Lifting distributions to tangent and jet bundles*

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Abstract

Two natural means are provided to lift a distribution from a manifold to its tangent bundle, and they are shown to agree if and only if the original distribution is integrable. Two special cases, the case when the manifold is the total space of a vector bundle, and the case when the manifold is the total space of a fibration over $\mathbb{R}$, are dealt with in particular. For the latter case, the two constructions interact with the affine structure of the corresponding jet bundles in the "same" way.

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1. Introduction

If one is provided with a distribution D on a manifold M, there are some natural criteria one might ask for a "lifted" distribution on TM. For example, three possible natural properties are:

L1. if $\operatorname{rank}(D) = m$ then the rank of the lifted distribution should be $2m$;

L2. if the tangent vector field to a curve c on M lies in D, then the tangent vector field to the curve c' on TM (c' is itself the tangent vector field of c) should lie in the lifted distribution;

L3. if D is an Ehresmann connection on a locally trivial fibre bundle $\pi\colon M \to B$ with total space M (thus $TM = D \oplus \ker(T\pi)$), then the lifted distribution should be an Ehresmann connection on $T\pi\colon TM \to TB$.

In this paper we provide *two* ways of lifting a distribution D to TM, both of which satisfy the three stated properties. The two methods are genuinely different in that they agree if and only if D is integrable. In the event that M is the total space of a fibre bundle over $\mathbb{R}$,[1] then we give conditions under which the lift to TM can be restricted to a distribution on J^1M, the bundle of one-jets of sections (recalling that J^1M is naturally a subset of TM

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[1]To avoid confusion, it should be stated that this investigation of fibrations over $\mathbb{R}$ is unrelated to the property L3 stated as desirable for lifted distributions.

in this case). We shall see that a compatibility condition must be imposed in order that D lift to J^1M.

The basic constructions we present for lifting vector fields and one-forms are well-known. For example, complete lifts of vector fields are discussed by Crampin and Pirani [1986], and it is no enormous stretch to extend this to distributions. Also, de León, Marrero, and Martín de Diego [1997] give a coordinate version of our lifting for a codistribution in Section 3. However, it is precisely these two constructions which are different, and we are able to completely characterise this difference in Section 4. A method for lifting connections on a fibre bundle to the tangent fibre bundle is given by Kolář, Michor, and Slovák [1993, §46.7]. However, it is not clear that their construction may be performed for distributions which are *not* connections. It is also interesting to note that the construction given by Kolář, Michor, and Slovák differs from *both* of the constructions we give. Moreover, even though certain elements of what we present are well-established, it is worth cataloguing the ideas methodically. In Section 5 the case is considered when M is the total space of a *vector* bundle, and the distribution or codistribution define a *linear* connection on this bundle. We show that, as expected, the two associated lifts provide linear connections on the tangent bundle. As a specialisation, this gives two ways of lifting an affine connection on a manifold to the tangent bundle of the manifold. One of the lifted affine connections is known in the existing literature as the complete lift [see Yano and Ishihara 1973], but the other, to the author's knowledge, is new. The constructions we discuss here appear in the treatment of mechanical systems with affine constraints. Indeed, this is the setting of [de León, Marrero, and Martín de Diego 1997]. In this context it is interesting to note that although we give two generally different ways of lifting a distribution from M to TM, the constructions are equivalent insofar as their application to constrained mechanical systems is concerned. This is one way to view the content of Proposition 6.3.

We shall adopt throughout the geometric and differential calculus notation of [Abraham, Marsden, and Ratiu 1988]. In particular, we work in the category of C^∞ Banach manifolds, but give finite-dimensional coordinate formulas where appropriate. Let us review some elements of our notation here.

If E and F are Banach spaces we denote by $\mathrm{L}(E;F)$ the continuous linear mappings from E to F. If U is open in E, we denote by $\boldsymbol{D}f(u) \in \mathrm{L}(E;F)$ the derivative of $f\colon U \to F$ at $u \in U$. By $\boldsymbol{D}f(u)\cdot e$ we mean the evaluation of the derivative at $e \in E$. If E is further a product, say $E = E_1 \times E_2$, then we write $u = (u_1, u_2)$ and denote the partial derivatives by $\boldsymbol{D}_i f(u_1, u_2) \in \mathrm{L}(E_i;F)$, $i = 1, 2$. If $A\colon U \to \mathrm{L}(E;F)$ and $f \in F$, then $\boldsymbol{D}A(u)\cdot(e, f)$ means the derivative with respect to u in the direction of e of the function from U to F given by $u \mapsto A(u)\cdot f$ (this resolves potential ambiguity which arises from the order of the arguments e and f). If S is a subset of a Banach space E we denote by $\mathrm{ann}(S)$ the subspace of E^* which annihilates F. Similarly, if T is a subset of E^* we denote by $\mathrm{coann}(T)$ the subspace of E annihilated by T.

If (U, ϕ) is a chart for a manifold M taking values in a Banach space E, we denote by $\boldsymbol{U} = \phi(U)$ the open subset of E which parameterises U. We denote by $\Gamma^\infty(V)$ the set of smooth sections of a vector bundle with total space V. $\pi_{TM}\colon TM \to M$ denotes the tangent bundle of a manifold M, and $Tf\colon TM \to TN$ is the derivative of $f\colon M \to N$. The restriction of Tf to T_xM we denote by T_xf. $\mathrm{T}\bigwedge^k(V)$ denotes the bundle of exterior k-forms associated with a vector bundle with total space V.

2. Lifting distributions

In this section we lift a distribution D on M to one on TM, and show that the lifted distribution has the properties L1–L3 presented in the introduction. The basic constructions for lifting vector fields which we present here are well-known (see [Crampin and Pirani 1986, §13.2]).

Let $X \in \Gamma^\infty(TM)$. For each $x_0 \in M$ there exists a neighbourhood U of x_0 on which the flow $t \mapsto \exp_X(t)\colon U \to M$ is defined and is a local diffeomorphism for t sufficiently small. Thus we may define a one-parameter family of local diffeomorphisms $t \mapsto \{v_x \mapsto T_x(\exp_X(t))(v_x)\}$ from TU to TM for t sufficiently small. We then define a vector field X^c on TM by defining

$$X^c(v_x) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} T_x \exp_X(t)(v_x).$$

Note that we do not require X to be complete to make sense of this definition. X^c is called the ***complete lift*** of X. It will be helpful to have a coordinate expression for X^c.

2.1 Proposition: *If $\boldsymbol{X}\colon \boldsymbol{U} \to E$ is the principal part of the vector field X in a chart (U, ϕ), then $(u, e) \mapsto (\boldsymbol{X}(u), \boldsymbol{D}\boldsymbol{X}(u) \cdot e)$ is the principal part of X^c in the natural chart $(TU, T\phi)$.*

Proof: Let $u_0 \in \boldsymbol{U}$ and select a neighbourhood V of u_0 sufficiently small that $\exp_X(t)\colon V \to \boldsymbol{U}$ is a local diffeomorphism for t sufficiently small. Let us write the local flow as $(t, u) \mapsto F(u, t)$ for $u \in V$. The flow of X^c is then $(t, (u, e)) \mapsto (F(u, t), \boldsymbol{D}_1 F(u, t) \cdot e)$. Thus we compute

$$X^c(u_0, e) = (\boldsymbol{D}_2 F(u_0, 0) \cdot 1, \boldsymbol{D}_1 \boldsymbol{D}_2 F(u_0, 0) \cdot (e, 1))$$

where we use the equality of mixed partial derivatives. The result now follows since $\boldsymbol{X}(u_0) = \boldsymbol{D}_2 F(u_0, 0)$. ■

In finite-dimensions, the coordinate expression for X^c is

$$X^c = X^i \frac{\partial}{\partial x^i} + \frac{\partial X^i}{\partial x^j} v^j \frac{\partial}{\partial v^i}$$

if X^i, $i = 1, \ldots, n$, are the components of X in a local chart, and we denote coordinates for TM by (x^i, v^j). Note that we shall employ the summation convention throughout the paper when writing coordinate formulas in finite-dimensions.

There is at least one more natural way of defining a vector field on TM given one on M. If $X \in \Gamma^\infty(TM)$ we define a vector field X^v, the ***vertical lift*** of X, on TM by

$$X^v(v_x) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (v_x + tX(x)).$$

In a natural chart for TM one readily verifies that the principal part of X^v is given by

$$X^v(u, e) = (0, \boldsymbol{X}(u))$$

if $\boldsymbol{X}$ is the principal part of X. In finite-dimensions we have

$$X^v = X^i \frac{\partial}{\partial v^i}$$

if X^i, $i = 1, \dots, n$, are the components of X.

Given a distribution D on M, we can now define a subset D^T of TTM whose fibre over $v_x \in TM$ is

$$D^T_{v_x} = \{X^c(v_x) \mid X \in \Gamma^\infty(D)\}.$$

This is in fact a subbundle of TTM.

2.2 Proposition: *The subset D^T of TTM is a distribution on TM.*

Proof: Let (U, ϕ) be a chart for M inducing natural charts $(TU, T\phi)$ for TM and $(TTU, TT\phi)$ for TTM. Since D is a distribution on M, this means that D_x splits in T_xM for each $x \in M$. Therefore, if we choose the chart domain U sufficiently small, we may find a vector bundle chart (TU, ψ) for TM with the following properties:

1. ψ is a bijection onto $\boldsymbol{U} \times F_1 \times F_2$ for Banach spaces F_1 and F_2;
2. $\psi(D_x) = \{\phi(x)\} \times F_1 \times \{0\}$;
3. the overlap map from $T\phi(TU) = \boldsymbol{U} \times E$ to $\psi(TU) = \boldsymbol{U} \times F_1 \times F_2$ has the form

$$h\colon (u, e) \mapsto (u, A_1(u) \cdot e, A_2(u) \cdot e)$$

for smooth maps $A_i\colon \boldsymbol{U} \to \mathrm{L}(E; F_i)$, $i = 1, 2$.

We shall write the inverse of the overlap map h as

$$h^{-1}\colon (u, f_1, f_2) \mapsto (u, B_1(u) \cdot f_1 + B_2(u) \cdot f_2)$$

for maps $B_i\colon \boldsymbol{U} \to \mathrm{L}(F_i; E)$, $i = 1, 2$. Note that

$$A_i(u) \circ B_j(u) = \begin{cases} 0, & i \neq j, \\ \mathrm{id}_{F_i}, & i = j. \end{cases} \tag{2.1}$$

We claim that the map

$$\begin{aligned} H\colon ((u, e), (e_1, e_2)) \mapsto ((u, e), (&A_1(u) \cdot e_1, A_2(u) \cdot e_1, \\ &A_1(u) \cdot e_2 - A_1(u) \cdot (\boldsymbol{D}B_1(u) \cdot (e, A_1(u) \cdot e_1) + \boldsymbol{D}B_2(u) \cdot (e, A_2(u) \cdot e_1)), \\ &A_2(u) \cdot e_2 - A_2(u) \cdot (\boldsymbol{D}B_1(u) \cdot (e, A_1(u) \cdot e_1) + \boldsymbol{D}B_2(u) \cdot (e, A_2(u) \cdot e_1)))) \end{aligned} \tag{2.2}$$

is a local vector bundle isomorphism from $\boldsymbol{U} \times E \times E \times E$ to $\boldsymbol{U} \times E \times F_1 \times F_2 \times F_1 \times F_2$. That it is a local vector bundle mapping is clear. That it is an isomorphism may be verified by checking, using (2.1), that its inverse is

$$\begin{aligned} H^{-1}\colon ((u, e), (f_{11}, f_{21}, f_{12}, f_{22})) \mapsto ((u, e), (&B_1(u) \cdot f_{11} + B_2(u) \cdot f_{21}, \\ &\boldsymbol{D}B_1(u) \cdot (e, f_{11}) + B_1(u) \cdot f_{12} + \boldsymbol{D}B_2(u) \cdot (e, f_{21}) + B_2(u) \cdot f_{22})). \end{aligned} \tag{2.3}$$

Next we claim that $(TTU, H \circ TT\phi)$ is a vector bundle chart for TTM which is adapted to D^T (thus showing D^T to be a subbundle). First let us get a local description of D^T in the chart $(TTU, TT\phi)$. The fibre of D at $u \in \boldsymbol{U}$ is locally given by

$$D_u = \{(u, B_1(u) \cdot f_1) \mid f_1 \in F_1\}.$$

Thus a section X of D will have the form

$$X(u) = (u, B_1(u) \cdot f(u))$$

for some $f\colon \boldsymbol{U} \to F_1$. Therefore, by Proposition 2.1 the fibre of D^T at (u, e) is given by

$$D^T_{(u,e)} = \{((u,e), (B_1(u)\cdot f(u), \boldsymbol{D}B_1(u)\cdot(e, f(u)) + B_1(u)\cdot(\boldsymbol{D}f(u)\cdot e))) \mid f\colon \boldsymbol{U} \to F_1\}. \quad (2.4)$$

