Notes on mechanics, symmetry, reduction, holonomy, and control

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1. Lagrangian mechanics

Here we give a brief survey of Lagrangian mechanics. For a more detailed exposition in the same vein, please see [Abraham and Marsden 1978]. For a more earthy, but still mathematical exposition see [Arnol′d 1978]. Also very good, but less well known is [Libermann and Marle 1987]. We say nothing of the related field of Hamiltonian mechanics since we will be interested only in a class of systems where the Lagrangian and Hamiltonian formalism "correspond".

We denote by Q the *configuration manifold*, and will assume it to be n-dimensional. We will refer to the tangent bundle, TQ, as the *phase space* for the problem, although some would dispute this terminology. The natural projection from TQ to Q will be denoted by τ_Q. Since most people are familiar with the "generalised coordinate" description of Lagrangian mechanics we introduce notation for coordinates for TQ. Let $(q^1, \dots, q^n)$ be coordinates for a chart (U, ϕ) on Q. A vector, v, in the tangent space at $q \in U$ will be of the form

$$v = v^i \frac{\partial}{\partial q^i}\bigg|_q$$

for some $v^1, \dots, v^n \in \mathbb{R}$. Thus any point in TQ whose base point lies in the chart U will have coordinates $(q^1, \dots, q^n, v^1, \dots, v^n)$. This defines a *natural chart* for TQ associated with the chart U. Unless we say otherwise, coordinates for TQ will always be from a natural chart.

A *Lagrangian* on Q is a function $L\colon TQ \to \mathbb{R}$. We will be interested in the case where

$$L(v) = \frac{1}{2}\langle v, v\rangle - V \circ \tau_Q(v)$$

where $\langle \cdot, \cdot \rangle$ is a Riemannian metric on Q and V is a potential function on Q. In coordinates this Lagrangian is

$$L(q^1, \dots, q^n, v^1, \dots, v^n) = \frac{1}{2} M_{ij} v^i v^j - V(q^1, \dots, q^n)$$

where

$$M_{ij} = \left\langle \frac{\partial}{\partial q^i}, \frac{\partial}{\partial q^j} \right\rangle$$

More concretely we have the following example.

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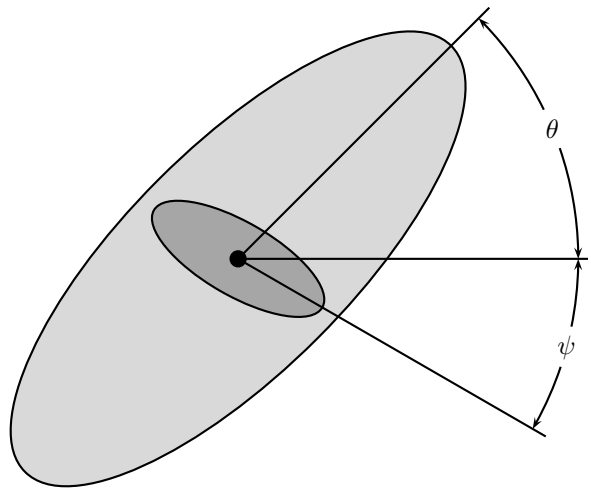

Figure 1. Elroy's beanie.

1.1 Example: (Elroy's beanie) We consider two rigid bodies in the plane which are hinged at their centres of mass. See Figure 1. The configuration space for the system is $Q = \mathbb{T}^2$, and we shall for the moment use coordinates $q^1 = \theta_1$ and $q^2 = \theta_2$, the angular positions of the bodies with respect to an inertial frame. The corresponding coordinates for TQ we shall write as $(\theta_1, \theta_2, v_1, v_2)$. We shall disregard the movement of the centre of mass of the system since this is uniform (i.e., we consider the system to have no initial motion of its centre of mass). The Lagrangian for the system is

$$L = \frac{1}{2} I_1 v_1^2 + \frac{1}{2} I_2 v_2^2 + V(\theta_2 - \theta_1)$$

Thus we allow a potential which is a function of the relative angle of the two bodies. The Riemannian metric is given by

$$\langle \cdot, \cdot \rangle = I_1 \mathrm{d}\theta_1 \otimes \mathrm{d}\theta_1 + I_2 \mathrm{d}\theta_2 \otimes \mathrm{d}\theta_2$$

This shall be a recurring example. •

Now we need to discuss how we get the equations of motion. Since we all know how to compute Lagrange's equations, we shall be purposefully pointed in our presentation. Suppose that we have a Lagrangian of the form

$$L(v) = \frac{1}{2} \langle v, v \rangle - V \circ \tau_Q(v)$$

Lagrange's equations in natural coordinates are

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial v^i} \right) - \frac{\partial L}{\partial q^i} = M_{ij} \ddot{q}^j + \left(\frac{\partial M_{ij}}{\partial q^k} - \frac{1}{2} \frac{\partial M_{jk}}{\partial q^i} \right) \dot{q}^j \dot{q}^k + \frac{\partial V}{\partial q^i} = 0$$

Since M_{ij} is invertible, this is equivalent to the system of first order equations

$$\begin{aligned}\dot{q}^i &= v^i \\ \dot{v}^i &= M^{ij}\left(\frac{1}{2}\frac{\partial M_{kl}}{\partial q^j} - \frac{\partial M_{jk}}{\partial q^l}\right)\dot{q}^k\dot{q}^l - M^{ij}\frac{\partial V}{\partial q^j}\end{aligned}$$

Here M^{ij} are the components of the inverse of M_{ij}. This then defines a vector field on TQ which is of a special type: a second order vector field. See [Abraham and Marsden 1978] for details. To be explicit this vector field is

$$\boldsymbol{X}_L = v^i\frac{\partial}{\partial q^i} + \left[M^{ij}\left(\frac{1}{2}\frac{\partial M_{kl}}{\partial q^j} - \frac{\partial M_{jk}}{\partial q^l}\right)\dot{q}^k\dot{q}^l - M^{ij}\frac{\partial V}{\partial q^j}\right]\frac{\partial}{\partial v^i}$$

