(1).

Legendre's Thm: VI.

Let p be an odd prime and ~~let~~ x, y, z be relatively prime integers such that $z = (y+1)$ and such that $x^p + y^p = z^p$.

Then $x^p = z^p - (z-1)^p$
and $x^p = (y+1)^p - y^p$.

So $\binom{p}{1} z^{(p-1)} - \binom{p}{2} z^{(p-2)} + \ldots - \binom{p}{p-1} z$
$= \binom{p}{1} y^{(p-1)} + \binom{p}{2} y^{(p-2)} + \ldots + \binom{p}{p-1} y$.

So $\binom{p}{1}(z^{(p-1)} - y^{(p-1)}) - \binom{p}{2}(z^{(p-2)} + y^{(p-2)}) + \ldots - \binom{p}{p-1}(z-y)$
$\div (z+y)$:
$= 0$.

$\binom{p}{1}(z^{(p-2)} - y z^{(p-3)} + y^2 z^{(p-4)} - \ldots - y^{(p-2)})$
$- \binom{p}{2}(z^{(p-3)} - y z^{(p-4)} + y^2 z^{(p-5)} - \ldots + y^{(p-3)})$
$+ \binom{p}{2}(z^{(p-4)}$ -----

$- \binom{p}{p-1}(1) = 0$ ----- (1)

Go to side (3)

This is

$$\frac{\{z^p-(z-1)^p-1\}}{z}-\frac{y}{z^2}\left\{z^p-(z-1)^p-1-\binom{p}{p-1}z\right\}+\dots$$

$$=0.$$

Try again!

(1st product in (1))

$$=\binom{p}{1}\left[z\left(z^{(p-3)}-yz^{(p-4)}+y^2z^{(p-5)}-\dots+y^{(p-3)}\right)-y^{(p-2)}\right]$$

But $z>(p+1)$ known, $y>p$ known.

So (sum of 1st 2 products in (1))

$$=\left\{\binom{p}{1}z-\binom{p}{2}\right\}\left(z^{(p-3)}-yz^{(p-4)}+\dots+y^{(p-3)}\right)$$

$$-\binom{p}{1}y^{(p-2)}$$

$$>\left\{p(p+1)-p\left(\frac{p-1}{2}\right)\right\}(\quad)-py^{(p-2)}$$

$$=\frac{p}{2}(p+3)(\ )-py^{(p-2)}$$

$$=p\left(\frac{p+3}{2}\right)\left(\frac{z^{(p-2)}+y^{(p-2)}}{z+y}\right)-py^{(p-2)}$$

Similarly for 3rd + 4th products in (1)?

(3)

Legendre's Thm: VI.

Look at (1), use $z-y=1$:

$$0 = (P_1)\left(z^{(p-3)} + y^2 z^{(p-5)} + \dots + y^{(p-3)}\right)$$

$$-(P_2)\left(z^{(p-4)} + y^2 z^{(p-6)} + \dots + zy^{(p-5)}\right) - (P_2)y^{(p-3)}$$

$$+(P_2)\left(z^{(p-5)} + y^2 z^{(p-7)} + \dots + y^{(p-5)}\right)$$

$$- \dots$$

$$= z(P_1)\left(z^{(p-4)} + y^2 z^{(p-6)} + \dots + zy^{(p-5)}\right) + (P_1)y^{(p-3)}$$

$$-(P_2)\left(z^{(p-4)} + y^2 z^{(p-6)} + \dots + zy^{(p-5)}\right) - (P_2)y^{(p-3)}$$

$$+ \dots$$

$$> 2(P_2)\left(z^{(p-4)} + y^2 z^{(p-6)} + \dots + zy^{(p-5)}\right) + \{(P_1) - (P_2)\}y^{(p-3)}$$

$$+ \dots$$

(because $z > 2P$.)

$$> 2(P_2)z\frac{\left(z^{(p-3)} - y^{(p-3)}\right)}{\left(z^2 - y^2\right)} - (P_2)y^{(p-3)}$$

$$+ \dots$$

$$= \frac{2(P_2)z}{(z+y)}\left(z^{(p-3)} - y^{(p-3)}\right) - (P_2)y^{(p-3)}$$

$$+ \dots$$

(because $z-y=1$)

$$> (P_2)(z^{(p-3)} - y^{(p-3)}) - (P_2)y^{(p-3)}$$

$$+ \cdots \quad \left(\text{because } \frac{2z}{(z+y)} > 1\right).$$

Better:

$$> \{z(P_1) - (P_2)\}\, z\, \frac{\{z^{(p-3)} - y^{(p-3)}\}}{(z+y)} - (P_2)y^{(p-3)}$$

$$+ \cdots$$

$$> 0.$$

Contradiction proves Legendre's Thm relating to FLT for odd prime exponent.

Legendre's Thm for composite exponent pq, $p+q$ odd, p prime, follows because $X^{pq} + Y^{pq} = Z^{pq}$

is $(X^q)^p + (Y^q)^p = (Z^q)^p$

and $Z^q > Y^q + 1$, so

NO! it doesn't follow! $Z \geq Y+1$.

Can use composite n in place of p.