

# Geometric sliding mode control: The linear and linearised theory\*

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## Abstract

The idea of sliding mode control for stabilisation is investigated to determine its geometric features. A geometric definition is provided for a sliding submanifold, and for various properties of a sliding submanifold. Sliding subspaces are considered for linear systems, where a pole placement algorithm is given that complements existing algorithms. Finally, it is shown that at an equilibrium for a nonlinear system with a controllable linearisation, the sliding subspace for a linearisation gives rise to many local sliding submanifolds for the nonlinear system. This theory is exhibited on the standard pendulum/cart system.

**Keywords.** sliding mode control, linear systems, linearisation.

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## 1. Introduction

In sliding mode control, the idea is that one provides a surface on which the dynamics of the system, when restricted to the surface, have desired properties, e.g., all trajectories asymptotically approach a desired equilibrium point. The surface should also have the property that one can, by a suitable control law, force trajectories to the surface in finite time. In this way, one can ensure that the system eventually behaves as behaves the system on the surface. Sometimes, in applications, the surface is obtained via the use of a linearising output. A rather thorough review of the classical notion of sliding mode control is provided by Utkin [1992]. The topic is also covered in the book by Slotine and Li [1990], and a recent book concerning sliding mode control is that of Edwards and Spurgeon [1998]. There is also a significant number of journal papers devoted to sliding mode control, many of them concerning specific applications of sliding mode control.

Our intent is to understand sliding mode control in a geometric way. In particular, we wish to place the emphasis on understanding the relationship between the surface and the

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ambient system dynamics, and, by exploiting this understanding, to develop some intuition for the design of sliding mode controllers. In this paper we consider this way of thinking about sliding mode control for linear systems, and for nonlinear systems near equilibria with controllable linearisations. The emphasis of the present paper is not so much to give new results, but to understand in a different light existing results concerning sliding mode control, and to better understand the very notion of sliding mode control. This insightful manner of presenting sliding mode control will be explored in subsequent papers where our different way of looking at things really does help to understand what one is doing.

In Section 2 we set up a general scheme for sliding mode control. We give general definitions for various concepts related to sliding mode control. A key idea is the notion of a “restriction” of a system, which helps to clarify what is typically referred to as the “equivalent control.” We also present a simple example that illustrates our approach by providing a simple, intuitive characterisation of *all* sliding mode controllers for the system. We investigate in detail the linear case in Section 3. Here we look for subspaces on which the dynamics can be specified to be stable. We provide a proof, different from that of [Utkin \[1992\]](#), that one can achieve *any* characteristic polynomial on the sliding subspace. As a corollary to our proof we provide a characterisation of all the ways in which one can obtain a sliding subspace with a certain characteristic polynomial. For linear systems we also provide algorithms, defined in a geometrically intuitive manner, that steer the system to a given sliding subspace in finite time. Although there are many ways to do this, we consider two, one being local in nature and using bounded controls, and the other being global, but perhaps requiring unbounded controls. In Section 5 we consider nonlinear systems at an equilibrium with a controllable linearisation. We show that for these systems the linear sliding subspace defines a many local sliding submanifolds for the nonlinear system. For the pendulum/cart system, we compare the performance of a sliding mode controller with a LQR controller.

There is, of course, a wealth of papers which use sliding mode control, as it is often quite effective in applications. The intent in this paper is to propose a way of thinking about sliding mode control, and to illustrate that this way of thinking about sliding mode control offers some nice insights into existing strategies. In future work we will explore how to use our geometric approach in cases that are more interesting than those that are linear, or have controllable linearisations. Since the preponderance of existing applications of sliding mode control employ some sort of linear controllability assumption, this future work will well exhibit the utility of a geometric approach such as is initiated in this paper.

## 2. A general picture of sliding mode control

In this section we develop our geometric approach to sliding mode control. We intend to develop a sufficiently general picture to allow for applications more general than are considered in this paper.

**2.1. Setup.** In this paper, for concreteness and to emphasise the geometric nature of our constructions, we consider control-affine systems. By a control-affine system we mean a triple  $\Sigma = (M, \mathcal{F} = \{f_0, f_1, \dots, f_m\}, U)$  where  $M$  is a smooth manifold of dimension  $n$ ,  $\mathcal{F}$  is a collection of smooth vector fields on  $M$ , and  $U \subset \mathbb{R}^m$  is the set within which controls will take their values. Thus a *controlled trajectory* is a pair  $(\xi: I \rightarrow M, u: I \rightarrow U)$  of

curves defined on the interval  $I \subset \mathbb{R}$  with  $u$  measurable and which together satisfy the differential equation

$$\dot{\xi}(t) = f_0(\xi(t)) + \sum_{a=1}^m u^a(t) f_a(\xi(t)).$$

Throughout the paper we make the assumption that the distribution  $\mathcal{F}$  whose fibre at  $x \in M$  is defined by

$$\mathcal{F}_x = \text{span}_{\mathbb{R}}(f_1(x), \dots, f_m(x))$$

has constant rank. We do not wish that considerations of the exact nature of the control set get in the way of our geometric constructions, so we ask that  $U = \mathbb{R}^m$ , unless we state otherwise. In practice, of course, one typically has only finite control action available. We will consider the consequences of this at various times, but to simplify parts of the general presentation we will take as default the case where control values are unrestricted. If  $\mathcal{X} \subset M$  is an open submanifold then  $\Sigma|\mathcal{X}$  denotes the control-affine system  $(\mathcal{X}, \mathcal{F}|\mathcal{X}, U)$ , where  $\mathcal{F}|\mathcal{X}$  is the collection of vector fields in  $\mathcal{F}$ , restricted to  $\mathcal{X}$ .

**2.2. Restrictions of control-affine systems.** We wish to have our sliding submanifold have the property that we may restrict the system to it. In cases when the restricted system is unique, the control giving rise to this restriction is called the “equivalent control.” In this section, we formulate this idea of restriction to a submanifold, and of characterising the equivalent control, in a slightly different manner than is the norm. We consider control-affine systems  $\Sigma = (M, \mathcal{F} = \{f_0, f_1, \dots, f_m\}, U = \mathbb{R}^m)$  and  $\tilde{\Sigma} = (S, \tilde{\mathcal{F}} = \{\tilde{f}_0, \tilde{f}_1, \dots, \tilde{f}_{\tilde{m}}\}, \tilde{U} = \mathbb{R}^{\tilde{m}})$  with the property that  $S$  is a submanifold of  $M$ . We shall say that  $\tilde{\Sigma}$  is a **restriction** of  $\Sigma$  if for every controlled trajectory  $(\tilde{\xi}, \tilde{u})$  for  $\tilde{\Sigma}$  there is a control  $u$  for which  $(i_S \circ \tilde{\xi}, u)$  is a controlled trajectory for  $\Sigma$ , where  $i_S: S \rightarrow M$  is the inclusion. A submanifold  $S$  is  **$\Sigma$ -rigid** if

1. there is only one control-affine system  $\tilde{\Sigma} = (S, \tilde{\mathcal{F}}, \tilde{U} = \mathbb{R}^{\tilde{m}})$  that is a restriction of  $\Sigma$  and if
2.  $\tilde{\mathcal{F}} = \{\tilde{f}_0\}$  for some smooth vector field  $\tilde{f}_0$  on  $S$ .

The following result characterises  $\Sigma$ -rigid submanifolds.

**2.1 Proposition:** *For a control-affine system  $\Sigma = (M, \mathcal{F}, U = \mathbb{R}^m)$  a submanifold  $S \subset M$  is  $\Sigma$ -rigid if for each  $x \in S$  we have  $T_x M = T_x S \oplus \mathcal{F}_x$ . Conversely, if  $S$  is  $\Sigma$ -rigid then  $T_x S \cap \mathcal{F}_x = \{0\}$ .*

**Proof:** First suppose that  $T_x M = T_x S \oplus \mathcal{F}_x$  for each  $x \in S$  and let  $(\xi, u)$  be a controlled trajectory for  $\Sigma$  passing through  $x$  at time, say,  $t \in \mathbb{R}$ , and having the property that  $\xi(I) \subset S$ . Then

$$\dot{\xi}(t) = f_0(x) + \sum_{a=1}^m u^a(t) f_a(x) \in T_x S.$$

This means

$$f_0(x) + \sum_{a=1}^m u^a(t) f_a(x)$$

lies in the subspace  $T_x S \subset T_x M$  as well as lying in the affine subspace

$$f_0(x) + \mathcal{F}_x = \left\{ f_0(x) + \sum_{a=1}^m u^a f_a(x) \mid u \in \mathbb{R}^m \right\}.$$

We claim that since  $T_x M = T_x S \oplus \mathcal{F}_x$  we have  $(f_0(x) + \mathcal{F}_x) \cap T_x S = \{v_x\}$  for some  $v_x \in T_x M$ . Indeed, let  $\{v_1, \dots, v_k\}$  be a basis for  $T_x S$  and let  $\{v_{k+1}, \dots, v_n\}$  be a basis for  $\mathcal{F}_x$ . The equation

$$c_1 v_1 + \dots + c_k v_k + c_{k+1} v_{k+1} + \dots + c_n v_n = f_0(x)$$

then has a unique solution for the constants  $c_1, \dots, c_n$ . We also have

$$T_x S \ni c_1 v_1 + \dots + c_k v_k = f_0(x) - c_{k+1} v_{k+1} - \dots - c_n v_n \in f_0(x) + \mathcal{F}_x,$$

so it follows that  $(f_0(x) + \mathcal{F}_x) \cap T_x S \neq \emptyset$ . To show that the intersection consists of a single vector also follows from elementary linear algebra. This shows that  $T_x M = T_x S \oplus \mathcal{F}_x$  implies that  $(f_0(x) + \mathcal{F}_x) \cap T_x S = \{v_x\}$  for some  $v_x \in T_x M$ , thus showing that there is a unique vector  $f_1(x) \in \mathcal{F}_x$  for which  $f_0(x) + f_1(x) \in T_x S$ . Thus  $S$  is indeed  $\Sigma$ -rigid by taking  $\tilde{\mathcal{F}} = \{f_0 + f_1\}$ , thinking of  $f_0 + f_1$  as being a vector field on  $S$ .

Conversely, suppose that  $S$  is  $\Sigma$ -rigid. Then for each  $x \in S$  there exists a unique  $f_1(x) \in \mathcal{F}_x$  with the property that  $f_0(x) + f_1(x) \in T_x S$ . Let us abbreviate  $v_x = f_0(x) + f_1(x)$ . Thinking of  $f_0(x) + \mathcal{F}_x$  and  $T_x S$  as submanifolds of  $T_x M$  this means that  $T_{v_x}(f_0(x) + \mathcal{F}_x)$  and  $T_{v_x}(T_x S)$  intersect in  $\{0\}$ . However, since  $T_{v_x}(f_0(x) + \mathcal{F}_x) \simeq \mathcal{F}_x$  and  $T_{v_x}(T_x S) \simeq T_x S$  this part of the proposition follows.  $\blacksquare$

In the case that  $T_x M = T_x S \oplus \mathcal{F}_x$ , let  $\text{pr}_S: TM|S \rightarrow TS$  be the projection onto the tangent bundle of  $S$  and let  $\text{pr}_{\mathcal{F}}: TM|S \rightarrow \mathcal{F}|S$  be the projection onto the input distribution on  $S$ , both defined with respect to the decomposition  $T_x M = T_x S \oplus \mathcal{F}_x$ .

**2.2 Remark:** In most of the literature on sliding mode control, sliding surfaces are assumed to be rigid. The equivalent control is then often characterised by considering the limits of closed-loop controls off the sliding surface as they approach the sliding surface. It is not perfectly obvious that the equivalent control defined in this way is unique, i.e., does not depend on the control off the surface. The characterisation of Proposition 2.1 makes this clear. Indeed, the equivalent control  $u_S: S \rightarrow \mathbb{R}^m$  is defined as satisfying

$$\text{pr}_{\mathcal{F}} \left( f_0(x) + \sum_{a=1}^m u_S^a(x) f_a(x) \right) = 0, \quad x \in S.$$

Note that for the equivalent control to really be unique, the vector fields  $\{f_1, \dots, f_m\}$  should be linearly independent. If this is not true, then one should really say that the vector field  $\sum_{a=1}^m u_S^a f_a$  is unique, even though the coefficients  $u_S^a$ ,  $a \in \{1, \dots, m\}$ , are not. In either case, the important thing is that the dynamics on  $S$  are simply prescribed by the vector field  $\tilde{f}_0$  that is defined by

$$\tilde{f}_0(x) = \text{pr}_S(f_0(x)), \quad x \in S. \quad \bullet$$

**2.3. Sliding submanifolds and their properties.** In this section we provide a general characterisation of sliding submanifolds and some of the properties they may possess. If  $S \subset M$ ,  $i_S: S \rightarrow M$  denotes the inclusion.

**2.3 Definition:** Let  $\Sigma = (M, \mathcal{F}, U)$  be a  $C^\infty$  control-affine system and let  $x_0 \in M$ .

- (i) A **sliding submanifold** for  $(\Sigma, x_0)$  is a  $C^\infty$  submanifold of  $M \setminus \{x_0\}$  with the following properties:

- (a)  $x_0 \in \text{cl}(S)$ ;
- (b)  $T_x S + \mathcal{F}_x = T_x M$ .

