1. Root Systems

Let $V$ be a Euclidean vector space with an inner product $(\cdot,\cdot)$. A root system is a finite collection of vectors $\Phi$ which satisfy the following:

1. The span of $\Phi$ is $V$.
2. If $\alpha \in \Phi$ then $\Phi$ is closed under the reflection $\sigma_\alpha$ in $\alpha^\perp$.
3. If $\alpha \in \Phi$ then $\lambda \alpha \in \Phi$ if and only if $\lambda = \pm 1$.
4. For $\alpha, \beta \in \Phi$ we have $(\beta, \alpha) \overset{\text{def}}{=} 2\frac{(\alpha, \beta)}{(\alpha, \alpha)} \in \mathbb{Z}$.

Remark. The pairing $(\cdot,\cdot)$ is not necessarily symmetric, only linear in the first variable.

The purpose of the integrality condition is to make it so that the root system can generate a lattice in a sensible way. (That it is integral and not rational says that the system is somehow ‘reduced’)

Using that $(\alpha, \beta)\langle \beta, \alpha \rangle = 4 \frac{(\alpha, \beta)^2}{(\alpha, \alpha)(\beta, \beta)} = 4 \cos^2(\theta)$ (where $\theta$ is the angle between $\alpha, \beta$). The integrality condition immediately imposes that the cosine of the angles are one of $\{0, \pm \frac{1}{2}, \pm \frac{\sqrt{2}}{2}, \pm \frac{\sqrt{3}}{2}, \pm 1\}$.

In particular the product is $0, 1, 2, 3$ they both have the same sign, and at least one of them is $\pm 1$.

The rank of the root system is the dimension of $V$.

Root systems are reducible if there exist $V_1, V_2 \subset V$ with $V = V_1 \oplus V_2$ and $\Phi = (\Phi \cap V_1) \cup (\Phi \cap V_2)$.

Two root systems are isomorphic if there exists an isomorphism between the spaces taking roots onto roots.

The Weyl group of $\Phi$ is the group of isometries of $V$ generated by $\sigma_\alpha$. This group is finite as it acts faithfully on the finite set of roots.

The Root lattice of $\Phi$ is the $\mathbb{Z}$ module generated by $\Phi$.

Example. 
• $(0,1), (1,0)$ ($A_1 \times A_1$)
• $(0,1), (1,0), (1,1)$ ($B_2$)
• Hexagon ($A_2$)
• Long+Short Hexagon ($G_2$)
• $E_6 = (12,12off)$

2. Simple roots

We say that a subset $\Phi^+$ is a system of positive roots if:

1. For each $\alpha \in \Phi$ exactly one of $\alpha, -\alpha \in \Phi^+$.
2. If $\alpha, \beta \in \Phi^+$ and $\alpha + \beta \in \Phi$ then $\alpha + \beta \in \Phi^+$.

An element of $\gamma \in \Phi^+$ is simple (with respect to $\Phi^+$) if $\gamma$ can not be written as $\alpha + \beta$ for $\alpha, \beta \in \Phi^+$.
Remark. Picking a collection $\Phi^+$ is equivalent to picking a connected component of $V \setminus (\cup \alpha \alpha^+)$. That is, picking a hyperplane $\gamma^+$ which contains none of the $\alpha$ and setting $\Phi^+ = \{ \alpha \in \Phi | \langle \gamma, \alpha \rangle > 0 \}$.

The collection $\Delta(\gamma)$ of simple roots of $\Phi^+(\gamma)$ forms a basis of $V$ and every $\alpha \in \Phi^+(\gamma)$ is a positive integral linear combination of elements of $\Delta(\gamma)$.

Generating set by induction relative to $(\gamma, \alpha)$. Next have $\alpha, \beta \in \Delta(\gamma)$ implies $\langle \alpha, \beta \rangle < 0$ otherwise one of $\alpha - \beta$ or $\beta - \alpha$ would be in $\Phi^+(\gamma)$. Now if $X = \sum_\alpha s_\alpha \alpha = \sum_\beta t_\beta \beta$ with all $s_i, t_i > 0$, then by the above $(X, X) = 0$ and so $X = 0$, then we consider $(X, \gamma) = 0$ and so $s_i, t_i = 0$.

We can recover a root system from a collection of simple roots. Indeed, inductively on height if $\alpha$ is a non-simple positive root there exists a $\beta \in \Delta$, such that $\alpha - \beta$ is also positive. (use linear independence of simple roots expressing alpha as a sum of them).