1.2 Example: (Elroy's beanie cont'd) We give Lagrange's equations for the beanie. In second order form the equations are

$$\begin{aligned}I_1\ddot{\theta}_1 - \frac{\partial V(\theta_2-\theta_1)}{\partial\theta_1} &= 0 \\ I_2\ddot{\theta}_2 + \frac{\partial V(\theta_2-\theta_1)}{\partial\theta_2} &= 0\end{aligned}$$

and in first order form we have

$$\begin{aligned}\dot{\theta}_1 &= v_1 \\ \dot{v}_1 &= \frac{1}{I_1}\frac{\partial V(\theta_2-\theta_1)}{\partial\theta_1} \\ \dot{\theta}_2 &= v_2 \\ \dot{v}_2 &= -\frac{1}{I_2}\frac{\partial V(\theta_2-\theta_1)}{\partial\theta_2}\end{aligned}$$

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Exercises:

1. Add translational degrees of freedom to Elroy's beanie and compute Lagrange's equations for the new system. Denote by m_1, m_2 the masses of the bodies. In the sequel we shall be calling this the translational beanie, so you know what I mean when I say that.

2. For an arbitrary Lagrangian, do Lagrange's equations define a vector field on TQ? Explain why or why not, and give conditions on the Lagrangian so that Lagrange's equations *do* give a vector field on TQ.

2. Symmetry and reduction in Lagrangian mechanics

A symmetry in a mechanical system is meant to reflect the fact that there are some conserved quantities which allow you to simplify the system. We will be interested in a special type of symmetry. Namely we consider the case where

1. The Lagrangian has the form $L(v) = \frac{1}{2}\langle v, v\rangle - V \circ \tau_Q(v)$.

2. A Lie group G acts on Q from the left with action Φ and where Φ_g is an isometry for each $g \in G$ and where the potential function V is G-invariant.

Interestingly, this is a very broad class of problems within the class of problems with symmetry. With the action of G on Q we have the corresponding action of G on TQ by lifts. This action is denoted by Φ^T and is defined by

$$\Phi_g^T(v_q) = T_q\Phi_g(v_q)$$

As you shall show below, the lifted action leaves L invariant under the above hypotheses.

2.1 Example: (Elroy's beanie cont'd) Consider the following group action of $\mathbb{S}^1$ on $\mathbb{T}^2$.

$$\Phi(\phi, (\theta_1, \theta_2)) = (\theta_1 + \phi, \theta_2 + \phi)$$

We shall show that the Lagrangian for Elroy's beanie is invariant under the lift of this group action. First we compute the lifted group action. Fix $\phi \in \mathbb{S}^1$. Then

$$\Phi_\phi(\theta_1, \theta_2) = (\theta_1 + \phi, \theta_2 + \phi)$$

Therefore

$$T_{(\theta_1,\theta_2)}\Phi_\phi(\theta_1, \theta_2, v_1, v_2) = (\theta_1 + \phi, \theta_2 + \phi, v_1, v_2)$$

Therefore, by definition, the lifted action is

$$\Phi^T(\phi, (\theta_1, \theta_2, v_1, v_2)) = (\theta_1 + \phi, \theta_2 + \phi, v_1, v_2)$$

Now, for each $\phi \in \mathbb{S}^1$ we have

$$\begin{aligned} L(\Phi_\phi^T(\theta_1, \theta_2, v_1, v_2)) &= L(\theta_1 + \phi, \theta_2 + \phi, v_1, v_2) \\ &= \frac{1}{2}I_1v_1^2 + \frac{1}{2}I_2v_2^2 + V((\theta_2 + \phi) - (\theta_1 + \phi)) \\ &= L(\theta_1, \theta_2, v_1, v_2) \end{aligned}$$

Thus L is G-invariant. •

Exercises:

1. Given the hypotheses on the group action and the Lagrangian above, show that the Lagrangian is G-invariant.

2. For the translated beanie find an action of $\mathbb{R}^2$ on Q so that the lifted action leaves the Lagrangian invariant. Don't try to be clever: choose the obvious one.

Our goal is to use this symmetry to reduce the study of the dynamics to a smaller manifold. This procedure is known as reduction. First we need to construct conserved quantities for a mechanical system with group symmetries as discussed above. One normally thinks of these conserved quantities as momenta of some sort (typically linear or angular momentum). One likes to think of momentum as having vector like qualities, but where is the vector space in which these conserved momenta live? The answer turns out to be that they live in the dual of the Lie algebra of the symmetry group. Let us give the definitions, then some examples may make it easier to swallow.

Given a group action as above, we define the *momentum mapping* for this group action to be the map $\boldsymbol{J}\colon TQ \to \mathfrak{g}^*$ defined by

$$\boldsymbol{J}(v_q) \cdot \xi = \langle v_q, \xi_Q(q)\rangle, \qquad \forall\ \xi \in \mathfrak{g}$$

where ξ_Q is the infinitesimal generator corresponding to ξ,

2.2 Example: (Elroy's beanie cont'd) We shall compute the momentum mapping for the $\mathbb{S}^1$ action given for Elroy's beanie. Since $G = \mathbb{S}^1$, we may identify $\mathfrak{g}$ with $\mathbb{R}$. With this identification, the exponential map looks like

$$\begin{aligned} \exp\colon \mathfrak{g} &\to G \\ \omega \cdot e_1 &\mapsto \omega \bmod 2\pi. \end{aligned}$$

Here e_1 is the standard basis element for $\mathbb{R} \simeq \mathfrak{g}$. Thus, for $\xi = \omega \cdot e_1 \in \mathfrak{g}$, the infinitesimal generator is

$$\begin{aligned} \xi_Q(\theta_1, \theta_2) &= \frac{\mathrm{d}}{\mathrm{d}t}\Phi(\exp(t\xi), (\theta_1, \theta_2))\Big|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Phi(t\omega, (\theta_1, \theta_2))\Big|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}(\theta_1 + t\omega, \theta_2 + t\omega)\Big|_{t=0} \\ &= \omega\frac{\partial}{\partial\theta_1} + \omega\frac{\partial}{\partial\theta_2} \end{aligned}$$