Note that the map $\Phi_{(u_0,e_0)}$ which sends the map $f\colon \boldsymbol{U} \to F_1$ to $(f(u_0), \boldsymbol{D}f(u_0)\cdot e_0) \in F_1 \times F_1$ is surjective for each $(u_0, e_0) \in \boldsymbol{U} \times E$. Indeed, for $(f_{11}, f_{12}) \in F_1 \times F_1$ let $A \in \mathrm{L}(E; F_1)$ have the property that $A(e_0) = f_{12}$. Then we see that the image of $f\colon u \mapsto f_{11} + A(u - u_0)$ under $\Phi_{(u_0,e_0)}$ is (f_{11}, f_{12}). Therefore we may take

$$D^T_{(u,e)} = \{((u,e), (B_1(u) \cdot f_{11}, \boldsymbol{D}B_1(u) \cdot (e, f_{11}) + B_1(u) \cdot f_{12})) \mid f_{11}, f_{12} \in F_1\}. \quad (2.5)$$

With this local description of D^T, we claim that

$$H \circ TT\phi(D^T_{v_x}) = \{(u,e)\} \times F_1 \times \{0\} \times F_1 \times \{0\}$$

where $(u, e) = T\phi(v_x)$. Substituting (2.5) into the expression (2.2) for H shows that

$$H \circ TT\phi(D^T_{v_x}) \subset \{(u,e)\} \times F_1 \times \{0\} \times F_1 \times \{0\}$$

The opposite inclusion follows immediately from (2.5), the definition (2.3) of H^{-1}, and the property (2.1) of the maps A_i, $i = 1, 2$, and B_i, $i = 1, 2$. ■

2.3 Remarks: 1. In finite-dimensions, the gist of the proof is as follows. Let (x^i, v^j, u^k, w^l) be natural coordinates for TTM and let $\{B_1, \dots, B_n\}$ be a local basis of vector fields on M for which $D_x = \mathrm{span}_{\mathbb{R}}(B_1(x), \dots, B_m(x))$. Then the change of coordinates

$$(x^i, v^j, v^k, w^l) \mapsto \left(x^i, v^j, B^k_r u^r, B^l_r w^r - B^l_s \frac{\partial A^s_t}{\partial x^p} v^p B^t_q u^q\right)$$

for TTM is adapted to the distribution D^T where B^i_j, $i = 1, \dots, n$ are the components of B_j, and A^i_j is the inverse matrix of B^i_j.

2. If we have a local basis of vector fields $\{X_a \mid a = 1, \dots, m\}$, for D, then the vector fields $\{X^c_a, X^v_b \mid a, b = 1, \dots, m\}$ form a local basis for D^T.

3. Note that if $D^v \subset TTM$ is the vertical lift of all tangent vectors in D then $D^v \subset D^T$.

4. Clearly we may inductively define the lift of D to the higher order tangent bundles T^kM. Let us denote the distribution so lifted to T^kM by D^{T^k}. •

We define the ***lift*** of D to be the distribution D^T. From Remark 2.3–2 we readily see that if $\mathrm{rank}(D) = m$ then $\mathrm{rank}(D^T) = 2m$. Thus we have satisfied the condition L1 in the introduction. Let us look at the condition L2 which concerns curves lying in D lifting to curves lying in D^T. In the statement of the following result we abuse notation slightly by thinking of c' as both the tangent vector field along a curve c *and* as a curve in TM. By c'' we mean the tangent vector field to c' when c' is thought of as a curve in TM.

2.4 Proposition: *Let $c\colon [a,b] \to M$ be a curve with $c'(t) \in D_{c(t)}$ for $t \in [a,b]$. If $c'\colon t \mapsto T_t c \cdot 1$ is its time-derivative which is a curve in TM, then $c''(t) \in D^T_{c'(t)}$ for $t \in [a,b]$.*

Proof: We use the notation of Proposition 2.2. A curve c whose tangent vector field lies in D will locally have the form $t \mapsto \boldsymbol{c}(t) \in \boldsymbol{U}$ where

$$\boldsymbol{D}\boldsymbol{c}(t) \cdot 1 = B_1(\boldsymbol{c}(t)) \cdot f(t)$$

for some curve $t \mapsto f(t)$ in F_1. The local form for c' is then $t \mapsto (\boldsymbol{c}(t), \boldsymbol{D}\boldsymbol{c}(t) \cdot 1)$. The local tangent vector to this curve at $(\boldsymbol{c}(t), \boldsymbol{D}\boldsymbol{c}(t) \cdot 1)$ is $(\boldsymbol{D}\boldsymbol{c}(t) \cdot 1, \boldsymbol{D}^2\boldsymbol{c}(t) \cdot (1,1))$. Note that

$$\boldsymbol{D}^2\boldsymbol{c}(t) \cdot (1,1) = \boldsymbol{D}B_1(\boldsymbol{c}(t)) \cdot (\boldsymbol{D}\boldsymbol{c}(t) \cdot 1, f(t)) + B_1(\boldsymbol{c}(t)) \cdot (\boldsymbol{D}f(t) \cdot 1)$$

By (2.5) this shows that the tangent vector to c' lies in D^T. ■

To see that D^T satisfies the condition L3 we stated in the introduction, we prove the following result. Recall that an Ehresmann connection for a fibration $\pi\colon M \to B$ is a complement HM to the vertical subbundle $VM = \ker(T\pi)$.

2.5 Proposition: *If HM is an Ehresmann connection on the locally trivial fibre bundle $\pi\colon M \to B$ then $(HM)^T$ is an Ehresmann connection on $T\pi\colon TM \to TB$.*

Proof: Let (U, ϕ) be an adapted chart for $\pi\colon M \to B$. We shall take $\boldsymbol{U} = \phi(U)$ to have the form $U_1 \times U_2 \subset E_1 \times E_2$ where U_i is an open subset of a Banach space E_i, $i = 1,2$. The chart being adapted implies that the local representative of the projection π is given by $(u_1, u_2) \mapsto u_1$. Obviously, the local form of the vertical subbundle $VM = \ker(T\pi)$ consists of those elements $((u_1,u_2),(0,e_2))$ in the local model for TM. An Ehresmann connection defines a complement HM to VM in TM so this locally is given by elements of the form

$$H_{(u_1,u_2)}\boldsymbol{U} = \{((u_1,u_2),(e_1, C(u_1,u_2) \cdot e_1)) \mid e_1 \in E_1\}$$

for some map $C\colon \boldsymbol{U} \to \mathrm{L}(E_1; E_2)$. Therefore a general section of $H\boldsymbol{U}$ has the form

$$(u_1,u_2) \mapsto ((u_1,u_2),(e(u_1,u_2), C(u_1,u_2) \cdot e(u_1,u_2))) \tag{2.6}$$

for some map $e\colon \boldsymbol{U} \to E_1$.

The adapted chart (U,ϕ) for $\pi\colon M \to B$ induces a natural chart $(TU, T\phi)$ for TM. We write coordinates in this chart as (u_1,u_2,e_1,e_2). The complete lift of a local section of HM of the form (2.6) is a local section of $(HM)^T$ given by

$$\begin{aligned}(u_1,u_2,e_1,e_2) \mapsto ((u_1,u_2,e_1,e_2),(e(u_1,u_2),\\ C(u_1,u_2) \cdot e(u_1,u_2), \boldsymbol{D}_1 e(u_1,u_2) \cdot e_1 + \boldsymbol{D}_2 e(u_1,u_2) \cdot e_2,\\ \boldsymbol{D}_1 C(u_1,u_2) \cdot (e_1, e(u_1,u_2)) + C(u_1,u_2) \cdot (\boldsymbol{D}_1 e(u_1,u_2) \cdot e_1) +\\ \boldsymbol{D}_2 C(u_1,u_2) \cdot (e_2, e(u_1,u_2)) + C(u_1,u_2) \cdot (\boldsymbol{D}_2 e(u_1,u_2) \cdot e_2))).\end{aligned}$$

Thus, by reasoning like that by which we derived (2.5) from (2.4), we see that

$$\begin{aligned}(H\boldsymbol{U})^T_{(u_1,u_2,e_1,e_2)} = \big\{((u_1,u_2,e_1,e_2),(e_{11}, C(u_1,u_2) \cdot e_{11}, e_{12},\\ \boldsymbol{D}_1 C(u_1,u_2) \cdot (e_1,e_{11}) + \boldsymbol{D}_2 C(u_1,u_2) \cdot (e_2,e_{11}) + C(u_1,u_2) \cdot e_{12})) \mid e_{11}, e_{12} \in E_1\big\}.\end{aligned} \tag{2.7}$$

Now note that the local representative of $T\pi$ has the form $(u_1, u_2, e_1, e_2) \mapsto (u_1, e_1)$. Thus

$$V_{(u_1,u_2,e_1,e_2)}\boldsymbol{TU} = \{((u_1, u_2, e_1, e_2), (0, e_{21}, 0, e_{22})) \mid e_{21}, e_{22} \in E_2\}. \tag{2.8}$$

Comparing (2.7) and (2.8) we see that $(HM)^T \cap VTM = \{0\}$. It is also evident that $TTM = (HM)^T + VTM$. Indeed we may write

$$\begin{aligned}(e_{11}, e_{21}, e_{12}, e_{22}) = \big(&e_{11}, C(u_1, u_2) \cdot e_{11}, e_{12},\\ &\boldsymbol{D}_1C(u_1, u_2) \cdot (e_1, e_{11}) + \boldsymbol{D}_2C(u_1, u_2) \cdot (e_2, e_{11}) + C(u_1, u_2) \cdot e_{12}\big)+\\ &\big(0, e_{21} - C(u_1, u_2) \cdot e_{11}, 0, e_{22}-\\ &\boldsymbol{D}_1C(u_1, u_2) \cdot (e_1, e_{11}) - \boldsymbol{D}_2C(u_1, u_2) \cdot (e_2, e_{11}) - C(u_1, u_2) \cdot e_{12}\big)\end{aligned}$$

where the first term on the right of the equals sign is horizontal, and the second is vertical. ■

In finite dimensions we denote adapted coordinates for $\pi\colon M \to B$ by (x^a, y^α), $a = 1, \dots, m$, $\alpha = 1, \dots, n - m$. A connection HM then locally has a basis

$$\frac{\partial}{\partial x^a} + C^\alpha_a(x, y)\frac{\partial}{\partial y^\alpha}, \quad a = 1, \dots, m$$

for some local matrix function C^α_a, $\alpha = 1, \dots, n - m$, $a = 1, \dots, m$. $(HM)^T$ then has the local basis

$$\begin{aligned}&\frac{\partial}{\partial u^a} + C^\alpha_a(x, y)\frac{\partial}{\partial v^\alpha}, && a = 1, \dots, m\\ &\frac{\partial}{\partial x^b} + C^\alpha_b(x, y)\frac{\partial}{\partial y^\alpha} + \left(\frac{\partial C^\alpha_b}{\partial x^c}u^c + \frac{\partial C^\alpha_b}{\partial y^\beta}v^\beta\right)\frac{\partial}{\partial v^\alpha}, && b = 1, \dots, m\end{aligned}$$

where we denote coordinates on TM by $((x^a, y^\alpha), (u^b, v^\beta))$. We remark that this does not agree with the construction in [Kolář, Michor, and Slovák 1993, §46.7]. However, the induced connection they give also satisfies Proposition 2.4; what is not clear is whether the procedure they give is applicable to distributions which are not connections.

3. Lifting codistributions

Next we look at the lifting of a codistribution on M (i.e., a subbundle of T^*M) to a codistribution on TM. The details of much of what we say in this section follow like those in Section 2, so our presentation is somewhat more abbreviated. However, there are some subtle notational changes, so we cannot afford to be too brief.

