Some two-dimensional manifolds with affine connections

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1. Introduction

Here we present some simple manifolds with affine connections to get some feeling for how things like curvature manifest themselves. All the affine connections we consider will be Levi-Civita connections and so are torsion free. Recall that the Levi-Civita connection is characterised by the Christoffel symbols being

$$\Gamma^i_{jk} = \frac{1}{2} g^{il} \left(\frac{\partial g_{lj}}{\partial x^k} + \frac{\partial g_{lk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l} \right)$$

where g_{ij} are the components of the metric tensor. Also recall that the components of the curvature tensor are

$$R^i_{jkl} = \frac{\partial \Gamma^i_{lj}}{\partial x^k} - \frac{\partial \Gamma^i_{kj}}{\partial x^l} + \Gamma^i_{km}\Gamma^m_{lj} - \Gamma^i_{lm}\Gamma^m_{kj}.$$

Now let's pull a few quantities out of thin air. Given a two-dimensional subspace $E_x \subset T_x M$, define the *sectional curvature* of E_x to be

$$K(E_x) = g(R(X_1, X_2)X_2, X_1)$$

where $\{X_1, X_2\}$ is an orthonormal basis for E_x. It may be shown that this does not depend on the choice of basis. The *Ricci curvature tensor* is of type $(0, 2)$ and is given in coordinates by

$$S_{ij} = R^k_{ikj}.$$

Now we define the *scalar curvature* by

$$\rho = g^{ij} S_{ij}.$$

We shall see the significance of these entities (mainly the sectional and scalar curvatures) later.

All computations were performed by the *Mathematica* package `Riemann.m`. We make the statement that the computations we perform are very "hands on." Less direct, more sophisticated treatments are possible (preferable?).

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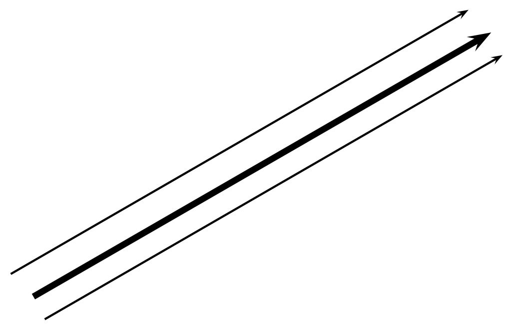

Figure 1. Parallel geodesics when curvature is zero. The bold line is the initial geodesic.

2. The Euclidean plane (curvature = 0)

Let's start with the easiest example first. Consider $\mathbb{R}^2$ with (x, y) the canonical coordinates. The standard metric is $g_{\mathbb{R}} = \mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y$. The Christoffel symbols are zero in these coordinates since the components of the metric tensor are constant. Therefore, geodesics are determined by the equations

$$\begin{aligned} \ddot{x} &= 0 \\ \ddot{y} &= 0 \end{aligned}$$

and so are simply "straight lines" in the plane. It is clear that the components of the curvature tensor are zero in these coordinates. Since the curvature tensor is, oddly enough, a tensor, its components are zero for all choices of coordinates. Thus all curvatures are zero.

2.1 Remarks: 1. As a, for the moment stupid, observation, note that $S = 0 \cdot g_{\mathbb{R}}$. We shall see a similar scenario below with the other spaces.

2. Note that there is a unique minimum length geodesic which joins any two points in $\mathbb{R}^2$.

3. Consider the following construction which we will also perform below for the other examples. Fix a geodesic and consider varying slightly its initial position (not its initial velocity). Observe that the geodesics obtained in this way maintain a constant distance from the original geodesic; they are simply lines "parallel" to the initial geodesic. See Figure 1. •

3. The two-dimensional sphere (curvature = 1)

Now we consider the sphere

$$\mathbb{S}^2 = \{\boldsymbol{x} \in \mathbb{R}^3 \mid \ \|\boldsymbol{x}\| = 1\}.$$