Now, if we let $\{e^1\}$ be the basis of $\mathfrak{g}^*$ dual to the basis e_1 of $\mathfrak{g}$, we may write

$$\boldsymbol{J}(\theta_1, \theta_2, v_1, v_2) = J_1(\theta_1, \theta_2, v_1, v_2) \cdot e^1$$

Therefore, recalling that $\xi = \omega \cdot e_1$, we have

$$\boldsymbol{J}(\theta_1, \theta_2, v_1, v_2) \cdot \xi = J_1(\theta_1, \theta_2, v_1, v_2) \cdot \omega$$

and

$$\left\langle v_1\frac{\partial}{\partial\theta_1} + v_2\frac{\partial}{\partial\theta_2}, \omega\frac{\partial}{\partial\theta_1} + \omega\frac{\partial}{\partial\theta_2}\right\rangle = I_1 v_1 \cdot \omega + I_2 v_2 \cdot \omega$$

Thus

$$\begin{aligned} & \boldsymbol{J}(\theta_1, \theta_2, v_1, v_2) \cdot \xi = \left\langle v_1\frac{\partial}{\partial\theta_1} + v_2\frac{\partial}{\partial\theta_2}, \omega\frac{\partial}{\partial\theta_1} + \omega\frac{\partial}{\partial\theta_2}\right\rangle \qquad \forall\ \xi = \omega \cdot e_1 \in \mathfrak{g} \\ \iff \quad & J_1(\theta_1, \theta_2, v_1, v_2) = I_1 v_1 + I_2 v_2 \end{aligned}$$

Therefore

$$\boldsymbol{J}(\theta_1, \theta_2, v_1, v_2) = (I_1 v_1 + I_2 v_2)e^1$$

We recognise the coefficient of e^1 as angular momentum! •

Exercises:

1. Show that the following diagram commutes

$$\begin{tikzcd} TQ \arrow[r, "\Phi_g^T"] \arrow[d, "\boldsymbol{J}"'] & TQ \arrow[d, "\boldsymbol{J}"] \\ \mathfrak{g} \arrow[r, "Ad^*_{g^{-1}}"'] & \mathfrak{g} \end{tikzcd}$$

for each $g \in G$ where $Ad^*\colon G \times \mathfrak{g}^* \to \mathfrak{g}^*$ is the coadjoint action (see [Abraham and Marsden 1978]). This property is known as Ad^*-*equivariance*.

2. Compute the momentum map for the action of $\mathbb{R}^2$ you found for the translated beanie. You should be able to identify things which look like linear momentum.

It is easy to show that if a Lagrangian, L, is invariant under the group action, then the momentum given by the momentum mapping is conserved. More precisely,

2.3 Proposition: *Let Φ be a left action of G on Q, and suppose that the lifted action leaves the Lagrangian, L, invariant. If F_t is the flow of the Lagrangian vector field, then $\boldsymbol{J}(F_t(x)) = \boldsymbol{J}(x)$.*

Since the momentum is left invariant by the flow of the Lagrangian vector field, integral curves which begin on $\boldsymbol{J}^{-1}(\mu)$ remain on $\boldsymbol{J}^{-1}(\mu)$ for each $\mu \in \mathfrak{g}^*$. That is to say, $\boldsymbol{J}^{-1}(\mu)$ is a subset of TQ which is invariant under the dynamics. If μ is a regular value of the momentum mapping, then $\boldsymbol{J}^{-1}(\mu)$ is a submanifold of TQ. By G_μ we denote the subgroup of G which leaves $\boldsymbol{J}^{-1}(\mu)$ invariant under the group action (i.e., G_μ is the isotropy subgroup of $\boldsymbol{J}^{-1}(\mu)$). More precisely

$$G_\mu = \{g \in G \mid \ \Phi(g, v) \in \boldsymbol{J}^{-1}(\mu) \quad \forall \ v \in \boldsymbol{J}^{-1}(\mu)\}$$

Since G_μ acts on $\boldsymbol{J}^{-1}(\mu)$ we may define the projection $\pi_\mu \colon \boldsymbol{J}^{-1}(\mu) \to P_\mu \triangleq \boldsymbol{J}^{-1}(\mu)/G_\mu$ from $\boldsymbol{J}^{-1}(\mu)$ to the set of orbits of G_μ.

2.4 Example: (Elroy's beanie cont'd) For the $\mathbb{S}^1$ action for the beanie we determine $\boldsymbol{J}^{-1}(\mu)$ for $\mu \in \mathfrak{g}^*$. With our established notation we may write $\mu = m \cdot e^1$ where e^1 is the standard basis vector for $\mathfrak{g}^* \simeq \mathbb{R}^*$. Therefore we easily see that

$$\boldsymbol{J}^{-1}(\mu) = \{(\theta_1, \theta_2, v_1, v_2) \in TQ \mid \ I_1 v_1 + I_2 v_2 = m\}$$

Observe that every value of the momentum map is regular, so $\boldsymbol{J}^{-1}(\mu)$ is always a three-dimensional submanifold. As coordinates for $\boldsymbol{J}^{-1}(\mu)$ we may use $(\theta_1, \theta_2, v_2 - v_1)$. The reason for choosing $v_2 - v_1$ will become clear below.

We claim that $G_\mu = G$ for each $\mu \in \mathfrak{g}^*$ for Elroy's beanie. This is quite clear by observing the two formulas

$$\Phi_\phi^T(\theta_1, \theta_2, v_1, v_2) = (\theta_1 + \phi, \theta_2 + \phi, v_1, v_2)$$

and

$$\boldsymbol{J}(\theta_1, \theta_2, v_1, v_2) = (I_1 v_1 + I_2 v_2) e^1$$

From these we easily see that if $(\theta_1, \theta_2, v_1, v_2)$ are such that $\boldsymbol{J}(\theta_1, \theta_2, v_1, v_2) = \mu$, then $\boldsymbol{J} \circ \Phi_\phi^T(\theta_1, \theta_2, v_1, v_2) = \mu$ for each $\phi \in G$. Therefore $G_\mu = G$.

