Notes on linear frame bundles

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The first part of these notes is a review of material from Kobayashi and Nomizu, vol. I, chapter II. We only present that material which is of direct relevance to the material in Chapters III and IV of the same volume. The second part is an overview of selected material from Chapters III and IV.

1. The bundle of linear frames

Let M be a manifold of dimension n and let $x \in M$. A *linear frame* at x is an ordered basis, $\{X_1, \ldots, X_n\}$, of T_xM. The collection of all linear frames at $x \in M$ shall be denoted $L_x(M)$. The collection of all linear frames on M will be denoted $L(M)$ and called the *bundle of linear frames* or the *linear frame bundle*. Denote the Lie group of invertible $n \times n$ matrices by $GL(n; \mathbb{R})$. This group acts on $L(M)$ on the right by

$$\begin{aligned} R_a \colon L(M) &\to L(M) \\ (x, \{X_1, \ldots, X_n\}) &\mapsto (x, \{a^i_1 X_i, \ldots, a^i_n X_i\}) \end{aligned}$$

for $a \in GL(n; \mathbb{R})$. Heuristically, if we think of $\{X_1, \ldots, X_n\}$ as a column vector, the action looks like

$$R_A \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} = A^t \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}$$

which explains why the action is a right action and not the left actions we see in most cases when we talk about mechanical systems with symmetry. If you are familiar with the concept, you may wish to think of the right action on the linear frame bundle as a "relabelling symmetry."

It is readily verified that this action is free (i.e., each orbit is diffeomorphic to the group) and proper (never mind) and so the quotient $L(M)/GL(n; \mathbb{R})$ inherits a manifold structure. The map from $L(M)/GL(n; \mathbb{R})$ to M which assigns to the orbit through $u \in L_x(M)$ the point $x \in M$ allows us to identify the orbit space with M. We will denote by $\pi \colon L(M) \to M$ the natural projection. So, the bottom line is that $\pi \colon L(M) \to M$ is a principal fibre bundle with structure group $GL(n; \mathbb{R})$. To verify that the bundle is locally trivial, we construct a local bundle chart for $L(M)$. Let (U, ϕ) be a chart for M with coordinates denoted $(x^1, \ldots, x^n)$. If $u = \{X_1, \ldots, X_n\}$ is a linear frame at $x \in U$, we may write

$$u = \left\{ X^i_1 \frac{\partial}{\partial x^i}, \ldots, X^i_n \frac{\partial}{\partial x^i} \right\}.$$

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X^i_j are the components of an invertible matrix (they map one basis to another) and so the coordinates (x^i, X^j_k) establish a diffeomorphism from $\pi^{-1}(U)$ to $U \times GL(n;\mathbb{R})$. This verifies that $L(M)$ is locally trivial. In fact, the coordinates (x^i, X^j_k) are *bundle coordinates* for $L(M)$ meaning they enjoy some nice properties with respect to the group action.

There is an important, alternative way to think about a linear frame. If we let $u = \{X_1, \dots, X_n\}$ be a linear frame at $x \in M$, we may regard u as an isomorphism from $\mathbb{R}^n$ to T_xM. If we denote by $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_n\}$ the standard basis for $\mathbb{R}^n$, this map is given by

$$\begin{aligned} u\colon \mathbb{R}^n &\to T_xM \\ v^i\boldsymbol{e}_i &\mapsto v^iX_i. \end{aligned}$$

Thought of in this way, what is the right action of $GL(n;\mathbb{R})$ on $L(M)$? Well, given $u \in L_x(M)$ and $a \in GL(n;\mathbb{R})$, we have to come up with $R_au\colon \mathbb{R}^n \to T_xM$ (i.e., a linear frame at x, in our current thinking). We may regard a as isomorphism on $\mathbb{R}^n$ in the natural way. It is an exercise to verify that R_au is simply the composition of $a\colon \mathbb{R}^n \to \mathbb{R}^n$ with $u\colon \mathbb{R}^n \to T_xM$. We will denote by $u\xi \in T_xM$ the image of $\xi \in \mathbb{R}^n$ under $u \in L_x(M)$.

> *It is easy to think of an element of $L(M)$ as an ordered basis. However, you will also have to be comfortable with regarding $u \in L_x(M)$ as an isomorphism from $\mathbb{R}^n$ to T_xM since Kobayashi and Nomizu (and hence we) do so without warning.*

Let $v \in T_xM$ and $u = \{X_1, \dots, X_n\} \in L_x(M)$. One way to think of $u^{-1}v \in \mathbb{R}^n$ is to regard it as the components of v in the basis $\{X_1, \dots, X_n\}$.

1.1 Example: The easiest example is $M = \mathbb{R}$. Note that $GL(1;\mathbb{R}) \simeq \mathbb{R}^*$, where $\mathbb{R}^*$ is the group of non-zero real numbers with the group operation of multiplication. Any linear frame at $x \in \mathbb{R}$ is given by

$$X = a\frac{\partial}{\partial x}$$

for some non-zero number a. To this frame we associate the point $(x, a) \in \mathbb{R} \times \mathbb{R}^*$. Thus $L(\mathbb{R}) \simeq \mathbb{R} \times \mathbb{R}^*$. We may represent $L(M)$ as in Figure 1. As coordinates for $L(M) \simeq \mathbb{R} \times \mathbb{R}^*$ we shall use (x, a). The action of $GL(1;\mathbb{R})$ in these coordinates appears like

$$((x, a), b) \mapsto (x, ab).$$

Now let's examine how we may think of an element of $L_x(M)$ as a linear map from $\mathbb{R}$ to T_xM. Let $\xi \in \mathbb{R}$ and let $u = (x, a) \in L_x(M)$. Note that (x, a) represents the frame $X = a\frac{\partial}{\partial x}$. Therefore

$$u\xi = \xi a\frac{\partial}{\partial x}. \qquad \bullet$$

2. Connections in $L(M)$

For $A \in \mathfrak{gl}(n;\mathbb{R})$, the Lie algebra of $GL(n;\mathbb{R})$ (i.e., the Lie algebra of $n \times n$ matrices), denote by A^* the *fundamental vector field* associated with A. In the language our group uses, A^* is the infinitesimal generator and so is defined by

$$A^*(u) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} R_{\exp(tA)}u.$$

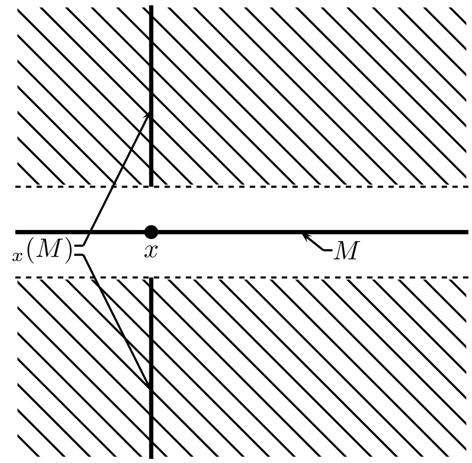

Figure 1. A representation of $L(M)$ when $M = \mathbb{R}$.

