①.

Legendre's Theorem: V.

Let p be an odd prime and let $0 < x < y < z = (y+1)$ be relatively prime integers and let $x^p + y^p = z^p = (y+1)^p$.

We obtain a contradiction.

$x^p \equiv x \pmod p$, $y^p \equiv y \pmod p$, $z^p \equiv z \pmod p$.

So $0 \equiv x^p + y^p - z^p \equiv x + y - z \equiv x - 1 \pmod p$.

So $x \equiv 1 \pmod p$.

So $x^p \equiv 1 \pmod{p^2}$.

Then
$$\begin{aligned} 0 &\equiv x^p + y^p - z^p \pmod{p^2} \\ &\equiv 1 + y^p - z^p \\ &\equiv 1 + (y + bp) - (z + b'p) \pmod{p^2} \end{aligned}$$

Where b, b' are integers.

So $b \equiv b' \pmod p$ and

$y^p = (y + bp)$ $\cancel{\pmod{p^2}}$

and $z^p = (z + b'p)$ $\cancel{\pmod{p^2}}$ for some integers b, b'.

$$\begin{aligned} \text{Now } x^p = z^p - y^p &= (z - y)(z^{(p-1)} + z^{(p-2)}y + \cdots + y^{(p-1)}) \\ &= z^{(p-1)} + z^{(p-2)}y + \cdots + y^{(p-1)} \\ &= (z^{(p-1)} + z^{(p-3)}y^2 + \cdots + y^{(p-1)}) \\ &\quad + yz(z^{(p-3)} + z^{(p-5)}y^2 + \cdots + y^{(p-3)}) \end{aligned}$$

$$= \frac{(z^{(p+1)} - y^{(p+1)})}{(z^2 - y^2)} + \frac{yz(z^{(p-1)} - y^{(p-1)})}{(z^2 - y^2)} \quad \text{----(1)}$$

(1) × (2y+1):

$$(2y+1)x^p = (z^{(p+1)} - y^{(p+1)}) + yz(z^{(p-1)} - y^{(p-1)})$$

$$= \{z(z+b'p) - y(y+bp)\}$$

$$+ \{y(z+b'p) - z(y+bp)\}$$

$$= (z^2 - y^2) + (z+y)b'p + (yz - yz) - (y+z)bp$$

$$= (2y+1)(1 + b'p - bp) \quad \text{----(2)}$$

So $x^p = 1$ unless $(2y+1) = 0$.

~~So x = 1 unless (2y+1) = 0.~~

But $y > 0$, so $2y > 0$ and $(2y+1) > 0$.

So $x^p = 1$.

So $1 = z^{(p-1)} + z^{(p-2)}y + \dots + y^{(p-1)} > p > 2$.

This contradiction proves ~~that if~~ FLT ~~fails for odd prime exponent~~ p then Legendre's Thm holds ($z \neq (y+1)$).

(2) gives $x^p = (y+1+b'p) - (y+bp) = z^p - y^p$.