- (ii) A sliding submanifold  $S$  for  $(\Sigma, x_0)$  is  **$C^r$** ,  $r \in \mathbb{Z}_+$ , if  $S \cup \{x_0\}$  is a  $C^r$  submanifold of  $M$ .
- (iii) A sliding submanifold  $S$  for  $(\Sigma, x_0)$  is **rigid** if it is  $\Sigma$ -rigid.
- (iv) For a rigid sliding submanifold  $S$  for  $(\Sigma, x_0)$ , let  $\tilde{f}_0$  be the vector field induced on  $S$ .  $S$  is **stabilising** if there exists a map  $T: S \rightarrow ]0, \infty]$  so that the solution to the initial value problem

$$\dot{\xi}(t) = \tilde{f}_0(\xi(t)), \quad \xi(0) = x,$$

is defined on  $[0, T(x)[$  and satisfies  $\lim_{t \uparrow T(x)} \xi(t) = x_0$ .

- (v) For a sliding submanifold  $S$  for  $(\Sigma, x_0)$ , let  $\tilde{\Sigma} = (S, \tilde{\mathcal{F}} = \{\tilde{f}_0, \tilde{f}_1, \dots, \tilde{f}_{\tilde{m}}, \tilde{U}\})$  be the restriction to  $S$ .  $S$  is  **$C^r$ -stabilisable**,  $r \in \mathbb{Z}_+$ , if there exists a  $C^r$  function  $\tilde{u}: S \rightarrow \tilde{U}$  and a map  $T: S \rightarrow ]0, \infty]$  so that the solution to the initial value problem

$$\dot{\xi}(t) = \tilde{f}_0(\xi(t)) + \sum_{a=1}^{\tilde{m}} \tilde{u}^a(\xi(t)) \tilde{f}_a(\xi(t)), \quad \xi(0) = x,$$

is defined on  $[0, T(x)[$  and satisfies  $\lim_{t \uparrow T(x)} i_S \circ \xi(t) = x_0$ .

- (vi) A sliding submanifold  $S$  is **locally attracting** if for any  $T_{\max} > 0$  there exists a neighbourhood  $\mathcal{N}$  of  $S$  in  $M$  and a map  $T: \mathcal{N} \setminus S \rightarrow ]0, T_{\max}]$  with the property that for each  $x \in \mathcal{N} \setminus S$  there is a control  $u: [0, T(x)[ \rightarrow U$  so that the solution to the initial value problem

$$\dot{\xi}(t) = f_0(\xi(t)) + \sum_{a=1}^m u^a(t) f_a(\xi(t)), \quad \xi(0) = x,$$

is defined on  $[0, T(x)[$  and  $\lim_{t \uparrow T(x)} \xi(t) \in S$ .

- (vii) A sliding submanifold  $S$  is **smoothly locally attracting** if there exists a  $C^\infty$  section  $f_{\text{cl}}$  of the vector bundle  $\mathcal{F}|(M \setminus S)$  so that for any  $T_{\max} > 0$  there exists a neighbourhood  $\mathcal{N}$  of  $S$  in  $M$  and a map  $T: \mathcal{N} \setminus S \rightarrow ]0, T_{\max}]$  with the property that for each  $x \in \mathcal{N} \setminus S$  the solution to the initial value problem

$$\dot{\xi}(t) = f_0(\xi(t)) + f_{\text{cl}}(\xi(t)), \quad \xi(0) = x,$$

is defined on  $[0, T(x)[$  and  $\lim_{t \uparrow T(x)} \xi(t) \in S$ .

- (viii) A **local sliding submanifold** for  $(\Sigma, x_0)$  is a sliding submanifold for  $(\Sigma|_{\mathcal{X}}, x_0)$  where  $\mathcal{X}$  is a neighbourhood of  $x_0$  in  $M$ . •

- 2.4 Remark:** 1. We exclude  $x_0$  from being in  $S$  in order to allow certain examples where it is helpful to be able to design  $S$  to have a singularity at  $x_0$ .
2. In part (v) of the definition, one can allow other forms of stabilisability of the restriction to  $S$ , if desired. •

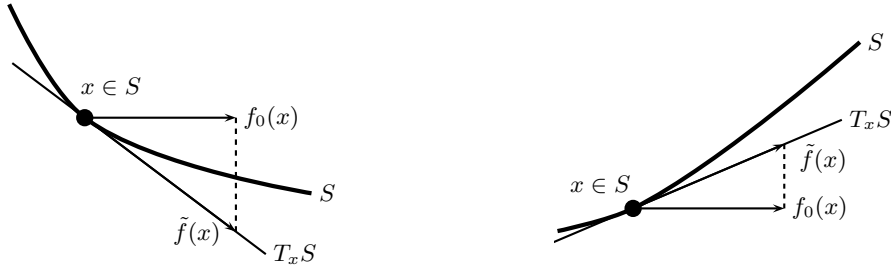
**2.4. A nonlinear controller for a simple linear system.** To give an indication of how one may use the geometric approach, we consider a rather simple example,

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= u,\end{aligned}\tag{2.1}$$

and we see what it might mean to design a “sliding mode controller” for the system that stabilises  $x_0 = (0, 0)$ . In terms of the definitions of the preceding section, we shall design a rigid stabilising smoothly locally attracting sliding submanifold  $S$ . Let us write (2.1) as a single-input control affine system  $\Sigma = (\mathbb{R}^2, \mathcal{F} = \{f_0, f_1\}, U)$  where

$$f_0(x_1, x_2) = x_2 \frac{\partial}{\partial x_1}, \quad f_1(x_1, x_2) = \frac{\partial}{\partial x_2}.$$

We first choose  $S$  so that it contains  $x_0$  in its closure. Next we must choose  $S$  so that the system (2.1) restricts to it. By Proposition 2.1 this means that we should choose  $S$  so that, away from  $x_0$ ,  $T_x S$  is not parallel to the vector  $(0, 1)$ . Another requirement of  $S$  is that the dynamics, when restricted to  $S$  as in Proposition 2.1, should have  $x_0$  as an asymptotically stable “equilibrium point.” The geometry underlying this second condition is more subtle to understand, but can still be done in an intuitive manner by considering the vector field  $f_0$ , and how it influences the restriction to  $S$ . In Figure 1 we show how this produces a



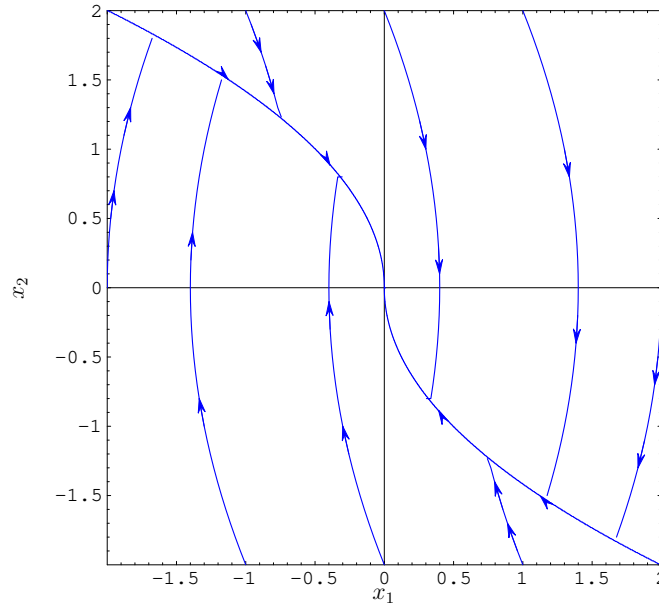
**Figure 1.** The character of the restricted dynamics in the  $(-, +)$ -quadrant (left) and in the  $(+, +)$ -quadrant (right)

vector field  $\tilde{f}_0$  on  $S$  for  $x_2 > 0$ . We see that when  $x_1 > 0$  the restricted dynamics on  $S$  renders  $x_0$  unstable, whereas the behaviour on  $S$  is desirable for  $x_1 < 0$ . Thus we are forced to ask that  $S$  intersect the upper half plane only in the  $(-, +)$ -quadrant. Similar arguments demand that  $S$  intersect the lower half plane in the  $(+, -)$ -quadrant.

At this point we have a pretty good idea of what a stabilising rigid sliding surface may look like. It must be the graph of a strictly decreasing function that passes through  $x_0 = (0, 0)$ . The only remaining issue to confront is how it should pass through  $x_0$ . It may

easily be checked that as long as  $S$  does not have a vertical tangency at  $x_0$ , the dynamics on  $S$ , and the control giving rise to these dynamics, is well behaved. However, vertical tangencies are also allowed for  $S$  at  $x_0$ , provided that they are not “too vertical.” We shall not address this in the present paper, although it will be investigated in a future paper.

Now that we have a sliding surface on which the dynamics behave in a satisfactory manner we should set about forcing the system trajectories near  $S$  to reach the sliding surface in finite time. In fact, in this example one can easily design bounded controls that steer the system to  $S$  in finite time, although this time will not be uniformly bounded. In Figure 2 we show the closed-loop phase portrait for the system having chosen



**Figure 2.** Phase portrait for closed-loop controller with a sliding surface  $S = \{(x_1, -x_1) \mid x_1 \in \mathbb{R}\}$

$$S = \{(x_1, x_2) \in \mathbb{R}^2 \setminus \{(0, 0)\} \mid x_1 = \frac{1}{2}x_2^2\},$$

and having chosen a constant control (of magnitude 5 in this case) away from  $S$  to steer to  $S$ . Note that  $S$  is a  $C^1$  rigid, stabilising submanifold. An interesting feature of this particular sliding submanifold is that the control required to stay on  $S$  has constant magnitude 1. Thus the closed-loop phase portrait of Figure 2 is obtained with controls taking values in the set  $\{-5, -1, 1, 5\}$ .

### 3. Geometric sliding mode control for linear systems

The ideas so far have been general in character. In this section we consider the particular case of linear systems and rigid sliding subspaces. The results in this case are known, of

course [e.g., [Utkin 1992](#), §7.2]. However, our approach is slightly different. This leads to, for example, a complete characterisation in Proposition 3.2 of rigid sliding subspaces and how to obtain them.

We let  $\mathcal{U}$  and  $\mathcal{V}$  be  $\mathbb{R}$ -vector spaces of dimension  $m$  and  $n$ , and say that a **linear system** on  $\mathcal{V}$  with inputs in  $\mathcal{U}$  is a pair  $\Lambda = (A, B)$  with  $A \in L(\mathcal{V}; \mathcal{V})$  and  $B \in L(\mathcal{U}; \mathcal{V})$ . Note that a linear system is also a control-affine system, so all the definitions of Section 2 apply. As usual the equations governing the linear system  $\Lambda$  are

$$\dot{\xi}(t) = A(\xi(t)) + B(u(t)). \quad (3.1)$$

We will assume without loss of generality that  $B$  is injective. (If it is not, let  $\tilde{\mathcal{U}} = \mathcal{U} / \ker(B)$  and  $\tilde{B}(u + \ker(B)) = B(u)$ . Then  $(A, \tilde{B})$  is a linear system on  $\mathcal{V}$  with inputs in  $\tilde{\mathcal{U}}$ , and  $\tilde{B}$  is injective. Furthermore, if  $(\xi, u)$  is a controlled trajectory for  $(A, B)$  then  $(\xi, u + \ker(B))$  is a controlled trajectory for  $(A, \tilde{B})$ .) A sliding submanifold for  $(\Lambda, 0)$  that is also a subspace is a **sliding subspace**.

Our objective in this section is to consider a systematic method for producing sliding subspaces on which the dynamics have a prescribed characteristic polynomial. This is done in Section 3.1. In Section 3.2 we consider the problem of steering the system to the sliding subspace in a systematic way. In doing so, we provide a geometric characterisation of the standard output-based sliding mode control law. Our characterisation does not rely on the presence of an output (typically denoted “ $s$ ” in the sliding mode control literature).

**3.1. Sliding subspaces for linear systems.** If  $\mathcal{S}$  is a rigid sliding subspace for  $\Lambda = (A, B)$  then we must have  $\mathcal{V} = \mathcal{S} \oplus \text{image}(B)$ . Denote  $\text{pr}_B: \mathcal{V} \rightarrow \text{image}(B)$  and  $\text{pr}_{\mathcal{S}}: \mathcal{V} \rightarrow \mathcal{S}$  the projections relative to the direct sum decomposition in this case. The inclusions we denote  $i_B: \text{image}(B) \rightarrow \mathcal{V}$  and  $i_{\mathcal{S}}: \mathcal{S} \rightarrow \mathcal{V}$ . The vector field on  $\mathcal{S}$  obtained by restriction of the system (3.1) will be obtained by projecting  $A(x)$  to  $\mathcal{S}$  for  $x \in \mathcal{S}$ . The control at  $x \in \mathcal{S}$  will then be the unique (since  $B$  is assumed injective)  $u \in \mathcal{U}$  satisfying  $B(u) = -\text{pr}_B(A(x))$ .

The following result provides the essential feature of the preceding constructions. If  $L \in L(\mathcal{V}; \mathcal{V})$  then  $\text{spec}(L) \subset \mathbb{C}$  denotes the eigenvalues of  $L$  and if  $P \in \mathbb{R}[\lambda]$  then  $\text{spec}(P) \subset \mathbb{C}$  denotes the roots of  $P$ .