For $u \in L(M)$ we define the *vertical subspace*, $V_u(L(M))$, at u to be the subspace of $T_u(L(M))$ generated by all fundamental vector fields at u. This is the same as defining

$$V_u(L(M)) = \ker(T_u\pi).$$

A *connection* in $L(M)$ is an assignment of a subbundle, $H(L(M))$, so that

1. $T(L(M)) = H(L(M)) \oplus V(L(M))$, and
2. $T_uR_a(H_u(L(M))) = H_{R_au}(L(M))$ for each $u \in L(M)$ and $a \in GL(n;\mathbb{R})$.

A connection in $L(M)$ is called a *linear connection*. Kobayashi and Nomizu also define *affine connections* which are connections on the bundle of *affine* frames. Please ignore these sections in Chapter III. We will reserve the name affine connection for a purpose to be disclosed later (i.e., the one we classically know). Associated with a linear connection is the *connection one-form*, ω, which is a one-form on $L(M)$ which takes its values in $\mathfrak{gl}(n;\mathbb{R})$. Intrinsically we define the connection one-form by

$$\omega(u)(X) = \{A \in \mathfrak{gl}(n;\mathbb{R}) \mid \quad A^*(u) = X^{\text{ver}}\}. \tag{2.1}$$

Here X^{ver} is the vertical part of X relative to the decomposition given by the connection. The condition 2 in the definition of a connection gives the condition

$$\omega(T_uR_a(X)) = \operatorname{Ad}_{a^{-1}} \omega(X) \tag{2.2}$$

for $u \in L(M)$, $X \in T_u(L(M))$, and $a \in GL(n;\mathbb{R})$. Here $\operatorname{Ad}_a \colon \mathfrak{gl}(n;\mathbb{R}) \to \mathfrak{gl}(n;\mathbb{R})$ is the image of $a \in GL(n;\mathbb{R})$ in $\operatorname{Aut}\mathfrak{gl}(n;\mathbb{R})$ under the adjoint representation. In fact, any $\mathfrak{gl}(n;\mathbb{R})$-valued one-form on P satisfying the conditions (2.1) and (2.2) defines a connection on $L(M)$ given by $H(L(M)) = \ker(\omega)$. We can also define the curvature form on $L(M)$ by

$$\Omega(X,Y) = D\omega(X,Y) = \boldsymbol{d}\omega(X^{\text{hor}}, Y^{\text{hor}})$$

where X^{hor} and Y^{hor} are the horizontal parts of X and Y, respectively.

2.1 Example: We continue the example above of $M = \mathbb{R}$ and derive the general form of a linear connection on M. We use coordinates (x, a) for $L(M) \simeq \mathbb{R} \times \mathbb{R}^*$. Let's compute the infinitesimal generator for $A \in \mathfrak{gl}(1;\mathbb{R})$. The Lie algebra of $GL(1;n)$, $\mathfrak{gl}(1;\mathbb{R})$, is isomorphic to the Lie algebra $\mathbb{R}$ with the zero bracket. For $A \in \mathfrak{gl}(1;\mathbb{R}) \simeq \mathbb{R}$, $\exp(A) = e^A \in \mathbb{R}^* \simeq GL(1;\mathbb{R})$. Therefore,

$$A^*(x,a) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (x, ae^{tA}) = (x, a, 0, aA).$$

Therefore, $V(L(M)) = \langle \frac{\partial}{\partial a} \rangle$. A complement to this vertical subbundle will be generated by a vector field which is not in the vertical subbundle. Thus, suppose that $H(L(M))$ is generated by the vector field

$$h(x,a) = R(x,a)\frac{\partial}{\partial x} + S(x,a)\frac{\partial}{\partial a} \quad \Big[= (x, a, R(x,a), S(x,a))\Big]$$

where R is non-zero (since the vector field must be complementary to the vertical subbundle). The condition of invariance of the horizontal distribution places restrictions on R and S. Let's figure out what those are. If we have $h(x, ab) = T_{(x,a)}R_b(h(x,a))$ then h will generate a connection. We compute

$$T_{(x,a)}R_b(h(x,a)) = (x, ab, R(x,a), bS(x,a)).$$

This gives

$$\begin{aligned} R(x,ab) &= R(x,a) \\ S(x,ab) &= bS(x,a). \end{aligned}$$

We may simplify this by supposing that $a = 1$ and that $R = 1$. We then see that the horizontal subbundle of a linear connection on $\mathbb{R}$ is generated by a vector field of the form

$$h(x,a) = \frac{\partial}{\partial x} + aS(x)\frac{\partial}{\partial a}.$$

Thus the "slope" of the horizontal subspaces increases as we increase in the positive group direction. At each point in the base space $M = \mathbb{R}$, the connection is determined by its slope at one point in the fibre. This is illustrated in Figure 2.

Now let's compute the connection one-form. Suppose that we have a vector $X \in T_{(x,a)}(L(M))$ given by

$$X = X_1\frac{\partial}{\partial x} + X_2\frac{\partial}{\partial a}.$$

We may write

$$X = X_1 h(x,a) + (X_2 - aS(x)X_1)\frac{\partial}{\partial a}$$

which gives the decomposition of X into its vertical and horizontal parts. Now suppose that $\omega(X) = A \in \mathfrak{gl}(1;\mathbb{R})$. Then we must have

$$A^*(x,a) = (x, a, 0, aA) = (x, a, 0, X_2 - aS(x)X_1)$$

from which we determine that

$$\omega(x,a)(X) = \frac{X_2}{a} - S(x)X_1 \in \mathfrak{gl}(1;\mathbb{R}) \simeq \mathbb{R}.$$

Note that the curvature is zero since M is one-dimensional. •

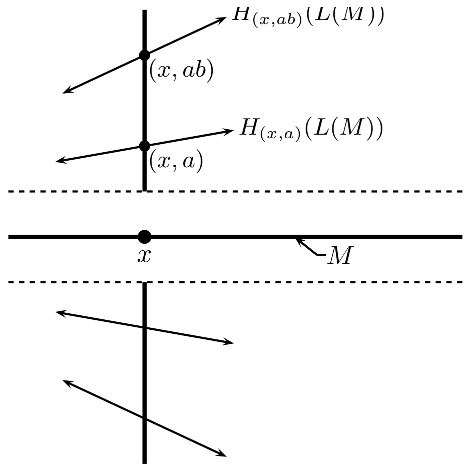

Figure 2. A representation of a linear connection in $L(M)$ when $M = \mathbb{R}$.