Now observe that the orbits for the G-action on $\boldsymbol{J}^{-1}(\mu)$ are one-dimensional. Indeed, the orbit through the point $p = (\theta_1, \theta_2, v_2 - v_1) \in \boldsymbol{J}^{-1}(\mu)$ is given by

$$\mathcal{O}_p = \{(\theta_1 + \phi, \theta_2 + \phi, v_2 - v_1) \mid \ \phi \in G\}$$

Note that each point in $\mathcal{O}_p$ is uniquely specified by the value of $\theta_2 - \theta_1$ and $v_2 - v_1$ (you can see now why we chose $v_2 - v_1$ as a coordinate for $\boldsymbol{J}^{-1}(\mu)$). Motivated by this define $\psi = \theta_2 - \theta_1$ and $v_\psi = v_2 - v_1$ as coordinates for $P_\mu = \boldsymbol{J}^{-1}(\mu)/G_\mu$. Then we have

$$\pi_\mu(\theta_1, \theta_2, v_2 - v_1) = (\theta_2 - \theta_1, v_2 - v_1) \qquad \bullet$$

2.5 Remark: In the example of Elroy's beanie we see some features in the structure of $\boldsymbol{J}^{-1}(\mu)$ which are general in momentum mappings of the type we are looking at. First of all note that, in the example, when $\mu = 0$, $\boldsymbol{J}^{-1}(\mu)$ is a subbundle of TQ. When $\mu \neq 0$, let $a, b \in \mathbb{R}$ be two numbers such that $I_1 a + I_2 b = -m$. Then the map

$$\begin{aligned} t_\mu \colon \boldsymbol{J}^{-1}(\mu) &\to \boldsymbol{J}^{-1}(0) \\ (\theta_1, \theta_2, v_1, v_2) &\mapsto (\theta_1, \theta_2, v_1 + a, v_2 + b) \end{aligned}$$

is a diffeomorphism of $\boldsymbol{J}^{-1}(\mu)$ onto the subbundle $\boldsymbol{J}^{-1}(0)$. It is simply "momentum shift" in the fibre of TQ. The direction along which we shift depends on the choice of a and b.

Also note that if we define $\psi = \theta_2 - \theta_1$ we may use ψ as a coordinate for Q/G_μ. Since we are using $v_2 - v_1 = v_\psi$ as the other coordinate for P_μ we have an identification of P_μ with $T(Q/G_\mu)$ in this case. This will in fact be possible exactly when G is abelian. If G is not abelian then we will not have $G = G_\mu$ and in this case it is true that P_μ is diffeomorphic to a subbundle of $T(Q/G_\mu)$. For more on this see [Marsden, Montgomery, and Ratiu 1990]. Those of you who know the reduction picture for the rigid body on $SO(3)$, you may want to think of what is going on there. In this case the fibre is 0-dimensional. •

Exercises:

1. Show that
$$G_\mu = \{g \in G \mid \quad Ad_g^*(\mu) = \mu\}$$
(*Hint:* Use Ad^*-equivariance of $\boldsymbol{J}$.)

2. For a fixed value of linear momentum for the translated beanie (thought of as an element of $(\mathbb{R}^2)^*$!) determine the level set of that linear momentum, and put coordinates on that subset. Compute the isotropy subgroup of $\boldsymbol{J}^{-1}(\mu)$. Find coordinates for the set of G_μ orbits and compute the projection in these coordinates. Interpret your answers in light of the remark on page 7.

We call P_μ the *reduced phase space*. We now wish to compute dynamics on P_μ induced by the full dynamics on TQ. Just as P_μ is the phase space with the group action modded out in an appropriate fashion, the dynamics on P_μ will be the full dynamics with the dynamics of the group action modded out. Hopefully this will be made clearer in our running example.

The general process goes as follows. Since the Lagrangian is invariant under the group action, so will be the Lagrangian vector field $\boldsymbol{X}_L$. (This, though not unexpected, is not obvious in our abbreviated presentation. See [Abraham and Marsden 1978].) Also, we have seen that the flow of the Lagrangian vector field leaves $\boldsymbol{J}^{-1}(\mu)$ invariant. It is further true that the flow of the Lagrangian vector field restricted to $\boldsymbol{J}^{-1}(\mu)$ commutes with the action of G_μ. All this combines to give us the following fact:
The flow, F_t, of the Lagrangian vector field on TQ gives rise to a flow, F_t^μ, on $P_\mu = \boldsymbol{J}^{-1}(\mu)/G_\mu$ such that $\pi_\mu \circ F_t = F_t^\mu \circ \pi_\mu$.
Corresponding to the flow F_t^μ is a vector field which we shall denote $\boldsymbol{X}_L^\mu$. Since P_μ will not, in general, be a tangent bundle, the vector field $\boldsymbol{X}_L^\mu$ will not in general be second order. Observe that

$$\begin{aligned} \pi_\mu \circ F_t(p) = F_t^\mu \circ \pi_\mu(p) \qquad &\Longleftrightarrow \qquad T\pi_\mu \circ \frac{\mathrm{d}F_t}{\mathrm{d}t}(p)\bigg|_{t=0} = \frac{\mathrm{d}F_t^\mu}{\mathrm{d}t} \circ \pi_\mu(p)\bigg|_{t=0} \\ &\Longleftrightarrow \qquad T\pi_\mu \circ \boldsymbol{X}_L(p) = \boldsymbol{X}_L^\mu \circ \pi_\mu(p) \end{aligned}$$