We have the following facts about

- The Weyl-group acts transitively on Weyl-champer and thus on all collections of positive and simple roots.
- Every root can be made to be simple.
- The Weyl-group is generated by the $\sigma_\alpha | \alpha \in \Delta$.
- If an element of the weyl-group maps $\Delta$ to $\Delta$ it is the identity.

3. Dynkin Diagram

We verify the following properties of root systems:

1. If $\Delta$ is a system of simple roots then $\Delta \setminus S$ generates a roots system in the span($\Delta \setminus S$).
2. there are at most $n$ pairs in $\Delta$ with $\langle \alpha, \beta \rangle \neq 0$.
3. If $\alpha, \beta \in \Delta$ and $\langle \alpha, \beta \rangle \neq 0$ then $\hat{\Delta}$ generates a root system in $V / (\alpha - \beta)$.
4. For $\alpha \in \Delta$ we have $\sum_{\beta \neq \alpha} \langle \beta, \alpha \rangle \geq -3$.

These correspond directly to the statements which allow us to define admissible Dynkin diagrams. Via some tedious dot product arguments we can then show the classification.

1. Every sub-diagram of admissible is admissible
2. fewer than $n$ pairs of connected vertices (implies acyclic) with above
3. Can delete vertices and glue.
4. No edge has more than 3 lines attached
5. A vertex with more than 2 lines imposes length of chain conditions.

Example. $A_n$

- $B_n$ (out arrow)
- $C_n$ (in arrow)
- $D_n$ (double head)
- $E_6, E_7, E_8$ (2,1,n)
- $F_4$ (double in middle)
- $G_2$ (triple)

4. 'Maximal Tori' in Lie... Algebras

A Cartan sub-algebra $h$ in a lie algebra $g$ is a maximal abelian subalgebra of $g$ ($[A, B] = 0$ for $A, B \in h$) such that $A \in h$ act semi-simply through on $g$ through the adjoint representation.

These correspond to the lie algebras of maximal tori in $G$. 
Definition 4.1. Let $H \in \mathfrak{g}$ consider:

$$\mathfrak{g}(H) \overset{\text{def}}{=} \{ X \in \mathfrak{g} | \text{ad}(H)X = 0 \text{ for some } k \}$$

$H$ is regular if $\dim(\mathfrak{g}(H)) = \min_{X \in \mathfrak{g}}(\dim \mathfrak{g}(X))$.

Theorem 4.2. Cartan sub-algebras exist! Moreover they are the all of the form $\mathfrak{g}(H)$ for regular elements $H$.

(This requires semi-simplicity (non-degenerate killing form))

Let $\alpha$ be a linear functional on the space $\mathfrak{h}$. Denote by $\mathfrak{g}^\alpha$ the linear subspace $\mathfrak{g}^\alpha = \{ X \in \mathfrak{g} : [H,X] = \alpha(H)X \}$ on which $\mathfrak{h}$ acts via $\alpha$.

We have the following:

- $\mathfrak{g}^0 = \mathfrak{h}$ and thus $\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \neq 0} \mathfrak{g}^\alpha$.
- $[\mathfrak{g}^\alpha, \mathfrak{g}^\beta] \subset \mathfrak{g}^{\alpha+\beta}$
  follows from the Jacobi identity:

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

- $\mathfrak{g}^\alpha, \mathfrak{g}^\beta$ are orthogonal if $\alpha + \beta \neq 0$.
  For $X_\alpha, Y_\beta$ we have $\text{ad}_{X_\alpha} \text{ad}_{Y_\beta} \mathfrak{g}^\gamma \subset \mathfrak{g}^{\gamma+\beta}$, Consequently the trace must be 0.
- The above implies that $B_\mathfrak{h}$ is non-degenerate, and that $\mathfrak{g}^\alpha \neq \{0\}$ implies $\mathfrak{g}^{-\alpha} \neq 0$.
- Let $H_\alpha$ be such that $B(H_\alpha, H) = \alpha(H)$ then $[\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}] = C H_\alpha$.
- $\dim(\mathfrak{g}^\alpha) = 1$ (or 0).
  For non-trivial spaces there exists $E_\alpha, E_{-\alpha}$ with $B(E_\alpha, E_{-\alpha}) = 1$. Fix $D_\alpha$ such that $B(D_\alpha, E_{-\alpha}) = 0$. Set $D_n = (\text{ad} E_\alpha)^n D_\alpha$ then $[E_{-\alpha}, D_n] = \frac{n(n+1)}{2} \alpha(H_\alpha) D_{n-1}$ and $D_n \in \mathfrak{g}^{(n+1)\alpha}$. If $D_n$ is non-zero this is a contradiction.

Let $\Phi$ be the set of all $\alpha \neq 0$ such that $\mathfrak{g}^\alpha \neq 0$.