**3.1 Theorem:** *Let  $\mathcal{V}$  and  $\mathcal{U}$  be  $\mathbb{R}$ -vector spaces of dimension  $n$  and  $m$ , respectively, and let  $A \in L(\mathcal{V}; \mathcal{V})$  and  $B \in L(\mathcal{U}; \mathcal{V})$  be linear maps with the property that  $(A, B)$  is controllable. If  $P \in \mathbb{R}[\lambda]$  is a monic polynomial of degree  $n - m$  then there exists a subspace  $\mathcal{S}$  with the following properties:*

- (i)  $\mathcal{V} = \text{image}(B) \oplus \mathcal{S}$ ;
- (ii)  $\text{spec}(A_{\mathcal{S}}) = \text{spec}(P)$  where  $A_{\mathcal{S}} = \text{pr}_{\mathcal{S}} \circ A \circ i_{\mathcal{S}}$ .

**Proof:** Let us choose a linear map  $F_0 \in L(\mathcal{V}; \mathcal{U})$  for which there exists bases  $\{v_1, \dots, v_n\}$  for  $\mathcal{V}$  and  $\{u_1, \dots, u_m\}$  for  $\mathcal{U}$  having the property that the representations of  $A + B \circ F_0$  and  $B$  in these bases are in Brunovsky form. Thus, letting the matrix representation of  $A + B \circ F_0$  be denoted  $\mathbf{A} \in \mathbb{R}^{n \times n}$  and the matrix representation of  $B$  be denoted  $\mathbf{B} \in \mathbb{R}^{n \times m}$  we have

$$\mathbf{A} = \begin{bmatrix} \mathbf{J}_{\kappa_1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_{\kappa_2} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{J}_{\kappa_m} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{e}_{\kappa_1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{e}_{\kappa_2} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{e}_{\kappa_m} \end{bmatrix},$$



where

$$\mathbf{J}_{\kappa_j} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}}_{\kappa_j} \kappa_j, \quad \mathbf{e}_{\kappa_j} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \kappa_j, \quad j \in \{1, \dots, m\},$$

and where  $\boldsymbol{\kappa} = (\kappa_1, \dots, \kappa_m)$  are the controllability indices. Now consider an arbitrary  $\mathbf{F} \in \mathbb{R}^{m \times n}$  and write it as

$$\mathbf{F} = \begin{bmatrix} \mathbf{f}_{11} & \mathbf{f}_{12} & \cdots & \mathbf{f}_{1m} \\ \mathbf{f}_{21} & \mathbf{f}_{22} & \cdots & \mathbf{f}_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{f}_{m1} & \mathbf{f}_{m2} & \cdots & \mathbf{f}_{mm} \end{bmatrix}$$

for  $\mathbf{f}_{jk} \in \mathbb{R}^{1 \times \kappa_k}$ ,  $j, k \in \{1, \dots, m\}$ . Let us write  $\mathbf{f}_{jk}$  as

$$\mathbf{f}_{jk} = [f_{jk1} \quad f_{jk2} \quad \cdots \quad f_{jk\kappa_k}].$$

We shall decompose  $\mathbb{R}^{m \times n}$  as  $\mathbb{R}^{(m-1) \times n} \oplus \mathbb{R}^{1 \times n}$  by writing  $\mathbf{F} \in \mathbb{R}^{m \times n}$  as

$$\mathbf{F} = \begin{bmatrix} \mathbf{f}_1 \\ \vdots \\ \mathbf{f}_{m-1} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \mathbf{f}_m \end{bmatrix}, \quad (3.2)$$

where  $\mathbf{f}_j \in \mathbb{R}^{1 \times n}$ ,  $j \in \{1, \dots, m\}$ . We denote by  $\pi_1(\mathbf{F})$  the first term on the right-hand side of the above equation, and by  $\pi_2(\mathbf{F})$  the second term.

We now consider the single input  $\mathbf{b}_m = (0, \dots, 0, 1)$ . We claim that the determinant of the controllability matrix for the single-input system  $(\mathbf{A} + \mathbf{B}\mathbf{F}, \mathbf{b}_m)$  depends only on  $\pi_1(\mathbf{F})$ . That is to say, if  $\mathbf{F}_1, \mathbf{F}_2 \in \mathbb{R}^{m \times n}$  have the property that  $\pi_1(\mathbf{F}_1 - \mathbf{F}_2) = \mathbf{0}$ , then

$$\begin{aligned} \det [\mathbf{b}_m \mid (\mathbf{A} + \mathbf{B}\mathbf{F}_1)\mathbf{b}_m \mid \cdots \mid (\mathbf{A} + \mathbf{B}\mathbf{F}_1)^{n-1}\mathbf{b}_m] \\ = \det [\mathbf{b}_m \mid (\mathbf{A} + \mathbf{B}\mathbf{F}_2)\mathbf{b}_m \mid \cdots \mid (\mathbf{A} + \mathbf{B}\mathbf{F}_2)^{n-1}\mathbf{b}_m]. \end{aligned}$$

Indeed, if  $\pi_1(\mathbf{F}_1 - \mathbf{F}_2) = \mathbf{0}$  then there exists  $\mathbf{f} \in \mathbb{R}^{1 \times n}$  so that

$$\mathbf{F}_2 = \mathbf{F}_1 + \begin{bmatrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \mathbf{f} \end{bmatrix}.$$

In this case we have

$$\mathbf{B}\mathbf{F}_2 = \mathbf{B}\mathbf{F}_1 + \mathbf{b}_m \mathbf{f},$$

and one easily shows by induction that

$$\begin{aligned} & [ \mathbf{b}_m \mid (\mathbf{A} + \mathbf{B}\mathbf{F}_2)\mathbf{b}_m \mid (\mathbf{A} + \mathbf{B}\mathbf{F}_2)^2\mathbf{b}_m \mid \cdots \mid (\mathbf{A} + \mathbf{B}\mathbf{F}_2)^{n-1}\mathbf{b}_m ] \\ &= [ \mathbf{b}_m \mid (\mathbf{A} + \mathbf{B}\mathbf{F}_1)\mathbf{b}_m \mid (\mathbf{A} + \mathbf{B}\mathbf{F}_1)^2\mathbf{b}_m \mid \cdots \mid (\mathbf{A} + \mathbf{B}\mathbf{F}_1)^{n-1}\mathbf{b}_m ] \\ &+ [ \mathbf{0} \mid \text{lc}(\mathbf{b}_m) \mid \text{lc}(\mathbf{b}_m, (\mathbf{A} + \mathbf{B}\mathbf{F}_1)\mathbf{b}_m) \mid \cdots \mid \text{lc}(\mathbf{b}_m, (\mathbf{A} + \mathbf{B}\mathbf{F}_1)\mathbf{b}_m, \dots, (\mathbf{A} + \mathbf{B}\mathbf{F}_1)^{n-1}\mathbf{b}_m) ], \end{aligned}$$

where  $\text{lc}(\mathbf{v}_1, \dots, \mathbf{v}_k)$  stands for some linear combination of the vectors  $\mathbf{v}_1, \dots, \mathbf{v}_k$ . By performing determinant preserving column operations we may reduce the expression on the right-hand side to that on the left-hand side, thus proving our claim.

The dependence of the controllability matrix for  $(\mathbf{A} + \mathbf{B}\mathbf{F}, \mathbf{b}_m)$  only on  $\pi_1(\mathbf{F})$  allows us to assert that there is an open, dense subset  $\mathcal{O}_\kappa$  of  $\mathbb{R}^{(m-1) \times n}$  with the property that  $(\mathbf{A} + \mathbf{B}\mathbf{F}, \mathbf{b}_m)$  is controllable if and only if  $\pi_1(\mathbf{F}) \in \mathcal{O}_\kappa$ . Now let  $P \in \mathbb{R}[\lambda]$  be the polynomial given in the statement of the theorem, and let  $Q \in \mathbb{R}[\lambda]$  be monic, degree  $m$ , and coprime with  $P$ . For each  $\mathbf{F}_1 \in \mathcal{O}_\kappa \subset \mathbb{R}^{(m-1) \times n} \subset \mathbb{R}^{m \times n}$  there then exists a unique  $\mathbf{F}_{\mathbf{F}_1} \in \pi_1^{-1}(\mathbf{F}_1)$  with the property that

$$\text{spec}(\mathbf{A} + \mathbf{B}\mathbf{F}_{\mathbf{F}_1}) = \text{spec}(P) \cup \text{spec}(Q).$$

Indeed, one constructs  $\mathbf{F}_{\mathbf{F}_1}$  by asking that  $\mathbf{f}_{\mathbf{F}_1} = \pi_2(\mathbf{F}_{\mathbf{F}_1})$  be chosen so that

$$\text{spec}(\mathbf{A} + \mathbf{B}\mathbf{F}_1 + \mathbf{b}_m \mathbf{f}_{\mathbf{F}_1}) = \text{spec}(P) \cup \text{spec}(Q),$$

this being possible since  $\mathbf{F}_1 \in \mathcal{O}_\kappa$ . Since

$$\mathbf{B}(\mathbf{F}_1 + \mathbf{f}_{\mathbf{F}_1}) = \mathbf{B}\mathbf{F}_1 + \mathbf{b}_m \mathbf{f}_{\mathbf{F}_1},$$

this does give the desired  $\mathbf{F}_{\mathbf{F}_1}$ . (Note the abuse of notation in the above equation. On the left,  $\mathbf{f}_{\mathbf{F}_1} \in \mathbb{R}^{1 \times n}$ , but one thinks of  $\mathbb{R}^{1 \times n} \subset \mathbb{R}^{m \times n}$  as per (3.2). On the right, one does not think of  $\mathbf{f}_{\mathbf{F}_1} \in \mathbb{R}^{1 \times n}$  as an element of  $\mathbb{R}^{m \times n}$ .) If one thinks for a moment about the pole-placement algorithm, one sees that  $\mathbf{F}_1 \mapsto \mathbf{F}_{\mathbf{F}_1}$  is a rational function in the  $(m-1)n$  components of the matrix  $\mathbf{F}_1$ . Thus we have a subset  $\mathcal{A}_{P,Q}$  of  $\mathbb{R}^{m \times n}$  with the following properties:

1. if  $\mathbf{F} \in \mathcal{A}_{P,Q}$  then  $\text{spec}(\mathbf{A} + \mathbf{B}\mathbf{F}) = \text{spec}(P) \cup \text{spec}(Q)$ ;
2.  $\mathcal{A}_{P,Q}$  is the graph of a rational function over the open dense subset  $\mathcal{O}_\kappa$  of  $\mathbb{R}^{(m-1) \times n}$ .

By virtue of our choosing  $Q$  coprime to  $P$ , for every  $\mathbf{F} \in \mathcal{A}_{P,Q}$  we have a decomposition  $\mathbb{R}^n = \mathcal{P}_\mathbf{F} \oplus \mathcal{Q}_\mathbf{F}$  into  $(\mathbf{A} + \mathbf{B}\mathbf{F})$ -invariant subspaces with the characteristic polynomial of  $(\mathbf{A} + \mathbf{B}\mathbf{F})|_{\mathcal{P}_\mathbf{F}}$  being  $P$ , and the characteristic polynomial of  $(\mathbf{A} + \mathbf{B}\mathbf{F})|_{\mathcal{Q}_\mathbf{F}}$  being  $Q$ . Furthermore, again since  $Q$  and  $P$  are coprime, the Cayley-Hamilton theorem (essentially) asserts that  $\mathcal{P}_\mathbf{F} = \ker(P(\mathbf{A} + \mathbf{B}\mathbf{F}))$ . We will now show that it is possible to choose  $\mathbf{F} \in \mathcal{A}_{P,Q}$  so that  $\mathcal{P}_\mathbf{F} \cap \text{image}(\mathbf{B}) = \{\mathbf{0}\}$ . Since  $\mathcal{P}_\mathbf{F} = \ker(P(\mathbf{A} + \mathbf{B}\mathbf{F}))$ ,  $\mathcal{P}_\mathbf{F} \cap \text{image}(\mathbf{B}) = \{\mathbf{0}\}$  if and only if for every  $\mathbf{b} \in \text{image}(\mathbf{B}) \setminus \{\mathbf{0}\}$  we have  $P(\mathbf{A} + \mathbf{B}\mathbf{F})\mathbf{b} \neq \mathbf{0}$ . Equivalently, for every  $\mathbf{u} \in \mathbb{R}^m \setminus \{\mathbf{0}\}$  we should have  $P(\mathbf{A} + \mathbf{B}\mathbf{F})\mathbf{B}\mathbf{u} \neq \mathbf{0}$ . This means that  $P(\mathbf{A} + \mathbf{B}\mathbf{F})\mathbf{B}$  should be injective. We let  $\Pi_m: \mathbb{R}^{n \times m} \rightarrow \mathbb{R}^{m \times m}$  be the map that extracts from an  $n \times m$  matrix its top  $m$  rows. A matrix  $\mathbf{M} \in \mathbb{R}^{n \times m}$  will be injective if (but not only if)  $\det(\Pi_m(\mathbf{M})) \neq 0$ . Thus, to show that we may choose  $\mathbf{F} \in \mathcal{A}_{P,Q}$  so that  $\mathcal{P}_\mathbf{F} \cap \text{image}(\mathbf{B}) = \{\mathbf{0}\}$  it suffices to show that the function

$$\det(\Pi_m(P(\mathbf{A} + \mathbf{B}\mathbf{F})\mathbf{B})),$$

thought of as a function on  $\mathcal{A}_{P,Q}$ , is not identically zero. The following technical lemma is devoted to proving this fact.