2.6 Example: (Elroy's beanie cont'd) We now compute the vector field on P_μ for Elroy's beanie. It will help to use coordinates for Q given by $(\psi = \theta_2 - \theta_1, \theta = \theta_1)$. Thus ψ is the relative angle of the bodies, and θ is the angle of body 1. In the beanie scenario, body 1 is supposed to be Elroy, and body 2 is supposed to be the propeller on the beanie. Thus ψ is the angle of the propeller relative to Elroy, and θ is the angle of Elroy relative to ground. We make some observations and computations in these new coordinates:

1. In the new coordinates we have $\boldsymbol{J}(\psi, \theta, v_\psi, v_\theta) = I_2 v_\psi + (I_1 + I_2) v_\theta$.
2. Recall above that we had decided to use $(\theta_1, \theta_2, v_2 - v_1)$ to parameterise $\boldsymbol{J}^{-1}(\mu)$. In our new coordinates this means that we use (ψ, θ, v_ψ) to parameterise $\boldsymbol{J}^{-1}(\mu)$.
3. Recall that we had decided to use $(\theta_2 - \theta_1, v_2 - v_1)$ to parameterise P_μ. In the (ψ, θ) coordinates this means that we will use (ψ, v_ψ) to parameterise P_μ.

Therefore we have

$$\pi_\mu(\psi, \theta, v_\psi) = (\psi, v_\psi)$$

In these coordinates the Lagrangian is

$$L = \frac{1}{2} I_1 \dot{\theta}^2 + \frac{1}{2} I_2 (\dot{\psi} + \dot{\theta})^2 - V(\psi)$$

Therefore the Lagrangian vector field is

$$\boldsymbol{X}_L = v_\psi \frac{\partial}{\partial \psi} + v_\theta \frac{\partial}{\partial \theta} + \frac{V'(\psi)}{I_1 + 2I_2} \frac{\partial}{\partial v_\theta} - \frac{(I_1 + I_2) V'(\psi)}{I_2 (I_1 + 2I_2)} \frac{\partial}{\partial v_\psi}$$

Since $\boldsymbol{X}_L$ is tangent to $\boldsymbol{J}^{-1}(\mu)$, we may write this vector field as a vector field on $\boldsymbol{J}^{-1}(\mu)$ by using the relation

$$I_2 v_\psi + (I_1 + I_2) v_\theta = m$$

A vector

$$v = a \frac{\partial}{\partial \psi} + b \frac{\partial}{\partial \theta} + c \frac{\partial}{\partial v_\psi} + d \frac{\partial}{\partial v_\theta}$$

will be tangent to $\boldsymbol{J}^{-1}(\mu)$ if and only if

$$(I_1 + I_2) c = -I_2 d$$

(Note that $\boldsymbol{X}_L$ satisfies this condition.) Now, since $\boldsymbol{X}_L$ is tangent to $\boldsymbol{J}^{-1}(\mu)$ we can write it in the coordinates (ψ, θ, v_ψ) as

$$\boldsymbol{X}_L \mid \boldsymbol{J}^{-1}(\mu) = v_\psi \frac{\partial}{\partial \psi} + v_\theta \frac{\partial}{\partial \theta} - \frac{(I_1 + I_2) V'(\psi)}{I_2 (I_1 + 2I_2)} \frac{\partial}{\partial v_\psi}$$

We now determine the vector field $\boldsymbol{X}_L^\mu$ on P_μ in the coordinates (ψ, v_ψ) using the relation

$$T\pi_\mu \circ \boldsymbol{X}_L(p) = \boldsymbol{X}_L^\mu \circ \pi_\mu(p)$$

In the coordinates $(\psi, \theta, v_\psi, v_\theta)$ for TQ and (ψ, v_ψ) for P_μ we have

$$T\pi_\mu \left(a \frac{\partial}{\partial \psi} + b \frac{\partial}{\partial \theta} + c \frac{\partial}{\partial v_\psi} \right) = a \frac{\partial}{\partial \psi} + c \frac{\partial}{\partial v_\psi}$$

If we write

$$\boldsymbol{X}_L^\mu = d\frac{\partial}{\partial\psi} + e\frac{\partial}{\partial v_\psi}$$

where d and e are to be determined then we have

$$\begin{aligned} & T\pi_\mu \circ \boldsymbol{X}_L(p) = \boldsymbol{X}_L^\mu \circ \pi_\mu(p) \\ \iff \qquad & v_\psi \frac{\partial}{\partial\psi} - \frac{(I_1+I_2)V'(\psi)}{I_2(I_1+2I_2)}\frac{\partial}{\partial v_\psi} = d\frac{\partial}{\partial\psi} + e\frac{\partial}{\partial v_\psi} \end{aligned}$$

Thus we have

$$\boldsymbol{X}_L^\mu = v_\psi \frac{\partial}{\partial\psi} - \frac{(I_1+I_2)V'(\psi)}{I_2(I_1+2I_2)}\frac{\partial}{\partial v_\psi}$$

which is equivalent to the second order equation on P_μ given by

$$\ddot{\psi} + \frac{(I_1+I_2)V'(\psi)}{I_2(I_1+2I_2)} = 0$$

This may be solved by quadratures.

Now, supposing that we have the solution $\psi(t)$ we have

$$\ddot{\theta} = \frac{V'(\psi(t))}{I_1+2I_2}$$

which may be directly integrated. The generalisation of this procedure is the subject of the next section. •

Exercises:

1. Compute the vector field on P_μ for the $\mathbb{R}^2$-action you have computed. You should notice that the reduced dynamics "decouple" from the total dynamics (providing you make the "obvious" choices of coordinates).

2. Prove the formula $F_t \circ \Phi_g = \Phi_g \circ F_t$ stating that the flow of the Lagrangian vector field commutes with the group action if the Lagrangian is G-invariant.