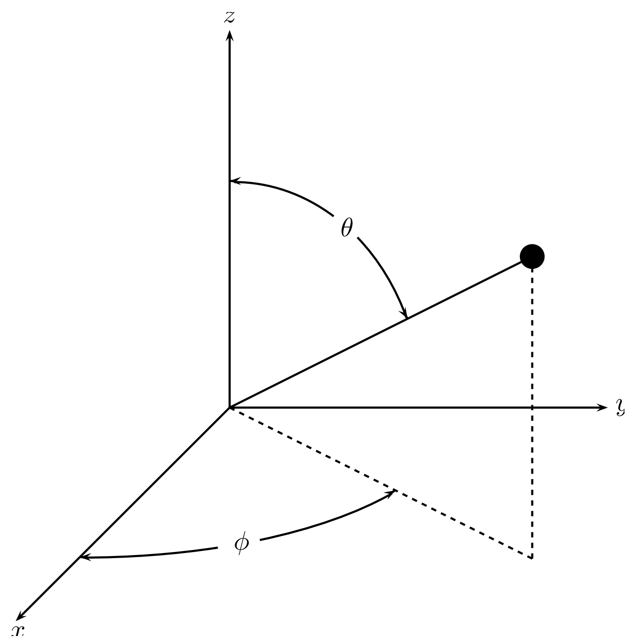

Figure 2. Latitude-longitude coordinates for $\mathbb{S}^2$.

The metric we consider on $\mathbb{S}^2$ is the canonical metric on $\mathbb{R}^3$ pulled back to $\mathbb{S}^2$. As coordinates for $\mathbb{S}^2$ we shall use latitude-longitude coordinates (θ, ϕ). These are defined by the relations

$$x = \sin\theta\cos\phi, \qquad y = \sin\theta\sin\phi, \qquad z = \cos\theta, \qquad (x, y, z) \in \mathbb{S}^2.$$

See Figure 2. We see that $\theta \in]0, \pi[$ and $\phi \in]0, 2\pi[$. We may now readily compute

$$g_{\mathbb{S}} = (\mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y + \mathrm{d}z \otimes \mathrm{d}z) \mid \mathbb{S}^2 = \mathrm{d}\theta \otimes \mathrm{d}\theta + \sin^2\theta \mathrm{d}\phi \otimes \mathrm{d}\phi.$$

Thus $g_{\theta\theta} = 1$, $g_{\phi\phi} = \sin^2\theta$, and $g_{\theta\phi} = g_{\phi\theta} = 0$. We may then compute the non-zero Christoffel symbols to be

$$\Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta, \qquad \Gamma^{\phi}_{\phi\theta} = \Gamma^{\phi}_{\theta\phi} = \cot\theta.$$

Thus geodesics are determined by the equations

$$\begin{aligned} \ddot{\theta} - \sin\theta\cos\theta\dot{\phi}^2 &= 0 \\ \ddot{\phi} + 2\cot\theta\dot{\theta}\dot{\phi} &= 0. \end{aligned}$$

We now present a sketchy argument which allows us to conclude what a general geodesic looks like if we are able to compute just a few geodesics. Note that the canonical metric on $\mathbb{R}^3$ is invariant under the usual action of $SO(3)$. Also, $\mathbb{S}^2$ is invariant under the action of $SO(3)$. Therefore, the metric on $\mathbb{S}^2$ in invariant under the action of $SO(3)$. This means that if we have a geodesic on $\mathbb{S}^2$, its image under multiplication by $A \in SO(3)$ is also a geodesic.

Now we compute a class geodesics on $\mathbb{S}^2$ with special initial conditions. The initial conditions we allow are such that $\dot{\phi}(0) = 0$. From the geodesic equations we easily see

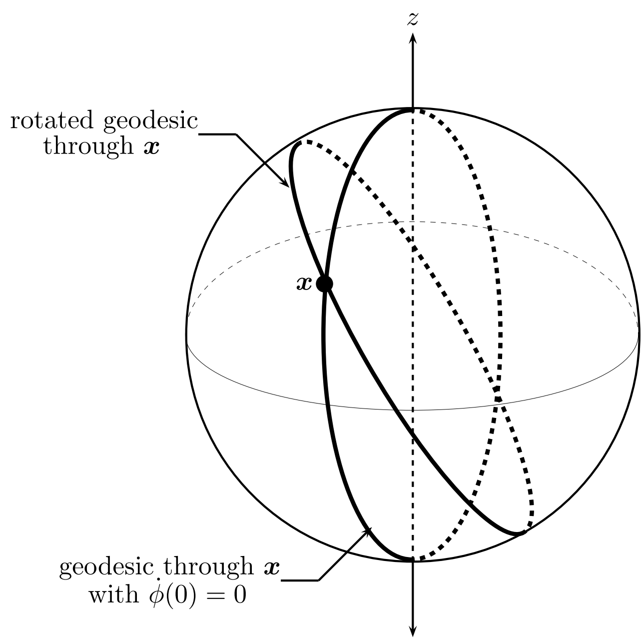

Figure 3. Obtaining arbitrary geodesics by rotating geodesics with $\dot\phi(0) = 0$ by an element of $SO(3)$.