3. Horizontal lift and parallel translation

With a connection comes the notion of parallel translation. Forget, for the moment, about parallel translation as you know it. In a minute we will relate what we are talking about with what you already know.

Let $X \in T_x M$ and let $u \in \pi^{-1}(x)$. There is a unique vector $X^* \in H_u(L(M))$ so that $T_u\pi(X^*) = X$. We shall call X^* the *horizontal lift* of X. Thus, for each $u \in \pi^{-1}(x)$, there is an isomorphism from $H_u(L(M))$ to $T_x M$.

Given a curve $\tau \colon [0, 1] \to M$ on M and $u \in \pi^{-1}(\tau(0))$, there exists a unique curve $\tau^* \colon [0, 1] \to L(M)$ such that:

1. $\tau^*(0) = u$;
2. $\pi \circ \tau^*(t) = \tau(t)$ for each $t \in [0, 1]$;
3. $\dot{\tau}^*(t) \in H_{\tau^*(t)}(L(M))$ for each $t \in [0, 1]$.

We call τ^* the *horizontal lift* of τ. Note that $\tau^*(1) \in L_{\tau(1)}(M)$. Thus, given the curve τ, we may define a map from $\pi^{-1}(\tau(0))$ to $\pi^{-1}(\tau(1))$. This map is a diffeomorphism and is called *parallel translation along* τ. Adopting the minimalist notation of Kobayashi and Nomizu we denote this diffeomorphism by $\tau \colon \pi^{-1}(\tau(0)) \to \pi^{-1}(\tau(1))$. Parallel translation enjoys the following properties:

1. $\tau \circ R_a = R_a \circ \tau$ for every $a \in GL(n; \mathbb{R})$;
2. if τ^{-1} is the inverse of the curve τ, then $(\tau)^{-1} = \tau^{-1}$ (here you see the true power of Kobayashi and Nomizu's notation...);
3. If τ is a curve which joins $x \in M$ with $y \in M$ and μ is a curve which joins y with $z \in M$, then parallel translation along the composed curve, $\mu \cdot \tau$, is the composition of τ and μ.

In practice, determining the horizontal lift, and hence parallel translation, amounts to solving an ordinary differential equation. The following example illustrates this.

3.1 Example: Let's see what the horizontal lift is for the example we have been considering where $M = \mathbb{R}$. Let us suppose that the connection is generated by a horizontal vector field

$$h(x,a) = \frac{\partial}{\partial x} + aS(x)\frac{\partial}{\partial a}$$

in the manner we described above. Suppose that we have a curve on M given by $\tau(t)$ for $t \in [0,1]$ and suppose that $\tau(0) = x_0$. Suppose that the horizontal lift through some point $(x_0, a_0) \in \pi^{-1}(x_0)$ is given by $\tau^*(t) = (x(t), a(t))$ for as yet undetermined functions $x(t)$ and $a(t)$. Since τ^* must project to τ we must have $x(t) = \tau(t)$. The curve must be horizontal which means that

$$\dot{x}(t)\frac{\partial}{\partial x} + \dot{a}(t)\frac{\partial}{\partial a} = C(t)h(x(t), a(t)).$$

This implies that

$$\dot{a}(t) = a(t)S(x(t))\dot{x}(t)$$

which is a non-autonomous linear ordinary differential equation for $a(t)$. In this case we simply integrate to obtain the answer as

$$a(t) = a_0 \exp\left(\int_0^t S(x(\eta))\dot{x}(\eta)\mathrm{d}\eta\right)$$

Note that when the horizontal subspace is horizontal in the naive sense, then $a(t) = a_0$. •

4. TM as an associated bundle of $L(M)$

When one thinks about affine connections in the "usual" way, they work on vector fields, i.e., sections of TM. Our initial development is on $L(M)$, so we need a way to relate the two ideas precisely. This is done by regarding TM as an *associated bundle* of $L(M)$. This is a specific example of a construction which may be made with general principal fibre bundles.

4.1. The basic association.

Observe that $GL(n;\mathbb{R})$ acts on $\mathbb{R}^n$ on the right by $(a,\xi) \mapsto a^{-1}\xi$. Therefore, $GL(n;\mathbb{R})$ acts on $L(M) \times \mathbb{R}^n$ on the right by $((u,\xi),a) \mapsto (R_a u, a^{-1}\xi)$. We will denote by $u\xi$ (!!) the orbit through $(u,\xi) \in L(M) \times \mathbb{R}^n$. The following lemma describes the orbits of this right action.

4.1 Lemma: *$(L(M) \times \mathbb{R}^n)/GL(n;\mathbb{R})$ is diffeomorphic to TM.*

Proof: Let $(u,\xi) \in L_x(M) \times \mathbb{R}^n$ and define $v = u\xi \in T_xM$. We claim that, so defined, v depends only on the orbit through (u,ξ). Let (u',ξ') be a point in $L_x(M) \times \mathbb{R}^n$ in the same orbit as (u,ξ). Then there exists $a \in GL(n;\mathbb{R})$ so that $(u',\xi') = (R_a u, a^{-1}\xi)$. Then $u'\xi' \in T_xM$ is the composition of $a^{-1}\colon \mathbb{R}^n \to \mathbb{R}^n$ with $R_a u\colon \mathbb{R}^n \to T_xM$. Since $R_a u$ is the composition of $a\colon \mathbb{R}^n \to \mathbb{R}^n$ with $u\colon \mathbb{R}^n \to T_xM$, we see that $u'\xi' = u\xi$.

