Notes on principal fibre bundles

Andrew D. Lewis[*]

1993/06/15

1. Group actions

We briefly review actions of groups on manifolds. Let G be a Lie group, and let M be a differentiable manifold. By the *Lie algebra* of G we shall mean the tangent space at the identity, e, of G *and* the set of left invariant vector fields on G. Which one we mean will depend on context. In either case we shall denote the Lie algebra by $\mathfrak{g}$. Here are some exercises for groups to warm you up:
Exercises:

1. Show that T_eG is isomorphic with $\mathscr{X}^L(G)$, the set of left invariant vector fields.
2. Let G be an abelian group. Show that the Lie bracket on the Lie algebra is the trivial one. That is to say, $[\xi, \eta] = 0$ for all $\xi, \eta \in \mathfrak{g}$.
3. Show that the Lie algebra of $GL(n, \mathbb{R})$ is isomorphic to the Lie algebra of $n \times n$ real matrices with the Lie bracket given by $[A, B] = AB - BA$. Be explicit.

We shall be interested in right actions.

1.1 Definition: Let G be a Lie group and M be a differentiable manifold. A *right action* of G on M is a map $\Phi \colon M \times G \to M$ such that

1. $\Phi(m, e) = m$ for all $m \in M$, and
2. $\Phi(\Phi(m, g_1), g_2) = \Phi(m, g_1 g_2)$.

We say that a right action Φ is

(i) *effective* or *faithful* if for all $g \in G \setminus \{e\}$ there exists $m \in M$ such that $\Phi(m, g) \neq m$,

(ii) *transitive* if for all $m_1, m_2 \in M$ there exists $g \in G$ such that $\Phi(m_1, g) = m_2$,

(iii) *free* if for all $g \in G \setminus \{e\}$ and for all $m \in M$, $\Phi(m, g) \neq m$,

(iv) *simply transitive* if it is both free and transitive. Thus for all $m_1, m_2 \in M$ there exists a unique $g \in G$ such that $\Phi(m_1, g) = m_2$. •

Understand free actions since they are important for principal fibre bundles.

[*]Professor, Department of Mathematics and Statistics, Queen's University, Kingston, ON K7L 3N6, Canada
Email: `andrew.lewis@queensu.ca`, URL: `http://www.mast.queensu.ca/~andrew/`

1.2 Definition: Let Φ be a right action of G on M, and let $m \in M$. The *orbit* though m is the subset of M given by

$$\mathcal{O}_m = \{\Phi(m, g) \mid \ g \in G\}$$

The isotropy subgroup of m is the subset of G given by

$$G_m = \{g \in G \mid \ \Phi(m, g) = m\} \qquad \bullet$$

If a group action is free then $G_m = \{e\}$ and $\mathcal{O}_m \simeq G$ for all $m \in M$. We shall denote by M/G the set of orbits. There are conditions for the set of orbits to be a smooth manifold and for the projection $\pi\colon M \to M/G$ to be a submersion.

1.3 Definition: Let $\xi \in \mathfrak{g}$ be thought of as a left invariant vector field with flow F_t. The *exponential map* is the map $\exp\colon \mathfrak{g} \to G$ given by $\exp(\xi) = F_1(e)$. $\bullet$

Now infinitesimal generators:

1.4 Definition: Let Φ be a right action of G on M, and let $\xi \in \mathfrak{g}$. The *infinitesimal generator* corresponding to ξ is the vector field on M defined by

$$\xi_M(m) = \frac{\mathrm{d}}{\mathrm{d}t}(\Phi(m, \exp(t\xi)))\bigg|_{t=0} \qquad \bullet$$

Note that if Φ is free then $T_m\mathcal{O}_m \simeq \mathfrak{g}$.
Exercises:

1. Think of a free right action of $(\mathbb{R}, +)$ on $\mathbb{R}^2$. Compute the infinitesimal generator for any Lie algebra element.
2. Let $G = SO(3)$ act on $M = \mathbb{R}^3$ in the "usual" manner. Compute infinitesimal generators. Is the action free? Why or why not?

Make sure you know about adjoint actions from [Abraham and Marsden 1978].