that this implies that $\dot\phi(t) = 0$ for all t for which the geodesic is defined. Therefore, the equations governing this class of geodesics are simply

$$\begin{aligned}\ddot\theta &= 0\\ \dot\phi &= 0\end{aligned}$$

which have the easy solution

$$\begin{aligned}\theta(t) &= \dot\theta(0)t + \theta(0)\\ \phi(t) &= \phi(0).\end{aligned}$$

Ignoring coordinate singularities, this gives the geodesic through any point on $\mathbb{S}^2$ with initial condition $\dot\phi(0) = 0$. To compute geodesics with other initial velocities, simply rotate the geodesics we have computed by an element of $SO(3)$ which leaves the initial point on $\mathbb{S}^2$ fixed. See Figure 3.

Now we may compute the components of the curvature tensor. The non-zero components are

$$R^{\theta}_{\phi\theta\phi} = -R^{\theta}_{\phi\phi\theta} = \sin^2\theta, \qquad R^{\phi}_{\theta\phi\theta} = -R^{\phi}_{\theta\theta\phi} = 1.$$

Since $\mathbb{S}^2$ is two-dimensional, there is only one sectional curvature. To compute it we may use the orthonormal basis

$$X_1 = \frac{\partial}{\partial\theta}, \qquad X_2 = \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}.$$

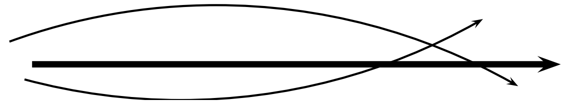

Figure 4. Converging geodesics when curvature is positive. The bold line is the initial geodesic.

It is now easily verified that $K = 1$. It is a simple matter to compute the Ricci tensor components as

$$S_{\theta\theta} = 1, \qquad S_{\phi\phi} = \sin^2\theta, \qquad S_{\theta\phi} = S_{\phi\theta} = 0.$$

We compute $\rho = 2$.

3.1 Remarks: 1. The geodesics are, at least in the range of our coordinate chart, the great circles that we are familiar with.

2. The vague arguments about the action of the rotation group reflect a general fact about geodesics on $\mathbb{S}^n$.

3. For objects embedded in Euclidean 3-space, we have an intuitive notion of what curvature should mean. In this intuitive notion, the sphere should have curvature 1. Let us call intuitive curvature *Gaussian curvature*. It is a theorem that the scalar curvature for objects embedded in $\mathbb{R}^3$ is twice the Gaussian curvature. Also, the sectional curvature of such an object (note there is only one sectional curvature!) is equal to the Gaussian curvature.

4. Note that in this example $S = 1 \cdot g_{\mathbb{S}}$. This is related to the fact that curvature is constant for $\mathbb{S}^2$.

5. Note that for antipodal points on the sphere,[1] there are an infinite number of equal length geodesics which connect them. In Riemannian geometry this means that these are *conjugate points*. Other pairs of points have a unique geodesic of least distance which connects them.

6. If we play the game of varying the initial position of a given geodesic, we see that all geodesics obtained will eventually intersect the given geodesic. Loosely speaking, geodesics with similar initial conditions "converge on each other." See the feeble representation of this in Figure 4. •

4. The hyperbolic plane (curvature = −1)

This example deals with the upper half-plane in $\mathbb{R}^2$. As coordinates we use (x, y) where y is restricted to be positive. We may define a metric on this manifold by

$$g_{\mathbb{H}} = \frac{1}{y^2}\left(\mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y\right).$$

[1] Two points $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{S}^2$ are said to be *antipodal* if $\boldsymbol{y} = -\boldsymbol{x}$.

The upper half-plane with this metric will be denoted $\mathbb{H}^2$ and we call it the *hyperbolic plane*. We shall give some motivation for the name later. Observe that $g_{xx} = g_{yy} = \frac{1}{y^2}$ and $g_{xy} = g_{yx} = 0$. The Christoffel symbols may be computed as

$$\Gamma^x_{xy} = \Gamma^x_{yx} = -\frac{1}{y}, \qquad \Gamma^y_{xx} = \frac{1}{y}, \qquad \Gamma^y_{yy} = -\frac{1}{y}$$

and all others are zero. Thus the equations for geodesics are

$$\begin{aligned} \ddot{x} - \frac{2}{y}\dot{x}\dot{y} &= 0 \\ \ddot{y} + \frac{1}{y}\dot{x}^2 - \frac{1}{y}\dot{y}^2 &= 0. \end{aligned}$$