3. Reconstruction

Hopefully at this point the reader has identified a familiar structure (or at least the possibility of that structure being present). Namely, the reader should recognise $\pi_\mu \colon \boldsymbol{J}^{-1}(\mu) \to P_\mu$ as perhaps being a principal fibre bundle. Indeed, we will *assume* that it is in this section. Because of the special structure of the Lagrangians and group actions we are looking at, the principal bundle structure on $\boldsymbol{J}^{-1}(\mu)(P_\mu, G_\mu)$ actually arises from a bundle structure on $Q(Q_\mu, G_\mu)$ where $Q_\mu = Q/G_\mu$. We are going to assume that this *is* a principal fibre bundle in the sequel. What we wish to do is show how to use a special connection on this principal bundle to *reconstruct* the full dynamics given integral curves of $\boldsymbol{X}_L^\mu$ on P_μ.

First we discuss the mechanical connection on the principal fibre bundle $Q(Q_\mu, G_\mu)$. Recall that a connection is a choice of horizontal subspace. We will define the *mechanical*

connection as the connection whose horizontal subspaces are the orthogonal complement of the vertical subspaces with respect to, $\langle\cdot,\cdot\rangle$, the kinetic energy metric.

To give coordinate expressions for the mechanical connection one-form we must introduce some intermediate entities, beginning with the *action tensor*. Let $\{e_1,\dots,e_m\}$ be a basis for $\mathfrak{g}_\mu$, the Lie algebra of G_μ. Then, for $\xi\in\mathfrak{g}_\mu$, we have $\xi=\xi^je_j$. The action tensor is defined for each $q\in Q$ by having the components $A^i_j(q)$ given by

$$[\xi_Q(q)]^i=A^i_j(q)\xi^j$$

Now we define the *locked inertia tensor*. For each $q\in Q$, the locked inertia tensor is the isomorphism from $\mathfrak{g}_\mu$ to $\mathfrak{g}_\mu^*$ defined by

$$\mathbb{I}(q)\eta\cdot\xi=\langle\eta_Q(q),\xi_Q(q)\rangle$$

for $\xi,\eta\in\mathfrak{g}_\mu$. In coordinates we have

$$\mathbb{I}_{ij}(q)=M_{kl}(q)A^k_i(q)A^l_j(q)$$

Now we may define the connection one-form for the mechanical connection. Recall that it is a $\mathfrak{g}$-valued one-form on Q. It is defined by

$$\alpha(v_q)=\mathbb{I}^{-1}(\boldsymbol{J}(v_q))$$

It is possible to derive coordinate formulas for the connection one-form for the mechanical connection. The expression is

$$\alpha^i(v_q)=\mathbb{I}^{ij}(q)M_{kl}(q)A^k_j(q)v^l$$

Here $\mathbb{I}^{ij}$ are the components of the inverse of $\mathbb{I}_{ij}$. For more on the mechanical connection see [Marsden 1992].

3.1 Example: (Elroy's beanie cont'd) We compute the mechanical connection for Elroy's beanie. First note that in the (ψ,θ) coordinates the metric on Q is given by

$$\langle\cdot,\cdot\rangle=I_2\mathrm{d}\psi\otimes\mathrm{d}\psi+(I_1+I_2)\mathrm{d}\theta\otimes\mathrm{d}\theta+I_2\mathrm{d}\psi\otimes\mathrm{d}\theta+I_2\mathrm{d}\theta\otimes\mathrm{d}\psi$$

The natural projection, ρ_μ, on the bundle $Q(Q_\mu,G_\mu)$ is given in coordinates as

$$\rho_\mu(\psi,\theta)=\psi$$

The vertical subspace is the kernel of $T\rho_\mu$. It is easy to see that

$$VQ=\left\langle\frac{\partial}{\partial\theta}\right\rangle_{C^\infty(Q)}$$

To compute the mechanical connection we must compute the $\langle\cdot,\cdot\rangle$-orthogonal complement of VQ. Suppose that $v\in HQ$ is given by

$$v=a\frac{\partial}{\partial\psi}+b\frac{\partial}{\partial\theta}$$

Then we must have

$$\begin{aligned}
& \left\langle v, \frac{\partial}{\partial\theta} \right\rangle = 0 \\
\iff \quad & (I_2 \mathrm{d}\psi \otimes \mathrm{d}\psi + (I_1 + I_2)\mathrm{d}\theta \otimes \mathrm{d}\theta + I_2 \mathrm{d}\psi \otimes \mathrm{d}\theta + I_2 \mathrm{d}\theta \otimes \mathrm{d}\psi)\left(a\frac{\partial}{\partial\psi} + b\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\theta}\right) = 0 \\
\iff \quad & (I_1 + I_2)b + I_2 a = 0
\end{aligned}$$

Therefore

$$HQ = \left\langle (I_1 + I_2)\frac{\partial}{\partial\psi} - I_2\frac{\partial}{\partial\theta} \right\rangle_{C^\infty(Q)}$$

Now we compute the action tensor relative to standard basis for $\mathfrak{g}_\mu = \mathfrak{g} \simeq \mathbb{R}$. Let $\xi = \omega \cdot e_1 \in \mathfrak{g}$. Then

$$\xi_Q = \omega\frac{\partial}{\partial\theta}$$

(To compute this you must compute the infinitesimal generator in the (ψ, θ) coordinates. This is straightforward.) Therefore

$$A_1^1 = 0, \quad A_1^2 = 1$$

Now we compute the locked inertia tensor. It is given by

$$\begin{aligned}
\mathbb{I}_{11} &= M_{11}A_1^1A_1^1 + M_{12}A_1^1A_1^2 + M_{21}A_1^2A_1^1 + M_{22}A_1^2A_1^2 \\
&= M_{22} = I_1 + I_2
\end{aligned}$$

(Notice that this is simply the inertia of the two bodies locked together!) Therefore

$$\begin{aligned}
\alpha^1(\psi, \theta, v_\psi, v_\theta) &= \mathbb{I}^{11}M_{11}A_1^1v_\psi + \mathbb{I}^{11}M_{12}A_1^1v_\theta + \mathbb{I}^{11}M_{21}A_1^2v_\psi + \mathbb{I}^{11}M_{22}A_1^2v_\theta \\
&= \frac{1}{I_1 + I_2}M_{21}v_\psi + \frac{1}{I_1 + I_2}M_{22}v_\theta \\
&= \frac{I_2}{I_1 + I_2}v_\psi + v_\theta
\end{aligned}$$