2. Principal fibre bundles

2.1 Definition: Let P and M be manifolds, and let G be a Lie group. A *principal fibre bundle* with *total space* P, *base space* M and *structure group* G consists of a right action, Φ, of G on P such that

(i) Φ is free,

(ii) $M = P/G$ and the projection $\pi\colon P \to M$ is a submersion, and

(iii) P is locally trivial. Thus for every $m \in M$ there exist a neighbourhood U of M such that $\pi^{-1}(U) \simeq U \times G$ via a diffeomorphism $\psi(p) = (\pi(p), \phi(p))$ where $\phi\colon \pi^{-1}(U) \to G$ is such that $\phi(\Phi(p, g)) = \phi(p) \cdot g$ for all $p \in \pi^{-1}(U)$ and $g \in G$.

We shall denote a principal fibre bundle by $P(M, G)$. $\bullet$

We shall call

$$V_pP \triangleq \{v \in T_pP \mid \ v = \xi_P(p) \text{ for some } \xi \in \mathfrak{g}\}$$

the *vertical* subspace at p. The subset of TP given by

$$VP \triangleq \overset{\circ}{\bigcup_{p\in P}} V_pP$$

is called the *vertical subbundle* of TP.
Exercises:

1. What do you think a trivial principal fibre bundle should be defined as? Is your proposed definition actually a principal fibre bundle? Check it explicitly.

2. Is the example of the action of $(\mathbb{R}, +)$ on $\mathbb{R}^2$ you gave above a principal fibre bundle? Why or why not?

3. Show that for a principal fibre bundle the map $\xi \mapsto \xi_P(p)$ is injective for each $p \in P$. Conclude that V_pP is isomorphic to $\mathfrak{g}$ for each $p \in P$.

3. Connections in principal fibre bundles

We now discuss an important structure in principal fibre bundles: connections. Keep in mind that given a principal fibre bundle, a connection is something that you specify, not something that comes along for free.

3.1 Definition: Let $P(M, G)$ be a principal fibre bundle with group action Φ. A *connection*, Γ, in P is a subbundle HP of TP which is complimentary to VP (i.e., $TP = HP \oplus VP$) and is such that if $p' = \Phi(p, g)$ then $H_{p'}P = T\Phi_g(H_pP)$. We shall call HP the *horizontal subbundle* of the connection Γ. •

We shall say that a vector $v \in T_pP$ is *vertical* (resp. *horizontal*) if $v \in V_pP$ (resp. $v \in H_pP$). Note that every vector $v \in T_pP$ may be uniquely written as $v = v_{hor} + v_{vert}$ where v_{hor} and v_{vert} are horizontal and vertical, respectively.
Exercises:

1. Consider the principal fibre bundle with total space $P = \mathbb{R}^2$, structure group $G = (\mathbb{R}, +)$. Let G act on P by

$$\Phi((x, y), g) = (x, y + g)$$

If you have not done this above, show that this is a principal fibre bundle with base space $M \simeq \mathbb{R}$. What is the vertical subbundle? Give the most general form for a connection on this principal fibre bundle. We shall be denoting this principal fibre bundle by **PFB1** in the sequel.

2. Answer the above questions when $P = \mathbb{R} \times \mathbb{R}^*$, $G = (\mathbb{R}^*, \times)$ and

$$\Phi((x, y), g) = (x, gy)$$

(This principal fibre bundle is isomorphic to the bundle of linear frames of $\mathbb{R}$.) We shall be denoting this principal fibre bundle by **PFB2** in the sequel.

3. Consider the principal fibre bundle with total space $P = \mathbb{R}^3$ and structure group $G = (\mathbb{R}, +)$. Let G act on P by

$$\Phi((x, y, z), g) = (x, y, z + g)$$

Answer the above questions for this principal fibre bundle. We shall be denoting this principal fibre bundle by **PFB3** in the sequel.

To talk about the connection form we need the notion of vector-valued form.

3.2 Definition: Let M be a differentiable manifold, and let V be a $\mathbb{R}$-Banach space. A *V-valued differential k-form* on M is an assignment, α, to each $m \in M$ a map from $T_mM \times \cdots \times T_mM$ to V such that the map

$$m \mapsto (\omega; \alpha(m)(v_1, \dots, v_m))$$

is a differential k-form for each $\omega \in V^*$. We shall denote the set of V-valued differential k-forms by $\Omega^k(M; V)$ •

It is easy to verify that if $\{e_1, \dots, e_m\}$ is a basis for V then we may write any V-valued differential k-form, α, as

$$\alpha = \alpha^1 e_1 + \cdots + \alpha^m e_m$$

where $\alpha^1, \dots, \alpha^m$ are differential k-forms. To define the exterior derivative of a V-valued differential k-form given in this way we simply define

$$\boldsymbol{d}\alpha \triangleq \boldsymbol{d}\alpha^1 e_1 + \cdots + \boldsymbol{d}\alpha^m e_m$$

Now we define the connection form.