First we note that if $\dot{x}(0) = 0$ then $\dot{x}(t) = 0$ for all t for which the geodesic is defined. Therefore, geodesics with initial velocities in the y-direction are lines in the y-direction. We compute the remaining geodesics with the advantage of knowing the answer. We claim that the curve

$$c\colon t \mapsto (x_c + R\cos t, R\sin t), \qquad t \in]0, \pi[$$

is, up to reparameterisation, a geodesic of $\mathbb{H}^2$ for $x_c, R \in \mathbb{R}$. Note that c parameterises a semi-circle in the upper-half plane of $\mathbb{R}^2$ centred on the x-axis. Suppose that the reparameterisation is given by $\tau(t)$. Along the curve we then compute

$$\begin{aligned} \ddot{x} - \frac{2}{y}\dot{x}\dot{y} &= -R\sin\tau\left(\ddot{\tau} - \dot{\tau}^2\cot\tau\right) \\ \ddot{y} + \frac{1}{y}\left(\dot{x}^2 - \dot{y}^2\right) &= R\cos\tau\left(\ddot{\tau} - \dot{\tau}^2\cot\tau\right). \end{aligned}$$

Therefore, choosing $\tau(t)$ to be a solution of $\ddot{\tau} - \dot{\tau}^2\cot\tau = 0$, we see that c is indeed a reparameterised geodesic. In fact, other than the vertical lines, these are the only geodesics. Indeed, let $(x, y) \in \mathbb{H}^2$ and let (u, v) be an initial velocity at (x, y) with $u \neq 0$. Then there exists a unique semi-circle centred on the x-axis passing through (x, y) and tangent to (u, v). See Figure 5.

Now we compute the components of the curvature tensor. The non-zero components are

$$R^x_{yyx} = -R^x_{yxy} = \frac{1}{y^2}, \qquad R^y_{xxy} = -R^y_{xyx} = \frac{1}{y^2}.$$

Again there is only one sectional curvature to compute since the manifold is two-dimensional. We may use the orthonormal basis

$$X_1 = \frac{1}{y}\frac{\partial}{\partial x}, \qquad X_2 = \frac{1}{y}\frac{\partial}{\partial y}$$

and compute $K = -1$. The Ricci tensor is given by

$$S_{xx} = S_{yy} = -\frac{1}{y^2}, \qquad S_{xy} = S_{yx} = 0.$$

We compute $\rho = -2$.

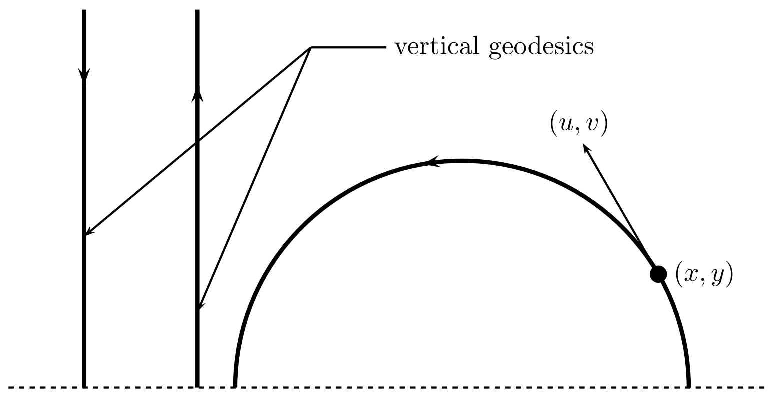

Figure 5. Geodesics for $\mathbb{H}^2$.

Let us now do some seemingly random constructions with $\mathbb{H}^2$. *Fix* a geodesic γ for $\mathbb{H}^2$ and let $p \in \mathbb{H}^2$. Then there exists a unique geodesic through p which is asymptotic to γ as time goes to infinity. Define a subspace $D_\gamma^+(p) \subset T_p\mathbb{H}^2$ as the orthogonal complement to the direction of this geodesic. Doing this for every $p \in \mathbb{H}^2$ defines a distribution D_γ^+ on $\mathbb{H}^2$. This distribution is integrable (in this case since it is one-dimensional) and its integral manifolds have the property that geodesics which are orthogonal to them are asymptotic to γ as time goes to infinity. We call these manifolds the *positive horocycles of* γ. It is simple to verify that the positive horocycles are the circles with the following two properties:

1. Their centres are on the vertical line which is tangent to γ as $t \to \infty$.

2. They are tangent to the x-axis.

See Figure 6. It is possible to construct *negative horocycles* as manifolds orthogonal to geodesics asymptotic to γ as time goes to minus infinity.

