^3b^3 = 4k^3$ ---------- (16),

$(a^6 + 2a^3b^3 + b^3) = (9k^2 + 4k^3)$

$(a^3+b^3)^2 = k^2(9+4k)$ ---------- (17),

$(9+4k)$ is a perfect square,

$(9+4k) = \lambda^2$ ---------- (18).

$27 + 4(3k) = 3\lambda^2$

$27 + 4(u_1-1)(u_2-1)(u_3-1) = 3\lambda^2$

$3\lambda^2 = 27 + 4\left\{u_1u_2u_3 - 1 + (u_1+u_2+u_3) - \begin{pmatrix} u_1u_2+u_2u_3 \\ +u_3u_1 \end{pmatrix}\right\}$

$= 27 + 4\{1 - 1 + 3 - (u_1u_2 + u_2u_3 + u_3u_1)\}$ ---------- (22)

Now

$(a^3+b^3) = k\lambda$ ---------- (19).

(14) + (19) give $a^3 = \frac{1}{2}(3k + k\lambda)$

So $a^3 = k\left(\frac{3}{2} + \frac{\lambda}{2}\right)$ ---------- (20)

and $b^3 = \frac{1}{2}(3k - k\lambda) = k\left(\frac{3}{2} - \frac{\lambda}{2}\right)$ ------ (21).

So to v, then u?

In (22), $3k = \{1 - 1 + 3 - (u_1 u_2 + u_2 u_3 + u_3 u_1)\}$

$$= 3 - u_1(u_2 + u_3) - u_2 u_3$$

$$= 3 - 3u_1 + u_1^2 - u_2 u_3$$

$$= \left(u_1 - \frac{3}{2}\right)^2 + \frac{3}{4} - u_2 u_3.$$

$$= \left(u_1 - \frac{3}{2}\right)^2 + \frac{3}{4} - \frac{1}{u_1}. \text{ ---------- (23)}$$

If $y > x > 0 > z$, $u_1 = \frac{z^{2p}}{x^p y^p} > 0$, $u_2 < 0$, $u_3 < 0$,

but $(u_1 + u_2 + u_3) = 3$, so $u_1 > 3$.

So by (23), $3k > \left(3 - \frac{3}{2}\right)^2 + \frac{3}{4} - \frac{1}{2}$

$$= \frac{9}{4} + \frac{3}{4} - \frac{2}{4} = \frac{10}{4}$$

$k > \frac{5}{6}$. ---------- (24)

So $3k > \frac{5}{2}$ ---------- (25).

I seem to remember getting $k < 0$ elsewhere.

Was that a different $k$?

$$u_2 + u_3 = \frac{x^{2p}}{y^p z^p} + \frac{y^{2p}}{z^p x^p} = \frac{1}{x^p y^p z^p}(x^{3p} + y^{3p})$$

$$= \frac{(x^p + y^p)}{x^p y^p z^p}(x^{2p} - x^p y^p + y^{2p}) = -\frac{1}{x^p y^p}(x^{2p} - x^p y^p + y^{2p})$$

$$= -u_1 + 3$$

So in (18), $(q+4k)=\lambda^2$, $\lambda$ is real.
So the cubic (26) has only one real root, and 2 complex roots (which are conjugate to each other).
But these 2 complex roots for v are $(u_2+1)$ and $(u_3+1)$, which are real.
Is this a proof?