3.3 Definition: Let Γ be a connection on a principal fibre bundle $P(M, G)$. We define the *connection form* $\omega \in \Omega^1(P; \mathfrak{g})$ by

$$\omega(p)(v) = \{\xi \in \mathfrak{g} \mid \xi_P(p) = v_{vert}\}$$

for $v \in T_pP$. •

One can easily verify that this uniquely defines ω. The reader should convince himself/herself that this definition of ω depends on the connection.

Exercises:

1. Compute the connection form for the connections you looked at above. Be as explicit as possible. Try to make sure that everything lives in the appropriate space.
2. What happens when we evaluate the connection form on horizontal vectors.

4. Horizontal lifts and parallel translation

For our purposes, one of the key notions in principal fibre bundles is that of a horizontal lift. It enable one to relate motions on the base space to motions on the total space using the group structure.

The first thing to notice is that for each $p \in P$, the map

$$T_p\pi \mid H_pP \colon H_pP \to T_{\pi p}M$$

is an isomorphism. Thus there is a 1–1 correspondence between vectors in H_pP and vectors in $T_{\pi(p)}M$. For each $m \in M$ and $p \in \pi^{-1}(m)$ we shall call the association of a vector $v \in T_mM$ to a vector in H_pP the *horizontal lift* of v to p.

4.1 Proposition: *Let $P(M,G)$ be a principal fibre bundle, and let $\boldsymbol{X}$ be a vector field on M. Then there exists a unique vector field on P which is π-related to $\boldsymbol{X}$. We shall denote this vector field by $\boldsymbol{X}_{horiz\ lift}$ and call it the* **horizontal lift** *of $\boldsymbol{X}$.*

Exercises:

1. Compute the horizontal lift for the general connections you found for **PFB1**, **PFB2**, and **PFB3**.
2. Prove Proposition 4.1.
3. Prove the following results:
   (a) $(\boldsymbol{X}+\boldsymbol{Y})_{horiz\ lift} = \boldsymbol{X}_{horiz\ lift} + \boldsymbol{Y}_{horiz\ lift}$,
   (b) $\pi^* f \cdot \boldsymbol{X}_{horiz\ lift} = (f \cdot \boldsymbol{X})_{horiz\ lift}$,
   (c) $[\boldsymbol{X}_{horiz\ lift}, \boldsymbol{Y}_{horiz\ lift}]_{hor} = [\boldsymbol{X}, \boldsymbol{Y}]_{horiz\ lift}$, and
   (d) Show that a vector field on P is G-invariant if and only if it is the horizontal lift of a vector field on M.

Now we discuss parallel translation. Suppose that we have a connection Γ is a principal fibre bundle $P(M,G)$, and a curve $c\colon I \to M$ on M. We shall call a curve $c_{horiz\ lift}\colon I \to P$ a *horizontal lift* of c if $\pi \circ c_{horiz\ lift} = c$ and if $c'_{horiz\ lift}(t)$ is horizontal for each $t \in I$. If $I = [a,b]$ and if c is such that $c(a) = m$, then, for each $p \in \pi^{-1}(m)$, there exists a unique horizontal lift, $c_{horiz\ lift}$, such that $c_{horiz\ lift}(a) = p$. We call this curve the *horizontal lift of c through p*.

Now let $c\colon [0,1] \to M$ be a curve in M such that $c(0) = m_0$ and $c(1) = m_1$. For $u_0 \in \pi^{-1}(m_0)$ we denote by $u_1 \in \pi^{-1}(m_1)$ the point such that $c_{horiz\ lift}(1) = u_1$ where $c_{horiz\ lift}$ is the horizontal lift of c through u_0. We call the map from $\pi^{-1}(m_0)$ to $\pi^{-1}(m_1)$ defined in this way *parallel displacement along* c and denote it by τ_c. Note that τ_c depends on the curve *and* the connection.