Now let $v \in T_xM$ and let $u \in \pi^{-1}(x)$. Define $\xi \in \mathbb{R}^n$ by $\xi = u^{-1}v$. We claim that the map which sends v to the orbit through (u,ξ) is well-defined. This follows by reversing the argument above. ■

Fixing $\xi \in \mathbb{R}^n$ and considering the map $\phi_\xi\colon L(M) \to TM$ which sends u to $u\xi$ has a useful interpretation. As we shall see below, it is the mechanism by which the geometry in $L(M)$ is transfered to TM.

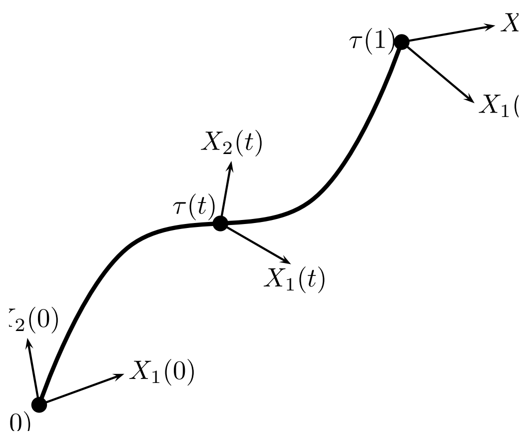

Figure 3. Parallel translation in $L(M)$ transfered to TM.

4.2 Example: Let's consider the case again when $M = \mathbb{R}$. Using the usual coordinates for $L(M) \times \mathbb{R}$, we see that the right action of $GL(1;\mathbb{R})$ on $L(M) \times \mathbb{R}$ is given by

$$(((x,a),\xi),b) \mapsto ((x,ab), b^{-1}\xi).$$

Let $u = (x,a) \in L(M)$ and let $\xi \in \mathbb{R}$. Then, as we computed above, $u\xi = \xi a\frac{\partial}{\partial x}$. Note that if $(u',\xi') = (R_b u, b^{-1}\xi)$ then $u'\xi' = (b^{-1}\xi)(ab)\frac{\partial}{\partial x} = \xi a\frac{\partial}{\partial x} = u\xi$ which explicitly verifies in this case that the set of orbits in $L(\mathbb{R}) \times \mathbb{R}$ are diffeomorphic to $T\mathbb{R}$.

Now let's examine $\phi_\xi \colon L(\mathbb{R}) \to T\mathbb{R}$ for $\xi \in \mathbb{R}$. We see that

$$\phi_\xi(x,a) = (x,\xi a)\left[= \xi a\frac{\partial}{\partial x}\right]. \qquad \bullet$$

4.2. Parallel translation and horizontal lift. We may define parallel translation in TM given a linear connection. We do this by transferring parallel translation in $L(M)$ in a geometrically meaningful manner. Here's how it goes. Let $\tau \colon [0,1] \to M$ be a curve in M and let $v \in T_{\tau(0)}M$. We wish to define parallel translation of v along τ. Let $u = \{X_1,\dots,X_n\} \in \pi^{-1}(\tau(0))$ be such that $v = X_1$. There are many such u's if $\dim(M) > 1$. Let τ^* be the horizontal lift of τ in $L(M)$ through u. This defines a frame, $u(t) = \{X_1(t),\dots,X_n(t)\}$, at each point in M along τ. We shall *define* the parallel translation in TM along τ to be the vector field $X_1(t)$ along τ. See Figure 3 for a two-dimensional representation.

The above construction does not appear to be intrinsic. However, the above construction is a specific example of the following general definition of parallel translation. Let τ be a curve and let $v \in T_{\tau(0)}M$ as above. Now let $(u,\xi) \in L_{\tau(0)}(M) \times \mathbb{R}^n$ be such that $u\xi = v$. Let τ^* be the horizontal lift of τ through u. We can then define parallel translation along τ in TM to be the image of τ^* under ϕ_ξ. This construction may be shown to be independent of the choice of (u,ξ) as long as $u\xi = v$. Our initial definition of parallel translation in TM corresponds to choosing $\xi = \boldsymbol{e}_1$.

4.3 Example: Let's see how we may determine parallel translation in our example when $M = \mathbb{R}$. Suppose that $\tau(t)$, $t \in [0,1]$, is a curve on M with $\tau(0) = x_0$. Let $v \in T_{x_0}M$. Choosing $\xi = \boldsymbol{e}_1 = 1$ and $u = \{v\}$ we see that $u\xi = v$. If $u = (x_0, a_0)$ in our natural coordinates for $L(\mathbb{R})$, we showed above that the horizontal lift of τ through u is given by $(x(t), a(t))$ where

$$a(t) = a_0 \exp\left(\int_0^t S(x(\eta))\dot{x}(\eta)\mathrm{d}\eta\right).$$

Here $S(x)$ is, as usual, the quantity used in defining the connection in $L(M)$. The image of this curve under ϕ_ξ (with $\xi = 1$) is given as

$$\left(x(t), a_0 \exp\left(\int_0^t S(x(\eta))\dot{x}(\eta)\mathrm{d}\eta\right)\right).$$

Note that (with an abuse of notation) if $v = v\frac{\partial}{\partial x}$ then $a_0 = v$. •

4.3. Covariant differentiation. With parallel translation defined, it is possible to define covariant differentiation. Let $v \in T_xM$ and let X be a vector field on M. Suppose that $\tau\colon [0,1] \to M$ is a curve on M with $\tau'(0) = v$. Denote by $\tau_t\colon T_{\tau(0)}M \to T_{\tau(t)}M$ the parallel translation along τ in TM. We define

$$\nabla_v X = \lim_{t\to 0} \frac{1}{t}\left(\tau_t^{-1}(X(\tau(t))) - X(\tau(0))\right).$$

In this way we can define covariant differentiation of a vector field with respect to a vector or with respect to a curve. We can also define parallel translation of a vector field Y with respect to a vector field X by

$$\left(\nabla_X Y\right)(x) = \nabla_{X(x)} Y.$$

One may check that, so defined, covariant differentiation has the usual properties:

1. $\nabla_{X+Y} Z = \nabla_X Z + \nabla_Y Z$;
2. $\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z$;
3. $\nabla_{fX} Y = f\nabla_X Y$;
4. $\nabla_X (fY) = f\nabla_X Y + \mathscr{L}_X f Y$.