If $\beta \in \Gamma^\infty(T^*M)$ we may define a function f_β on TM by $f_\beta(v_x) = \beta(x) \cdot v_x$. We define the ***complete lift*** of β to be the one-form $\beta^c(v_x) = \boldsymbol{d}f_\beta(v_x)$ on TM. One readily verifies that the local representative of β^c in a natural chart for TM is

$$\beta^c(u, e) = ((u, e), (\boldsymbol{D\beta}(u) \cdot (\cdot, e), \boldsymbol{\beta}(u))) \tag{3.1}$$

where $\boldsymbol{\beta}$ is the principal part of β in the chart. We have used here the "place holder" notation which we will employ to advantage in the remainder of the paper. To explain, by

$\boldsymbol{D\beta}(u)\cdot(\cdot,e)$ we mean that element of E^* which is the derivative at $u\in\boldsymbol{U}$ of the $\mathbb{R}$-valued function $u\mapsto\boldsymbol{\beta}(u)\cdot e$ for fixed $e\in E$. In finite-dimensions (3.1) reads

$$\beta^c=\frac{\partial\beta_i}{\partial x^j}v^i\mathrm{d}x^j+\beta_i\mathrm{d}v^i.$$

We may now associate to a codistribution Λ a subset Λ^T of T^*TM by

$$\Lambda^T_{v_x}=\{\beta^c(v_x)\mid\ \beta\in\Gamma^\infty(\Lambda)\}.$$

We call Λ^T the ***lift*** of Λ to TM. The following result asserts that this is in fact a subbundle.

3.1 Proposition: *If Λ is a codistribution on M then Λ^T is a codistribution on TM.*

Proof: Suppose M is modelled on a Banach space E. Let (U,ϕ) be a chart for M with $(T^*U,T^*\phi)$ the induced natural chart for T^*M. Since Λ is a subbundle of T^*M we may choose U so that there exists a vector bundle chart (T^*U,χ) for T^*M with the following properties:

1. χ is a bijection onto $\boldsymbol{U}\times V_1^*\times V_2^*$ for Banach spaces V_1 and V_2;
2. $\chi(\Lambda_x)=\{\phi(x)\}\times\{0\}\times V_2^*$;
3. the overlap map from $T^*\phi(T^*U)=\boldsymbol{U}\times E^*$ to $\chi(T^*U)=\boldsymbol{U}\times V_1^*\times V_2^*$ is given by

$$g\colon(u,\alpha)\mapsto(u,P_1(u)\cdot\alpha,P_2(u)\cdot\alpha)$$

for maps $P_i\colon\boldsymbol{U}\to\mathrm{L}(E^*;V_i^*)$, $i=1,2$.

We denote the inverse of the overlap map by

$$g^{-1}\colon(u,\nu^1,\nu^2)\mapsto(u,Q_1(u)\cdot\nu^1+Q_2(u)\cdot\nu^2)$$

for maps $Q_i\colon\boldsymbol{U}\to\mathrm{L}(V_i^*;E^*)$, $i=1,2$. The relations

$$P_i(u)\circ Q_j(u)=\begin{cases}0, & i\neq j\\ \mathrm{id}_{V_i^*}, & i=j\end{cases}\tag{3.2}$$

hold.

With respect to the chart (U,ϕ) for M we write natural coordinates for T^*TM as $((u,e),(\alpha^1,\alpha^2))\in\boldsymbol{U}\times E\times E^*\times E^*$. We claim that the map

$$\begin{aligned}G\colon((u,e),(\alpha^1,\alpha^2))\mapsto((u,e),(P_1(u)\cdot\alpha^1-&\\ \boldsymbol{D}Q_1(u)\cdot(P_1^*(u)\cdot(\cdot),P_1(u)\cdot\alpha^2,e)-\boldsymbol{D}Q_2(u)\cdot(P_1^*(u)\cdot(\cdot),P_2(u)\cdot\alpha^2,e),&\\ P_2(u)\cdot\alpha^1-\boldsymbol{D}Q_1(u)\cdot(P_2^*(u)\cdot(\cdot),P_1(u)\cdot\alpha^2,e)-&\\ \boldsymbol{D}Q_2(u)\cdot(P_2^*(u)\cdot(\cdot),P_2(u)\cdot\alpha^2,e),P_1(u)\cdot\alpha^2,P_2(u)\cdot\alpha^2))&\end{aligned}\tag{3.3}$$

is a local vector bundle isomorphism from $\boldsymbol{U}\times E\times E^*\times E^*$ to $\boldsymbol{U}\times E\times V_1^*\times V_2^*\times V_1^*\times V_2^*$ which provides a chart for T^*TM which is a subbundle chart for Λ^T. Here, for example,

by $\boldsymbol{D}Q_1(u)\cdot(P_1^*(u)\cdot(\cdot),P_1(u)\cdot\alpha^2,e)$ we mean that element of V_1^* whose value at $v_1\in V_1$ is defined to be the derivative of the function

$$\boldsymbol{U}\ni u'\mapsto(Q_1(u')\circ P_1(u)\cdot\alpha^2)\cdot e\in V_1^*$$

in the direction of $P_1^*(u)\cdot v_1\in E$. That (3.3) defines a local vector bundle mapping is obvious. Furthermore, its inverse may be verified to be

$$\begin{aligned}G^{-1}\colon((u,e),(\nu^{11},\nu^{21},\nu^{12},\nu^{22}))\mapsto((u,e),(\boldsymbol{D}Q_1(u)\cdot(\cdot,\nu^{12},e)+Q_1(u)\cdot\nu^{11}+\\ \boldsymbol{D}Q_2(u)\cdot(\cdot,\nu^{22},e)+Q_2(u)\cdot\nu^{21},Q_1(u)\cdot\nu^{12}+Q_2(u)\cdot\nu^{22})),\end{aligned}\quad(3.4)$$

using (3.2).

Now we show that G provides a subbundle chart for Λ^T. We need to compute the local form of sections of Λ^T. In our local model $(T^*U,T^*\phi)$ for T^*M a section of Λ has the form

$$u\mapsto(u,Q_2(u)\cdot\nu(u))$$

for some $\nu\colon\boldsymbol{U}\to V_2^*$. Therefore, using (3.1) and arguments very much like those used in the proof of Proposition 2.2 we derive

$$\Lambda^T_{(u,e)}=\{((u,e),(\boldsymbol{D}Q_2(u)\cdot(\cdot,\nu^{21},e)+Q_2(u)\cdot\nu^{22},Q_2(u)\cdot\nu^{21}))\mid\ \nu^{21},\nu^{22}\in V_2^*\}.\quad(3.5)$$

The relation

$$G(\Lambda^T_{(u,e)})\subset\{(u,e)\}\times\{0\}\times V_2^*\times\{0\}\times V_2^*$$

is derived by substituting the form (3.5) for Λ^T into the definition (3.3) for G. The opposite inclusion follows directly from the definition (3.4) of G^{-1}. ■

3.2 Remarks: 1. In finite-dimensions the result shows that if $(x^i,v^j,\alpha_k,\beta_l)$ are natural coordinates for T^*TM and $Q^1,\dots,Q^n$ form a local basis for one-forms on M with $\Lambda_x=\operatorname{span}_{\mathbb{R}}(Q^{m+1}(x),\dots,Q^n(x))$, then adapted coordinates for Λ^T are given by

$$\left(x^i,v^j,P^r_k\alpha_r-P^r_k\frac{\partial Q^s_t}{\partial x^r}v^tP^q_s\beta_q,P^r_l\beta_r\right).$$

Here Q^i_j, $j=1,\dots,n$, are the components of Q^i, and P^i_j is the inverse matrix of Q^i_j.

2. If $\{\beta^{m+1},\dots,\beta^n\}$ form a local basis of one-forms for Λ, then the one-forms $\{(\beta^a)^c,\pi^*_{TM}\beta^b\mid\ a,b=m+1,\dots,n\}$ form a local basis for Λ^T.

3. Note that the subbundle $\pi^*_{TM}\Lambda$ of T^*TM is contained in Λ^T.

4. As we did with distributions, we can lift a codistribution Λ to the kth tangent bundle T^kM. Let us denote the codistribution so lifted as Λ^{T^k}. •

The following result further characterises Λ^T.

3.3 Proposition: *Let Λ be a codistribution on M and let $c\colon [a,b] \to M$ be a curve on TM for which $c'(t) \lrcorner \Lambda_{c(t)} = 0$ for $t \in [a,b]$. Then $c''(t) \lrcorner \Lambda^T_{c'(t)} = 0$ for $t \in [a,b]$.*

Proof: We use the notation of Proposition 3.1. If $\boldsymbol{c}$ is the local representative of c then we must have

$$(Q_2(\boldsymbol{c}(t)) \cdot \nu^2) \cdot (\boldsymbol{D}\boldsymbol{c}(t) \cdot 1) = 0$$

for every $\nu^2 \in V_2^*$ since Λ must annihilate c'. Differentiating this with respect to t gives

$$\boldsymbol{D}Q_2(\boldsymbol{c}(t)) \cdot (\boldsymbol{D}\boldsymbol{c}(t) \cdot 1, \nu^2, \boldsymbol{D}\boldsymbol{c}(t) \cdot 1) + (Q_2(\boldsymbol{c}(t)) \cdot \nu^2) \cdot \boldsymbol{D}^2\boldsymbol{c}(t) \cdot (1,1) = 0$$

for every $\nu^2 \in V_2^*$. By (3.5), and since the local representative of c' is $t \mapsto (\boldsymbol{c}(t), \boldsymbol{D}\boldsymbol{c}(t) \cdot 1)$ this means that c'' is annihilated by Λ^T as was desired. ■

Remark 3.2–2 and Proposition 3.3 show that for the distribution $\operatorname{coann}(\Lambda^T)$ the properties L1 and L2 of the introduction hold. Now let us show that L3 holds as well. It is convenient to use an alternate, equivalent, definition of an Ehresmann connection from that used in Section 2. Note that $H^*M = \operatorname{ann}(VM)$ is a naturally defined subbundle of T^*M. Thus an Ehresmann connection may be regarded as a complement V^*M to H^*M in TM.

3.4 Proposition: *If V^*M is a connection on $\pi\colon M \to B$ then the codistribution $(V^*M)^T$ is a connection on $T\pi\colon TM \to TB$.*

Proof: We must show that $T^*TM = H^*TM \oplus (V^*M)^T$. We shall take advantage of the coordinate notation introduced in the proof of Proposition 2.5. Thus we let (U, ϕ) be an adapted chart for $\pi\colon M \to B$ taking values in $E_1 \times E_2$, and we recall that the horizontal subspace at $(u_1, u_2) \in \boldsymbol{U}$ is given by

$$H_{(u_1,u_2)}\boldsymbol{U} = \{((u_1,u_2),(e_1, C(u_1,u_2) \cdot e_1)) \mid e_1 \in E_1\}$$

for some $C\colon \boldsymbol{U} \to \mathrm{L}(E_1; E_2)$. One easily sees that the annihilating codistribution V^*M has the local representative

$$V^*_{(u_1,u_2)}\boldsymbol{U} = \big\{((u_1,u_2),(-C^*(u_1,u_2) \cdot \alpha^2, \alpha^2)) \mid \alpha^2 \in E_2^*\big\}. \tag{3.6}$$

In the natural chart $(TU, T\phi)$ for TM the local representation (3.5) of $(V^*M)^T$ gives

$$\begin{aligned}(V^*\boldsymbol{U})^T_{(u_1,u_2,e_1,e_2)} = \big\{((u_1,u_2,e_1,e_2),(-\boldsymbol{D}_1C^*(u_1,u_2) \cdot (\cdot, \alpha^{22}, e_1) - C^*(u_1,u_2) \cdot \alpha^{21},\\ \alpha^{21}, -\boldsymbol{D}_2C^*(u_1,u_2) \cdot (\cdot, \alpha^{22}, e_1), -C^*(u_1,u_2) \cdot \alpha^{22}, \alpha^{22})) \mid \alpha^{21}, \alpha^{22} \in E_2^*\big\}.\end{aligned} \tag{3.7}$$

We also have

$$H^*_{(u_1,u_2,e_1,e_2)}\boldsymbol{TU} = \{((u_1,u_2,e_1,e_2),(\alpha^{11}, 0, \alpha^{12}, 0)) \mid \alpha^{11}, \alpha^{12} \in E_1^*\}.$$