4.1 Remarks: 1. Since the metric for $\mathbb{H}^2$ is simply the standard metric for $\mathbb{R}^2$ with a scaling factor, angles are the same as they are in $\mathbb{R}^2$. Thus orthogonal has its usual meaning.

2. It is possible to represent $\mathbb{H}^2$ is many ways. We make a few remarks about some of these representations here.

   (a) Denote by $A(1)$ the group of proper affine transformations of $\mathbb{R}$. Thus an element of $A(1)$ has the form

$$\mathbb{R} \ni s \mapsto ys + x \in \mathbb{R}, \qquad y > 0, x \in \mathbb{R}. \tag{4.1}$$

   Thus we may establish a bijection to $A(1)$ from $\mathbb{H}^2$ by assigning to $(x, y) \in \mathbb{H}^2$ the proper affine transformation given by (4.1). The identity element is $(0, 1)$ and the inverse of (x, y) is $(-x/y, 1/y)$. It turns out that the metric $\frac{1}{y^2}(\mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y)$ is a left-invariant metric on this group. This is explained in [Arnol′d and Avez 1968].

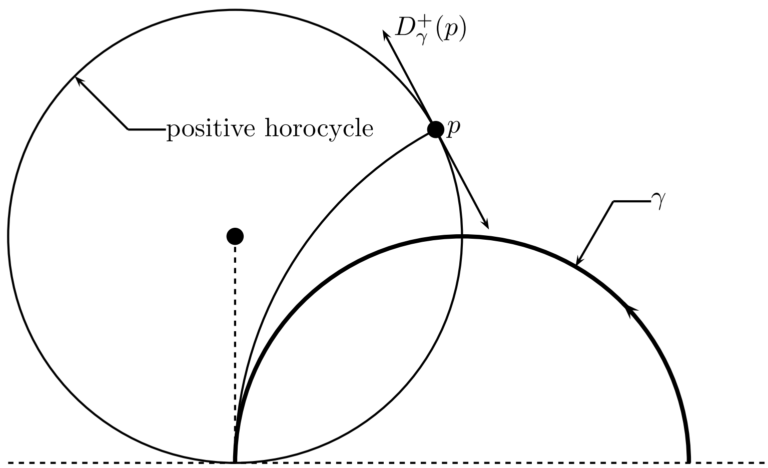

Figure 6. Positive horocycles of γ.

(b) We may also think of $\mathbb{H}^2$ as being the upper half of the complex plane. Doing this enables one to identify the orientation preserving part of the isometry group of $\mathbb{H}^2$ with a three-dimensional subgroup of the group of linear fractional transformations of $\mathbb{C}$.[2]

(c) So where does the name *hyperbolic space* come from? Consider H, the top half of the hyperboloid $z^2 - x^2 - y^2 = 1$ in $\mathbb{R}^3$ (Figure 7). On $\mathbb{R}^3$ consider the *pseudo-Riemannian* metric

$$\mathrm{d}z \otimes \mathrm{d}z - \mathrm{d}y \otimes \mathrm{d}y - \mathrm{d}x \otimes \mathrm{d}x.$$

This metric may be restricted to the H and defines a *negative-definite* pseudo-Riemannian metric on H. Denote this pseudo-Riemannian metric on H by g_H. From the point of view of geodesics, this may as well be a positive-definite metric. It is possible to establish an isometry from $(H, -g_H)$ to $(\mathbb{H}^2, g_{\mathbb{H}})$. This is where the name hyperbolic space comes from. This is explained in [Dubrovin, Fomenko, and Novikov 1991].

(d) Notice that all of the references given for $\mathbb{H}^2$ are Russian. It is common to see $\mathbb{H}^2$ called the *Lobachevski plane*.[3]

3. Note that the representation 2c allows us to say similar things about $\mathbb{H}^2$ as we said about $\mathbb{S}^2$ in Remark 3.1.3. That is to say, the Gaussian curvature of $\mathbb{H}^2$ is -1.

4. Note that $S = -1 \cdot g_{\mathbb{H}}$. As we observed in Remark 3.1.4, this is a consequence of $\mathbb{H}^2$ having constant curvature.

[2] Recall that a linear fractional transformation of $\mathbb{C}$ has the form $z \mapsto \frac{az+b}{cz+d}$ where $a, b, c, d \in \mathbb{C}$ are such that $ad - bc \neq 0$. Those linear fractional transformations with $a, b, c, d \in \mathbb{R}$ such that $ad - bc = 1$ form the orientation preserving part of the isometry group of $\mathbb{H}^2$. The non-orientation preserving part comes from the reflections $(x, y) \mapsto (-x, y)$.

