On Legendre's Thm relating to FLT.

Fermat's Last Thm states that for $n>2$ and integral, there are no integer solutions x, y, z of $x^n+y^n=z^n$ with $xyz\neq 0$. That this is so has been proved by research in Wiles and in Taylor and Wiles. It is not necessary to read these two long and difficult papers in order to understand this paper.

Legendre's Thm was a conjecture that if there were integer solutions x, y, z of $x^n+y^n=z^n$ with $z>y>x>0$ and $n>2$ and integral, then $z\neq(y+1)$. The idea was to prove this without proving Fermat's Last Thm. This paper achieves this.

Lemma 1 (due to Fermat). If $z>y>x>0$ are integers, then $z^4\neq y^4+x^4$.

A proof was found in the effects of the seventeenth century French mathematician, Pierre de Fermat.

Lemma 2. Let n be an even integer, $n=2m$, m an odd integer. Then $x^n+y^n=z^n$ has no relatively prime solutions x, y, z such that $z>y>x>0$, and $z=(y+1)$.

Proof. $x^n=z^n-y^n$ Let there be such a solution.

$$=(z-y)(z^{(n-1)}+yz^{(n-2)}+\dots+y^{(n-1)})$$

$$x^n=z^{(n-1)}+yz^{(n-2)}+\dots+y^{(n-1)} \quad \text{------} (1)$$

So $x^n = (z-y)(z^{(n-2)} + 2yz^{(n-3)} + 3y^2z^{(n-4)} + \dots + (n-1)y^{(n-2)}) + ny^{(n-1)}$

$x^n = (z^{(n-2)} + 2yz^{(n-3)} + 3y^2z^{(n-4)} + \dots + (n-1)y^{(n-2)}) + ny^{(n-1)}$ -------- (2).

Similarly, from (1),

$x^n = (y-z)(y^{(n-2)} + 2zy^{(n-3)} + \dots + (n-1)z^{(n-2)}) + nz^{(n-1)}$

$x^n = -(y^{(n-2)} + 2zy^{(n-3)} + 3z^2y^{(n-4)} + \dots + (n-1)z^{(n-2)}) + nz^{(n-1)}$ ----(3)

Add (2) and (3):

$2x^n = n(y^{(n-1)} + z^{(n-1)})$
$+ (n-2)y^{(n-2)} + (n-4)y^{(n-3)}z + (n-6)y^{(n-4)}z^2 + \dots + (-n+2)z^{(n-2)}$ -------- (4)

Put $n = 2m$, $m > 2$:

~~$x^n = m(y^{(2m-1)} + z^{(n-1)}) + (m-1)y^{(n-2)} + (m-2)y^{n-3}z + \dots + (-m+1)z$~~

$x^{2m} = m(y^{(2m-1)} + z^{(2m-1)})$
$+ \{(m-1)y^{(2m-2)} + (m-2)y^{(2m-3)}z + \dots + (1)y^m z^{(m-2)}\}$
$- \{(m-1)z^{(2m-2)} + (m-2)z^{(2m-3)}y + \dots + (1)z^m y^{(m-2)}\}$ ----(5)

x is odd because if x even, x is not relatively prime to y or z whichever is even.

So $x^{2m} \equiv 1 \pmod 8$; $z^{(2m-3)}y \equiv (1)zy \pmod 8$.

Suppose y even, z odd. Then, modulo 4, we get

$$1 \equiv mz + \{0\} - \{(m-1) + (m-2)yz\} \pmod 4$$

$$0 \equiv mz - m - (m-2)yz \pmod 4$$

$$0 \equiv my - (m-2)yz \pmod 4$$

$$\equiv y\{m - (m-2)z\} \pmod 4$$

$$\equiv y\{m - (m-2)(y+1)\} \equiv my - (m-2)y \pmod 4$$

$$\equiv 2y \pmod 4$$, because $y^2 \equiv 0 \pmod 4$, y even. Known!

Try again!

(3) mod 4: $1 \equiv -(n-1)z^{(n-2)} + nz^{(n-1)} \equiv nz^{(n-2)}(z-1) + z^{(n-2)}$

$$\equiv 0 + z^{(n-2)} \equiv 1 \pmod 4$$

Try again! Similarly for (2) mod 4.

$$x^{2m} = z^{2m} - y^{2m}$$

$$= (z+y)(z-y)(z^{(2m-2)} + y^2 z^{(2m-4)} + \dots + y^{(2m-2)})$$

$z - y = 1$; $(z+y)$ is $2m^{th}$ power??