**1 Lemma:** *There exists a curve  $\alpha \mapsto \mathbf{F}_\alpha$  in  $\mathcal{A}_{P,Q}$  with the property that*

$$\alpha \mapsto \det(\Pi_m(P(\mathbf{A} + \mathbf{B}\mathbf{F}_\alpha)\mathbf{B}))$$

*is a nontrivial polynomial in  $\alpha$ .*

**Proof:** We take

$$\mathbf{F}_\alpha = \begin{bmatrix} 0 & \cdots & \alpha & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & \alpha & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & \alpha & 1 & \cdots & 0 \\ f_1 & \cdots & f_{\rho_1} & f_{\rho_1+1} & \cdots & f_{\rho_2} & f_{\rho_2+1} & \cdots & f_{\rho_{m-1}} & f_{\rho_{m-1}+1} & \cdots & f_{\rho_m} \end{bmatrix}$$

where  $\rho_j = \sum_{i=1}^j \kappa_i$ , and where the last row of  $\mathbf{F}_\alpha$  is chosen so that  $\mathbf{A}_\alpha \triangleq \mathbf{A} + \mathbf{B}\mathbf{F}_\alpha$  has characteristic polynomial  $P(\lambda)Q(\lambda)$ .

For  $i \in \{1, \dots, m\}$  define integers  $\ell_i$  by

$$\ell_1 = \kappa_m - 1, \ell_2 = \ell_1 + \kappa_{m-1} - 1, \dots, \ell_m = \ell_{m-1} + \kappa_1 - 1.$$

For  $k \in \{0, 1, \dots, n-m\}$  define integers  $r_i(k)$ ,  $i \in \{1, \dots, m\}$ , as follows. For  $k = 0$  we let

$$r_i(0) = \sum_{j=1}^i \kappa_j, \quad i \in \{1, \dots, m\}.$$

For  $k \in \{1, \dots, \ell_1\}$  we let

$$\begin{aligned} r_i(k) &= \kappa_1 + \cdots + \kappa_i, & i \in \{1, \dots, m-1\} \\ r_m(k) &= n - k. \end{aligned}$$

Generally, for  $k \in \{\ell_j + 1, \dots, \ell_{j+1}\}$  define

$$\begin{aligned} r_i(k) &= \kappa_1 + \cdots + \kappa_i, & i \in \{1, \dots, m-j-1\} \\ r_{m-j}(k) &= n - k - j \\ &\vdots \\ r_m(k) &= n - k. \end{aligned}$$

If

$$P(\lambda) = \lambda^{n-m} + p_{n-m-1}\lambda^{n-m-1} + \cdots + p_1\lambda + p_0,$$

for  $k \in \{0, 1, \dots, n-m\}$  we denote by  $P_k$  the polynomial

$$P_k(\lambda) = \lambda^k + p_{n-1}\lambda^{k-1} + \cdots + p_{n-k}\lambda + p_{n-k-1}.$$

With this notation, we let  $\mathbf{M}_\alpha(k)$ ,  $k \in \{0, 1, \dots, n-m\}$ , be the  $m \times m$  matrix whose  $i$ th row is the  $r_i(k)$ th row of  $P_k(\mathbf{A}_\alpha)\mathbf{B}$ . Since  $r_i(n-m) = i$ ,  $i \in \{1, \dots, m\}$ ,  $\mathbf{M}_\alpha(n-m) = \Pi_m(P(\mathbf{A}_\alpha)\mathbf{B})$ . A messy iterative computation shows that  $\det(\mathbf{M}_\alpha(k))$  is a monic polynomial in  $\alpha$  of degree  $\ell_1 + \cdots + \ell_j$  if  $k \in \{\ell_j, \dots, \ell_{j+1}\}$ . In particular, this calculation shows that  $\det(\Pi_m(P(\mathbf{A}_\alpha)\mathbf{B}))$  is a monic polynomial of degree  $\ell_1 + \cdots + \ell_m$ , and from this the lemma follows.  $\blacktriangledown$

Continuing with the proof, since  $\dim(\mathcal{P}_F) = n - m$  and since  $\mathcal{P}_F \cap \text{image}(\mathbf{B}) = \{\mathbf{0}\}$ , we now have  $\mathbb{R}^n = \mathcal{P}_F \oplus \text{image}(\mathbf{B})$ . Going back now to the basis independent setting of  $\mathcal{V}$  and  $\mathcal{U}$  we have shown the existence of  $F_1 \in L(\mathcal{V}; \mathcal{U})$  so that the linear map  $A + B \circ F$ ,  $F = F_0 + F_1$ , has the following properties:

1. there exists a decomposition  $\mathcal{V} = \mathcal{P}_F \oplus \mathcal{Q}_F$  of  $(A + B \circ F)$ -invariant subspaces so that the characteristic polynomial of  $(A + B \circ F)|_{\mathcal{P}_F}$  is  $P$  and the characteristic polynomial of  $(A + B \circ F)|_{\mathcal{Q}_F}$  is  $Q$ ;
2.  $\mathcal{P}_F$  forms a complement to  $\text{image}(B)$  in  $\mathcal{V}$ .

We now claim that  $\mathcal{S} = \mathcal{P}_F$  has property (ii) in the statement of the theorem. Let  $v \in \mathcal{S}$ . Since  $\mathcal{S}$  is  $(A + B \circ F)$ -invariant,  $(A + B \circ F) \circ i_{\mathcal{S}}(v) \in \mathcal{S}$ . Also, since  $\text{pr}_{\mathcal{S}} \circ B = 0$  we have

$$(A + B \circ F) \circ i_{\mathcal{S}}(v) = \text{pr}_{\mathcal{S}} \circ (A + B \circ F) \circ i_{\mathcal{S}}(v) = \text{pr}_{\mathcal{S}} \circ A \circ i_{\mathcal{S}}(v).$$

Thus the characteristic polynomial of  $\text{pr}_{\mathcal{S}} \circ A \circ i_{\mathcal{S}}$  is the same as that of  $(A + B \circ F)|_{\mathcal{S}}$ , giving the result.  $\blacksquare$

The proof of the theorem suggests that it is actually quite easy to find the desired subspace  $\mathcal{S}$ . Indeed, the first part of the following result follows directly from the proof of the theorem.

**3.2 Proposition:** *Let  $\mathcal{V}, \mathcal{U}, A, B$ , and  $P$  be as in Theorem 3.1.*

*Let  $b \in L(\mathbb{R}; \text{image}(B))$  be nonzero and let  $Q \in \mathbb{R}[\lambda]$  be a monic polynomial of degree  $m$  coprime with  $P$ . Denote by  $L(\mathcal{V}; \mathcal{U})_b$  the open dense subset of  $L(\mathcal{V}; \mathcal{U})$  with the property that if  $F \in L(\mathcal{V}; \mathcal{U})_b$  then  $(A + B \circ F, i_B \circ b)$  is controllable.*

*Consider the algorithm taking  $F_0 \in L(\mathcal{V}; \mathcal{U})_b$  and producing a subspace  $\mathcal{S}(F_0)$  as follows:*

- (i) choose (by, say, Ackermann's formula) the unique  $f_{F_0} \in L(\mathcal{V}; \mathbb{R})$  so that*

$$\text{spec}(A + B \circ F_0 + b \circ f_{F_0}) = \text{spec}(P) \cup \text{spec}(Q);$$

- (ii) define  $F_{F_0} \in L(\mathcal{V}; \mathcal{U})$  by  $F_{F_0}(v) = F_0(v) + f_{F_0}(v)b(1)$ ;*

- (iii) define  $\mathcal{S}(F_0) = \ker(P(A + B \circ F_{F_0}))$ .*

*Then,  $\mathcal{S}$  satisfies conditions (i) and (ii) of Theorem 3.1 for all choice of  $F_0$ , except for a subset that is locally the intersection of zeros of a finite collection of analytic functions.*

*Conversely, if  $\mathcal{S}$  is a rigid sliding subspace for which the characteristic polynomial of  $A_{\mathcal{S}}$  is  $P$ , then there exists a monic degree  $m$  polynomial  $Q \in \mathbb{R}[\lambda]$  coprime with  $P$ , a nonzero  $b \in L(\mathbb{R}; \text{image}(B))$ , and  $F_0 \in L(\mathcal{U}; \mathcal{V})_b$  so that  $\mathcal{S} = \mathcal{S}(F_0)$ .*

**Proof:** The only part that does not follow directly from the proof of Theorem 3.1 is the second assertion. So let  $\mathcal{S}$  have the properties that  $\mathcal{V} = \mathcal{S} \oplus \text{image}(B)$  and that  $\text{spec}(A_{\mathcal{S}}) = \text{spec}(P)$ . Also let  $Q \in \mathbb{R}[\lambda]$  be monic, degree  $m$  and coprime with  $P$ . We claim that this implies that  $\mathcal{S}$  is an invariant subspace for  $A + B \circ F$  for some  $F \in L(\mathcal{U}; \mathcal{V})$ , and that furthermore  $F$  may be chosen so that  $\text{spec}(A + B \circ F) = \text{spec}(P) \cup \text{spec}(Q)$ . To see this, note that relative to the decomposition  $\mathcal{V} = \mathcal{S} \oplus \text{image}(B)$  we may write

$$A = \begin{bmatrix} A_{\mathcal{S}} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ B_2 \end{bmatrix},$$

where  $B_2$  is invertible by our assumption that  $B$  is injective. Taking

$$F = [-B_2^{-1} \circ A_{21} \quad F_2]$$

gives

$$A + B \circ F = \begin{bmatrix} A_8 & A_{12} \\ 0 & A_{22} + B_2 \circ F_2 \end{bmatrix}.$$

Note that  $(A_{22}, B_2)$  is controllable, so our claim follows by taking  $F_2 \in L(\text{image}(B); \text{image}(B))$  so that  $A_{22} + B_2 \circ F_2$  has characteristic polynomial  $Q$ . It remains to show that  $F \in L(\mathcal{U}; \mathcal{V})_b$  for some nonzero  $b \in L(\mathbb{R}; \text{image}(B))$ . This, however, follows since  $(A + B \circ F, B)$  is controllable, and so there is some vector  $v \in \text{image}(B)$  for which  $A + B \circ F$  is cyclic on  $v$ .  $\blacksquare$

**3.2. Steering to the sliding subspace.** Since  $\mathcal{S}$  as constructed in Proposition 3.2 is transversal to  $\text{image}(B)$  it follows immediately that it is possible to find a control law around  $\mathcal{S}$  that renders  $\mathcal{S}$  attractive in finite time. In this section we explicitly indicate how to do this in two ways. Of course, there are many ways to design a controller that renders  $\mathcal{S}$  attracting, at least near 0. Our intent is to design a specific control law in a manner that benefits from our geometric way of thinking of things. One of our controllers is designed to locally attract to  $\mathcal{S}$  using bounded controls. The other gives a geometric interpretation of the “usual” sliding mode control law.

First let us provide a locally valid controller that attracts trajectories near 0 to  $\mathcal{S}$  in finite time. To make clear the geometry behind our construction we endow the input space  $\mathcal{U}$  with an inner product  $\langle \cdot, \cdot \rangle$  and we denote by  $\|\cdot\|$  the norm defined by this inner product. Since  $B$  defines an isomorphism of  $\mathcal{U}$  with  $\text{image}(B)$  (we are still assuming that  $B$  is injective), the inner product may also be thought of as being on  $\text{image}(B) \subset \mathcal{V}$ . We shall, without indicating that we are doing so, write  $\langle \cdot, \cdot \rangle$  and  $\|\cdot\|$  for the inner product and norm on both  $\mathcal{U}$  and  $\text{image}(B)$ . For  $K > 0$  we define  $u_{\text{loc}, K}(x) \in \mathcal{U}$  by asking that  $B(u_{\text{loc}, K}(x))$  be given by

$$B(u_{\text{loc}, K}(x)) = -K \frac{\text{pr}_B(x)}{\|\text{pr}_B(x)\|}.$$

Thus  $B(u_{\text{loc}, K}(x))$  should be thought of as a tangent vector of length  $K$  at  $T_x \mathcal{V}$  and pointing along  $\text{image}(B)$  towards the point  $\text{pr}_{\mathcal{S}}(x) \in \mathcal{S}$ . Note that on  $\mathcal{V} \setminus \mathcal{S}$  the dynamical system

$$\dot{\xi}(t) = A(\xi(t)) + B(u_{\text{loc}, K}(\xi(t)))$$

is  $C^\infty$  and so it possesses maximal integral curves in the usual sense. When we refer to maximal integral curves for this equation below, we mean for the system on the open subset  $\mathcal{V} \setminus \mathcal{S}$ .

The control law  $u_{\text{loc}, K}$  is sufficient to locally attract to  $\mathcal{S}$  no matter the nature of  $A$ . What’s more, the time to reach  $\mathcal{S}$  can be uniformly bounded for points near 0.

**3.3 Proposition:** *Let  $\mathcal{S} \subset \mathcal{V}$  be a rigid sliding subspace for  $\Lambda = (A, B)$ . For any  $K, T_{\max} > 0$  and for any neighbourhood  $\mathcal{X}_1$  of  $0 \in \mathcal{V}$ , we may find a neighbourhood  $\mathcal{X}_0(K, T_{\max}, \mathcal{X}_1)$  of  $0 \in \mathcal{V}$  with the property that if  $\xi: [0, T(x)[ \rightarrow \mathcal{V} \setminus \mathcal{S}$  is the maximal integral curve for the initial value problem*

$$\dot{\xi}(t) = A(\xi(t)) + B(u_{\text{loc}, K}(\xi(t))), \quad \xi(0) = x \in \mathcal{X}_0(K, T_{\max}, \mathcal{X}_1) \setminus \mathcal{S}, \quad (3.3)$$

then

- (i)  $T(x) \leq T_{\max}$ ,
- (ii)  $\xi(t) \in \mathcal{X}_1$  for  $t \in [0, T(x)[$ , and
- (iii)  $\lim_{t \uparrow T(x)} \xi(t) \in \mathcal{S}$ .