[3] Nikolai Ivanovich Lobachevski, 1792–1856.

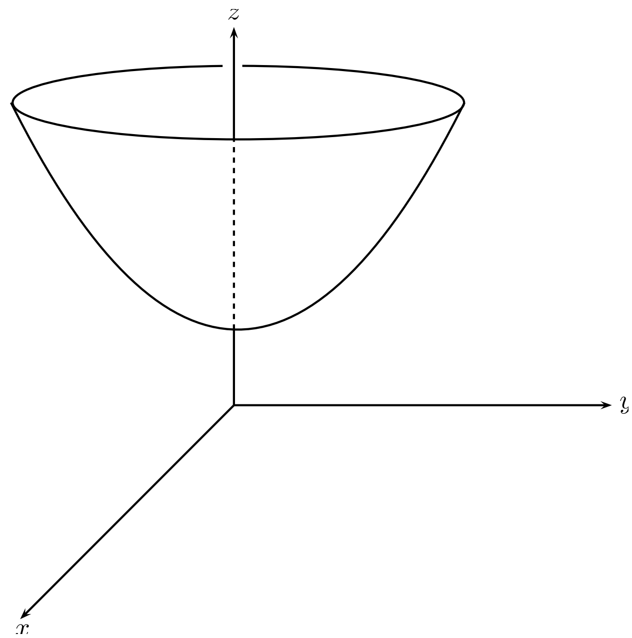

Figure 7. A hyperboloid in $\mathbb{R}^3$.

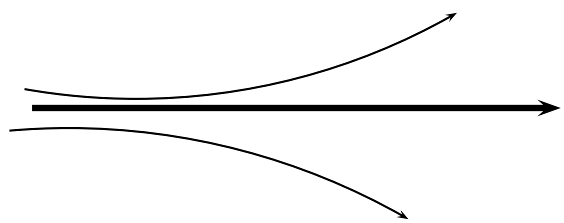

Figure 8. Diverging geodesics when curvature is negative. The bold line is the initial geodesic.

5. Note that in $\mathbb{H}^2$ there is a unique shortest length geodesic joining *any* two points. Thus $\mathbb{H}^2$ has no conjugate points.

6. The varying of initial conditions for a given geodesic results in nearby geodesics "diverging." This is illustrated in Figure 8. (The astute reader already sees that Figure 8 is wrong for some geodesics in $\mathbb{H}^2$.) We shall have more to say about this in Section 6.

7. What's the deal with these horocycles? It turns out that the construction we made (i.e., constructing manifolds orthogonal to geodesics which are asymptotic to a given geodesic) can be generalised to a class of Riemannian manifolds with negative curvature. Doing so enables one to prove a global structural stability result about the geodesic flow of such systems. Results of this nature may be found in [Arnol′d and Avez 1968]. •

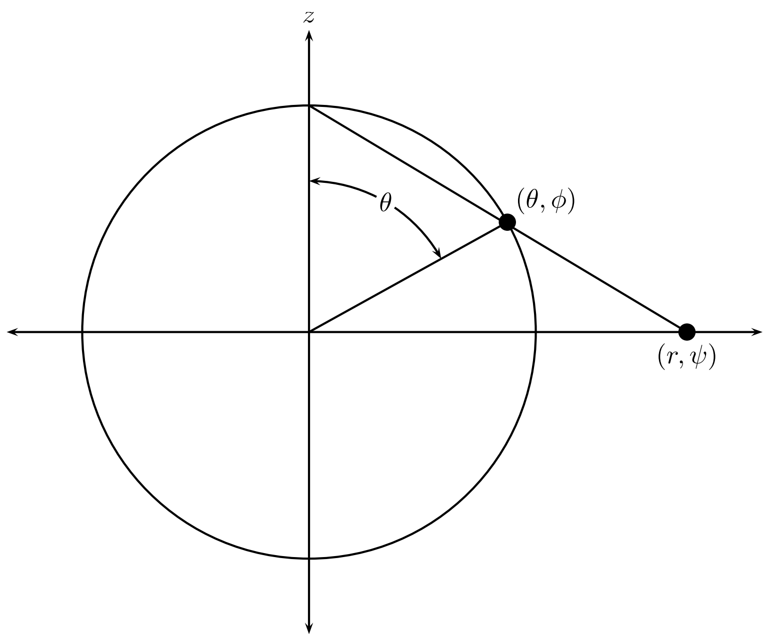

Figure 9. Stereographic coordinates for $\mathbb{S}^2$.