Thus we may write

$$\alpha = \left(\frac{I_2}{I_1 + I_2}\mathrm{d}\psi + \mathrm{d}\theta\right)e^1$$

Observe that α vanishes on horizontal vectors as it is supposed to. •

3.2 Remark: To add some intuition to the procedure, let's look at conservation of angular momentum when $\mu = 0$. This reads

$$\begin{aligned}
& I_2 v_\psi + (I_1 + I_2)v_\theta = 0 \\
\iff \quad & I_2\frac{\mathrm{d}\psi}{\mathrm{d}t} + (I_1 + I_2)\frac{\mathrm{d}\theta}{\mathrm{d}t} = 0 \\
\iff \quad & I_2\mathrm{d}\psi + (I_1 + I_2)\mathrm{d}\theta = 0 \\
\iff \quad & \alpha = 0
\end{aligned}$$

Thus conservation of zero angular momentum is equivalent to the constraint $\alpha = 0$. •

Exercises:

1. Show that the mechanical connection as defined above is indeed a connection on the bundle $Q(Q_\mu, G_\mu)$. (You must show that the horizontal subspaces satisfy the group invariance property in the definition of a connection.)

2. Compute the mechanical connection and the connection one-form for the translated beanie. Remember to be explicit about where all objects live.

Now we state an algorithm which uses the mechanical connection to reconstruct the dynamics on TQ given the dynamics on P_μ. The algorithm is implicit in [Abraham and Marsden 1978], but is made explicit in [Marsden, Montgomery, and Ratiu 1990].

1. Given the contents of the remark made on page 7, we may consider P_μ as a subbundle of TQ_μ. Therefore it makes sense to speak of a base integral curve for $\boldsymbol{X}_L^\mu$ as a curve on Q_μ by projection.

2. Let $q_\mu(t)$ be such a base integral curve, and let $v_q \in \boldsymbol{J}^{-1}(\mu)$ be such that $\pi_\mu(v_q)$ is the initial condition for $q_\mu(t)$.

3. Using the mechanical connection, lift $q_\mu(t)$ to a horizontal curve $q_h(t)$ in Q with $q_h(0) = q$.

4. Solve $\xi(t) \in \mathfrak{g}_\mu$ via the algebraic equation

$$\langle \xi(t)_Q(q_h(t)), \eta_Q(q_h(t))\rangle = \mu \cdot \eta \quad \text{for all } \eta \in \mathfrak{g}_\mu$$

   This means that $q_h'(0)$ and $\xi(0)_Q(q)$ are the horizontal and vertical parts, respectively, of the initial condition v_q. Notice that if the initial condition is horizontal then $\xi(0)_Q(q) = 0$.

5. Solve the differential equation

$$g'(t) = T_e L_{g(t)} \xi(t)$$

   in G_μ with initial condition $g(0) = e$. Here L_g is left translation by g.

6. The base integral curve on Q is then given by $q(t) = \Phi(g(t), q_h(t))$. The integral curve of the Lagrangian vector field on TQ is given by $c(t) = q'(t) = \Phi^T(g(t), q_h'(t) + \xi(t)_Q(q_h(t)))$.

3.3 Example: (Elroy's beanie cont'd) We now carry out the above procedure for Elroy's beanie. For concreteness we use $V = 0$. The vector field on P_μ in coordinates is then

$$\boldsymbol{X}_L^\mu = v_\psi \frac{\partial}{\partial \psi}$$

which has integral curves $t \mapsto (\psi(0) + v_\psi(0)t, v_\psi(0))$ where $(\psi(0), v_\psi(0))$ is the initial condition. Thus, in the general notation above, $q_\mu(t)$ is the curve on $Q_\mu \simeq \mathbb{S}^1$ given by $t \mapsto \psi(0) + v_\psi(0)t$. Let $(\psi(0), \theta(0), v_\psi(0), v_\theta(0)) \in \boldsymbol{J}^{-1}(\mu)$. (Thus we must have

$I_2 v_\psi(0) + (I_1 + I_2)V_\theta(0) = m$.) The horizontal lift of $q_\mu(t)$ through $(\psi(0), \theta(0), v_\psi(0), v_\theta(0))$ is given by

$$q_h \colon t \mapsto \left(\psi(0) + v_\psi(0)t, \theta(0) - \frac{I_2}{I_1 + I_2} v_\psi(0)t\right)$$

Now we use the algebraic relation

$$\langle \xi(t)_Q(q_h(t)), \eta_Q(q_h(t))\rangle = \mu \cdot \eta \quad \text{for all } \eta \in \mathfrak{g}$$

to determine $\xi(t)$. Suppose that $\xi(t) = \omega(t) \cdot e_1$ and $\eta = \delta \cdot e_1$. Then

$$\begin{aligned} \xi(t)_Q(\psi, \theta, v_\psi, v_\theta) &= \omega(t)\frac{\partial}{\partial \theta} \\ \eta_Q(\psi, \theta, v_\psi, v_\theta) &= \delta \frac{\partial}{\partial \theta} \end{aligned}$$

Using

$$\langle \cdot, \cdot \rangle = I_2 \mathrm{d}\psi \otimes \mathrm{d}\psi + (I_1 + I_2)\mathrm{d}\theta \otimes \mathrm{d}\theta + I_2 \mathrm{d}\psi \otimes \mathrm{d}\theta + I_2 \mathrm{d}\theta \otimes \mathrm{d}\psi$$

we compute

$$\langle \xi(t)_Q(q_h(t)), \eta_Q(q_h(t))\rangle = (I_1 + I_2)\omega(t)\delta$$

Therefore

$$\begin{aligned} & \langle \xi(t)_Q(q_h(t)), \eta_Q(q_h(t))\rangle = \mu \cdot \eta \quad \text{for all } \eta \in \mathfrak{g} \\ \iff \quad & (I_1 + I_2)\omega(t)\delta = m\delta \quad \text{for all } \delta \in \mathbb{R} \\ \iff \quad & \omega(t) = \frac{m}{I_1 + I_2} \end{aligned}$$