We then easily see that $H^*TM \cap (V^*M)^T = \{0\}$. We may also write

$$\begin{aligned}(\alpha^{11}, \alpha^{21}, \alpha^{12}, \alpha^{22}) = \big(&-\boldsymbol{D}_1C^*(u_1,u_2) \cdot (\cdot, \alpha^{22}, e_1) - C^*(u_1,u_2) \cdot \alpha^{21} -\\ &\boldsymbol{D}_2C^*(u_1,u_2) \cdot (C(u_1,u_2) \cdot (\cdot), \alpha^{22}, e_1), \alpha^{21}, -C^*(u_1,u_2) \cdot \alpha^{22}, \alpha^{22}\big) +\\ &\big(\alpha^{11} + \boldsymbol{D}_1C^*(u_1,u_2) \cdot (\cdot, \alpha^{22}, e_1) + C^*(u_1,u_2) \cdot \alpha^{21} +\\ &\boldsymbol{D}_2C^*(u_1,u_2) \cdot (C(u_1,u_2) \cdot (\cdot), \alpha^{22}, e_1), 0, \alpha^{12} + C^*(u_1,u_2) \cdot \alpha^{22}, 0\big)\end{aligned}$$

where the first term is in $(V^*\boldsymbol{U})^T$ and the second is in $H^*\boldsymbol{TU}$, which proves the result. ■

Let us examine how this looks in finite-dimensions. We let (x^a, y^α), $a = 1, \dots, m$, $\alpha = 1, \dots, n-m$, be coordinates in an adapted chart, with $((x^a, y^\alpha), (u^b, v^\beta))$ the coordinates for the natural chart for TM. Then we may choose one-forms of the form

$$\mathrm{d}y^\alpha - C^\alpha_a(x, y)\mathrm{d}x^a, \quad \alpha = 1, \dots, n-m$$

as local generators for V^*M. The corresponding local generators for $(V^*M)^T$ are then

$$\begin{aligned} &\mathrm{d}y^\alpha - C^\alpha_a(x, y)\mathrm{d}x^a, && \alpha = 1, \dots, n-m \\ &\mathrm{d}v^\beta - C^\beta_a(x, y)\mathrm{d}u^a - \frac{\partial C^\beta_a}{\partial x^b}u^a\mathrm{d}x^b - \frac{\partial C^\beta_a}{\partial y^\gamma}u^a\mathrm{d}y^\gamma, && \beta = 1, \dots, n-m. \end{aligned}$$

Note that, as with the construction of Section 2, this differs from the formulas given in [Kolář, Michor, and Slovák 1993, §46.7] for an induced connection on TM.

4. When the codistribution and distribution annihilate one another

In Sections 2 and 3 we gave two methods for lifting a distribution on M to one on TM, in the latter case the definition being made via the annihilating codistribution. It might be natural to expect that the two constructions are the same in that $(\mathrm{ann}(D))^T = \mathrm{ann}(D^T)$. Interestingly, however, this is generally not the case. In this section we let D be a distribution and denote by Λ its annihilating codistribution.

First let us just look at the case of a single one-form and vector field which vanish when paired. When performing the exterior derivative in infinite-dimensions we use the characterisation in terms of the Lie derivative given by Palais [1954].

4.1 Proposition: *If $\beta \in \Gamma^\infty(T^*M)$ and $X \in \Gamma^\infty(TM)$ have the property that $\beta \cdot X = 0$, then $\beta^c \cdot X^c(v_x) = \boldsymbol{d}\beta(X(x), v_x)$ for all $v_x \in T_xM$ and $x \in M$.*

Proof: We work locally with a chart (U, ϕ) for M with corresponding natural charts for TM, T^*M, TTM, and T^*TM. We suppose that β and X have local forms

$$\beta(u) = (u, \boldsymbol{\beta}(u)), \quad X(u) = (u, \boldsymbol{X}(u)).$$

We then have

$$\begin{aligned} \beta^c(u, e) &= ((u, e), (\boldsymbol{D\beta}(u) \cdot (\cdot, e), \boldsymbol{\beta}(u))) \\ X^c(u, e) &= ((u, e), (\boldsymbol{X}(u), \boldsymbol{DX}(u) \cdot e)). \end{aligned}$$

Therefore,

$$\beta^c \cdot X^c(u, e) = \boldsymbol{D\beta}(u) \cdot (\boldsymbol{X}(u), e) + \boldsymbol{\beta} \cdot (\boldsymbol{DX}(u) \cdot e). \tag{4.1}$$

If we differentiate the expression $\boldsymbol{\beta}(u) \cdot \boldsymbol{X}(u) = 0$ in the direction of $e \in E$ we get

$$\boldsymbol{D\beta}(u) \cdot (e, \boldsymbol{X}(u)) + \boldsymbol{\beta}(u) \cdot (\boldsymbol{DX}(u) \cdot e) = 0$$

which, upon substitution into (4.1), gives

$$\beta^c \cdot X^c(u, e) = \boldsymbol{D\beta}(u) \cdot (\boldsymbol{X}(u), e) - \boldsymbol{D\beta}(u) \cdot (e, \boldsymbol{X}(u)).$$

The right-hand side is exactly the local representation of $\boldsymbol{d}\beta(X(x), v_x)$. ■

With this formula, we may easily compare Λ^T with D^T. While we are at it, let us also relate integrability of D to that of D^T.

4.2 Proposition: *Let D be a distribution on M with Λ its annihilating codistribution. The following are equivalent:*

(i) $\Lambda^T = \mathrm{ann}(D^T)$*;*

(ii) *D is integrable;*

(iii) *D^T is integrable.*

Proof: (i) $\Longleftrightarrow$ (ii) Assume (i). Define

$$\Lambda^2 = \{\Omega \in \mathrm{T}\bigwedge^2(T_xM) \mid \Omega(v_1, v_2) = 0 \text{ for all } v_1, v_2 \in D_x \text{ and } x \in M\}$$

and recall that D is integrable if and only if $\boldsymbol{d}\beta \in \Gamma^\infty(\Lambda^2)$ for all $\beta \in \Gamma^\infty(\Lambda)$ [Abraham, Marsden, and Ratiu 1988, page 439]. If $\Lambda^T = \mathrm{ann}(D^T)$ then $\beta^c \cdot X^c = 0$ for every $\beta \in \Gamma^\infty(\Lambda)$ and $X \in \Gamma^\infty(D)$. By Proposition 4.1 this means that $\boldsymbol{d}\beta(u_x, v_x) = 0$ for every $u_x \in D_x$ and every $v_x \in T_xM$, and for every $x \in M$. In particular, D is integrable.

Now suppose D integrable and let (U, ϕ) be a chart adapted to the foliation associated with D. Thus we can take $\boldsymbol{U} = U_1 \times U_2 \subset E_1 \times E_2$ so that

$$D_{(u_1,u_2)} = \{((u_1, u_2), (e_1, 0)) \mid e_1 \in E_1\}.$$

Thus we also have

$$\Lambda_{(u_1,u_2)} = \{((u_1, u_2), (0, \alpha^2)) \mid \alpha^2 \in E_2^*\}.$$

One then readily ascertains that

$$D^T_{(u_1,u_2,e_1,e_2)} = \{((u_1, u_2, e_1, e_2), (e_{11}, 0, e_{12}, 0)) \mid e_{11}, e_{12} \in E_1\}. \tag{4.2}$$

and

$$\Lambda^T_{(u_1,u_2,e_1,e_2)} = \{((u_2, u_2, e_1, e_2), (0, \alpha^{21}, 0, \alpha^{22})) \mid \alpha^{21}, \alpha^{22} \in E_2^*\}.$$

From this we immediately see that (i) must hold.

(ii) $\Longleftrightarrow$ (iii) Assume D integrable and let (U, ϕ) be a chart adapted to the foliation associated with D as above. From (4.2) it immediately follows that D^T is integrable.

Now suppose that D^T is integrable. We shall employ the notation introduced in Proposition 2.2. Consider local sections of D^T given by

$$\begin{aligned} X(u, e) &= ((u, e), (B_1(u) \cdot f_1, \boldsymbol{D}B_1(u) \cdot (e, f_1) + B_1(u) \cdot f_2)) \\ Y(u, e) &= ((u, e), (B_1(u) \cdot f_3, \boldsymbol{D}B_1(u) \cdot (e, f_3) + B_1(u) \cdot f_4)) \end{aligned}$$

for $f_1, f_2, f_3, f_4 \in F_1$. For the computations we perform here, it is sufficient to take these elements of F_1 to be constant. A computation then gives

$$\begin{aligned} [X, Y](u, 0) = ((u, 0), (&\boldsymbol{D}B_1(u) \cdot (B_1(u) \cdot f_3, f_1) - \boldsymbol{D}B_1(u) \cdot (B_1(u) \cdot f_1, f_3), \\ &\boldsymbol{D}B_1(u) \cdot (B_1(u) \cdot f_3, f_2) - \boldsymbol{D}B_1(u) \cdot (B_1(u) \cdot f_2, f_3) + \\ &\boldsymbol{D}B_1(u) \cdot (B_1(u) \cdot f_4, f_1) - \boldsymbol{D}B_1(u) \cdot (B_1(u) \cdot f_1, f_4). \end{aligned}$$

Note that

$$D^T_{(u,0)} = \{((u, 0), (B_1(u) \cdot f_5, B_1(u) \cdot f_6)) \mid f_5, f_6 \in F_1\}.$$

Therefore, since D^T is integrable we must therefore have

$$\boldsymbol{D}B_1(u) \cdot (B_1(u) \cdot f_3, f_1) - \boldsymbol{D}B_1(u) \cdot (B_1(u) \cdot f_1, f_3) = B_1(u) \cdot f_5$$

for some $f_5 \in F_1$. However, this is exactly the condition that Lie brackets of sections of D should remain sections of D—that is to say, D is integrable. ■

5. When M is the total space of a vector bundle

In this section we specialise to the interesting case when M is the total space of a vector bundle, and we denote this vector bundle as $\pi\colon E \to B$. An especially interesting, and well-studied, case of this occurs when the linear connection is derived from an affine connection. We treat this in Section 5.4.

5.1. Linear connections. Suppose that we have a connection HE on π, and let $\mathrm{ver}\colon TE \to VE$ be the projection onto the vertical subbundle. If ver is a vector bundle mapping with respect to the following diagram,

$$\begin{tikzcd} TE \arrow[rr, "\mathrm{ver}"] \arrow[dr, "T\pi"'] & & VE \arrow[dl, "T\pi|VE"] \\ & TB & \end{tikzcd}$$

then we say that HE is a ***linear connection***. The following local characterisation of a linear connection will also be useful. We let (U, ϕ) be a vector bundle chart for E taking values in the Banach space $E_1 \times E_2$. We regard E_2 as the local model for the fibres so that $\phi(U) = \boldsymbol{U} \times E_2$ for some open subset $\boldsymbol{U}$ of E_1. As in the proof of Proposition 2.5, an Ehresmann connection is defined locally by

$$H_{(u,e)}\boldsymbol{U} \times E_2 = \{((u,e),(e_1, C(u,e)\cdot e_1)) \mid e_1 \in E_1\},$$

for some map $C\colon \boldsymbol{U} \times E_2 \to \mathrm{L}(E_1; E_2)$. One may verify that the Ehresmann connection is linear exactly when C has the form $C(u,e) = \Gamma(u)\cdot e$ for a map $\Gamma\colon \boldsymbol{U} \to \mathrm{L}(E_2; \mathrm{L}(E_1; E_2))$. We call Γ the ***Christoffel map*** for the linear connection.

We recall that if one has a linear connection HE on a vector bundle $\pi\colon E \to B$ then this defines, for each $X \in \Gamma^\infty(TB)$, a derivation ∇_X on the sections of E. Thus, for example, given a section $\sigma\colon B \to E$, one can define a section $\nabla_X\sigma$. In local coordinates we have

$$\nabla_{\boldsymbol{X}}\boldsymbol{\sigma}(u) = (u, \boldsymbol{D\sigma}(u)\cdot \boldsymbol{X}(u) + (\Gamma(u)\cdot \boldsymbol{X}(u))\cdot \boldsymbol{\sigma}(u)).$$

In the case when $E = TM$, $B = M$, and $\pi = \pi_{TM}$, this gives the usual definition of an affine connection on M.

As a final trivial note, we observe that if $\pi\colon E \to B$ is a vector bundle then so too it $T\pi\colon TE \to TB$. If (u,e) are adapted coordinates for E with $((u,e),(e_1,e_2))$ natural coordinates for TM, then the coordinates $((u,e_1),(e,e_2))$ are adapted for the vector bundle $T\pi\colon TE \to TB$.

5.2. Lifting linear connections using distributions. The result in this section is straightforward and hopefully not unexpected.