If we define $\nabla_X f = \mathscr{L}_X f$ where f is a function on M, we may then use Willmore's theorem to extend the derivation ∇_X to a derivation on the entire tensor algebra.

4.4. Connections in TM. Given a linear connection, we can also define a connection on TM. First we define an *Ehresmann connection* on TM, or a splitting of TTM into vertical and horizontal subspaces. The vertical subspace is, as usual, defined by $V_vTM = \ker(T_v\pi_{TM})$ where $\pi_{TM}\colon TM \to M$ is the tangent bundle projection. Given a linear connection on M, we define a horizontal subbundle of TTM as follows. Let $v \in TM$ and let $(u, \xi) \in L(M) \times \mathbb{R}^n$ be such that $u\xi = v$. Fix ξ and consider the map ϕ_ξ from $L(M)$ to TM given by $\phi_\xi(u) = u\xi$. We define the horizontal subspace at $v \in T_xM$ to be the image of the horizontal subspace at $u \in L_x(M)$ under $T\phi_\xi$. One may show (please do so if

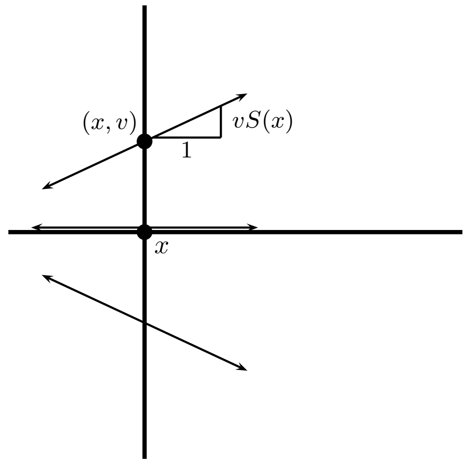

Figure 4. The Ehresmann connection induced in $T\mathbb{R}$ by a linear connection on $M = \mathbb{R}$.

you are interested) that this definition does not depend on the initial choice of (u, ξ) and that the horizontal subspace defined in this way is complementary to the vertical subspace. Observe that the above definition may be motivated by observing that horizontal curves in $L(M)$ are mapped to horizontal curves in TM by ϕ_ξ. Therefore, it stands to reason that horizontal vectors in $T(L(M))$ will be mapped to horizontal vectors in TTM by $T\phi_\xi$.

4.4 Example: Let's look at the connection induced in TM in the example we have been looking at, $M = \mathbb{R}$. Suppose that the linear connection in $L(M)$ is generated by a vector field of the form

$$h(x, a) = \frac{\partial}{\partial x} + aS(x)\frac{\partial}{\partial a}$$

as we determined above. To compute the connection in $T\mathbb{R}$, it suffices to compute the image of h under $T\phi_\xi \colon T(L(M)) \to TM$. We compute

$$T_{(x,a)}\phi_\xi(h(x, a) = T_{(x,a)}\phi_\xi(x, a, 1, aS(x)) = (x, \xi a, 1, \xi aS(x)).$$

In other words, at $(x, v) \in T\mathbb{R}$, the horizontal subspace is generated by

$$\tilde{h}(x, v) = \frac{\partial}{\partial x} + vS(x)\frac{\partial}{\partial v}.$$

Figure 4 tells the tale. It is pretty clear that this definition does not depend on the choice of (u, ξ) as long as $u\xi = (x, v)$ •

5. The canonical form and the torsion form on $L(M)$

On $L(M)$ there is a canonical $\mathbb{R}^n$-valued one-form which will be used to define torsion later on. It is given by

$$\theta(X) = u^{-1}(T_u\pi(X))$$

for $X \in T_u(L(M))$. The canonical one-form has the following property under group action:

$$(R_a^*\theta)(X) = a^{-1}(\theta(X))$$

for $X \in T_u(L(M))$. We may define the *torsion form* as the $\mathbb{R}^n$-valued two-form on $L(M)$ determined by

$$\Theta(X, Y) = D\theta(X, Y) = \boldsymbol{d}\theta(X^{\text{hor}}, Y^{\text{hor}})$$

where X^{hor} and Y^{hor} are the horizontal parts of X and Y, respectively. Note that θ does not depend on a choice of connection, but that Θ does.

The following are the *structure equations* for a linear connection:

$$\begin{aligned} &\boldsymbol{d}\theta(X, Y) = \omega(Y)\theta(X) - \omega(X)\theta(Y) + \Theta(X, Y) && \text{1st structure equation} \\ &\boldsymbol{d}\omega(X, Y) = -[\omega(X), \omega(Y)] + \Omega(X, Y) && \text{2nd structure equation.} \end{aligned}$$

Here $X, Y \in T_u(L(M))$ and $u \in L(M)$. Note that these equations differ from those in Kobayashi and Nomizu by factors of 2 in some places. This is a consequence of their using different definitions of wedge product than we use. This will not bother us too much in our presentation, but the readers should be aware of it. These structure equations indicate that not every collection of equivariant $\mathbb{R}^n$ and $\mathfrak{gl}(n; \mathbb{R})$-valued forms, θ, Θ, ω, and Ω are candidates to be canonical, torsion, connection, and curvature forms.

We also have *Bianchi's identities:*

$$\begin{aligned} &D\Theta(X, Y, Z) = \Omega(X, Y)\theta(Z) + \Omega(Z, X)\theta(Y) + \Omega(Y, Z)\theta(X) && \text{1st identity} \\ &D\Omega = 0 && \text{2nd identity.} \end{aligned}$$

Remember what we said above about factors of 2 in the structure equation? Ditto the factor of 3 in Bianchi's 1st identity.

5.1 Example: Let's look at the canonical one-form when $M = \mathbb{R}$. If $u = (x, a)$ in our natural coordinates for $L(M)$ and if $v\frac{\partial}{\partial x} \in T_xM$ then $u^{-1}(v) = v/a \in \mathbb{R}^n$. If $X = (x, a, X_1, x_2) \in T_{(x,a)}(L(M))$ then $T_{(x,a)}\pi(X) = (x, X_1)$. Thus

$$\theta(X) = X_1/a \in \mathbb{R}.$$

Note that the torsion form is zero since $\dim(M) = 1$. •

6. Standard horizontal vector fields

Fix $\xi \in \mathbb{R}^n$ and suppose that $L(M)$ has a linear connection. At $u \in L(M)$ let $B(\xi)$ be the unique horizontal vector which projects to $u\xi \in T_{\pi(u)}M$. The vector field $u \mapsto B(\xi)(u)$ is called the *standard horizontal vector field* associated with ξ. Clearly we may think of $\Sigma\colon L(M) \times \mathbb{R}^n \to T(L(M) \times \mathbb{R}^n)\colon (u, \xi) \mapsto (B(\xi)(u), 0)$ as a vector field on $L(M) \times \mathbb{R}^n$. We shall see how this is related to a vector field on TM later on.