5. More fun and games with $\mathbb{S}^2$ and $\mathbb{H}^2$

In our presentation of the geometry of $\mathbb{S}^2$ and $\mathbb{H}^2$, we have not emphasised their similarity as much as is possible. So let's do that now by using appropriate coordinates. This presentation may be found in [Dubrovin, Fomenko, and Novikov 1991].

Using latitude-longitude coordinates, (θ, ϕ), for $\mathbb{S}^2$ we have already seen that the metric is

$$g_{\mathbb{S}} = \mathrm{d}\theta \otimes \mathrm{d}\theta + \sin^2\theta \mathrm{d}\phi \otimes \mathrm{d}\phi.$$

Now let's use stereographic coordinates for $\mathbb{S}^2 \setminus (0, 0, 1)$. These coordinates are illustrated in Figure 9 and are given explicitly by $(r = \cot(\theta/2), \psi = \phi)$. One may readily compute that in these coordinates we have

$$g_{\mathbb{S}} = \frac{4}{(1+r^2)^2}\left(\mathrm{d}r \otimes \mathrm{d}r + r^2 \mathrm{d}\psi \otimes \mathrm{d}\psi\right).$$

Now the next obvious thing to do is to use Cartesian coordinates for the plane onto which we are projecting. Thus we use $(x = r\cos\psi, y = r\sin\psi)$ and compute

$$g_{\mathbb{S}} = \frac{4}{(1+x^2+y^2)^2}\left(\mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y\right).$$

Thus, in stereographic coordinates, we see that the metric for $\mathbb{S}^2$ is a scaling of the standard metric for $\mathbb{R}^2$. This implies that the notion of orthogonality is the same for $g_{\mathbb{S}}$ as for the standard metric. This feature of a set of coordinates for $\mathbb{S}^2$ makes them *conformal* in the

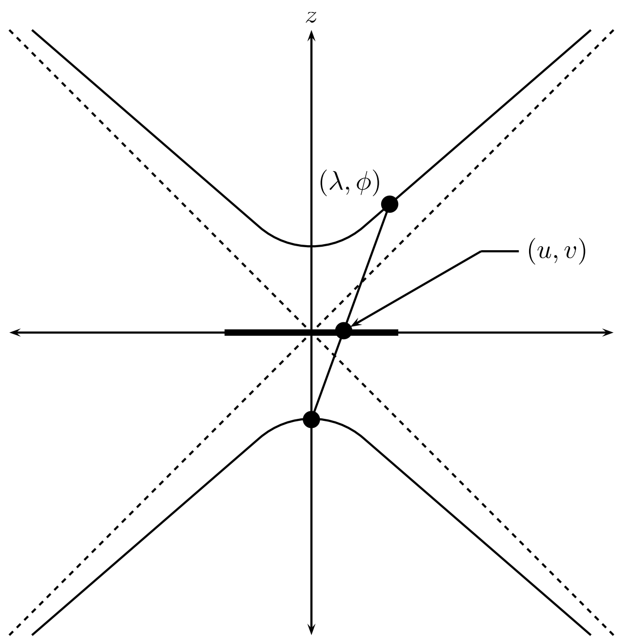

Figure 10. Stereographic coordinates for H.

language of cartographers.[4]

Now we turn our attention to $\mathbb{H}^2$. Recall (Remark 4.1.2c) that we denote by H the upper half of the hyperboloid $z^2 - x^2 - y^2 = 1$ in $\mathbb{R}^3$. We also defined the negative-definite pseudo-Riemannian metric g_H on H and we stated that $(H, -g_H)$ is isometric to $(\mathbb{H}, g_{\mathbb{H}})$. We will deal here with (H, g_H), understanding its relationship with $(\mathbb{H}, g_{\mathbb{H}})$. First we introduce "pseudo-spherical" coordinates for H, (λ, ϕ), which are defined by

$$x = \sinh\lambda\cos\phi, \qquad y = \sinh\lambda\sin\phi, \qquad z = \cosh\lambda, \qquad (x, y, z) \in H.$$

Here $\lambda \in]0, \infty[$ and $\phi \in]0, 2\pi[$. It is easy to show that

$$-g_H = \mathrm{d}\lambda \otimes \mathrm{d}\lambda + \sinh^2\lambda \mathrm{d}\phi \otimes \mathrm{d}\phi.$$

