6.

Let $p > 2$ be prime and let x, y, z be relatively prime integers such that $x^p + y^p + z^p = 0$.

So $(x+y+z) \equiv 0 \pmod{p}$ ——— (1)

This follows from Fermat's little Theorem.

Square both sides of (1) to obtain

$(x^2+y^2+z^2+2xy+2yz+2zx) \equiv 0 \pmod{p}$ — (2)

Now look at $(xy+yz+zx)$. By using (1), we obtain

$$\begin{aligned}(xy+yz+zx) &\equiv (xy-z^2) \pmod{p} \\ &\equiv (yz-x^2) \pmod{p} \\ &\equiv (zx-y^2) \pmod{p} \quad \text{—— (3)}\end{aligned}$$

Use 3x(2) and 2x(3):

$$\begin{aligned}0 &\equiv 3(x^2+y^2+z^2)+6(xy+yz+zx) \pmod{p} \\ &\equiv 3(x^2+y^2+z^2)+2(xy-z^2) \\ &\quad +2(yz-x^2) \\ &\quad +2(zx-y^2) \pmod{p}\end{aligned}$$

Add all three versions of (3):

$$3(xy+yz+zx) \equiv [(xy-z^2)+(yz-x^2)+(zx-y^2)] \pmod{p}$$

This is

$$2(xy+yz+zx)+(x^2+y^2+z^2) \equiv 0 \pmod{p}.$$

From (1), $(x+y) \equiv -z \pmod{p}$.

Square this:

$$(x^2 + 2xy + y^2) \equiv z^2 \pmod{p} \quad \text{——(4)}$$

Similarly,

$$(y^2 + 2yz + z^2) \equiv x^2 \pmod{p} \quad \text{——(5)}$$

and $(z^2 + 2zx + x^2) \equiv y^2 \pmod{p}$ ——(6).

Subtract (4) from (2):

$$z^2 + 2zx + 2yz \equiv -z^2 \pmod{p}.$$

$$2z^2 + 2zx + 2zy \equiv 0 \pmod{p}.$$

This is $2z \times (1)$.

Let $S \equiv (xy + yz + zx) \pmod{p}$ ——(7)

This $S \equiv (xy - z^2) \pmod{p}$ ——(8)

$\equiv (yz - x^2) \pmod{p}$ ——(9)

$\equiv (zx - y^2) \pmod{p}$ ——(10).

These use $(x+y)z \equiv -z^2 \pmod{p}$, etc, from (1).

~~Add (8) + (9) + (10):~~

~~$3S \equiv (xy + yz + zx - x^2 - y^2 - z^2) \pmod{p}$~~

Look at (1) and $(x+y-z) \equiv -2z \pmod{p}$ ——(11)

Square both sides of (11):

$(x^2+y^2+z^2+2xy-2yz-2zx) \equiv +4z^2 \pmod{p}$

Using $y+x \equiv -z \pmod{p}$ from (1), ——(12)

(12) becomes

$(x^2+y^2+z^2+2xy+2z^2) \equiv 4z^2 \pmod{p}$ ——(12')

This is

$(x^2+y^2+z^2+2xy) \equiv 2z^2 \pmod{p}$ ——(13)

Similarly,

$(x^2+y^2+z^2+2yz) \equiv 2x^2 \pmod{p}$ ——(14)

and $(x^2+y^2+z^2+2zx) \equiv 2y^2 \pmod{p}$ ——(15)

~~[illegible]~~

By (8), $xy \equiv (S+z^2) \pmod{p}$ ——(8')

So in (12'), we get $(x^2+y^2+z^2+2S) \equiv 0 \pmod{p}$.

(13) is $(x^2+y^2-z^2+2xy) \equiv 0 \pmod{p}$ ——(13')

This by (8) is

$(x^2+y^2+S[illegible]+xy) \equiv 0 \pmod{p}$ ——(16)

However, (13) also gives from (13') and (8):

$(x^2+y^2+2S+z^2) \equiv 0 \pmod{p}$ KNOWN!

An analogue of (16) is

$(y^2+z^2+S+yz) \equiv 0 \pmod{p}$ ——(17)

Subtract

(16) − (17) : $(x^2 - z^2 + xy - yz) \equiv 0 \pmod{p}$

This is $(x-z)(x+z+y) \equiv 0 \pmod{p}$ KNOWN! ——— (18)

Now look at

$$(x^2+y^2+z^2+2xy+2yz+2zx) \equiv 0 \pmod{p} \quad \text{——— (2)}$$

and multiply by 3:

$$3(x^2+y^2+z^2)+(6xy+6yz+6zx) \equiv 0 \pmod{p}$$

Add (8) + (9) + (10): ——— (25).

$$3S \equiv (xy+yz+zx-x^2-y^2-z^2) \pmod{p}$$

$$\equiv (S - x^2 - y^2 - z^2) \pmod{p}.$$

$$2S \equiv (-x^2-y^2-z^2) \pmod{p}.$$

$$6S \equiv 3(-x^2-y^2-z^2) \pmod{p}.$$

Put it in (25) to obtain $0 \equiv 0 \pmod{p}$ KNOWN!