6.1 Example: What does a standard horizontal vector field look like when $M = \mathbb{R}$? Well, let's just see, shall we. Let $\xi \in \mathbb{R}$ and let $u = (x, a) \in L(M)$. Then $u\xi = (x, a\xi) \in T_x\mathbb{R}$. The horizontal vector at u which projects to $u\xi$ is given by $a\xi h(x, a)$ where h is the horizontal vector field which generates the connection. Thus, in coordinates we have

$$B(\xi)(x, a) = (x, a, a\xi, a^2\xi S(x))$$

where $S(x)$ defines the connection as usual. •

7. Covariant differential

When we discuss linear holonomy groups later we will need the notion of the *covariant differential*. Let K be a tensor field of type (r, s). Thus we regard K as a $C^\infty(M)$-multilinear mapping from $\mathscr{T}^*(M) \times \cdots r \text{ times} \cdots \times \mathscr{T}^*(M) \times \mathscr{T}(M) \times \cdots s \text{ times} \cdots \times \mathscr{T}(M)$ into $C^\infty(M)$. Therefore, for $X \in \mathscr{T}(M)$, $\nabla_X K$ is also a type (r, s) tensor field. We define the covariant differential of K to be the type $(r, s+1)$ tensor field defined by

$$\nabla K(X_1, \ldots, X_s; X) = (\nabla_X K)(X_1, \ldots, X_s).$$

Here we think of $\nabla K(X_1, \ldots, X_s; X)$ as a type $(r, 0)$ tensor field. There are various formulas regarding the covariant differential, but I don't care to regurgitate them.

8. Holonomy groups for linear frame bundles

It will be interesting to us to know about so-called holonomy groups for the bundle of linear frames. This is a general concept in principal fibre bundle theory, but we shall present it only in the context of frame bundles.

8.1. Holonomy groups and restricted holonomy groups. Let $x \in M$ and denote by $C(x)$ the set of loops in M whose initial and final point is x. Denote by $C_0(x)$ those loops which are null homotopic. Choose $u \in \pi^{-1}(x)$. Let $\mathrm{Diff}(\pi^{-1}(x))$ denote the group of diffeomorphisms of the fibre π^{-1}. Denote by $\Psi(x)$ the subgroup of $\mathrm{Diff}(\pi^{-1}(x))$ whose elements are obtained by parallel translation by curves in $C(x)$. In other words, every element of $\Psi(x)$ is of the form $\tau\colon \pi^{-1}(x) \to \pi^{-1}(x)$ for $\tau \in C(x)$. We call $\Psi(x)$ the *holonomy group with reference point x*. Similarly we define $\Psi_0(x)$ to be the subgroup of $\mathrm{Diff}(\pi^{-1}(x))$ of elements of the form $\tau\colon \pi^{-1}(x) \to \pi^{-1}(x)$ for $\tau \in C_0(x)$. We call $\Psi_0(x)$ the *restricted holonomy group with reference point x*.

For each $u \in \pi^{-1}(x)$ we may identify these groups with a subgroup of $GL(n; \mathbb{R})$ as follows. Define

$$\Psi(u) = \{a \in GL(n; \mathbb{R}) \mid \quad R_a(u) = \tau(u) \text{ for some } \tau \in \Psi(x)\}.$$

This may easily be shown to be a subgroup of $GL(n; \mathbb{R})$ since parallel translation and the right action of $GL(n; \mathbb{R})$ on $L(M)$ commute. We call $\Psi(u)$ the *holonomy group with reference point u*. In like manner we may define $\Psi_0(u)$ as the *restricted holonomy group with reference point u*. If we choose a different reference point, say $u' \in \pi^{-1}(x)$, then the holonomy groups $\Psi(u)$ and $\Psi(u')$ are related. Indeed, if $a \in GL(n; \mathbb{R})$ is such that $u' = R_a(u)$ then $\Psi(u') = a^{-1}\Psi(u)a$.

8.2. Local holonomy groups. We now construct the local holonomy group which we denote $\Psi^*(u)$ for $u \in \pi^{-1}(x)$. Let $\{U_k\}_{k \in \mathbb{Z}^+}$ be a sequence of neighbourhoods of x such that

1. $U_{k+1} \subset U_k$ for $k \in \mathbb{Z}^+$, and
2. $\displaystyle\bigcap_{k=1}^{\infty} U_k = \{x\}$.

Denote by $\Psi_0(u, U_k)$ the subset of $\Psi_0(u)$ consisting of those parallel translation by curves which remain in U_k. Clearly $\Psi_0(u, U_k) \subset \Psi_0(u, U_{k+1})$ and so we define

$$\Psi^*(u) = \bigcap_{k=1}^{\infty} \Psi_0(u, U_k)$$

which is the local holonomy group with reference point u. It is true that $\Psi^*(u) = \Psi_0(u, U_k)$ for every $k > K$ where K is sufficiently large. If $\dim(\Psi^*(u))$ is constant then $\Psi^*(u) = \Psi_0(u)$ for each $u \in L(M)$.

8.3. Infinitesimal holonomy groups. Now we define infinitesimal holonomy group. Define $\mathfrak{m}_0(u)$ to be the subspace of $\mathfrak{gl}(n;\mathbb{R})$ generated by $\Omega_u(X, Y)$ for horizontal vectors $X, Y \in T_u(L(M))$. Now inductively define $\mathfrak{m}_k(u)$ to be the subspace of $\mathfrak{gl}(n;\mathbb{R})$ generated by elements from $\mathfrak{m}_{k-1}$ and by elements of the form

$$V_k \cdots V_1(\Omega(X, Y))$$

where $X, Y, V_1, \ldots, V_k$ are horizontal vectors at u. Now define

$$\mathfrak{g}'(u) = \bigcup_{k=0}^{\infty} \mathfrak{m}_k.$$

This may be shown to be a subalgebra of $\mathfrak{gl}(n;\mathbb{R})$ and we call the connected subgroup of $GL(n;\mathbb{R})$ generated by $\mathfrak{g}'(u)$ the *infinitesimal holonomy group with reference point u* and denote it by $\Psi'(u)$. If $\dim(\Psi'(v))$ is constant for v in a neighbourhood of u then $\Psi'(u) = \Psi^*(u)$.