With $\xi = (m/(I_1 + I_2))e_1$ the differential equation

$$g'(t) = T_e L_{g(t)} \xi(t), \qquad g(0) = e$$

reads

$$\dot{\phi} = \frac{m}{I_1 + I_2}, \qquad \phi(0) = 0$$

which has the solution

$$\phi(t) = \frac{mt}{I_1 + I_2}$$

Therefore the solution curve in Q with initial conditions $(\psi(0), \theta(0), v_\psi(0), v_\theta(0))$ is given by

$$\begin{aligned} t \mapsto & \Phi(\phi(t), q_h(t)) \\ &= \left(\psi(0) + v_\psi(0)t, \theta(0) - \frac{I_2}{I_1 + I_2} v_\psi(0)t + \frac{mt}{I_1 + I_2}\right) \\ &= \left(\psi(0) + v_\psi(0)t, \theta(0) + \left[\frac{I_2 v_\psi(0) + (I_1 + I_2)v_\theta(0)}{I_1 + I_2} - \frac{I_2}{I_1 + I_2} v_\psi(0)\right]t\right) \\ &= (\psi(0) + v_\psi(0)t, \theta(0) + v_\theta(0)t) \end{aligned}$$

as anyone could have computed in the first place! •

Exercises:

1. Determine the base integral curves for the reduced system (i.e., determine $q_\mu(t)$) for the translated beanie. Use the algorithm presented above to construct the solutions in the total space. You should know what the answer is before you begin, but don't let this stop you from going through the formal procedure.

2. Show that if the initial condition, $v_q \in \boldsymbol{J}^{-1}(\mu)$ is horizontal, then the solution will remain horizontal for all time. (*Hint:* Look at the intrinsic formula for the connection one-form.)

4. Control of systems with symmetry

One of our objectives is to use the tools of geometric mechanics in a control setting. The basic idea is that we are allowed to control the system on the base space, Q_μ. With this control, the symmetry gives us a certain amount of control in vertical directions, depending on the structure of the group. Since this is still an active research area, let's just look at our beanie example, and see what the situation is there.

4.1 Example: (Elroy's beanie cont'd) The natural control problem for the beanie is to suppose that we are allowed to *specify* the angle ψ as a function of t in any manner we choose. We know that this control input does not disturb the conservation of momentum (or at least we believe we can show this). Therefore the relation $I_2 v_\psi + (I_1 + I_2) v_\theta = m$ must hold at all times. For $m = 0$ this is equivalent to saying that solution curves on Q must be horizontal by the remark on page 11. Therefore, given a curve on Q_μ, and given that angular momentum is zero, to compute the solution on Q we simply horizontally lift the curve on Q_μ. For example, suppose that $\psi(t)$ undergoes a rotation of 2π. That is to say, with our control we apply a rotation of the two bodies relative to each other through an angle of 2π. We can ask: "What will be the resulting angle that body 1 undergoes as a consequence of the relative motion?" To answer this let us be explicit and say that $q_\mu(t)$ is given by $t \mapsto 2\pi t$ for $t \in [0, 1]$. Then, as we computed above

$$q_h \colon t \mapsto \left(2\pi t, \theta(0) - \frac{I_2}{I_1 + I_2} 2\pi t\right)$$

therefore

$$\theta(1) - \theta(0) = \theta(0) - \frac{I_2}{I_1 + I_2} 2\pi - \theta(0) = -\frac{I_2}{I_1 + I_2} 2\pi$$

If $\mu \neq 0$ we must account for vertical motion determined by the vertical part of the initial condition. The way this is done is to compute $\xi(t)$ as in step 4 of the reconstruction algorithm, determine $g(t)$ as in step 5 of the reconstruction algorithm, then determine $\theta(t)$ by step 6 of the reconstruction algorithm. The idea is that the reconstruction algorithm preserves the symmetries. Thus, with $\psi(t) = 2\pi t$ we and $\mu = m \cdot e^1$ we would get

$$q \colon t \mapsto \left(2\pi t, \theta(0) + \left[\frac{m}{I_1 + I_2} - \frac{I_2}{I_1 + I_2} 2\pi\right] t\right)$$

which is essentially the zero angular momentum solution with a uniform rotation superimposed because of the non-zero value of angular momentum. •

5. Constraints defining Ehresmann connections

Because this subject has not really been fully worked out, we will simply pose an exercise for the reader after some brief discussion of Ehresmann connections.

Suppose that $\pi\colon E \to B$ is a fibre bundle (not necessarily principal). The vertical subbundle is defined in the usual manner as $VE = \ker(T\pi)$. Any distribution HE, on E which is complimentary to VE (i.e., is such that $TE = VE \oplus HE$) is called an *Ehresmann connection* on E. This type of connection is not as structured as a principal connection due to the lack of group structure. However, it is still interesting in mechanics as the following exercises are supposed to show.

1. Write down the equations of motion and constraints for the rolling penny. Let x, y be the Cartesian coordinates of the point of contact of the penny with the rolling plane, let θ be the "angle of Lincoln's head", and let ϕ be the heading angle of the penny with respect to the plane. Thus $Q = \mathbb{R}^2 \times \mathbb{T}^2$.

2. Consider the fibre bundle $\pi\colon Q \to B$ where $B = \mathbb{T}^2$ given by $\pi(x, y, \theta, \phi) = (\phi, \theta)$. Show that the constraints define an Ehresmann connection on this bundle.

3. Show that the system is controllable. That is: For any two points $q_1 = (x_1, y_1, \theta_1, \phi_1)$, $q_2 = (x_2, y_2, \theta_2, \phi_2) \in Q$ there exists a curve $c\colon t \mapsto (\theta(t), \phi(t))$ in B so that the horizontal lift of c joins q_1 and q_2.

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