Exercises:

1. If $c_1\colon a_1, b_1 \to M$ and $c_2\colon [a_2, b_2] \to M$ are two smooth curves on M such that $c_1(b_1) = c_2(a_2)$ then the concatenated curve $c_2 * c_1$ is a piecewise smooth curve from $c_1(a_1)$ to $c_2(b_2)$. Show that
$$\tau_{c_2 * c_1} = \tau_{c_2} \circ \tau_{c_1}$$
2. Suppose that the action of G on P is denoted by Φ. Show that for any curve, c, on M we have $\tau_c \circ \Phi_g = \Phi_g \circ \tau_c$. (*Hint:* First show that Φ_g maps horizontal curves to horizontal curves.)
3. Show that τ_c is a diffeomorphism.
4. Do you think that parallel translation is dependent on the parameterisation of the curve c?
5. Compute the parallel translation along a general curve for the general connections you found for **PFB1**, **PFB2**, and **PFB3**. If necessary, make simplifications to the connections so that you may solve the resulting ordinary differential equations. Illustrate your conclusions (i.e., make pictures). Where applicable, interpret the above exercises in the context of your examples.

5. Holonomy groups

Since we are interested in how periodic motions generate holonomy, we will look at the relevant entity in the principal fibre bundle case: the holonomy group. Let $P(M, G)$ be a principal fibre bundle and let Γ be a connection on P. For $m \in M$, let $\Omega(m)$ be the set of smooth curves in M with m as the initial and final point. Recall that $\Omega(m)$ is a group with the group operation given by concatenation of curves. Therefore,

$$\Psi(m) \triangleq \{\tau_c \mid \quad c \in \Omega(m)\}$$

is a subgroup of the group of diffeomorphisms of the fibre $\pi^{-1}(m)$ by the exercises of the previous section. We shall call $\Phi(m)$ the *holonomy group of* Γ *with reference point* m. Now let $\Omega^0(m)$ be the subset of $\Omega(m)$ consisting of those curves which are homotopic to the identity. Note that this is a subgroup of $\Omega(m)$. Therefore

$$\Psi^0(m) \triangleq \{\tau_c \mid \quad c \in \Omega^0(m)\}$$

is a subgroup of $\Psi(m)$ which we call the *restricted holonomy group of* Γ *with reference point* m.

It is possible to realise these groups as subgroups of the structure group G. We may do this as follows: Fix $p \in \pi^{-1}(m)$. For each $c \in \Omega(m)$, $\tau_c \in \Psi(m)$, therefore $\tau_c(p) \in \pi^{-1}(m)$. Since the group action is free, there must exist a unique $g \in G$ so that $\Phi(p, g) = \tau_c(p)$. Also, since parallel translation commutes with the group action, if $c_1, c_2 \in \Omega(m)$ have corresponding group elements $g_1, g_2 \in G$, we have

$$\Phi(p, g_1 g_2) = \tau_{c_1 * c_2}(p) \tag{5.1}$$

In this manner we establish an injection of $\Psi(m)$ into G. We denote the image of this injection by $\Psi(p)$ and call it the *holonomy group of* Γ *with reference point* p. In like fashion we may construct $\Psi^0(p)$, the *restricted holonomy group of* Γ *with reference point* p.

Note that $\Phi(p)$ is a subgroup of G which *depends on the choice of* p. It is easy to show that if $p' \in \pi^{-1}(m)$ is such that $p' = \Phi(g, p)$ for some (unique) $g \in G$, then we have

$$\Psi(p') = g^{-1} \cdot \Psi(p) \cdot g, \quad \text{and} \quad \Psi^0(p') = g^{-1} \cdot \Psi^0(p) \cdot g \tag{5.2}$$

Thus, although the subgroup of G changes as we change the reference point p, the groups are isomorphic (in fact conjugate).
Exercises:

1. Compute the holonomy groups of all types for **PFB1**, **PFB2**, and **PFB3**.
2. Compute the holonomy group of **PFB1** but make the base space $\mathbb{S}$ rather than $\mathbb{R}$. The calculations will be the same except everything will have to be done $\bmod 2\pi$.
3. Prove equation 5.1.
4. Prove equation 5.2.

References

Abraham, R. and Marsden, J. E. [1978] *Foundations of Mechanics*, 2nd edition, Addison Wesley: Reading, MA, ISBN: 978-0-8218-4438-0.