We may also define stereographic coordinates for H. We do this as in Figure 10. Note that it is expedient to bypass the polar coordinate version of the stereographic coordinates and go straight to the Cartesian coordinates which we denote by (u, v). Using elementary geometry we may compute

$$u = \frac{\sinh\lambda\cos\phi}{1 + \cosh\lambda}, \qquad v = \frac{\sinh\lambda\sin\phi}{1 + \cosh\lambda}.$$

[4]Conformal has a meaning in Riemannian geometry as well. Let (M, g) be a pseudo-Riemannian manifold. The *conformal group* of (M, g) is the set of diffeomorphisms $f\colon M \to M$ such that $f^*g = \lambda \cdot g$ for a positive function λ. For reference, the conformal group of $\mathbb{R}^2 \simeq \mathbb{C}$ is the group of conformal transformations from the theory of complex functions.

It may be verified that, given the domain of λ and ϕ, we must have $u^2 + v^2 < 1$. It is a straightforward calculation to arrive at

$$-g_H = \frac{4}{(1 - u^2 - v^2)^2} \left(\mathrm{d}u \otimes \mathrm{d}u + \mathrm{d}v \otimes \mathrm{d}v\right).$$

As with the metric induced by $\mathbb{S}^2$, we see that this metric is a scaling of the standard metric on $\mathbb{R}^2$ and so orthogonality is preserved.

Let's summarise our computations of this section in a table.

	$\mathbb{S}^2$	$\mathbb{H}^2$
Domain for Stereographic Coordinates	All of $\mathbb{R}^2$	$u^2 + v^2 < 1$
"Spherical" Coordinates	$\mathrm{d}\theta \otimes \mathrm{d}\theta + \sin^2\theta \mathrm{d}\phi \otimes \mathrm{d}\phi$	$\mathrm{d}\lambda \otimes \mathrm{d}\lambda + \sinh^2\lambda \mathrm{d}\phi \otimes \mathrm{d}\phi$
Stereographic Coordinates	$\frac{4}{(1+x^2+y^2)^2}\left(\mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y\right)$	$\frac{4}{(1-u^2-v^2)^2}\left(\mathrm{d}u \otimes \mathrm{d}u + \mathrm{d}v \otimes \mathrm{d}v\right)$

6. Discussion

In the course of our presentation, some interesting tidbits were revealed, and it is worth recapping them here.

Curvature and stability of geodesics

In each of the examples we said a few vague things about stability of geodesics. Our statements may be roughly summarised as:

1. When curvature is zero, geodesics are neutrally stable.
2. When curvature is positive, geodesics are stable.
3. When curvature is negative, geodesics are unstable.

These statements seem to be mythically held to be true, but nowhere is there any precise statement along the lines of the above observations. The closest thing is a theorem of [Anosov 1963] stating that on a compact Riemannian manifold with negative sectional curvatures, the geodesic flow on each energy level is "ergodic" (for example, it has a dense trajectory).

Curvature and topology

This is a real subject in Riemannian geometry (take a look at any text on Riemannian geometry). We cannot make any really strong statements in this direction, but a few observations can be made. Note that both $\mathbb{R}^2$ and $\mathbb{H}^2$ have the property that every pair of points can be joined by a unique minimum length geodesic whereas this is not true for $\mathbb{S}^2$. Also notice that $\mathbb{S}^2$ is compact and that neither $\mathbb{R}^2$ nor $\mathbb{H}^2$ are compact. This is not a coincidence. The existence of conjugate points has something to do with the topology of the manifold.

Remarks on Ricci curvature

In all of our examples we observed that $S = \lambda g$ for some constant λ (0 for $\mathbb{R}^2$, 1 for $\mathbb{S}^2$, and -1 for $\mathbb{H}^2$). Manifolds with this property are called *Einstein manifolds* and are characterised by having constant sectional curvature. It is further true that, when the dimension of the manifold is at least 3, if $S = \lambda g$ for some *function* λ, then λ must be constant.

Generalisations

The three examples, $\mathbb{R}^2$, $\mathbb{S}^2$, and $\mathbb{H}^2$ may be naturally generalised to $\mathbb{R}^n$, $\mathbb{S}^n$, and $\mathbb{H}^n$. Perhaps the only non-obvious generalisation is $\mathbb{H}^n$. Here we use $(x^1, \dots, x^n) \in \mathbb{R}^n$ where $x^n > 0$. The metric is simply $\frac{1}{(x^n)^2}$ times the standard metric for $\mathbb{R}^n$. Most of the conclusions made in these notes regarding geodesics, curvature, etc. may be naturally extended to these higher dimensional generalisations.

References

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