5.1 Proposition: *If HE is a linear connection on a vector bundle $\pi\colon E \to B$ then $(HE)^T$ is a linear connection on the vector bundle $T\pi\colon TE \to TB$.*

Proof: We adopt the notation of Section 5.1 for the local representation of the linear connection HE. Thus the local model for HE is given by

$$H_{(u,e)}(\boldsymbol{U} \times E_2) = \{((u,e),(e_1,(\Gamma(u)\cdot e)\cdot e_1)) \mid e_1 \in E_1\}. \tag{5.1}$$

If we refer to the proof of Proposition 2.5, particularly to (2.7), we see that $(HE)^T$ has the local model

$$\begin{aligned}(H(\boldsymbol{U}\times E_2))^T_{(u,e,e_1,e_2)} = \big\{&((u,e,e_1,e_2),(e_{11},(\Gamma(u)\cdot e)\cdot e_{11},e_{12},\\ &(\boldsymbol{D}\Gamma(u)\cdot(e_1,e))\cdot e_{11}+(\Gamma(u)\cdot e_2)\cdot e_{11}+(\Gamma(u)\cdot e)\cdot e_{12}))\mid\ e_{11},e_{12}\in E_1\big\}.\end{aligned}\tag{5.2}$$

Note that in this expression the local coordinates (u,e,e_1,e_2) that we use are the natural tangent bundle coordinates for TE associated with the vector bundle coordinates (u,e) for E. These are *not* adapted coordinates for the vector bundle $T\pi\colon TE\to TB$; these would be (u,e_1,e,e_2). If we write the local model for $(HE)^T$ in these coordinates adapted to the vector bundle $T\pi\colon TE\to TB$, then we have

$$\begin{aligned}(H(\boldsymbol{U}\times E_2))^T_{(u,e_1,e,e_2)} = \big\{&((u,e_1,e,e_2),(e_{11},e_{12},(\Gamma(u)\cdot e)\cdot e_{11},\\ &(\boldsymbol{D}\Gamma(u)\cdot(e_1,e))\cdot e_{11}+(\Gamma(u)\cdot e_2)\cdot e_{11}+(\Gamma(u)\cdot e)\cdot e_{12}))\mid\ e_{11},e_{12}\in E_1\big\}.\end{aligned}$$

Now we simply observe that this does indeed have the form of a linear connection in adapted coordinates as per (5.1). ■

5.2 Remarks: 1. In coordinates in finite-dimensions, suppose that we denote vector bundle coordinates for E by (x^i,u^a), and that the coordinates for TE adapted to the vector bundle $T\pi\colon TE\to TB$ are (x^i,v^j,u^a,w^b). The ***Christoffel symbols*** for a linear connection are defined in these coordinates by

$$\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial u^a}=\Gamma^b_{ia}\frac{\partial}{\partial u^b},\quad i=1,\dots,n,\ a=1,\dots,m.$$

In this case, if

$$\frac{\partial}{\partial x^i}-\Gamma^a_{ib}u^b\frac{\partial}{\partial u^a},\quad i=1,\dots,n,$$

form a basis for HE, then

$$\begin{aligned}&\frac{\partial}{\partial v^i}-\Gamma^a_{ib}u^b\frac{\partial}{\partial w^a}, && i=1,\dots,n\\ &\frac{\partial}{\partial x^j}-\Gamma^a_{jb}u^b\frac{\partial}{\partial u^a}-\Big(\frac{\partial\Gamma^a_{jb}}{\partial v^k}u^b+\Gamma^a_{jb}w^b\Big)\frac{\partial}{\partial w^a}, && j=1,\dots,n,\end{aligned}$$

form a basis for $(HE)^T$.

2. Note that the lifted linear connection, as a distribution on TE, must be thought of as a connection with the proper base space, which is TB and not E. In the tangent bundle situation, $TB=E=TM$, so one expects to be able to establish a relationship between the two types of connections. We do this in Section 5.4. •

5.3. Lifting linear connections using codistributions. We may now mirror Proposition 5.1, except we now use codistributions to do the lifting. In this setting, of course, it is convenient to use the dual notion of a linear connection. Thus we say that a subbundle V^*E of T^*E is a ***linear connection*** if $\operatorname{coann}(V^*E)$ is a linear connection in the sense of Section 5.1.

5.3 Proposition: *If V^*E is a linear connection on a vector bundle $\pi\colon E \to B$ then $(V^*E)^T$ is a linear connection on the vector bundle $T\pi\colon TE \to TB$.*

Proof: We adopt the notation of Proposition 5.1. Referring to the proof of Proposition 3.4, particularly to (3.6), we see that the local model for V^*E has the form

$$V^*_{(u,e)}(\boldsymbol{U} \times E_2) = \{((u,e), (-(\Gamma^*(u)\cdot e)\cdot \alpha^2, \alpha^2)) \mid \alpha^2 \in E_2^*\}, \tag{5.3}$$

where $\Gamma^*\colon \boldsymbol{U} \to \mathrm{L}(E_2; \mathrm{L}(E_2^*; E_1^*))$ is specified by asking that $\Gamma^*(u)\cdot e \in \mathrm{L}(E_2^*; E_1^*)$ be the dual endomorphism of $\Gamma(u)\cdot e \in \mathrm{L}(E_1; E_2)$. We then make an application of (3.7) to ascertain that the local model for $(V^*E)^T$ is

$$\begin{aligned}(V^*(\boldsymbol{U} \times E_2))^T_{(u,e,e_1,e_2)} = \big\{((u,e,e_1,e_2), &-((\boldsymbol{D}\Gamma^*(u)\cdot(\cdot,e))\cdot\alpha^{22})\cdot e_1 - (\Gamma^*(u)\cdot e)\cdot\alpha^{21},\\ &\alpha^{21} - ((\Gamma^*(u)\cdot(\cdot))\cdot\alpha^{22})\cdot e_1 - (\Gamma^*(u)\cdot e)\cdot\alpha^{22}, \alpha^{22}) \mid \ \alpha^{21}, \alpha^{22} \in E_2^*\big\}.\end{aligned}$$

The result now follows in the same way as did Proposition 5.1, after we use coordinates (u, e_1, e, e_2) that are adapted to the vector bundle $T\pi\colon TE \to TB$. Indeed, writing the local model for $(V^*E)^T$ in these coordinates yields

$$\begin{aligned}(V^*(\boldsymbol{U} \times E_2))^T_{(u,e_1,e,e_2)} = \big\{((u,e_1,e,e_2), &-((\boldsymbol{D}\Gamma^*(u)\cdot(\cdot,e))\cdot\alpha^{22})\cdot e_1 - (\Gamma^*(u)\cdot e)\cdot\alpha^{21},\\ &-((\Gamma^*(u)\cdot(\cdot))\cdot\alpha^{22})\cdot e_1 - (\Gamma^*(u)\cdot e)\cdot\alpha^{22}, \alpha^{21}, \alpha^{22}) \mid \ \alpha^{21}, \alpha^{22} \in E_2^*\big\}.\end{aligned}$$

All one need do is observe that this has the form of (5.3). ■

5.4 Remarks: 1. Recall the coordinates used in Remark 5.2–1. If the one-forms

$$\mathrm{d}u^a + \Gamma^a_{ib} u^b \mathrm{d}x^i, \quad a = 1, \dots, m,$$

form a basis for V^*E, then the one-forms

$$\begin{aligned}&\mathrm{d}u^a + \Gamma^a_{ic} u^c \mathrm{d}x^i, && a = 1, \dots, m\\ &\mathrm{d}w^b + \frac{\partial \Gamma^b_{ic}}{\partial x^j} v^i u^c \mathrm{d}x^j + \Gamma^b_{ic} v^i \mathrm{d}u^c + \Gamma^b_{ic} u^c \mathrm{d}v^i, && b = 1, \dots, m,\end{aligned}$$

form a basis for $(V^*E)^T$.

2. Of course, Remark 5.2–2 holds equally well here. •

5.4. When the linear connection comes from an affine connection. It is now a simple matter to consider the results of Sections 5.2 and 5.3 when applied to the case when $E = TM$, $B = M$, and $\pi = \pi_{TM}$. In this case, a linear connection HTM on $\pi_{TM}\colon TM \to M$ is exactly specified by an affine connection ∇ on M. One would then like to lift this affine connection to an affine connection on TM using the two constructions we have provided. Note that a direct application of the two constructions $(HTM)^T$ and $(V^*TM)^T$ will not give what is desired in this respect. These connections are on the bundle $T\pi_{TM}\colon TTM \to TM$, whereas an affine connection on TM will define a linear connection on $\pi_{TTM}\colon TTM \to TM$.

To repair the discrepancy, we use the canonical involution of the double tangent bundle $\Phi_M\colon TTM \to TTM$ that is given in local coordinates by

$$\boldsymbol{\Phi_M}((u,e),(e_1,e_2)) = ((u,e_1),(e,e_2)).$$

Of course, an intrinsic definition of Φ_M exists, but we shall not give it here. The appropriate manner in which to use the involution Φ_M to modify the connections of the preceding two sections turns out to be as follows. For a distribution D on TTM we define a distribution $\Phi_M^* D$ on TTM by declaring that the fibre at $X_{v_x} \in T_{v_x}TQ$ be given by

$$\Phi_M^* D_{X_{v_x}} = \{\Phi_M^* Z(X_{v_x}) \mid \ Z \in \Gamma^\infty(D)\}.$$

Thus $\Phi_M^* D$ is defined by declaring that its sections be pull-backs of sections of D by Φ_M.

Our main result in this section is the following.

5.5 Proposition: *Let HTM be a linear connection on the bundle $\pi_{TM}\colon TM \to M$ with $V^*TM = \operatorname{ann}(HTM)$. The following statements hold:*

(i) $\Phi_M^*((HTM)^T)$ is a linear connection on the bundle $\pi_{TTM}\colon TTM \to TM$;

(ii) $\Phi_M^*(\operatorname{coann}((V^*TM)^T))$ is a linear connection on the bundle $\pi_{TTM}\colon TTM \to TM$.

Furthermore, the linear connections of (i) and (ii) agree if and only if the affine connection on M associated with HTM is flat.

Proof: Part (i) of the proposition will follow from Proposition 5.1 after we provide the local form of a linear connection on $\pi_{TTM}\colon TTM \to TM$. Let us use coordinates $(u,e) \in \boldsymbol{U} \times E$ as natural coordinates for TM, noting then that the natural tangent bundle coordinates $((u,e),(e_1,e_2))$ for TTM are adapted to the bundle $\pi_{TTM}\colon TTM \to TM$. Thus a linear connection on $\pi_{TTM}\colon TTM \to TM$ will have the local form

$$\begin{aligned} H(TTM) = \big\{((u,e,e_1,e_2),(e_{11},e_{12}, \\ (\Gamma_{111}(u,e)\cdot e_1)\cdot e_{11} + (\Gamma_{121}(u,e)\cdot e_2)\cdot e_{11} + (\Gamma_{112}(u,e)\cdot e_1)\cdot e_{12} + \\ (\Gamma_{122}(u,e)\cdot e_2)\cdot e_{12}, (\Gamma_{211}(u,e)\cdot e_1)\cdot e_{11} + (\Gamma_{221}(u,e)\cdot e_2)\cdot e_{11} + \\ (\Gamma_{212}(u,e)\cdot e_1)\cdot e_{12} + (\Gamma_{222}(u,e)\cdot e_2)\cdot e_{12})) \mid \ e_{11}, e_{12} \in E\big\}, \end{aligned}$$

for appropriate functions $\Gamma_{abc}\colon \boldsymbol{U} \times E \to \mathrm{L}(E;\mathrm{L}(E;E))$, $a,b,c = 1,2$. For (i), a reference to (5.2) and the definition of pull-back shows that

$$\begin{aligned} \boldsymbol{\Phi_M^*}\big((HTM)^T_{(u,e,e_1,e_2)}\big) = \big\{((u,e,e_1,e_2),(e_{11},e_{12},(\Gamma(u)\cdot e_1)\cdot e_{11}, \\ (\boldsymbol{D}\Gamma(u)\cdot(e,e_1))\cdot e_{11} + (\Gamma(u)\cdot e_2)\cdot e_{11} + (\Gamma(u)\cdot e_1)\cdot e_{12})) \mid \ e_{11}, e_{12} \in E_1\big\}. \end{aligned}$$

This shows that, indeed, $\Phi_M^*((HTM)^T)$ is a linear connection on $\pi_{TTM}\colon TTM \to TM$, thus proving (i). In fact, the above computations are readily generalised to show that if D is a linear connection on $T\pi_{TM}\colon TTM \to TM$, then $\Phi_M^* D$ is a linear connection on $\pi_{TTM}\colon TTM \to TM$. This directly implies (ii).