- Let $\alpha \in \Phi$ and $\beta$ be a root, then $\beta + n \alpha \in \Phi$ for $p \leq n \leq q$ where $p + q = -2 \frac{\beta H_\alpha}{\alpha H_\alpha}$.
  Compute the trace of $H_\alpha$ on $\sum \mathfrak{g}^{\alpha+\beta} n$ for a maximal string of $n$.
- The only multiples of $\alpha$ such that $\lambda \alpha \in \Phi$ are $\pm \alpha$ and 0.
  Uses the above + tricks to rule out non-integers.
- $[\mathfrak{g}^\alpha, \mathfrak{g}^\beta] = \mathfrak{g}^{\alpha+\beta}$.
  Similar arguments with traces and invariance.
- $[X_{-\alpha}, [X_\alpha, X_\beta]] = \frac{q(1-p)}{2} \alpha(H_\alpha) B(X_\alpha, X_{-\alpha}) X_\beta$
  Induction on the $\alpha$ series of $\beta$.

Let $\mathfrak{h}_\mathbb{R} = \sum_{\alpha \in \Phi} \mathbb{R} H_\alpha$.

Theorem 4.3. $B$ is strictly positive definite on $\mathfrak{h}_\mathbb{R}$.

$$B(H, H') = \sum_{\beta \in \Phi} \beta(H) \beta(H')$$

$$\beta(H_\alpha) = -\alpha(H_\alpha) (p_{\beta, \alpha} + q_{\beta, \alpha})$$

$$\alpha(H_\alpha) \in \mathbb{R} \text{ and positive.}$$

- $\Phi$ is a root system in $\mathfrak{h}_\mathbb{R}$ relative to $B$.

5. The Root System

Fix $X_\alpha \in \mathfrak{g}^\alpha$ such that $B(X_\alpha, X_{-\alpha}) = 1$. Let $N_{\alpha, \beta}$ be such that $[X_\alpha, X_\beta] = N_{\alpha, \beta}$

Lemma 5.1. $\bullet$ if $\alpha + \beta + \gamma = 0$ and all terms defined then $N_{\alpha, \beta} = N_{\beta, \gamma} = N_{\gamma, \alpha}$

$\bullet$ if $\alpha + \beta + \gamma + \delta = 0$ and all terms defined then $N_{\alpha, \beta} N_{\gamma, \delta} + N_{\beta, \gamma} N_{\alpha, \delta} + N_{\gamma, \alpha} N_{\beta, \delta} = 0$.
Theorem 5.2. Let $\Phi$ be a root system. Define a Lie algebra with generators $H_\lambda, X_\lambda, Y_\lambda$ to have the relations:

1. $[H_\lambda, H_\mu] = 0$ for all $\lambda, \mu \in \Phi$.
2. $[H_\lambda, X_\mu] = (\lambda, \mu)X_\mu$ for all $\lambda, \mu \in \Phi$.
3. $[H_\lambda, Y_\mu] = -(\lambda, \mu)X_\mu$ for all $\lambda, \mu \in \Phi$.
4. $[X_\lambda, Y_\mu] = \delta_{\mu\lambda}H_\mu$ for all $\lambda, \mu \in \Phi$.
5. $\text{ad}_{X_\lambda}^{-(\mu,\lambda)+1}X_\mu = 0$ for all $\lambda \neq \mu \in \Phi$.
6. $\text{ad}_{Y_\lambda}^{-(\mu,\lambda)+1}Y_\mu = 0$ for all $\lambda \neq \mu \in \Phi$.

Then the generated Lie algebra is semi-simple with root system $\Phi$.

Theorem 5.3. If two semi-simple lie algebra’s have isomorphic root systems then they are isomorphic.

We can obtain an isomorphism of the cartan algebra for free, it is easy to check it is an isometry. The cartan algebra and the system of roots determines a decomposition of the space. One uses induction and the above lemma to get the result.

Theorem 5.4. Any two cartan algebras are conjugate, and thus give the same root system. And thus we have converse to the above.

6. The Classification Theorem

Theorem 6.1. Classification of complex semi-simple adjoint algebraic groups

- $A_n$ - $\text{SL}_{n+1}$
- $B_n$ - $\text{SO}_{2n+1}$
- $C_n$ - $\text{Sp}_{2n}$
- $D_n$ - $\text{SO}_{2n}$

Theorem 6.2. Classification of non-adjoint groups.

(this uses things about fundamental groups)

Theorem 6.3. Classification of forms.

This is galois descent in general, for $\mathbb{R}$ one can work it out using extended Dynkin diagrams.