1.

Legendre's Theorem.

Let x, y, z be relatively prime (positive) integers and let p be an odd prime and let $z = (y+1)$ and $z^p = y^p + x^p$ and $p \mid xyz$. Examine the case $p^k \mid y$, $p^{(k+1)} \nmid y$, $k \geq 1$ and k integral. Note that $(y+1) \equiv z \equiv (y+x) \pmod{p}$,
So $x \equiv 1 \pmod{p}$, so $p \nmid x$.

Then let $y = Bp^k$, $p \nmid B$, B integral.

So $(z-x) \mid y^p = B^p p^{pk}$

Put $(z-x) = b^p p^{(pk-1)}$, b integral, $b \mid B$, $B = b\beta$.

$$(y+1-x) = b^p p^{(pk-1)} \quad \text{------ (1)}$$

$$y = b\beta p^k \quad \text{------ (2)}$$

(1) and (2) show that

$$x = [1 + b\beta p^k - b^p p^{(pk-1)}] \quad \text{------ (3)}$$

$$y = b\beta p^k \quad \text{------ (2)}$$

$$z = (1 + b\beta p^k) \quad \text{------ (4)}$$

But $z^p = y^p + x^p$. So

$$(1+b\beta p^k)^p = b^p \beta^p p^{kp} + [(1+b\beta p^k) - b^p p^{(pk-1)}]^p$$

$$\text{------ (5)}.$$

OVER.

Expand:

$$0 \equiv b^p\beta^p\rho^{pk} - (1+b\beta\rho^k)^{(p-1)}(b^p\rho^{(pk-1)})\binom{p}{1}$$

$$+\binom{p}{2}(1+b\beta\rho^k)^{(p-2)}(b^p\rho^{(pk-1)})^2$$

$$-\binom{p}{3}(1+b\beta\rho^k)^{(p-3)}(b^p\rho^{(pk-1)})^3$$

$$+ \cdots$$

$$-(b^p\rho^{(pk-1)})^p \text{ ---------- } (6).$$

By (6), $[b^p\beta^p\rho^{pk} - (b^p\rho^{(pk-1)})^p] \equiv 0 \pmod{1+b\beta\rho^k}$ ---- (7).

Divide (7) by $b^p\rho^{pk}$,

$$[\beta^p - b^{p(p-1)}\rho^{(p(p-1)k-p)}] \equiv 0 \pmod{z} \text{ --- } (8)$$

$$[\beta^p - (z-x)^{(p-1)}\rho^{-1}] \equiv 0 \pmod{z} \text{ --- } (9)$$

(9) × ρ,

$$[\rho\beta^p - (z-x)^{(p-1)}] \equiv 0 \pmod{z}.$$

So $[\rho\beta^p - x^{(p-1)}] \equiv 0 \pmod{z}$ --- (10).

(10) × x, $[x\rho\beta^p - x^p] \equiv 0 \pmod{z}$.

So $[x\rho\beta^p - (z^p - y^p)] \equiv 0 \pmod{z}$.

Legendre's Theorem.

So $[x\rho\beta^p + y^p] \equiv 0 \pmod{z}$,

$$[x\rho\beta^p + b^p\beta^p\rho^{kp}] \equiv 0 \pmod{z} \quad \text{------ (11)}$$

(11) $\div\ \rho\beta^p$:

$$[x + b^p\rho^{(kp-1)}] \equiv 0 \pmod{z} \quad \text{------ (12)}$$

$$z \equiv 0 \pmod{z}, \text{ known!}$$

From (6), modulo $b^{2p}\rho^{(2pk-1)}$,

$$0 \equiv \left[b^p\beta^p\rho^{pk} - (1+b\beta\rho^k)^{(p-1)}(b^p\rho^{(pk-1)})(P_1)\right] (\text{mod}) \quad \text{------ (13)}$$

(13) $\div\ b^p\rho^{pk}$:

$$\beta^p \equiv (1+b\beta\rho^k)^{(p-1)} \pmod{b^p\rho^{(pk-1)}} \quad \text{---- (14)}$$

More later on this!

Also from (6), modulo β:

$$0 \equiv \left\{\begin{array}{l} -(P_1)(b^p\rho^{(pk-1)}) \\ +(P_2)(b^p\rho^{(pk-1)})^2 \\ -(P_3)(b^p\rho^{(pk-1)})^3 \\ \pm \cdots (b^p\rho^{(pk-1)})^p \end{array}\right\} \pmod{\beta} \quad \text{----- (15)}$$

$$1^p \equiv \{1 - b^p\rho^{(pk-1)}\}^p \pmod{\beta} \quad \text{------ (16)}$$

Taking pth roots modulo γ, a prime factor of β,

$$1 \equiv \alpha\left(1 - b^p p^{(pk-1)}\right) \pmod{\gamma} \qquad (17)$$

Where $\alpha^p \equiv 1 \pmod{\gamma}$, $\alpha\bar{\alpha} \equiv 1 \pmod{\gamma}$, so $\bar{\alpha} = \alpha^{(p-1)}$,

$$\alpha^{(p-1)} \equiv \left(1 - b^p p^{(pk-1)}\right) \pmod{\gamma} \qquad (18)$$

$$b^p p^{(pk-1)} \equiv \left(1 - \alpha^{(p-1)}\right) \pmod{\gamma} \qquad (19)$$

More on this later!