8.4. Linear representations. We shall need a few simple ideas from representation theory.

Let V be a vector space and let G be a Lie group. A *representation* of G in V is a homomorphism ψ from G to $\mathrm{Aut}V$. Thus

$$\begin{aligned} \psi(e) &= \mathrm{id}_V \\ \psi(g_1 g_2) &= \psi(g_1)\psi(g_2). \end{aligned}$$

For example, let $G = GL(n;\mathbb{R})$ and let V be an n-dimensional vector space with a basis $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$. The map

$$\begin{aligned} \psi\colon\ GL(n;\mathbb{R}) &\to \mathrm{Aut}V \\ a &\mapsto \{v^i \boldsymbol{e}_i \mapsto a^i_j v^j \boldsymbol{e}_i\} \end{aligned}$$

is a representation of $GL(n;\mathbb{R})$ in V which depends on the basis.

Let $U \subset V$ be a subspace and let ψ be a representation of G in V. We shall say U is *invariant* under ψ if U is an invariant subspace of $\psi(g)$ for each $g \in G$. We say the representation ψ is *irreducible* if there are no non-trivial subspaces of V which are invariant under the representation. A representation is *reducible* if it is not irreducible. Note that if U is invariant under ψ then there is a natural representation of G in U induced by ψ. In this way, a representation is irreducible if there are no non-trivial subspaces of U onto which the representation ψ can be restricted.

Although we do not need them for linear connections, we shall need to talk about orthogonal representations for Riemannian connections. So let V be a vector space with an inner product $\langle\cdot,\cdot\rangle$. A representation ψ of G in $\mathrm{Aut}V$ is called *orthogonal* if $\psi(g)$ is an orthogonal transformation of $(V, \langle\cdot,\cdot\rangle)$ for each $g \in G$. The following result will be useful to us.

8.1 Lemma: *Let ψ be an orthogonal representation of $G \subset O(n)$ in $(V, \langle\cdot,\cdot\rangle)$. If U is an invariant subspace of ψ then $U^{\perp}$ is also invariant under ψ.*

Proof: Let $g \in G$ and let $u \in U$ and $v \in U^{\perp}$. We have

$$\langle \psi(g)(v), u\rangle = \langle v, (\psi(g))^t(u)\rangle = \langle v, \psi(g^{-1})u\rangle = 0$$

since U is invariant under ψ. This proves that $\psi(g)(v)$ is orthogonal to any $u \in U$ for any $g \in G$. That is, $U^{\perp}$ is invariant under ψ. ■

8.5. The holonomy groups as subgroups of $\mathrm{Aut}(T_xM)$. What is really interesting to us are the groups $\Psi(x)$, $\Psi_0(x)$, $\Psi^*(x)$, and $\Psi'(x)$, i.e., the holonomy groups with reference point in M and not in $L(M)$. We may actually realise these as subgroups of $\mathrm{Aut}T_xM$. We will do this for $\Psi(x)$ and the others follow in exactly the same manner. Recall that $\Psi(u)$ is the holonomy group with reference point u and is a subgroup of $GL(n;\mathbb{R})$. Recall that $u \in L_x(M)$ can be thought of as a linear map from $\mathbb{R}^n$ to T_xM. Also, since $\Psi(u)$ is a subgroup of $GL(n;\mathbb{R})$ and since $GL(n;\mathbb{R})$ is the automorphism group of $\mathbb{R}^n$, we may regard $\Psi(u)$ as a subgroup of the automorphism group of $\mathbb{R}^n$. Using $u\colon \mathbb{R}^n \to T_xM$ we can think of $\Psi(u)$ as having a representation in T_xM. Denote this representation by $\psi_u\colon \Psi(u) \to \mathrm{Aut}T_xM$. Explicitly,

$$\psi_u(a) = u \circ a \circ u^{-1} \in \mathrm{Aut}T_xM.$$

In this way we may regard $\Psi(u)$ as a subgroup of $\mathrm{Aut}T_xM$. We now show that this subgroup does not depend on the choice of $u \in \pi^{-1}(x)$. Let $u' = R_b(u) \in \pi^{-1}(x)$ so that $\Psi(u') = b^{-1}\Psi(u)b$. Suppose that $a \in \Psi(u)$ and let $a' = b^{-1}ab$ be the corresponding element in $\Psi(u')$. We then have

$$\psi_{u'}(a') = u' \circ a' \circ (u')^{-1} = u \circ b \circ b^{-1} \circ a \circ b \circ b^{-1} \circ u^{-1} = u \circ a \circ u^{-1} = \psi_u(a).$$

This then unambiguously defines $\Psi(x)$ as a subgroup of $\mathrm{Aut}T_xM$. It is now clear that the Lie algebra of $\Psi(x)$ is then naturally a subalgebra of $\mathrm{End}T_xM$.

It is clear that in this way we can also regard $\Psi_0(x)$, $\Psi^*(x)$, and $\Psi'(x)$ as subgroups of $\mathrm{Aut}T_xM$.

9. Riemannian connections

Next we discuss Riemannian connections and their holonomy groups.

9.1. Connections on the bundle of orthonormal frames. Let (M, g) be a Riemannian manifold. Thus g is a Riemannian metric on M. We denote by $O(M) \subset L(M)$ the set of orthonormal frames on M. The action of $O(n) \subset GL(n;\mathbb{R})$ on $O(n)$ makes $O(M)$ a principal fibre bundle with structure group $O(n)$. A connection on $O(M)$ for which parallel translation is an isometry is called a *Riemannian connection* on $O(M)$. If the connection is further torsion-free, it is called a *Levi-Civita* connection. It may be shown that there exists a unique Levi-Civita connection on the orthonormal frame bundle of a Riemannian manifold. Since a Riemannian connection specifies a horizontal subspace on a subbundle of $L(M)$, it defines a connection on all of $L(M)$ by the translation invariance property of linear connections. Thus, associated with a Riemannian connection (and, in particular, a Levi-Civita connection) there exists a unique linear connection agreeing with the original connection on $O(M)$.

