Legendre's Theorem.

Let $n > 2$ be an integer and let $0 < x < y < z = (y+1)$ be integers and let $z^n = x^n + y^n$.

Then $2y^n > z^n = (y+1)^n$.

$$\left(1+\frac{1}{y}\right)^n < 2 = e^{\ln(2)}$$

$$\left(1+\frac{1}{y}\right) < e^{\frac{1}{n}\ln(2)} < \left(1 + \frac{\ln(2)}{n} + \frac{(\ln(2))^2}{n^2}\right)$$

$$\left(1+\frac{1}{y}\right) < \frac{1 - \left(\frac{\ln(2)}{n}\right)^3}{\left(1 - \frac{\ln(2)}{n}\right)}$$

$$= 1 + \frac{\frac{\ln(2)}{n} - \left(\frac{\ln(2)}{n}\right)^3}{\left(1 - \frac{\ln(2)}{n}\right)}$$

$$\frac{1}{y} < \frac{\ln(2)}{n}\left(1 + \frac{\ln(2)}{n}\right)$$

$$y > \frac{n}{\ln(2)\left(1 + \frac{\ln(2)}{n}\right)} \quad \text{------------} (1)$$

This does not go towards proving Legendre's Theorem. Nor is it the best for y, because there is $y > (2n+1)$ by other method.

Try $2x^n < z^n = (y+1)^n$.

$$e^{\frac{1}{n}\ln(2)} x < (y+1).$$

$$\left(1+\frac{1}{n}\ln(2)\right)x < e^{\frac{1}{n}\ln(2)}x < (y+1) \text{ ---- } (2)$$

$$\cancel{\frac{x}{y} < \frac{n}{\ln(2)}} \text{ ---- } (1)$$

~~Add this to my Illinois talk?~~

$$x\,\frac{\ln(2)}{n} < (y-x+1) = (z-x).$$

$$\frac{\ln(2)}{n} < \frac{z-x}{x}$$

$$\frac{z}{x} > \left(1+\frac{\ln(2)}{n}\right) \text{ ---- } (2).$$

But on other sheets, I have proved

$$\frac{y}{z} > \left\{1-\frac{\ln(2)}{n}\right\} \text{ for } z=(y+1) \text{ and } z\neq(y+1)$$

So $$\frac{y}{(y+1)} > \left\{1-\frac{\ln(2)}{n}\right\} \text{ for } z=(y+1).$$

$$\left\{1-\frac{1}{(y+1)}\right\} > \left\{1-\frac{\ln(2)}{n}\right\}$$

$$\frac{1}{(y+1)} < \frac{\ln(2)}{n}.$$

$$(y+1) > \frac{n}{\ln(2)} \text{ (For } z=(y+1)) \text{ ---- } (4).$$

(4) is like (1), not as good as $y>(2n+1)$ but it may be useful.