1. Q.

Similar to Fermat's equation for exponent 4, has $x^2 + y^4 = z^4$ a solution in integers?

Must be presumably $x = kmn$,

$y^2 = k(m^2 - n^2)$, $y^2 = k(m^2 + n^2)$ for some integers $m$ and $n$.

So $z^2 = y^2 + 2nk$.

This is similar to Pell's equation when written $x^2 - 2nk = y^2$ or $1$ solution.

From $y^2 = k(m^2 + n^2)$, we obtain that $z = \sqrt{2nk}$, where $a$, $b$, $k$ are integers.

Also, from $m^2 = (x^2 + y^2)$, we presumably $n = \sqrt{k}$.

Also $(m^2 + 3) = 2a^2$, $z = 2b$. So

$(m + 3) = 2a^2$, $z = 2b$. So
\( x^2 + y^4 = 3 \) becomes
\[
(2k mn)^2 + y^4 = 4(a^2 + b^2)4
\]
\[
2(2k mn)^2 + a^2 = 4k a b y^2
\]
\[
+ k^4 (p^2 q^2)4 = 4(a^2 + b^2)4
\]
\[
2(2k mn)^2 + a^2 = 4k a b y^2
\]
\[
16k^4 p^2 q^2 (p^2 + q^2)^2 + k^4 (p^2 q^2)4
\]
\[
= 4(a^2 + b^2)4
\]

L.H.S. be an exact 4th power.

\[
\begin{align*}
k^4 &= \left\{ \begin{array}{l}
(p^2)^4 - 4(p^2)^3 (q^2) + 6(p^2)^2 (q^2)^2 - 4(p^2) (q^2)^3 + (q^2)^4 \\
+ 16(p^2)^3 (q^2) + 32(p^2)^2 (q^2)^2 + 16(p^2) (q^2)^3
\end{array} \right.
\end{align*}
\]

\[
= \{1, 12, 38, 12, 1\}
\]

are the coefficients + exact 4th power except in special cases maybe.
Find if with that set of coefficients it could be a 4th power.

Presumably there is no special case because (sum of coefficients) = 64 = (2 + \sqrt{2})^4 not a 4th power of integers such as \( 2^4 (a^2 + b^2)^4 \) vs...

A better proof of there being no special case is if \( p = 3 \) and \( g = 2 \) and then

\[
\frac{\text{LHS}}{k^4} = \left\{ \begin{array}{l}
94 + 12 \times 9 \times 4^2 + 32 \times 9 \times 4^3 + 4^4 \\
+ 3 \end{array} \right.
\]

\[
\equiv \{1 + 2 + 2 + 2 + 6^2 \pmod{10} \} \equiv 13 \pmod{10} \equiv 3 \pmod{10}
\]

whereas no 4th power is congruent to 3 modulo 10.

In a special case pair \( p, q \), then

\[
\frac{\text{LHS}}{k^4} = \left\{ \begin{array}{l}
(2^2)^4 + 12(2^2)^3 (2^2) \pmod{9^4} \\
= \{ (2^2+3q^2)^4 \pmod{9^4} \}
\end{array} \right.
\]

but

\[
\frac{\text{LHS}}{k^4} = \left\{ \begin{array}{l}
(2^2)^4 + 12(2^2)^3 (2^2)^2 + 3(2^2)^2 \pmod{9^4} \pmod{9^4} \end{array} \right.
\]
(FROM OVERLEAF)

modulo \( q^2 \) = 9

So \( \frac{\text{LHS}}{K^4} \neq \left[ (p^2) + 36^2 \right] ^{4/2} \) (modulo 9).

This is sufficient to show that \( \text{LHS} \) is not a 4th power.

Thus, the contradiction from assuming \( x^2 + 4y^4 = 36 \) has no solutions in integers.

A simpler counterexample is \( x^2 + 4y^4 = 36 \).