The final assertion of the proposition follows directly from Proposition 4.2, recalling that an affine connection has zero curvature if and only if its associated linear connection is integrable [Kobayashi and Nomizu 1963]. ■

Let us provide the finite-dimensional formulae for the Christoffel symbols for the two lifted affine connections. Let us first develop the notation for doing this. We suppose that we have a fixed affine connection ∇ on M defining and defined by a linear connection HTM on $\pi_{TTM}\colon TTM \to TM$. We shall denote the Christoffel symbols for ∇ by γ^i_{jk}, $i,j,k = 1,\dots,n$. The Christoffel symbols for the affine connection ∇^T on TM defined by the linear connection $\Phi_M^*((HTM)^T)$ will be denoted $\Gamma^\alpha_{\beta\sigma}$, $\alpha,\beta,\sigma = 1,\dots,2n$. The Christoffel symbols for the affine connection $\overline{\nabla}^T$ on TM defined by the linear connection $\Phi_M^*(\mathrm{coann}((V^*TM)^T))$ will be denoted $\overline{\Gamma}^\alpha_{\beta\sigma}$, $\alpha,\beta,\sigma = 1,\dots,2n$. One may then verify that for $i,j,k = 1,\dots,n$ we have

$$\begin{aligned}
&\Gamma^i_{jk} = \gamma^i_{jk}, && \Gamma^{n+i}_{jk} = \frac{\partial \gamma^i_{jk}}{\partial x^\ell} v^\ell,\\
&\Gamma^i_{j,k+n} = 0, && \Gamma^{n+i}_{j,k+n} = \gamma^i_{jk},\\
&\Gamma^i_{j+n,k} = 0, && \Gamma^{n+i}_{j+n,k} = \gamma^i_{jk},\\
&\Gamma^i_{j+n,k+n} = 0, && \Gamma^{n+i}_{j+n,k+n} = 0,
\end{aligned}$$

and

$$\begin{aligned}
&\overline{\Gamma}^i_{jk} = \gamma^i_{jk}, && \overline{\Gamma}^{n+i}_{jk} = \frac{\partial \gamma^i_{\ell k}}{\partial x^j} v^\ell - \gamma^i_{\ell m}\gamma^m_{jk} v^\ell,\\
&\overline{\Gamma}^i_{j,k+n} = 0, && \overline{\Gamma}^{n+i}_{j,k+n} = 0,\\
&\overline{\Gamma}^i_{j+n,k} = 0, && \overline{\Gamma}^{n+i}_{j+n,k} = \gamma^i_{jk},\\
&\overline{\Gamma}^i_{j+n,k+n} = 0, && \overline{\Gamma}^{n+i}_{j+n,k+n} = 0.
\end{aligned}$$

5.6 Remarks: 1. If Z denotes the geodesic spray for ∇ and if Z^T denotes the geodesic spray for ∇^T, then one readily verifies that $Z^T = \Phi_M^* Z^c$.

2. We note that the affine connection ∇^T is the same as the "complete lift" of ∇ as defined in [Yano and Ishihara 1973]. As such, it is the unique affine connection on TM satisfying

$$(\nabla_X Y)^c = \nabla^T_{X^c} Y^c$$

for all vector fields X and Y on M. The author knows of no occurrence of $\overline{\nabla}^T$ in the existing literature, but it is entirely possible that one exists. •

6. When M is the total space of a bundle over $\mathbb{R}$

We now let M be the total space of a locally trivial fibre bundle $\pi\colon M \to \mathbb{R}$. We are not here interested in connections on this bundle as in Propositions 2.5 and 3.4. Rather we are interested in looking at when a distribution is "compatible" with the process of taking local sections $c\colon [a,b] \to M$ of π. Local sections are to be regarded as special classes of curves, so we are interested in distributions which admit this class of curves as integral curves. Also associated with sections of π are jets of sections. When a distribution is compatible with the process of taking sections, it also interacts nicely with the jet bundles.

Now let us make this discussion precise.

6.1. Jet bundle notation. We denote by J^kM the bundle of k-jets of sections of π (we refer to [Saunders 1989] for jet bundles). For $k > l$ we let $\tau_{k,l}\colon J^kM \to J^lM$ be the projection which forgets the higher order equivalence. We take $J^0M = M$ and denote $\tau_k = \tau_{k,0}$. Denote by $VM \subset TM$ the kernel of the projection $T\pi$ and denote by $\nu_M\colon VM \to M$ the projection. Recall that the fibre bundle $\tau_{k,k-1}\colon J^kM \to J^{k-1}M$ is an affine bundle modelled on the pull-back vector bundle $\tau_{k-1}^*\nu_M\colon \tau_{k-1}^*VM \to J^{k-1}M$. If $\xi \in J^{k-1}M$, we denote by $J_\xi^kM = \tau_{k,k-1}^{-1}(\xi)$. The bundle $\pi\colon M \to \mathbb{R}$ admits natural adapted charts since the base space $\mathbb{R}$ has a natural set of coordinates. Let us always assume we are using such an adapted chart, and we denote by (t,u) the coordinates in this chart where $u \in U'$ and U' is an open subset of a Banach space E. We denote natural coordinates for J^kM by $(t,u,u_1,\dots,u_k)$. Recall that J^kM is naturally a submanifold of $T(J^{k-1}M)$. For $k = 1,2$ these inclusions are given in natural coordinates by

$$\begin{aligned}(t,u,u_1) &\mapsto ((t,u),(1,u_1))\\ (t,u,u_2,u_2) &\mapsto ((t,u,u_1),(1,u_1,u_2)).\end{aligned}$$

Let us denote the inclusion of J^kM in $T(J^{k-1}M)$ by ι_k. Note also that $J^2M \subset TTM$. In natural coordinates this inclusion is given by

$$(t,u,u_1,u_2) \mapsto ((t,u,1,u_1),(1,u_1,0,u_2)).$$

In this section we shall make these identifications of jet bundles as subsets of tangent bundles without warning.

6.2. Restricting lifted distributions to J^1M. Now let us suppose D to be a distribution on M. We wish to ascertain under what conditions the lifted distribution D^T restricts to a distribution on $J^1M \subset TM$. We further wish to give some properties of the restriction of the lifted distribution in this case. Let us denote $D_1 = J^1M \cap D$ and $D_2 = J^2M \cap D^T$. The following result tells the tale.

6.1 Proposition: *If D is a distribution on M then the following are equivalent:*

(i) $\boldsymbol{d}t \notin \operatorname{ann}(D_x)$ *for every* $x \in M$*;*

(ii) $D_x \cap J_x^1M \neq \varnothing$ *for every* $x \in M$*;*

(iii) for every $x \in M$ *there exists a local section* $c\colon [a,b] \to M$ *so that* $c(t) = x$ *for some* $t \in [a,b]$ *and so that* $c'(t) \in D_{c(t)}$ *for every* $t \in [a,b]$*.*

Furthermore, any of the above statements implies each of the following:

(iv) $(D^T|J^1M) \cap T(J^1M)$ *is a subbundle of* $D^T|J^1M$ *of codimension* 1*.*

(v) D_1 *is an affine subbundle of* J^1M *modelled on the subbundle* $D \cap VM$ *of* VM*;*

(vi) $D_2|D_1$ *is an affine subbundle of* $J^2M|D_1$ *modelled on the pull-back of* $D \cap VM$ *to* D_1 *by* $\tau_1|D_1$*.*

Proof: Let us first get our local representations down. We let (U,ϕ) be an adapted chart for $\pi\colon M \to \mathbb{R}$ and we use the induced charts for TM and J^1M. Let us also adapt the local notation for D introduced in the proof of Proposition 2.2. Thus we write

$$D_{(t,u)} = \{((t,u),(B_{01}(t,u)\cdot f_1, B_{11}(t,u)\cdot f_1)) \mid f_1 \in F_1\}$$

where F_1 is a Banach space and the map $f_1 \mapsto (B_{01}(t,u)\cdot f_1, B_{11}(t,u)\cdot f_1) \in \mathbb{R}\times E$ is injective with split image. Here $B_{01}\colon \boldsymbol{U}\to \mathrm{L}(F_1;\mathbb{R})$ and $B_{11}\colon \boldsymbol{U}\to \mathrm{L}(F_1;E)$.

(i) $\Longleftrightarrow$ (ii) We have

$$D_{1,(t,u)} = \{((t,u),(1,B_{11}(t,u)\cdot f_1)) \mid B_{01}(t,u)\cdot f_1 = 1\}. \tag{6.1}$$

This set is non-empty if and only if $B_{01}(t,u)\neq 0$. However, (i) is clearly also equivalent to having $B_{01}(t,u)\neq 0$.

(ii) $\Longleftrightarrow$ (iii) Suppose (ii) holds. Let $f_1\in F_1$ be defined by

$$(B_{01}(t_0,u_0)\cdot f_1, B_{11}(t_0,u_0)\cdot f_1) = (1,e)$$

for some $(t_0,u_0)\in \boldsymbol{U}$ and some $e\in \mathrm{image}(B_{11}(t_0,u_0))$. Let $t\mapsto (s(t),\boldsymbol{c}(t))\in \mathbb{R}\times U'$ be the solution of the differential equation

$$\begin{aligned}\dot{s}(t) &= 1\\ \dot{\boldsymbol{c}}(t) &= B_{11}(t,\boldsymbol{c}(t))\cdot f_1\end{aligned}$$

with initial condition $(s(t_0),\boldsymbol{c}(t_0)) = (t_0,u_0)$. Clearly the tangent vector field to the local section $t\mapsto (t,\boldsymbol{c}(t))$ lies in D. In other words, we have shown that (iii) holds. Now suppose (iii) holds and let c be a local section whose tangent vector field lies in D. If we locally write c as $t\mapsto (t,\boldsymbol{c}(t))$ then c' has local representative $t\mapsto ((t,\boldsymbol{c}(t)),(1,\boldsymbol{D}\boldsymbol{c}(t)\cdot 1))$ which means that $(1,\boldsymbol{D}\boldsymbol{c}(t)\cdot 1)\in D_{(t,\boldsymbol{c}(t))}$. This implies that $D_x\cap J^1_xM$ is non-empty for each $x\in M$ since such a section can be constructed so as to pass through any point $x\in M$.

Now we move on to the second part of the proposition. We first need some notation. We have the local representation of $T\iota_1$ as

$$T\iota_1((t,u,u_1),(\tau,e_1,e_2)) = ((t,u,1,u_1),(\tau,e_1,0,e_2)). \tag{6.2}$$

Using (2.5) we easily compute

$$\begin{aligned}D^T_{(t,u,\tau,e)} = \big\{((t,u,\tau,e),(&B_{01}(t,u)\cdot f_{11}, B_{11}(t,u)\cdot f_{11},\\ &\boldsymbol{D}_1B_{01}(t,u)\cdot(\tau,f_{11}) + \boldsymbol{D}_2B_{01}(t,u)\cdot(e,f_{11}) + B_{01}(t,u)\cdot f_{12},\\ &\boldsymbol{D}_1B_{11}(t,u)\cdot(\tau,f_{11}) + \boldsymbol{D}_2B_{11}(t,u)\cdot(e,f_{11}) + B_{11}(t,u)\cdot f_{12})) \mid \ f_{11},f_{12}\in F_1\big\}.\end{aligned} \tag{6.3}$$

To restrict D^T to J^1M set $\tau = 1$.

(i) $\Longrightarrow$ (iv) We first claim that (i) implies that the subspace

$$\{(f_{11},f_{12})\in F_1\times F_1 \mid \boldsymbol{D}_1B_{01}(t,u)\cdot(1,f_{11}) + \boldsymbol{D}_2B_{01}(t,u)\cdot(e,f_{11}) + B_{01}(t,u)\cdot f_{12} = 0\} \tag{6.4}$$

has codimension 1 in $F_1\times F_1$ for each $(t,u,e)\in \boldsymbol{U}\times E$. Indeed, since (i) is equivalent to $B_{01}(t,u)$ being non-zero for each $(t,u)\in \boldsymbol{U}$, this claim follows. Now since the map

$$(f_1,f_2)\mapsto \boldsymbol{D}_1B_{01}(t,u)\cdot(1,f_{11}) + \boldsymbol{D}_2B_{01}(t,u)\cdot(e,f_{11}) + B_{01}(t,u)\cdot f_{12}$$

is continuous and $\mathbb{R}$-valued, it has closed kernel. This shows that the fibres of $(D^T|J^1M)\cap T(J^1M)$ are closed and of finite-codimension. This implies that these fibres split in the fibres of $D^T|J^1M$ and are of constant finite-codimension. This in turn implies (iv).

