ON THE NUMBER OF WAYS AN INTEGER CAN BE THE SUM OF TWO SQUARES OF POSITIVE INTEGERS

John H. Ursell (zero is not positive).

Department of Mathematics and Statistics, Queen's University, Kingston, Ontario, Canada, K7L 3N6

Introduction. Diophantus and later writers discussed the solution of equations in integers. Certain positive integers are each the sum of the squares of two integers. We show that this can only be achieved in one way or zero ways and not in two or more ways.

The main result. We state and prove:

THEOREM 1. A particular positive integer can be expressed as the sum of the squares of two positive integers in exactly one way or not at all.

Proof. Let the integer \( N \) be expressed as the sum of the squares of the positive integers \( x \) and \( y \). Thus...
\[ N = x^2 + y^2 \]

where \( x > 0, y > 0, x > y \).

Assume that the theorem is not true for \( N \) and thus that there exist positive integers \( a \) and \( b \) such that

\[ N = a^2 + b^2 \]

where \( a > 0, b > 0, a > x, a > y, b < x \), \( b < y \).

Let \( a = (x-k), b = (y+k+l) \), where \( k > 0, l > 0 \). Then

\[ N = x^2 + y^2 = (x-k)^2 + (y+k+l)^2 \]

\[ = (x^2 - 2xk + k^2) + (y^2 + 2yk + k^2 + l^2) \]

Thus,

\[ 2xk = k^2 + (k+l)^2 + 2y(k+l) \]

This is

\[ 2xk = 2(k^2 + (k+l)^2 + 2y(k+l)) \]

This becomes

\[ 2xk = 2(k+l)(k+y) + l^2 \]

Which can be written

\[ 2(k+l)(x) = 2(k+l)(k+y) + l^2 + 2x \]

Which becomes
\[2(k+l)(x-k-y) = t^2 + 2tx. \quad (3)\]

This can be re-expressed as
\[a(b-y)(a-y) = t^2 + 2tx \quad (4).\]

Equation (4) is
\[ay^2 - \{x(a+b)\}y + \{xab - t^2 - 2tx\} = 0 \quad (5)\]

The two values of \(y\) sum to \((a+b)\).

Let them be \(y_1\) and \(y_2\). Thus
\[(y_1 + y_2) = (a+b) = (x+y+t) \quad (6)\]

Thus we may select that \(y_2 = y\) and \(y_1 = (x+t)\).

But \(l > 0\), so \(y_1 > x > a > b > 0\). Thus \(y_1\) cannot be associated with \(-\) being equal to \(x\) or \(-\) to \(y\) or \(-\) to \(a\) or \(-\) to \(b\). Instead \(y_1\) is associated with a third sum of squares \(x^2 + y_2^2\) where \(x > x\) and \(x = (x+t) = y\).

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Thus, the second (and the third, etc.) ways of expressing \(N\) as the sum of two squares of positive integers does not provide any such largest \(x\) and smallest \(y\), there is no such larger number \(X\) with that \(x\), \(y\). Thus, the second (and the third, etc.) ways of expressing \(N\) as the sum of two squares of positive integers does not provide any such largest \(x\) and smallest \(y\), there is no such larger number \(X\) with that \(x\), \(y\).
This proves that there is either exactly one way or no way. This proves the theorem.

Parenthetically, we note that the analogous result for the sum of three squares of integers is not true. The example
\[13^2 + 3^2 + 4^2 = 12^2 + 5^2 + 5^2\]
suffices as a counterexample.

Furthermore, a result of Diophantus is that each positive integer can be expressed as the sum of at most four squares of positive integers. However, this representation is not unique and the example
\[13^2 + 3^2 + 4^2 + 1^2 = 12^2 + 5^2 + 5^2 + 1^2\]
suffices here.

Again, a positive integer can be expressed as the sum of three squares of positive integers and as the sum of four.

Assume no existence.
squares of positive integers. An example is $2^2 + 2^2 + 2^2 = 3^2 + 1^2 + 1^2 + 1^2$.

Also, a particular positive integer can be expressed as the sum of two squares of positive integers and also as the sum of four squares of positive integers. An example is $3^2 + 1^2 = 2^2 + 2^2 + 1^2 + 1^2$.

Another example is $13^2 + 5^2 = 12^2 + 5^2 = 4^2 + 3^2$.

Furthermore, a particular positive integer can be expressed as the sum of two squares of positive integers and also as the sum of three squares of positive integers. An example is $13^2 + 5^2 = 13^2 + 4^2 + 3^2$.

It is possible to believe that a particular positive integer which can be expressed as the sum of four squares of positive integers and also as the sum of three squares of positive integers can only be expressed as the sum of three squares of positive integers in exactly one way.
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Also, similarly, for a particular positive integer which can be expressed as the sum of two such squares and also as the sum of four such squares,

Again, similarly, for a particular positive integer which can be expressed as the sum of two such squares and also as the sum of three such squares,

Additionally, a particular positive integer can be the square of a positive integer and also the sum of the squares of two positive integers.

The example 25 = 5^2 = 4^2 + 3^2 suffices.

Moreover, a particular positive integer can be a square of a positive integer and also the sum of the squares of three positive integers. The example 169 = 13^2 = 12^2 + 4^2 + 3^2 suffices.