**Proof:** For the proof let us extend the inner product  $\langle \cdot, \cdot \rangle$  on  $\text{image}(B)$  to all of  $\mathcal{V}$ , and do so in such a way that  $\mathcal{S}$  is orthogonal to  $\text{image}(B)$ . The extended inner product and norm we still denote by  $\langle \cdot, \cdot \rangle$  and  $\|\cdot\|$ , respectively. The induced norm on  $L(\mathcal{V}; \mathcal{V})$  we denote by  $|||\cdot|||$ . Denote by  $B_R(x)$  the open ball of radius  $R$  centred at  $x$ .

Let us first obtain an estimate on the growth of  $\xi(t)$ . We let  $\psi(x) = \frac{1}{2}\|x\|^2$  and compute

$$\begin{aligned} \frac{d\psi}{dt}(\xi(t)) &= \langle A(\xi(t)), \xi(t) \rangle - \frac{K}{\|\text{pr}_B(\xi(t))\|} \langle \text{pr}_B(\xi(t)), \xi(t) \rangle \\ &= \langle A(\xi(t)), \xi(t) \rangle - K\|\text{pr}_B(\xi(t))\| \\ &\leq |||A|||\|\xi(t)\|^2 = 2|||A|||\psi(\xi(t)), \end{aligned}$$

where we have used the Cauchy-Bunyakovsky-Schwarz inequality. By Gronwall's lemma it now follows that

$$\begin{aligned} \psi(t) &\leq \psi(0)e^{2|||A|||t} \\ \implies \|\xi(t)\| &\leq \|\xi(0)\|e^{|||A|||t}. \end{aligned} \tag{3.4}$$

Now let us obtain an estimate governing the behaviour of the distance of  $\xi(t)$  from  $\mathcal{S}$ . Define  $\phi: \mathcal{V} \rightarrow \mathbb{R}$  by  $\phi(x) = \frac{1}{2}\|\text{pr}_B(x)\|^2$ . We then have

$$\begin{aligned} \frac{d\phi}{dt}(\xi(t)) &= \langle \text{pr}_B \circ A(\xi(t)), \text{pr}_B(\xi(t)) \rangle - K\|\text{pr}_B(\xi(t))\| \\ &\leq \|\text{pr}_B \circ A(\xi(t))\|\|\text{pr}_B(\xi(t))\| - K\|\text{pr}_B(\xi(t))\| \\ &\leq (|||\text{pr}_B \circ A|||\|\xi(t)\| - K)\sqrt{\phi(\xi(t))} \\ &\leq (|||\text{pr}_B \circ A|||\|\xi(0)\|e^{|||A|||t} - K)\sqrt{\phi(\xi(t))}, \end{aligned}$$

where we have used the Cauchy-Bunyakovsky-Schwarz inequality along with (3.4). Applying Gronwall's lemma to the last inequality gives

$$\begin{aligned} \phi(\xi(t)) &\leq \frac{(C_1\|\xi(0)\|(e^{C_2t} - 1) + C_2(2\sqrt{\phi(\xi(0))} - Kt))^2}{4C_2^2} \\ &\leq \frac{(C_1\|\xi(0)\|(e^{C_2t} - 1) + C_2(\sqrt{2}\|\xi(0)\| - Kt))^2}{4C_2^2}, \end{aligned}$$

where

$$C_1 = |||\text{pr}_B \circ A|||, \quad C_2 = |||A|||.$$

We define  $f_\alpha: \mathbb{R} \rightarrow \mathbb{R}$  by

$$f_\alpha(t) = C_1\alpha(e^{C_2t} - 1) + C_2(\sqrt{2}\alpha - Kt),$$

We now show that for fixed positive constants  $C_1$  and  $C_2$ , for any  $\epsilon > 0$  it is possible to choose  $\alpha$  so that  $f_\alpha$  has a root in  $[0, \epsilon[$ . To show this note that for  $t \in [0, \epsilon]$  we have

$$f_\alpha(t) \leq f_{\alpha,\epsilon}(t) \triangleq C_1\alpha(e^{C_2\epsilon} - 1) + C_2(\sqrt{2}\alpha - Kt).$$

Thus it suffices to show that we may choose  $\alpha$  sufficiently small that  $f_{\alpha,\epsilon}$  has a root in  $[0, \epsilon[$ . This, however, is clear. Indeed,  $f_{\alpha,\epsilon}$  is a linear function of  $t$  satisfying

$$f_{\alpha,\epsilon}(0) = \alpha(C_1(e^{C_2\epsilon} - 1) + \sqrt{2}C_2), \quad f'_{\alpha,\epsilon}(0) = -KC_2.$$

By choosing  $\alpha$  sufficiently small,  $f_{\alpha,\epsilon}$  can be made to have a positive root as small as one likes.

To complete the proof, let  $R_1 > 0$  have the property that  $B_{Re^{C_2}T_{\max}}(0) \subset \mathcal{X}_1$  and let  $R_2 > 0$  have the property that  $f_{R_2}$  has a root in  $]0, T_{\max}[$ . Define  $R = \min\{R_1, R_2\}$ . Taking  $\mathcal{X}_0(K, T_{\max}, \mathcal{X}_1) = B_R(0)$  gives the result.  $\blacksquare$

**3.4 Remark:** If  $A$  is Hurwitz, then the condition that the time to reach  $\mathcal{S}$  be uniformly bounded can be relaxed (i.e., take  $T_{\max} = \infty$ ) with the ensuing advantage of being able to take  $\mathcal{X}_0(T_{\max}) = \mathcal{V}$ . On the other hand if one demands at least of the following properties of the control law:

- (i) the time to reach  $\mathcal{S}$  be bounded uniformly in  $x$  by  $T_{\max}$ ;
- (ii)  $A$  be allowed to be non-Hurwitz;

then  $\mathcal{X}_0(T_{\max})$  will have to be contained in a compact subset containing 0.  $\bullet$

Now let us define a global control law, again in a geometric manner. For  $x \in \mathcal{V} \setminus \mathcal{S}$  we define  $u_{\text{glob},K}(x) \in \mathcal{U}$  by

$$B(u_{\text{glob},K}(x)) = u_{\text{loc},K}(x) - \text{pr}_B \circ A(x).$$

The idea here is simply that we augment our local control law with a “cancellation of as much of the uncontrolled dynamics as possible.” Such interpretations aside, let us show that this control law does in fact globally attract trajectories to  $\mathcal{S}$ .

**3.5 Proposition:** *Let  $\mathcal{S} \subset \mathcal{V}$  be a rigid sliding subspace for  $\Lambda = (A, B)$ . For each  $x \in \mathcal{V} \setminus \mathcal{S}$  there exists  $T(x) > 0$  so that the maximal integral curve for the initial value problem*

$$\dot{\xi}(t) = A(\xi(t)) + B(u_{\text{glob},K}(\xi(t))), \quad \xi(0) = x$$

*is defined on  $[0, T(x)[$  and so that  $\lim_{t \uparrow T(x)} \xi(t) \in \mathcal{S}$ .*

**Proof:** As in the proof of Proposition 3.3, we extend the inner product on  $\text{image}(B)$  to all of  $\mathcal{V}$  so that  $\mathcal{S}$  is orthogonal to  $\text{image}(B)$ , and we adopt the notation introduced in that proof. We again define  $\psi(x) = \frac{1}{2}\|x\|^2$  and a similar estimate to that performed in the proof of Proposition 3.3 gives

$$\|\xi(t)\| \leq \|\xi(0)\|e^{Ct}$$

for some  $C > 0$ . Thus trajectories do not blow up in finite time. Also again defining  $\phi(x) = \frac{1}{2}\|\text{pr}_B(x)\|^2$  we have

$$\begin{aligned} \frac{d\phi}{dt}(\xi(t)) &= \langle \text{pr}_B \circ A(\xi(t)), \text{pr}_B(\xi(t)) \rangle - K\|\text{pr}_B(\xi(t))\| - \langle \text{pr}_B \circ A(\xi(t)), \text{pr}_B(\xi(t)) \rangle \\ &= -K\|\text{pr}_B(\xi(t))\| = -K\sqrt{\phi(\xi(t))}. \end{aligned}$$

This gives

$$\phi(\xi(t)) = \frac{1}{4}(Kt - 2\sqrt{\phi(\xi(0))})^2,$$

and from this the result follows immediately. ■

**3.3. An example.** Let's look first at a simple example that we treat in some detail to aid in providing some intuition behind the constructions of Sections 3.1 and 3.2. We take  $\mathcal{V} = \mathbb{R}^2$  and  $\mathcal{U} = \mathbb{R}$  and consider

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u.$$

For this system we wish to find a codimension one subspace  $\mathcal{S}$  to serve as a rigid sliding subspace. Rigidity demands that  $\mathcal{S} = \mathcal{S}_a = \text{span}_{\mathbb{R}}((1, a))$  for some  $a \in \mathbb{R}$ . Let us denote  $v_a = (1, a)$  so that

$$A_{\mathcal{S}_a}(v_a) = \text{pr}_{\mathcal{S}_a} \circ A \circ i_{\mathcal{S}_a}(v_a) = (1 + a)v_a.$$

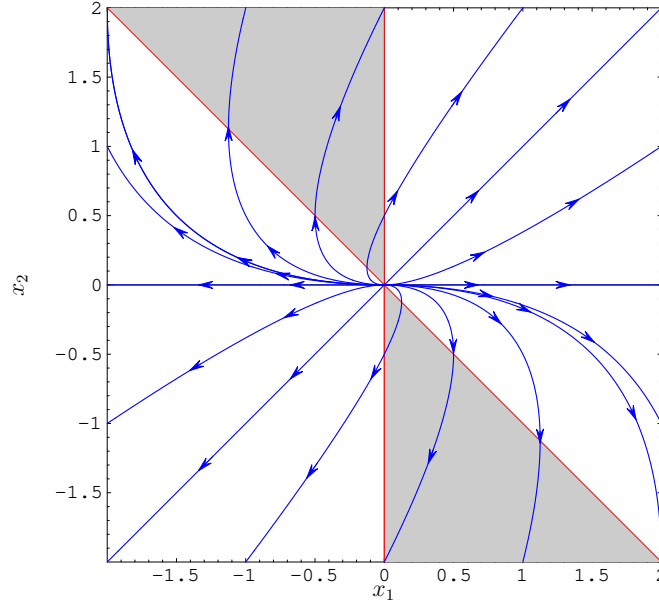
Therefore  $\mathcal{S}_a$  is a stabilising sliding subspace provided that  $a < -1$ . In Figure 3 we show the phase portrait for the system  $\dot{x} = A(x)$ , and the subspaces  $\mathcal{S}_{-1}$  and  $\mathcal{S}_{-\infty}$ . Stabilising rigid sliding subspaces lie within the shaded region in the figure. Note that there is a simple interpretation one can make here that really illustrates what one does in choosing a stabilising rigid sliding subspace. At each tangent space one makes the decomposition into the input direction and the tangent space to the sliding subspace. The sliding subspace will be stabilising when the drift vector field, projected onto the sliding subspace, has the origin as an asymptotically stable equilibrium point. In this example, this amounts to the projection pointing towards the origin. The control exerted to maintain a trajectory on  $\mathcal{S}_a$  is readily verified to be  $u_{\mathcal{S}}(x) = a(a - 1)x_1$ .

Now that we have a stabilising rigid sliding subspace for the linear system on which the restricted dynamics can be specified to be stable, let us look into the control laws  $u_{\text{loc}, K}$  and  $u_{\text{glob}, K}$  defined in Section 3.2. We take as inner product on the input space the usual Euclidean inner product. One then verifies that

$$u_{\text{loc}, K}(x_1, x_2) = K \frac{ax_1 - x_2}{|ax_1 - x_2|}.$$

This is then a control of magnitude  $K$  that points “down” when we are “above”  $\mathcal{S}_a$  and that points “up” when we are “below”  $\mathcal{S}_a$ , these notions making sense provided that  $a < -1$ , as required for stability. In Figure 4 we show the closed-loop phase portrait off  $\mathcal{S}_a$  with the local controller. Note that with the chosen gain for the controller off  $\mathcal{S}_a$ , the basin of attraction is a region shaped as shown in Figure 5. In particular, we see that with bounded controls, our local controller is not able to globally stabilise  $x_0$ . The closed-loop phase portrait for the global controller is shown in Figure 6.





**Figure 3.** The uncontrolled phase portrait (in blue), the subspaces  $\mathcal{S}_{-1}$  and  $\mathcal{S}_{-\infty}$  (in red), and the region occupied by stabilising sliding subspaces (shaded)

#### 4. Output stabilisation using the sliding mode philosophy

Sliding mode control is often presented in the context of output tracking, and the sliding surface is then one on which the dynamics can at least partially be prescribed. In this section we investigate this for SISO linear systems, and compare what comes out of this approach with what comes out of Theorem 3.1.