(i) $\Longrightarrow$ (v) By the local form (6.1) of D_1 we must show that the set

$$\{(1,u_1)\in\mathbb{R}\times E\mid\ u_1=B_{11}(t,u)\cdot f_1 \text{ where } B_{01}(t,u)\cdot f_1=1\}$$

is an affine space modelled on the subspace

$$\{e\in E\mid\ e=B_{11}(t,u)\cdot f_1 \text{ where } B_{10}(t,u)\cdot f_1=0\}$$

with the affine action given by $(e,(1,u_1))\mapsto(1,u_1+e)$. This is a simple matter of verifying the axioms of an affine space (see [Berger 1987]) using the fact that the map $f_1\mapsto(B_{01}(t,u)\cdot f_1,B_{11}(t,u)\cdot f_1)$ is injective.

(i) $\Longrightarrow$ (vi) Let $(t,u,u_1)\in\boldsymbol{U}\times E$ be the image of a point $v\in D_1$. We have

$$\begin{aligned}D_{2,(t,u,u_1)}=\big\{((t,u,1,u_1),(1,u_1,0,&\\ \boldsymbol{D}_1B_{11}(t,u)\cdot(1,f_1)+\boldsymbol{D}_2B_{11}(t,u)\cdot(u_1,f_1)+B_{11}(t,u)\cdot f_2))\mid&\\ B_{01}(t,u)\cdot f_1=1,\ \ B_{11}(t,u)\cdot f_1=u_1,&\\ \boldsymbol{D}_1B_{01}(t,u)\cdot(1,f_1)+\boldsymbol{D}_2B_{01}(t,u)\cdot(u_1,f_1)+B_{01}(t,u)\cdot f_2=0\big\}.&\end{aligned}$$

The relations

$$B_{01}(t,u)\cdot f_1=1,\ \ B_{11}(t,u)\cdot f_1=u_1$$

are satisfied since $(t,u,u_1)\in D_{1,(t,u)}$. Note that these relations uniquely determine $f_1\in F_1$. We shall write $f_1(t,u,u_1)$ to denote this element of F_1 for $(t,u,u_1)\in\boldsymbol{U}\times E$. The relation

$$\boldsymbol{D}_1B_{01}(t,u)\cdot(1,f_1)+\boldsymbol{D}_2B_{01}(t,u)\cdot(u_1,f_1)+B_{01}(t,u)\cdot f_2=0$$

is satisfied since we are restricting to $J^2M\subset T(J^1M)$. Our assertion will be proved if we can show that, for each $(t,u,u_1)\in\boldsymbol{U}\times E$, the set

$$\begin{aligned}\big\{(1,u_1,0,u_2)\in\mathbb{R}\times E\times\mathbb{R}\times E\mid\ \ u_2=\boldsymbol{D}_1B_{11}(t,u)\cdot(1,f_1(t,u,u_1))+&\\ \boldsymbol{D}_2B_{11}(t,u)\cdot(u_1,f_1(t,u,u_1))+B_{11}(t,u)\cdot f_2) \text{ where}&\\ \boldsymbol{D}_1B_{01}(t,u)\cdot(1,f_1(t,u,u_1))+\boldsymbol{D}_2B_{01}(t,u)\cdot(u_1,f_1(t,u,u_1))+B_{01}(t,u)\cdot f_2=0\big\}&\end{aligned}$$

is an affine space modelled on the subspace

$$\{e\in E\mid\ \ e=B_{11}(t,u)\cdot f_1 \text{ where } B_{01}(t,u)\cdot f_1=0\}$$

with the affine action given by $(e,(1,u_1,0,u_2))\mapsto(1,u_1,0,u_2+e)$. This verification of the axioms for an affine space follows from the fact that the map $f_1\mapsto(B_{01}(t,u)\cdot f_1,B_{11}(t,u)\cdot f_1)$ is injective. ■

When a distribution satisfies the condition (i), we shall say it is ***compatible with $\boldsymbol{\pi}$***, and we denote the associated distribution $(D^T|J^1M)\cap T(J^1M)$ on J^1M by j^1D. If D is compatible with π then it is not difficult to see that we may inductively define j^kD as a distribution on J^kM by suitably restricting the distribution D^{T^k} on T^kM to the submanifold J^kM.

In finite-dimensions, the situation is as follows. Let us denote by (t, x^i) coordinates in an adapted chart, and by (t, x^i, τ, v^i) the associated natural coordinates for TM. Suppose we have a local basis

$$X_a = X_a^0 \frac{\partial}{\partial t} + X_a^i \frac{\partial}{\partial x^i}, \quad a = 1, \dots, m$$

for D. Then $\boldsymbol{d}t \notin \operatorname{ann}(D)$ means exactly that X_a^0 should be non-zero for at least one $a = 1, \dots, m$. The restricted vector bundle $D^T|J^1M$ is then locally generated by

$$\begin{aligned} X_a^v|J^1M &= X_a^0 \frac{\partial}{\partial \tau} + X_a^i \frac{\partial}{\partial v^i}, && a = 1, \dots, m \\ X_b^c|J^1M &= X_b^0 \frac{\partial}{\partial t} + X_b^i \frac{\partial}{\partial x^i} + \left(\frac{\partial X_b^0}{\partial t} + \frac{\partial X_b^0}{\partial x^j} v^j \right) \frac{\partial}{\partial \tau} + \\ &\quad \left(\frac{\partial X_b^i}{\partial t} + \frac{\partial X_b^i}{\partial x^j} v^j \right) \frac{\partial}{\partial v^i}, && b = 1, \dots, m. \end{aligned} \tag{6.5}$$

A local basis for j^1D consists of linear combinations of these $2m$ vector fields which satisfy the single linear equation (6.4). In finite-dimensions this equation reads

$$X_a^0 A^a + \left(\frac{\partial X_a^0}{\partial t} + \frac{\partial X_a^0}{\partial x^i} v^i \right) B^a = 0$$

which we may solve for A^a, B^a, $a = 1, \dots, m$, these then being coefficients for the vector fields (6.5) ensuring that they lie in j^1D. Our assumption that $\boldsymbol{d}t \notin \operatorname{ann}(D)$ implies that this linear system has rank 1. Without loss of generality we may suppose that $X_1^0(x) \neq 0$ for all x in our coordinate neighbourhood on M. With this supposition one may verify that the $2m - 1$ vector fields

$$\begin{gathered} X_2^0(X_1^v|J^1M) - X_1^0(X_2^v|J^1M) \\ \vdots \\ X_m^0(X_1^v|J^1M) - X_1^0(X_m^v|J^1M) \\ \left(\frac{\partial X_1^0}{\partial t} + \frac{\partial X_1^0}{\partial x^i} v^i \right) (X_1^v|J^1M) - X_1^0(X_1^c|J^1M) \\ \vdots \\ \left(\frac{\partial X_m^0}{\partial t} + \frac{\partial X_m^0}{\partial x^i} v^i \right) (X_1^v|J^1M) - X_1^0(X_m^c|J^1M) \end{gathered}$$

then form a local basis for j^1D.

6.3. Restricting lifted codistributions to J^1M. Now we turn to the case when Λ is a codistribution on the total space M of a locally trivial fibre bundle $\pi\colon M \to \mathbb{R}$. The following result gives the analogue of Proposition 6.1 in this case. We denote $\Lambda_1^\perp = \operatorname{coann}(\Lambda) \cap J^1M$ and $\Lambda_2^\perp = \operatorname{coann}(\Lambda^T) \cap J^2M$.

6.2 Proposition: *Let Λ be a codistribution on M. The following are equivalent:*

(i) $\Lambda_x \wedge \boldsymbol{d}t \neq 0$ *(meaning $\beta_x \wedge \boldsymbol{d}t \neq 0$ for $\beta_x \in \Lambda_x \setminus \{0\}$) for every $x \in M$;*

(ii) $\operatorname{coann}(\Lambda_x) \cap J^1_x M \neq \varnothing$ *for every $x \in M$;*

(iii) for every $x \in M$ there exists a local section $c\colon [a,b] \to M$ so that $c(t) = x$ for some $t \in [a,b]$ and so that $c'(t) \lrcorner \Lambda_{c(t)} = 0$ for every $t \in [a,b]$.

Furthermore, any of the above statements implies each of the following:

(iv) $T^*_v\iota_1|\Lambda^T_v$ is an injection for every $v \in J^1M$;

(v) $\Lambda_1^\perp$ is an affine subbundle of J^1M modelled on the subbundle $\operatorname{coann}(\Lambda)\cap VM$ of VM;

(vi) $\Lambda_2^\perp|\Lambda_1^\perp$ is an affine subbundle of $J^2M|\Lambda_1^\perp$ modelled on the pull-back of $\operatorname{coann}(\Lambda)\cap VM$ to $\Lambda_1^\perp$ by $\tau_1|\Lambda_1^\perp$.

Proof: We use the notation of Proposition 3.1. Thus we let (U,ϕ) be an adapted chart for M with natural chart $(TU, T\phi)$ for TM. We suppose ϕ is $(\mathbb{R}\times E)$-valued. Adapting the notation of Proposition 3.1 to our present situation of a locally trivial fibre bundle, we have

$$\Lambda_{(t,u)} = \{((t,u),(Q_{10}(t,u)\cdot\nu^2, Q_{11}(t,u)\cdot\nu^2)) \mid \nu^2 \in V_2^*\}$$

for some Banach space V_2, and where $Q_{10}\colon \boldsymbol{U} \to \mathrm{L}(V_2^*;\mathbb{R}^*)$ and $Q_{11}\colon \boldsymbol{U} \to \mathrm{L}(V_2^*;E^*)$. Note that since Λ is a subbundle, the map $\nu^2 \mapsto (Q_{10}(t,u)\cdot\nu^2, Q_{11}(t,u)\cdot\nu^2)$ is injective with split image.

The crux of the proof is the following lemma.

1 Lemma: *In our local coordinate representation, (i) is equivalent to $Q_{11}(t,u)$ being injective for each $(t,u) \in \boldsymbol{U}$.*

Proof: In our local coordinate representation $\boldsymbol{d}t = (1,0)$. Using the definition of the wedge product, (i) is equivalent to

$$(Q_{11}(t,u)\cdot\nu^2)\cdot(\tau_2 e_1 - \tau_1 e_2) \neq 0$$

for every $\nu^2 \in V_2^*\setminus\{0\}$, $\tau_1,\tau_2 \in \mathbb{R}$, and $e_1, e_2 \in E$. This is clearly equivalent to $(Q_{11}(t,u)\cdot\nu^2)\cdot e \neq 0$ for every $\nu^2 \in V_2^*\setminus\{0\}$ and $e \in E$. But this is equivalent to the statement that $Q_{11}(t,u)\cdot\nu^2 = 0$ if and only if $\nu^2 = 0$, i.e., that $Q_{11}(t,u)$ be injective. ▼

(i) $\Longleftrightarrow$ (ii) This follows from Proposition 6.1(i) since (i) is equivalent to $\boldsymbol{d}t \notin \Lambda_x$ for every $x \in M$.

(ii) $\Longleftrightarrow$ (iii) This follows in exactly the same manner as did the similar step in the proof of Proposition 6.1.

(i) $\Longrightarrow$ (iv) By (3.5) we have

$$\begin{aligned}\Lambda^T_{(t,u,\tau,e)} = \big\{((t,u,\tau,e),(&\boldsymbol{D}_1Q_{10}(t,u)\cdot(\cdot,\nu^{21},\tau) + \boldsymbol{D}_1Q_{11}(t,u)\cdot(\cdot,\nu^{21},\tau)+\\ &Q_{10}(t,u)\cdot\nu^{22}, \boldsymbol{D}_2Q_{10}(t,u)\cdot(\cdot,\nu^{21},e) + \boldsymbol{D}_2Q_{11}(t,u)\cdot(\cdot,\nu^{21},e) + Q_{11}(t,u)\cdot\nu^{22},\\ &Q_{10}(t,u)\cdot\nu^{21}, Q_{11}(t,u)\cdot\nu^{21})) \mid \nu^{21},\nu^{22} \in V_2^*\big\}.\end{aligned}$$

We also have the local representative of $T^*_v\iota_1$ as

$$T^*_{(t,u,1,u_1)}\iota_1(\lambda^1,\alpha^1,\lambda^2,\alpha^2) = ((t,u,u_1),(\lambda^1,\alpha^1,\alpha^2)).$$

Therefore, $T^*_{(t,u,u_1)}\iota_1|\Lambda^T$ is injective if and only if the map

$$\begin{aligned}(\nu^{21},\nu^{22}) \mapsto (&\boldsymbol{D}_1Q_{10}(t,u)\cdot(\cdot,\nu^{21},1)+\boldsymbol{D}_1Q_{11}(t,u)\cdot(\cdot,\nu^{21},1)+Q_{10}(t,u)\cdot\nu^{22},\\ &\boldsymbol{D}_2Q_{10}(t,u)\cdot(\cdot,\nu^{21},u_1)+\boldsymbol{D}_2Q_{11}(t,u)\cdot(\cdot,\nu^{21},u_1)+Q_{11}(t,u)\cdot\nu^{22},Q_{11}(t,u)\cdot\nu^{21})\end{aligned}$$

is injective. By Lemma 1 and by the fact that the map $\nu^2 \mapsto (Q_{01}(t,u)\cdot\nu^2, Q_{11}(t,u)\cdot\nu^2)$ is injective, the result follows.

