(1).

Lagrange's Problem.

$z = (y+1)$, $z^p = y^p + x^p$, where x, y, z rel prime & positive, p prime >2.

So $x^p = z^p - y^p = (z-y)(z^{(p-1)} + z^{(p-2)}y + \dots + y^{(p-1)})$

$$= z^{(p-1)} + z^{(p-2)}y + \dots + zy^{(p-2)} + y^{(p-1)}$$

$$= (y+1)^{(p-1)} + (y+1)^{(p-2)}y + \dots + (y+1)y^{(p-2)} + y^{(p-1)}$$

$$= py^{(p-1)} + \left\{\binom{p-1}{1} + \binom{p-2}{1} + \dots + \binom{1}{1}\right\}y^{(p-2)}$$

$$+ \left\{\binom{p-1}{2} + \binom{p-2}{2} + \dots + \binom{2}{2}\right\}y^{(p-3)}$$

$$+ \dots + 1.$$

$$= py^{(p-1)} + \frac{(p-1)p}{2}y^{(p-2)} + \binom{p}{3}y^{(p-3)} + \dots + 1$$

Try again! = OK. Nothing new!

Mod y, $x^p \equiv z^{(p-1)} \equiv 1 \pmod{y}$.

Mod z, $x^p \equiv y^{(p-1)} \equiv 1 \pmod{z}$.

If this is so, $x^p \equiv 1 \pmod{yz}$.

So $x^p \equiv (z^{(p-1)} + y^{(p-1)}) \pmod{yz}$

$$\equiv \left(z(y+1)^{(p-2)} + y(z-1)^{(p-2)}\right) \pmod{yz}$$

$$\equiv (z-y) \pmod{yz} \equiv 1 \pmod{yz}.$$

Now, $x^p \equiv (z^p - y^p) \pmod{yz}$

$$\equiv \left[z(y+1)^{(p-1)} - y(z-1)^{(p-1)}\right] \pmod{yz}$$

$$\equiv (z-y) \pmod{yz} \equiv 1 \pmod{yz}$$

Use n in place of p, n composite. Then $x^n \equiv 2 \pmod{yz}$

if $(n-2)$ is even, i.e. if n even. Impossible.

$x^n = z^n - y^n$ for n odd ≥ 2, $z = (y+1)$.

$x^{2n} = z^{2n} - 2z^n y^n + y^{2n}$

Mod yz, $x^{2n} \equiv (z^{2n} + y^{2n}) \pmod{yz}$

$\equiv (z(y+1)^{(2n-1)} + y(z-1)^{(2n-1)}) \pmod{yz}$

$\equiv (z-y) \equiv 1 \pmod{yz}$. So?

Mod $y^n z^n$, $x^{2n} \equiv (z^{2n} + y^{2n}) \pmod{y^n z^n}$

$\equiv (z^{2n} + 2y^n z^n + y^{2n}) \pmod{y^n z^n}$

$\equiv (z^n + y^n)^2 \pmod{y^n z^n}$

$\pm x^n \equiv (z^n + y^n) \pmod{y^n z^n}$.

But $x, y, z > 1$. So, bec x, y, z rel prime, $\mathrm{Max}(y, z) \geq 3$. So $y^n z^n > (z^n + y^n) > 0$.

Also $y^n z^n > x^n > 0$.

So $(z^n + y^n) = \pm x^n$

So $z^n + y^n = x^n$.

But $z^n - y^n = x^n$.

So $2y^n = 0$. So $y = 0$. Contradiction.