9.2. Holonomy groups for the Levi-Civita connection. Since the Levi-Civita connection can be regarded as a linear connection, our statements above about linear connections remain valid. Let $\Psi(x)$ be the holonomy group with reference point $x \in M$ and recall that we are able to think of $\Psi(x)$ as a subgroup of $\mathrm{Aut}T_xM$. Since we are talking about a Riemannian connection, it turns out that this subgroup of $\mathrm{Aut}M$ acts orthogonally. That is to say the inclusion of $\Psi(x)$ in $\mathrm{Aut}M$ is an orthogonal representation of $\Psi(x)$ in T_xM. We shall say that M is *irreducible* (resp. *reducible*) if this representation of $\Psi(x)$ is irreducible (resp. reducible). We shall suppose that this statement makes sense in that it does not depend on $x \in M$.

Suppose that M is reducible and let $T'_x \subset T_xM$ be an invariant subspace under $\Psi(x)$. For $y \in M$, let τ be a curve from x to y (I guess we are assuming that M is connected). Define T'_y to be the parallel translation of T'_x along τ.

9.1 Lemma: *This definition of T'_y is independent of τ.*

Proof: Let μ be another curve from x to y. Then $\mu^{-1} \cdot \tau$ is a loop based at x. Since T'_x is invariant under $\Psi(x)$, $\mu^{-1} \cdot \tau(T'_x) = T'_x$ which implies that $\mu(T'_x) = \tau(T'_x)$. ■

From the lemma we see that we can define a distribution T' on M.

A injectively immersed submanifold N of M is said to be *totally geodesic* if any geodesic whose initial condition is tangent to N evolves on N.

9.2 Proposition: *(i) T' is differentiable and involutive;*

(ii) The maximal integral manifolds of T' are totally geodesic.

The proof is conceptually easy. By construction, if X and Y are sections of T', then so is $\nabla_X Y$. Since the Levi-Civita connection is torsion free,

$$[X, Y] = \nabla_X Y - \nabla_Y X.$$

Therefore $[X, Y]$ is a section of T' and so T' is involutive. It is also evident from the construction of T' that if c is the geodesic with initial condition $v_x \in T'_x$, then $c'(t) \in T'_{c(t)}$ for each t for which the geodesic is defined.

Kobayashi and Nomizu also prove the following intuitive lemma.

9.3 Lemma: *Let N be a totally geodesic submanifold of M. Then every geodesic of N with the induced Riemannian metric is also a geodesic of M.*

9.4 Example: Consider $M = \mathbb{R}^3$ with its standard metric

$$\mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y + \mathrm{d}z \otimes \mathrm{d}z$$

in which the geodesics are straight lines. Let $N \subset M$ be the (x, y)-plane. If we use the natural coordinates (x, y) for N the induced metric is

$$\mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y.$$

The geodesics for this metric in N are straight lines in the (x, y)-plane. Clearly these are also geodesics for M. •

Now we have constructed a distribution T' from a subspace T'_x invariant under $\Psi(x)$. By virtue of Lemma 8.1, the orthogonal complement of T'_x, which we denote T''_x, is also invariant under $\Psi(x)$. Thus, we obtain two involutive distributions, T' and T'', which are orthogonal to one another. Therefore, locally M has coordinates $(x^1, \ldots, x^k, x^{k+1}, \ldots, x^n)$ such that

1. $T' = \mathrm{span}(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^k})$ and $T'' = \mathrm{span}(\frac{\partial}{\partial x^{k+1}}, \ldots, \frac{\partial}{\partial x^n})$
2. $g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = 0$ for $i = 1, \ldots, k$ and $j = k+1, \ldots, n$.
3. Geodesics which start with $x^{k+1} = \cdots = x^n = 0$ (resp. $x^1 = \ldots, x^k = 0$) will evolve with $x^{k+1} = \cdots = x^n = 0$ (resp. $x^1 = \ldots, x^k = 0$).

That is to say, locally M looks like the product of two Riemannian manifolds and the Riemannian metric on M is the direct product of the metrics on the components.

This decomposition also reflects itself in the holonomy group. It turns out that if M is reducible and if we construct two distributions T' and T'' as above, then the holonomy group $\Psi(x)$ is the direct sum of normal subgroups $\Psi'(x)$ and $\Psi''(x)$ and the representation of $\Psi'(x)$ on T''_x and the representation of $\Psi''(x)$ on T'_x is trivial.

With these notions in hand, the next step is quite clear. Let $T_x^{(0)}$ be the subset of elements of T_xM which are fixed by every automorphism in $\Psi(x)$. Potentially $T_x^{(0)} = \{0\}$. Observe that $T_x^{(0)}$ is invariant under $\Psi(x)$ and therefore so is its orthogonal complement T'_x invariant under $\Psi(x)$. If T'_x is irreducible then define $T_x^{(1)} = T'_x$ and we have

$$T_xM = T_x^{(0)} \oplus T_x^{(1)}.$$

If T'_x is reducible, we may choose from it an irreducible, invariant subspace $T_x^{(1)}$. Now define T''_x to be the orthogonal complement in T'_x of $T_x^{(1)}$. Again, T''_x is invariant, etc. We proceed in this way and define invariant irreducible subspaces $T_x^{(1)}, \ldots, T_x^{(k)}$ of T'_x such that

$$T_xM = \bigoplus_{i=0}^{k} T_x^{(i)}.$$

Denote by $T^{(i)}$ the distribution associated with $T_x^{(i)}$. These distributions are involutive and their maximal integral manifolds are geodesically invariant. This defines a local decomposition of M into components $V_0 \times \cdots \times V_k$. It turns out that V_0 is locally *isometric* to a Euclidean space. That is to say, as a Riemannian manifold V_0 is flat.

A similar thing happens to the structure group. We may write

$$\Psi(x) = \Psi^{(0)} \times \cdots \times \Psi^{(i)}(x)$$

where the representation of $\Psi^{(i)}(x)$ in $T_x^{(j)}$ is trivial for $i \neq j$. Also $\Psi^{(0)} = \{e\}$.

If M is simply connected the above decompositions are unique up to reordering. If we make stronger assumptions about M, we can make a much stronger statement. Suppose that M is connected, simply connected and complete. Denote by $M_0, \ldots, M_k$ the maximal integral manifolds of $T^{(0)}, \ldots, T^{(k)}$, respectively, through x. Then M is isometric to $M_0 \times \cdots \times M_k$. Note that this is a global statement.