(i) $\Longrightarrow$ (v) We have

$$\Lambda_1^\perp = \{((t,u),(1,u_1)) \mid \ Q^*_{10}(t,u)\cdot 1 + Q^*_{11}(t,u)\cdot u_1 = 0\}. \tag{6.6}$$

Our assertion will follow if we can show that for fixed (t,u) the set

$$\{(1,u_1)\in\mathbb{R}\times E \mid \ Q^*_{11}(t,u)\cdot u_1 + Q^*_{10}(t,u)\cdot 1 = 0\}$$

is an affine space modelled on the set

$$\{e\in E \mid \ Q^*_{11}(t,u)\cdot e = 0\}$$

with the affine action given by $(e,(1,u_1))\mapsto(1,u_1+e)$. This is a simple matter of verifying the axioms for an affine space.

(i) $\Longrightarrow$ (vi) Using the definition (6.6) of $\Lambda_1^\perp$, we may see that we must verify that for fixed (t,u,u_1) the set

$$\begin{aligned}\big\{(1,u_1,0,u_2)\in\mathbb{R}\times E\times\mathbb{R}\times E \mid \ & Q^*_{11}(t,u)\cdot u_2 + \boldsymbol{D}_2Q^*_{10}(t,u)\cdot(u_1,1)+\\ &\boldsymbol{D}_2Q^*_{11}(t,u)\cdot(u_1,u_1)+\boldsymbol{D}_1Q^*_{10}(t,u)\cdot(1,1)+\boldsymbol{D}_1Q^*_{11}(t,u)\cdot(1,u_1)=0\big\}\end{aligned}$$

is an affine space modelled on the subspace

$$\{e\in E \mid \ Q^*_{11}(t,u)\cdot e = 0\}$$

with the affine action given by $(e,(1,u_1,0,u_2))\mapsto(1,u_1,0,u_2+e)$. Again, this is a simple matter of verifying the affine space axioms. ■

If a codistribution Λ satisfies (i) we shall say it is ***compatible with $\boldsymbol{\pi}$***. The codistribution on J^1M defined by $v\mapsto T^*_v\iota_1(\Lambda^T_v)$ we will denote as $j^1\Lambda$. As with distributions compatible with π, if Λ is compatible with π then we may lift Λ to a codistribution $j^k\Lambda$ on J^kM by restricting Λ^{T^k} to $J^kM\subset T^kM$.

Let us see what this looks like in finite-dimensions. We write a local basis for Λ as

$$\beta^a = \beta^a_i\mathrm{d}x^i + \beta^a_0\mathrm{d}t, \quad a=1,\dots,m.$$

The condition (i) is equivalent in this case to the matrix with components β^a_i having maximal rank. A local basis for $j^1\Lambda$ is then

$$\begin{aligned}&\beta^a_0\mathrm{d}t+\beta^a_i\mathrm{d}x^i, && a=1,\dots,m\\ &\left(\frac{\partial\beta^b_0}{\partial t}+\frac{\partial\beta^b_i}{\partial t}v^i\right)\mathrm{d}t+\left(\frac{\partial\beta^b_0}{\partial x^j}+\frac{\partial\beta^b_i}{\partial x^j}v^i\right)\mathrm{d}x^j+\beta^b_i\mathrm{d}v^i, && b=1,\dots,m.\end{aligned}$$

These formulas are also produced by de León, Marrero, and Martín de Diego [1997].

6.4. When the codistribution and distribution annihilate one another. Of course, the results of Section 4 still hold when D and $\Lambda = \mathrm{ann}(D)$ are defined on the total space of the fibration $\pi\colon M \to \mathbb{R}$. However, the jet bundle structure comes into play to provide extra and interesting structure, even when $(\mathrm{ann}(D))^T \neq \mathrm{ann}(D^T)$. The gist of the following result is that even though one cannot generally expect $\Lambda^T = \mathrm{ann}(D^T)$, the jet bundle constructions of the previous two sections are unable to distinguish between when one uses distributions or when one uses codistributions. That is to say, Λ^T and $\mathrm{ann}(D^T)$ differ on that part of $T(J^1M)$ which does not intersect J^2M.

6.3 Proposition: *Let D be a distribution on the total space M of a locally trivial fibre bundle $\pi\colon M \to \mathbb{R}$ and let $\Lambda = \mathrm{ann}(D)$ be its annihilating codistribution. The following statements hold:*

(i) D is compatible with π if and only if Λ is compatible with π;

(ii) $D_1 = \Lambda_1^\perp$;

(iii) $D_2|D_1 = \Lambda_2^\perp|\Lambda_1^\perp$.

Proof: (i) This is a consequence of the fact that condition (i) of Proposition 6.1 and condition (i) of Proposition 6.2 are equivalent if $\Lambda = \mathrm{ann}(D)$.

(ii) This is obvious since $\Lambda = \mathrm{ann}(D)$.

(iii) We resume with the notation of the proof of Propositions 6.1 and 6.2. We also use some notation from the proof of Proposition 2.2 which we did not use in Proposition 6.1: we denote by

$$(f_1, f_2) \mapsto (B_{01}(t,u) \cdot f_1 + B_{02}(t,u) \cdot f_2, B_{11}(t,u) \cdot f_1 + B_{12}(t,u) \cdot f_2)$$

the full parameterisation of the tangent space to M at (t,u) in a vector bundle chart adapted to D. Thus $B_{0i}\colon \boldsymbol{U} \to \mathrm{L}(F_i;\mathbb{R})$ and $B_{1i}\colon \boldsymbol{U} \to \mathrm{L}(F_i;E)$ for $i = 1,2$. We denote the inverse of this map by

$$(\tau, e) \mapsto (A_{10}(t,u) \cdot \tau + A_{11}(t,u) \cdot e, A_{20} \cdot \tau + A_{21}(t,u) \cdot e)$$

where $A_{i0}\colon \boldsymbol{U} \to \mathrm{L}(\mathbb{R};F_i)$ and $A_{i1}\colon \boldsymbol{U} \to \mathrm{L}(E;F_i)$, $i = 1,2$. Thus

$$D_{(t,u)} = \{((t,u), (B_{01}(t,u) \cdot f_1, B_{11}(t,u) \cdot f_1)) \mid f_1 \in F_1\}$$

as before. Since we are assuming $\Lambda = \mathrm{ann}(D)$ we may take

$$((t,u), (\lambda, \alpha)) \mapsto ((t,u), (B_{01}^*(t,u) \cdot \lambda + B_{11}^*(t,u) \cdot \alpha, B_{02}^*(t,u) \cdot \lambda + B_{11}^*(t,u) \cdot \alpha))$$

as a vector bundle chart for T^*M adapted to Λ. Thus

$$\Lambda_{(t,u)} = \{((t,u), (A_{20}^*(t,u) \cdot \nu^2, A_{21}^*(t,u) \cdot \nu^2)) \mid \nu^2 \in F_2^*\}.$$

We must then have

$$(A_{20}^*(t,u) \cdot \nu^2) \cdot (B_{01}(t,u) \cdot f_1) + (A_{21}^*(t,u) \cdot \nu^2) \cdot (B_{11}(t,u) \cdot f_1) = 0$$

for all $f_1 \in F_1$ and $\nu^2 \in F_2^*$. Differentiating this in the direction of $(\tau, e) \in \mathbb{R} \times E$ gives

$$\begin{aligned}(\boldsymbol{D}_1 A_{20}^*(t,u)\cdot(\tau,\nu^2))\cdot(B_{01}(t,u)\cdot f_1) + (\boldsymbol{D}_2 A_{20}^*(t,u)\cdot(e,\nu^2))\cdot(B_{01}(t,u)\cdot f_1)+\\ (A_{20}^*(t,u)\cdot\nu^2)\cdot(\boldsymbol{D}_1 B_{01}(t,u)\cdot(\tau,f_1)) + (A_{20}^*(t,u)\cdot\nu^2)\cdot(\boldsymbol{D}_2 B_{01}(t,u)\cdot(e,f_1))+\\ (\boldsymbol{D}_1 A_{21}^*(t,u)\cdot(\tau,\nu^2))\cdot(B_{11}(t,u)\cdot f_1) + (\boldsymbol{D}_2 A_{21}^*(t,u)\cdot(e,\nu^2))\cdot(B_{11}(t,u)\cdot f_1)+\\ (A_{21}^*(t,u)\cdot\nu^2)\cdot(\boldsymbol{D}_1 B_{11}(t,u)\cdot(\tau,f_1)) + (A_{21}^*(t,u)\cdot\nu^2)\cdot(\boldsymbol{D}_2 B_{11}(t,u)\cdot(e,f_1)) = 0. \quad (6.7)\end{aligned}$$

Now let (t, u, u_1) lie in the intersection of D with J^1M. Therefore, for some uniquely determined $f_1 \in F_1$ we have

$$(B_{01}(t,u)\cdot f_1, B_{11}(t,u)\cdot f_1) = (1, u_1). \quad (6.8)$$

We denote this value of f_1 by $f_1(t, u, u_1)$. Since $\Lambda = \operatorname{ann}(D)$ we also have

$$(A_{20}^*(t,u)\cdot\nu^2)\cdot u_1 + (A_{21}^*(t,u)\cdot\nu^2)\cdot 1 = 0 \quad (6.9)$$

for $\nu^2 \in F_2^*$. We have

$$\begin{aligned}D^T_{(t,u,1,u_1)} = \big\{((t,u,1,u_1),(1,u_1,\boldsymbol{D}_1 B_{01}(t,u)\cdot(1,f_1(t,u,u_1))+\\ \boldsymbol{D}_2 B_{01}(t,u)\cdot(u_1,f_1(t,u,u_1)) + B_{01}(t,u)\cdot f_2, \boldsymbol{D}_1 B_{11}(t,u)\cdot(1,f_1(t,u,u_1))+\\ \boldsymbol{D}_2 B_{11}(t,u)\cdot(u_1,f_1(t,u,u_1)) + B_{11}(t,u)\cdot f_2)) \mid \ f_1 \in F_1\big\}. \quad (6.10)\end{aligned}$$

and

$$\begin{aligned}\Lambda^T_{(t,u,1,u_1)} = \big\{((t,u,1,u_1),(\boldsymbol{D}_1 A_{20}^*(t,u)\cdot(\cdot,\nu^{21},1) + \boldsymbol{D}_1 A_{21}^*(t,u)\cdot(\cdot,\nu^{21},1)+\\ A_{20}^*(t,u)\cdot\nu^{22}, \boldsymbol{D}_2 A_{20}^*(t,u)\cdot(\cdot,\nu^{21},u_1) + \boldsymbol{D}_2 A_{21}^*(t,u)\cdot(\cdot,\nu^{21},u_1) + A_{21}^*(t,u)\cdot\nu^{22},\\ A_{20}^*(t,u)\cdot\nu^{21}, A_{21}^*(t,u)\cdot\nu^{21})) \mid \ \nu^{21},\nu^{22} \in F_2^*\big\}. \quad (6.11)\end{aligned}$$

Combining (6.7)–(6.11) with a tedious but straightforward calculation gives $D^T_{(t,u,1,u_1)} \lrcorner \Lambda^T_{(t,u,1,u_1)} = 0$ which proves the result, given that $(t, u, u_1) \in D_{1,(t,u)}$. ■

6.4 Remark: As stated in the introduction, our constructions involving the lifting of distributions are applicable in the investigation of mechanical systems with affine constraints [de León, Marrero, and Martín de Diego 1997]. Within this context, one may interpret the previous result as stating that our two ways of lifting a distribution are equally valid when applied to constrained mechanical systems. •

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