We consider a system with state space  $\mathcal{V}$ , an  $n$ -dimensional  $\mathbb{R}$  vector space, and with input and output spaces being  $\mathbb{R}$ . The system satisfies the equations

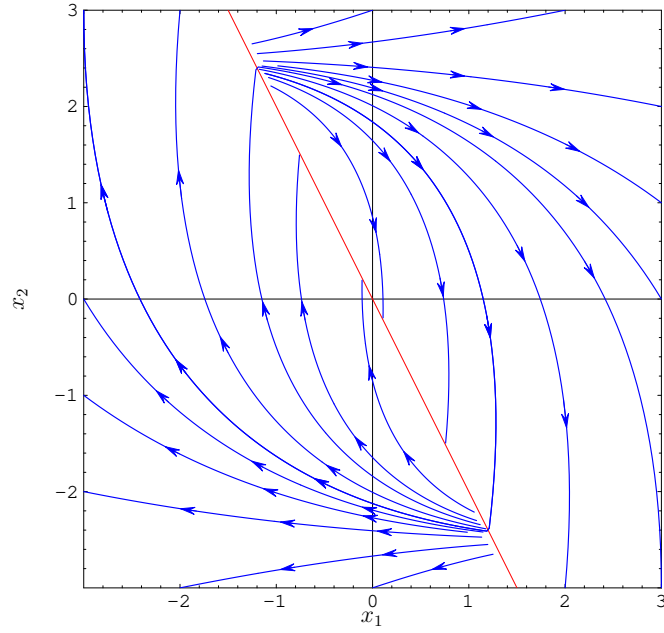
$$\begin{aligned}\dot{\xi}(t) &= A(\xi(t)) + bu(t) \\ \eta(t) &= c(\xi(t)),\end{aligned}$$

where  $b \in \mathcal{V}$  and  $c \in \mathcal{V}^*$ . The **relative degree** of  $(A, b, c)$  is the unique integer  $r$  for which  $c \circ A^{r-1}(b) \neq 0$  and  $c \circ A^{r-2}(b) = 0$ . We shall assume that the relative degree lies in the range  $1 \leq r \leq n$  (it can happen that the relative degree is undefined if the transfer function for the system is identically zero, and we exclude this possibility). We then choose a monic, degree  $r - 1$  Hurwitz polynomial

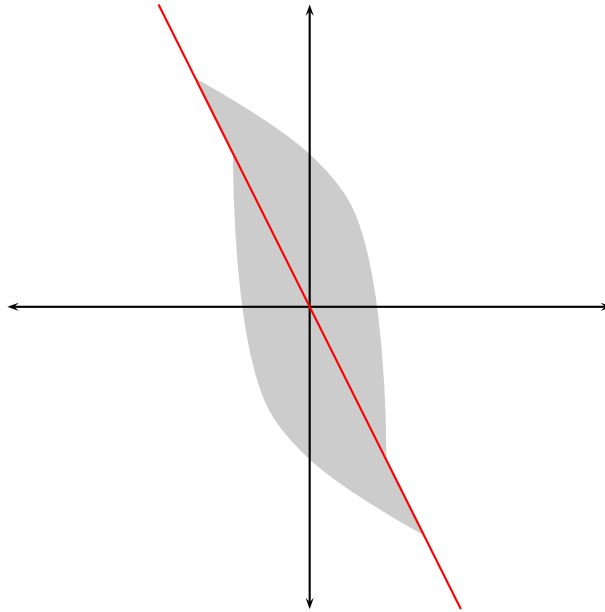
$$P(\lambda) = \lambda^{r-1} + p_{r-2}\lambda^{r-2} + \cdots + p_1\lambda + p_0 \in \mathbb{R}[\lambda],$$

and define a sliding subspace using  $P$ , this being given by

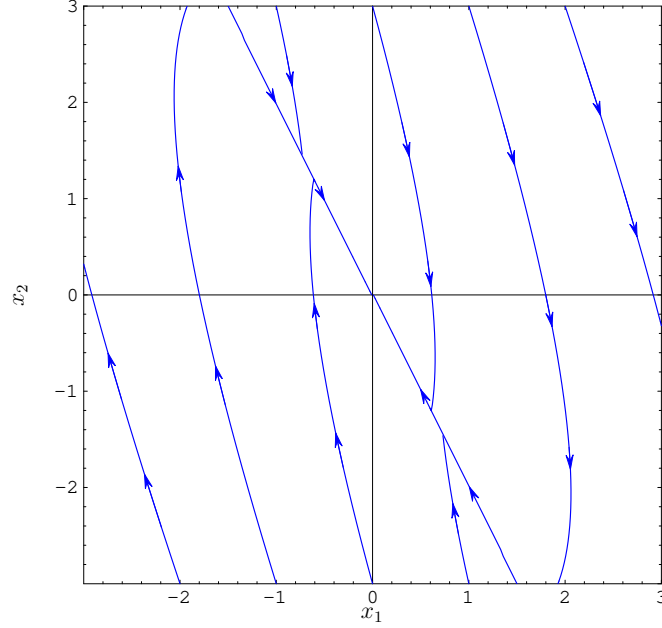
$$\mathcal{S}_P = \ker(c \circ P(A)) \subset \mathcal{V}.$$



**Figure 4.** The closed-loop phase portrait on  $\mathcal{V} \setminus \mathcal{S}_a$  for  $u_{\text{loc},K}$  with  $a = -2$  and  $K = 5$



**Figure 5.** The domain of attraction for the local controller



**Figure 6.** The closed-loop phase portrait on  $\mathcal{V} \setminus \mathcal{S}_a$  for  $u_{\text{glob},K}$  with  $a = -2$  and  $K = 5$

The definition of relative degree ensures both that  $\dim(\mathcal{S}_P) = n - 1$ , and that  $b \notin \mathcal{S}_P$ . Thus  $\mathcal{S}_P$  is indeed a *bona fide* rigid sliding subspace for the system  $\Lambda = (A, b)$ . As such, the dynamics on  $\mathcal{S}_P$  are uniquely determined by Proposition 2.1. Our wish is to characterise these dynamics.

As a first step in doing this, we recall that a subspace  $\mathcal{W} \subset \mathcal{V}$  is  **$(A, b)$ -invariant** if there exists  $f \in \mathcal{V}^*$  so that  $\mathcal{W}$  is an invariant subspace for  $A + b \circ f$ . We denote by  $\mathcal{Z}_{(A,b,c)}$  the largest  $(A, b)$ -invariant subspace contained in  $\ker(c)$ . The following description of  $\mathcal{Z}_{(A,b,c)}$  will be useful to us, and does not seem to follow immediately from existing characterisations in the literature, as far as we know.

**4.1 Lemma:**  $\mathcal{Z}_{(A,b,c)} = \ker(c) \cap \ker(c \circ A) \cap \cdots \cap \ker(c \circ A^{r-1})$ .

**Proof:** Let

$$\mathcal{S} = \ker(c) \cap \ker(c \circ A) \cap \cdots \cap \ker(c \circ A^{r-1}).$$

Clearly  $\mathcal{S} \subset \ker(c)$ . Let us define

$$f = -\frac{1}{c \circ A^{r-1}(b)} c \circ A^r \in \mathcal{V}^*. \quad (4.1)$$

We claim that  $\mathcal{S}$  is an invariant subspace for  $A + b \circ f$ . Indeed, let  $v \in \mathcal{S}$  and compute

$$c \circ (A + b \circ f)(v) = c \circ A(v) - \frac{c(b)}{c \circ A^{r-1}(b)} c \circ A^r(v) = c \circ A(v) = 0,$$

where we have used the definition of relative degree and the fact that  $v \in \mathcal{S}$ . In like manner we compute

$$c \circ A^k \circ (A + b \circ f)(v) = 0, \quad v \in \mathcal{S}, \quad k \in \{1, \dots, r-2\}.$$

We finally compute

$$c \circ A^{r-1} \circ (A + b \circ f)(v) = c \circ A^r(v) - \frac{c \circ A^{r-1}(b)}{c \circ A^{r-1}(b)} c \circ A^r(v) = 0,$$

thus showing that  $\mathcal{S}$  is  $(A, b)$ -invariant. This means that  $\mathcal{S} \subset \mathcal{Z}_{(A, b, c)}$ . To show that  $\mathcal{S} = \mathcal{Z}_{(A, b, c)}$  we recall that  $\dim(\mathcal{Z}_{(A, b, c)}) = n - r = \dim(\mathcal{S})$ , with the assumption that  $(A, b)$  is controllable and  $(A, c)$  is observable.  $\blacksquare$

If

$$\mathcal{F}_{(A, b, c)} = \{f \in V^* \mid (A + b \circ f)(\mathcal{Z}_{(A, b, c)}) \subset \mathcal{Z}_{(A, b, c)}\},$$

one may show that  $\text{spec}((A + b \circ f)|_{\mathcal{Z}_{(A, b, c)}})$  is independent of  $f \in \mathcal{F}_{(A, b, c)}$ . We denote this spectrum by  $\zeta_{(A, b, c)}$  (this is the spectrum of the zero dynamics). The following theorem now provides the spectrum for the dynamics on  $\mathcal{S}_P$ .

**4.2 Theorem:** *With  $\mathcal{S}_P$  constructed as above we have*

$$\text{spec}(A_P) = \text{spec}(P) \cup \zeta_{(A, b, c)},$$

where  $A_P = \text{pr}_{\mathcal{S}_P} \circ A \circ i_{\mathcal{S}_P}$ .

**Proof:** First let us find  $f \in V^*$  so that  $A + b \circ f$  has  $\mathcal{S}_P$  as an invariant subspace. We claim that

$$f = -\frac{1}{c \circ P(A)(b)} c \circ P(A) \circ A \quad (4.2)$$

does the job. To see this, let  $v \in \mathcal{S}_P$  and compute

$$c \circ P(A) \circ (A + b \circ f)(v) = c \circ P(A) \circ A(v) - \frac{c \circ P(A)(b)}{c \circ P(A)(b)} c \circ P(A) \circ A(v) = 0.$$

We also claim that  $f \in \mathcal{F}_{(A, b, c)}$ . Indeed, for  $v \in \mathcal{Z}_{(A, b, c)}$  we compute

$$c \circ (A + b \circ f)(v) = c \circ A(v) - \frac{c(b)}{c \circ P(A)(b)} c \circ P(A)(v) = c \circ A(v) = 0,$$

by Lemma 4.1 and the definition of relative degree. In like manner we compute

$$c \circ A^k \circ (A + b \circ f)(v) = 0, \quad v \in \mathcal{Z}_{(A, b, c)}, \quad k \in \{1, \dots, r-2\}. \quad (4.3)$$

We further compute

$$c \circ A^{r-1} \circ (A + b \circ f)(v) = c \circ A^r(v) - \frac{c \circ A^{r-1}(b)}{c \circ P(A)(b)} c \circ P(A) \circ A(v).$$

Now we note that  $c \circ P(A)(b) = c \circ A^{r-1}(b)$  by definition of relative degree, and we compute  $c \circ P(A) \circ A(v) = c \circ A^r(v)$  by Lemma 4.1. Thus we have  $c \circ A^{r-1}(A_P(v)) = 0$  if  $v \in \mathcal{Z}_{(A, b, c)}$ .

This, along with (4.3), shows that  $A_P(v) \in \mathcal{Z}_{(A,b,c)}$  whenever  $v \in \mathcal{Z}_{(A,b,c)}$ . Thus  $A_P$  restricts to  $\mathcal{Z}_{(A,b,c)}$ , and the spectrum of the restriction is exactly  $\zeta_{(A,b,c)}$ .

Let us denote by  $\tilde{A}_P \in L(\mathcal{S}_P/\mathcal{Z}_{(A,b,c)}; \mathcal{S}_P/\mathcal{Z}_{(A,b,c)})$  the linear map induced by  $A_P$ . We claim that  $\text{spec}(\tilde{A}_P) = \text{spec}(P)$ . To look at this we define a basis for  $\mathcal{V}^*$  by  $\{\omega^0, \dots, \omega^{r-2}, \omega^{r-1}, \beta^1, \dots, \beta^{n-r}\}$  where  $\omega^k = c \circ A^k$ . Also let  $\{v_0, \dots, v_{r-1}, w_1, \dots, w_{n-r}\}$  be the basis for  $\mathcal{V}$  dual to the given basis for  $\mathcal{V}^*$ . Note that with respect to this basis,  $\mathcal{S}_P$  is defined by those vectors whose components satisfy the relation

$$v_{r-1} + p_{r-2}v_{r-2} + \dots + p_1v_1 + p_0v_0 = 0. \quad (4.4)$$

We claim that  $\omega^0, \dots, \omega^{r-2}$  are linearly independent on  $\mathcal{S}_P$ . Indeed, suppose that there are  $\alpha_0, \dots, \alpha_{r-2} \in \mathbb{R}$  so that

$$\sum_{j=0}^{r-2} \alpha_j \omega^j(v) = 0, \quad v \in \mathcal{S}_P.$$

Note that  $v_j = p_j v_{r-1} + v_j$ ,  $j \in \{0, 1, \dots, r-2\}$ , lies in  $\mathcal{S}_P$  by (4.4). But we also have

$$\sum_{k=0}^{r-2} \alpha_k \omega^k(v_j) = \alpha_j = 0,$$

thus showing linear independence of  $\{\omega_0, \dots, \omega_{r-2}\}$  restricted to  $\mathcal{S}_P$ . Now note that  $\omega^0, \omega^1, \dots, \omega^{r-2} \in \text{ann}(\mathcal{Z}_{(A,b,c)})$ , and so naturally form a basis for  $(\mathcal{S}_P/\mathcal{Z}_{(A,b,c)})^*$ , where  $\text{ann}(\cdot)$  denotes the annihilator. We denote by  $\{y_0, y_1, \dots, y_{r-2}\}$  the dual basis for  $\mathcal{S}_P/\mathcal{Z}_{(A,b,c)}$ . First let us determine the representation for  $\tilde{A}_P^*$  in the basis  $\{\omega^0, \omega^1, \dots, \omega^{r-2}\}$ . Thinking of these as elements of  $\mathcal{V}^*$  we have

$$\begin{aligned} (A + b \circ f)^* \omega^0 &= \omega^0 \circ (A + b \circ f) = \omega^1 \\ (A + b \circ f)^* \omega^1 &= \omega^1 \circ (A + b \circ f) = \omega^2 \\ &\vdots \\ (A + b \circ f)^* \omega^{r-2} &= \omega^{r-2} \circ (A + b \circ f) = \omega^{r-1}, \end{aligned}$$

using the fact that  $c \circ A^k(b) = 0$ ,  $k \in \{0, \dots, r-2\}$ . Restriction to  $\mathcal{S}_P$  gives

$$\omega^{r-1} = -p_0\omega^0 - p_1\omega^1 - \dots - p_{r-2}\omega^{r-2}.$$

Therefore the matrix representative of  $\tilde{A}_P^*$  is given by

$$\begin{bmatrix} 0 & 0 & 0 & \dots & 0 & -p_0 \\ 1 & 0 & 0 & \dots & 0 & -p_1 \\ 0 & 1 & 0 & \dots & 0 & -p_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & -p_{n-3} \\ 0 & 0 & 0 & \dots & 1 & -p_{n-2} \end{bmatrix}.$$

From this we immediately see that the characteristic polynomial of  $\tilde{A}_P^*$ , and therefore that of  $\tilde{A}_P$ , is  $P$ , as desired.  $\blacksquare$

## 5. Sliding submanifolds for systems with controllable linearisations

As one might expect, the linear results from the preceding section can be applied locally around an equilibrium point with a controllable linearisation. In this section we address this matter explicitly and indicate some of the relevant issues involved.

We consider control-affine system  $\Sigma = (M, \mathcal{F}, U)$  as in Section 2.1. An **equilibrium point** for  $\Sigma$  is a pair  $(x_0, u_0) \in M \times U$  for which

$$f_0(x_0) + \sum_{a=1}^m u_0^a f_a(x_0) = 0_{x_0},$$

where  $0_{x_0}$  denotes the zero vector in  $T_{x_0}M$ . The **linearisation** of  $\Sigma$  at an equilibrium point  $(x_0, u_0)$  is defined by  $A_\Sigma(x_0) \in L(T_{x_0}M; T_{x_0}M)$  and  $B_\Sigma(x_0) \in L(\mathbb{R}^m; T_{x_0}M)$  where

$$A_\Sigma(x_0)(v) = f_0^T(v) + \sum_{a=1}^m u_0^a f_a^T(v), \quad B_\Sigma(x_0)(u) = \sum_{a=1}^m u^a f_a(x_0),$$

and where  $X^T$  denotes the complete lift to  $TM$  of a vector field  $X$  on  $M$ . In natural coordinates  $(x, v)$  for  $TM$  we have

$$X^T = X^i \frac{\partial}{\partial q^i} + \frac{\partial X^i}{\partial x^j} v^j \frac{\partial}{\partial v^i},$$

so this reduces to the usual notion of linearisation in coordinates where one uses the Jacobian. We assume that  $U$  is convex and contains  $u_0$  in its interior. Thus one has some control “left over” once one has established the equilibrium.

**5.1. From linear to local.** We wish to use a sliding mode control law for the linearisation at an equilibrium to control the nonlinear system in a neighbourhood of the equilibrium point. This requires a means of “transferring” the control law from the tangent space to the state manifold. In practice, this is done naturally when one chooses coordinates. Here we briefly describe what is going on geometrically when one does this. Although this is simple, we could not find this procedure described explicitly in the literature.

We consider a manifold  $M$  of dimension  $n$  and fix a point  $x_0 \in M$ . If  $\phi: M \rightarrow N$  is a smooth mapping between manifolds,  $T_x\phi: T_xM \rightarrow T_{\phi(x)}N$  denotes the derivative at  $x \in M$ .

**5.1 Definition:** A **near identity diffeomorphism at  $x_0$**  is a triple  $(\phi, \mathcal{X}_0, \mathcal{X}_1)$  where

- (i)  $\mathcal{X}_0 \subset T_{x_0}M$  is a neighbourhood of  $0_{x_0}$ ,
- (ii)  $\mathcal{X}_1 \subset M$  is a neighbourhood of  $x_0$ , and
- (iii)  $\phi: \mathcal{X}_0 \rightarrow \mathcal{X}_1$  is a diffeomorphism satisfying
  - (a)  $\phi(0_{x_0}) = x_0$  and
  - (b)  $T_{0_{x_0}}\phi = \text{id}_{T_{x_0}M}$  (where we make the natural identification of  $T_{0_{x_0}}(T_{x_0}M)$  with  $T_{x_0}M$ ). •

In practice, a near identity diffeomorphism arises from a coordinate chart as described by the following lemma. The lemma also tells us that this is the most general way to obtain such a diffeomorphism.

**5.2 Lemma:** *Let  $(\mathcal{A}, \chi)$  be a coordinate chart for  $M$  satisfying  $\chi(x_0) = \mathbf{0} \in \mathbb{R}^n$ . Then the triple  $(\phi = \chi^{-1} \circ T_{x_0}\chi, \mathcal{X}_0 = \phi(\mathcal{A}), \mathcal{X}_1 = \mathcal{A})$  is a near identity diffeomorphism at  $x_0$ .*

*Conversely, let  $(\phi, \mathcal{X}_0, \mathcal{X}_1)$  be a near identity diffeomorphism at  $x_0$  and let  $L: T_{x_0}M \rightarrow \mathbb{R}^n$  be an isomorphism. Then  $(\mathcal{A} = \mathcal{X}_1, \chi = L \circ \phi^{-1})$  is a coordinate chart for  $M$  satisfying  $\chi(x_0) = \mathbf{0} \in \mathbb{R}^n$  and  $\phi = \chi^{-1} \circ T_{x_0}\chi$ .*

**Proof:** With  $(\phi, \mathcal{X}_0, \mathcal{X}_1)$  as defined in the first part of the lemma we compute

$$\phi(0_{x_0}) = \chi^{-1} \circ T_{x_0}\chi(0_{x_0}) = \chi^{-1}(\mathbf{0}) = x_0$$

and, using the chain rule,

$$\begin{aligned} T_{0_{x_0}}\phi(v) &= T_{0_{x_0}}(\chi^{-1} \circ T_{x_0}\chi)(v) \\ &= T_{T_{x_0}\chi(0_{x_0})}\chi^{-1} \circ T_{0_{x_0}}(T_{x_0}\chi)(v) \\ &= T_{\mathbf{0}}\chi^{-1} \circ T_{x_0}\chi(v) \\ &= T_{x_0}(\chi^{-1} \circ \chi)(v) = v, \end{aligned}$$

showing that  $(\phi, \mathcal{X}_0, \mathcal{X}_1)$  is indeed a near identity diffeomorphism at  $x_0$ .

Now let  $(\chi, \mathcal{A})$  be as defined in the second part of the lemma. We then have

$$\chi(x_0) = L \circ \phi^{-1}(x_0) = L(0_{x_0}) = \mathbf{0}$$

and

$$\chi^{-1} \circ T_{x_0}\chi = \phi \circ L^{-1} \circ T_{x_0}(L \circ \phi) = \phi \circ L^{-1} \circ T_{\phi(x_0)}L \circ T_{x_0}\phi = \phi,$$

thus giving the desired assertion. ■

**5.2. From sliding subspaces to sliding submanifolds.** The main result in this section is the following result which essentially tells us that any implementation of a locally stabilising linear sliding mode controller for the linearisation of a nonlinear system will be locally stabilising for the full system. There are three components to this: (1) showing that a rigid sliding subspace for the linearisation gives rise to many rigid sliding submanifolds; (2) that if the sliding subspace for the linearisation is stabilising, then locally so are all corresponding sliding submanifolds; (3) that one can reach the sliding submanifold in finite time.

**5.3 Theorem:** *Consider a control-affine system  $\Sigma = (M, \mathcal{F}, U)$  with a controllable linearisation at  $(x_0, u_0) \in M \times U$  defined by  $A_\Sigma(x_0)$  and  $B_\Sigma(x_0)$ . If  $\mathcal{S}_{x_0} \subset T_{x_0}M$  is a rigid sliding subspace for the linearisation, and if  $S \subset M$  is a  $C^1$  submanifold passing through  $x_0$  and having the property that  $T_{x_0}S = \mathcal{S}_{x_0}$ , then there exists a neighbourhood  $\mathcal{X}$  of  $x_0$  for which  $S \cap \mathcal{X}$  is a rigid local sliding submanifold for  $\Sigma$ . If  $\mathcal{S}_{x_0}$  is stabilising, then so too is  $S \cap \mathcal{X}$ , provided that  $\mathcal{X}$  is taken sufficiently small.*

*Furthermore, suppose that  $(\phi, \tilde{\mathcal{X}}_0, \tilde{\mathcal{X}}_1)$  is a near identity diffeomorphism at  $x_0$ . Then, for any  $K, T_{\max} > 0$  and for any neighbourhood  $\mathcal{X}_1$  of  $x_0 \in \tilde{\mathcal{X}}_1$ , we may find a neighbourhood  $\mathcal{X}_0(K, T_{\max}, \mathcal{X}_1)$  of  $x_0 \in \tilde{\mathcal{X}}_1$  with the property that if  $\xi: [0, T(x)[ \rightarrow \tilde{\mathcal{X}}_1 \setminus S$  is the maximal integral curve for the initial value problem*

$$\dot{\xi}(t) = f_0(\xi(t)) + \sum_{a=1}^m (u_0^a + u_{\text{loc}, K}^a(\phi^{-1}(\xi(t)))) f_a(\xi(t)), \quad \xi(0) = x \in \mathcal{X}_0(K, T_{\max}, \mathcal{X}_1)$$

*then*

- (i)  $T(x) \leq T_{\max}$ ,
- (ii)  $\xi(t) \in \mathcal{X}_1$  for  $t \in [0, T(x)[$ , and
- (iii)  $\lim_{t \uparrow T(x)} \xi(t) \in \mathcal{S}$ .

**Proof:** If  $S$  is a submanifold of  $M$  for which  $T_{x_0}S = \mathcal{S}_{x_0}$  then it is clear that it  $T_{x_0}S$  is transverse to  $\mathcal{F}_{x_0}$  (as will be the case if  $\mathcal{S}_{x_0}$  is a rigid sliding subspace for the linearisation), then  $T_xS$  is transverse to  $\mathcal{F}_x$  for  $x$  in a neighbourhood  $\mathcal{X}$  of  $x_0$ . This is the first part of the first assertion.

Let  $\text{pr}_S: TM|(S \cap \mathcal{X}) \rightarrow TS|(S \cap \mathcal{X})$  be the projection onto the tangent bundle of  $S$  and let  $\text{pr}_{\mathcal{F}}: TM|(S \cap \mathcal{X}) \rightarrow \mathcal{F}|(S \cap \mathcal{X})$ , both defined with respect to the decomposition  $T_xM = T_xS \oplus \mathcal{F}_x$  for  $x \in \mathcal{X}$ . Let  $\tilde{f}_0$  be the vector field induced by  $\Sigma$  on  $S \cap \mathcal{X}$  by rigidity of  $S$ , as in Proposition 2.1. Let  $u_S: S \cap \mathcal{X} \rightarrow U$  be the control with the property that

$$\tilde{f}_0(x) = f_0(x) + \sum_{a=1}^m u_S^a(x) f_a(x). \quad (5.1)$$

Let us extend the controls  $u_S$  to a neighbourhood of  $S$  in an arbitrary way so that we may use (5.1) to define a vector field in a neighbourhood of  $S$ . By rigidity we must have  $u_S(x_0) = u_0$ . Therefore,  $x_0$  is an equilibrium point for  $\tilde{f}_0$ . What's more, for  $v \in \mathcal{S}_{x_0}$  we may compute

$$\begin{aligned} \tilde{f}_0^T(v) &= \text{pr}_S \left( f_0^T(v) + \sum_{a=1}^m u_0^a f_a^T(v) + \sum_{a=1}^m du^a(v) f_a(x_0) \right) \\ &= \text{pr}_S \left( f_0^T(v) + \sum_{a=1}^m u_0^a f_a^T(v) \right). \end{aligned}$$

Thus the linearisation of  $\tilde{f}_0$  at  $x_0$  is the restriction of  $A_{\Sigma}(x_0)$  to  $\mathcal{S}_{x_0}$ . Since this restriction is Hurwitz provided  $\mathcal{S}_{x_0}$  is stabilising,  $x_0$  is an asymptotically stable equilibrium point for  $\tilde{f}_0$ , meaning that  $S \cap \mathcal{X}$  is stabilising for a sufficiently small neighbourhood  $\mathcal{X}$  of  $x_0$ .

Now we show that we may choose  $\mathcal{X}_0(K, T_{\max}, \mathcal{X}_1)$  sufficiently small that trajectories with initial conditions in the set will reach  $S$  in finite time. To do this, choose a chart  $(\mathcal{A}, \chi)$  about  $x_0$  with coordinates  $(x^1, \dots, x^n)$  having the following properties:

1.  $\chi$  takes values in  $\mathbb{R}^{n-m} \times \mathbb{R}^m$ ;
2.  $\chi(S \cap \mathcal{A}) = \chi(\mathcal{A}) \cap (\mathbb{R}^{n-m} \times \{\mathbf{0}\})$ ;
3.  $\chi(x_0) = (\mathbf{0}, \mathbf{0})$ ;
4.  $\text{span}_{\mathbb{R}}(\frac{\partial}{\partial x^{n-m+1}}|_{x_0}, \dots, \frac{\partial}{\partial x^n}|_{x_0}) = \mathcal{F}_{x_0}$ ;
5. the inner product on  $\mathcal{F}_{x_0}$  induced by the inner product on the control space  $\mathbb{R}^m$  is exactly the standard inner product in the coordinates  $(x^{n-m+1}, \dots, x^n)$ .

With these coordinates, let us now agree to identify  $M$  with  $\chi(\mathcal{A})$ . Thus we write a point in  $M$  as  $\mathbf{x}$ . We let  $\mathbf{A}$  and  $\mathbf{B}$  be the matrix representations for the linearisation. Let  $\text{pr}_1: \mathbb{R}^n \rightarrow \mathbb{R}^{n-m} \times \{\mathbf{0}\}$  and  $\text{pr}_2: \mathbb{R}^n \rightarrow \{\mathbf{0}\} \times \mathbb{R}^m$  be the projections. Now write

$$f_0(\mathbf{x}) + \sum_{a=1}^m u_0^a f_a(\mathbf{x}) = \mathbf{A}\mathbf{x} + \tilde{f}(\mathbf{x}),$$



so defining  $\tilde{f}(\mathbf{x})$ . Similarly write

$$\sum_{a=1}^m u_{\text{loc},K}^a(\phi^{-1}(\mathbf{x}))f_a(\mathbf{x}) = -K \frac{\text{pr}_2(\mathbf{x})}{\|\text{pr}_2(\mathbf{x})\|} + \tilde{g}(\mathbf{x}),$$

so defining  $\tilde{g}(\mathbf{x})$ . Note that with  $\tilde{f}$  and  $\tilde{g}$  defined in this way we have  $\tilde{f}(\mathbf{0}) = \tilde{g}(\mathbf{x}) = \mathbf{0}$ . Therefore, there exists  $R_1 > 0$  so that

$$\|\tilde{f}(\mathbf{x}) + \tilde{g}(\mathbf{x})\| \leq M\|\mathbf{x}\|, \quad \mathbf{x} \in B_{R_1}(\mathbf{0}).$$

Define  $\psi(\mathbf{x}) = \frac{1}{2}\|\mathbf{x}\|^2$  and compute

$$\begin{aligned} \frac{d\psi}{dt}(\mathbf{x}(t)) &= \langle \mathbf{A}(\mathbf{x}(t)), \mathbf{x}(t) \rangle - \frac{K}{\|\text{pr}_2(\mathbf{x}(t))\|} \langle \text{pr}_2(\mathbf{x}(t)), \mathbf{x}(t) \rangle \\ &\quad + \langle \tilde{f}(\mathbf{x}(t)) + \tilde{g}(\mathbf{x}(t)), \text{pr}_2(\mathbf{x}(t)) \rangle \\ &= \langle \mathbf{A}\mathbf{x}(t), \mathbf{x}(t) \rangle - K\|\text{pr}_2(\mathbf{x}(t))\| + \langle \tilde{f}(\mathbf{x}(t)) + \tilde{g}(\mathbf{x}(t)), \text{pr}_2(\mathbf{x}(t)) \rangle \\ &\leq \| \mathbf{A} \| \|\mathbf{x}(t)\|^2 + M\|\mathbf{x}(t)\| \|\text{pr}_2(\mathbf{x}(t))\| \\ &\leq (\| \mathbf{A} \| + M)\|\mathbf{x}(t)\|^2 = 2(\| \mathbf{A} \| + M)\psi(\mathbf{x}(t)), \end{aligned}$$

where we have twice use the Cauchy-Bunyakovsky-Schwarz inequality. Gronwall's lemma then gives

$$\|\mathbf{x}(t)\| \leq \|\mathbf{x}(0)\| e^{(\| \mathbf{A} \| + M)t}. \quad (5.2)$$

Now we define  $\phi(\mathbf{x}) = \frac{1}{2}\|\text{pr}_2(\mathbf{x})\|^2$  and compute

$$\begin{aligned} \frac{d\phi}{dt}(\mathbf{x}(t)) &= \langle \text{pr}_2(\mathbf{A}(\mathbf{x}(t))), \text{pr}_2(\mathbf{x}(t)) \rangle - K\|\text{pr}_2(\mathbf{x}(t))\| \\ &\quad + \langle \text{pr}_2(\tilde{f}(\mathbf{x}(t)) + \tilde{g}(\mathbf{x}(t))), \text{pr}_2(\mathbf{x}(t)) \rangle \\ &\leq \|\text{pr}_2(\mathbf{A}(\mathbf{x}(t)))\| \|\text{pr}_2(\mathbf{x}(t))\| - K\|\text{pr}_2(\mathbf{x}(t))\| + M\|\mathbf{x}(t)\| \|\text{pr}_2(\mathbf{x}(t))\| \\ &\leq ((\|\text{pr}_2 \mathbf{A}\| + M)\|\mathbf{x}(t)\| - K)\sqrt{\phi(\mathbf{x}(t))} \\ &\leq ((\|\text{pr}_2 \mathbf{A}\| + M)\|\mathbf{x}(0)\| e^{\| \mathbf{A} \| t^2} - K)\sqrt{\phi(\mathbf{x}(t))}, \end{aligned}$$

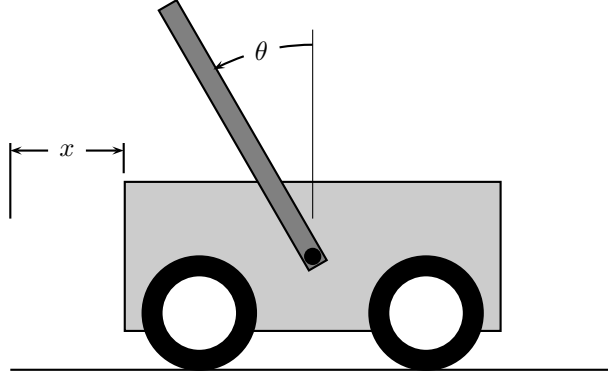
again using the Cauchy-Bunyakovsky-Schwarz inequality several times. The result now follows from a repetition of the calculations at the end of the proof of Proposition 3.3. ■

## 6. An example

We consider in this section the pendulum/cart problem as depicted in Figure 7, where the pendulum is assumed to be a rod of uniform mass density. The state space for the system is  $M = T(\mathbb{R} \times \mathbb{S}^1)$ , the tangent bundle of the cylinder. We use coordinates  $(x, \theta)$  for the cylinder as shown in Figure 7, and use as coordinates for  $M$  the induced natural coordinates  $(x, \theta, v_x, v_\theta)$ . The input to the system is a force  $u$  pushing the cart. In these coordinates, the Euler-Lagrange equations are

$$\begin{aligned} \ddot{x} &= \frac{6mg \cos \theta \sin \theta - m\ell \sin \theta \dot{\theta}^2}{2m + 5M + 3M \cos(2\theta)} + \frac{5 + 3 \cos(2\theta)}{2m + 5M + 3M \cos(2\theta)} u \\ \ddot{\theta} &= \frac{12(M + m)g \sin \theta + 3M\ell \sin(2\theta)\dot{\theta}^2}{(2m + 5M + 3M \cos(2\theta))\ell} + \frac{12 \cos \theta}{(2m + 5M + 3M \cos(2\theta))\ell} u, \end{aligned}$$

where



**Figure 7.** Pensulum on a cart

- $M$  mass of cart
- $m$  mass of pendulum
- $\ell$  length of pendulum arm
- $g$  acceleration due to gravity.

The equilibrium state we consider is that corresponding to the pendulum pointing straight up with no force exerted. The linearisation at this equilibrium is

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{3mg}{4M+m} & 0 & 0 \\ 0 & \frac{6(M+m)g}{(4M+m)\ell} & 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \frac{4}{4M+m} \\ \frac{6}{(4M+m)\ell} \end{bmatrix}$$

We think of the matrices  $\mathbf{A}$  and  $\mathbf{B}$  as being the representatives of respective linear maps relative to the basis  $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial v_x}, \frac{\partial}{\partial v_\theta}\}$  for the tangent space at the equilibrium position.

**6.1. Output stabilisation.** We shall consider a large class of outputs, and consider the nature of the output stabilisation problem by determining the relative degree and the character of the zero dynamics. Thus we take as output

$$\mathbf{c} = [c_x \quad c_\theta \quad 0 \quad 0]$$

where  $c_x^2 + c_\theta^2 \neq 0$ . Thus we take as output a linear function of the positions of the system. We compute

$$\begin{aligned} \mathbf{c}\mathbf{b} &= 0 \\ \mathbf{c}\mathbf{A}\mathbf{b} &= \frac{6c_\theta + 4\ell c_x}{(4M+m)\ell} \\ \mathbf{c}\mathbf{A}^2\mathbf{b} &= 0 \\ \mathbf{c}\mathbf{A}^3\mathbf{b} &= \frac{18g(m\ell c_x + 2(M+m)c_\theta)}{(4M+m)^2\ell^2}. \end{aligned}$$

If  $\mathbf{cAb} = 0$  then one can verify that  $\mathbf{cA}^3\mathbf{b} \neq 0$ , so the system has relative degree 4. In this case, the spectrum of the zero dynamics is empty, and all dynamics on the sliding subspace  $\mathcal{S}_P$  are determined by the roots of the degree 3 monic polynomial  $P$ . Thus the more interesting case is the generic one, where  $\mathbf{cAb} \neq 0$ . Here one determines that

$$\mathcal{Z}_{(A,b,c)} = \text{span}_{\mathbb{R}}((c_\theta, -c_x, 0, 0), (0, 0, c_\theta, -c_x)).$$

Using the two vectors in the above expression as a basis for  $\mathcal{Z}_{(A,b,c)}$  we ascertain that using (4.1) we have the matrix for  $(A + b \circ f)|_{\mathcal{Z}_{(A,b,c)}}$  given by

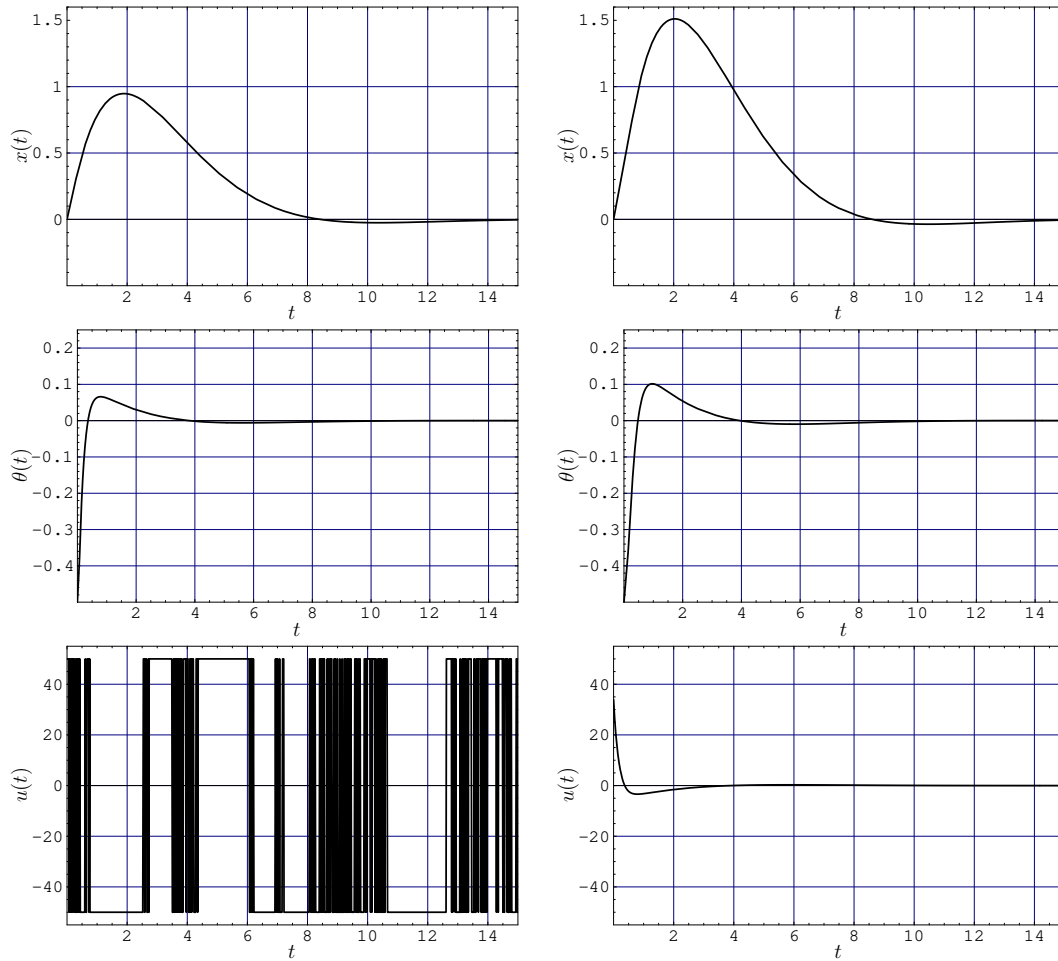
$$\begin{bmatrix} 0 & 1 \\ \frac{3gc_x}{3c_\theta + 2\ell c_x} & 0 \end{bmatrix}.$$

Since this matrix always has a positive real eigenvalue, this means that the only choice of output involving only configurations of the system, and that will stabilise the state along with the output, is the special one that renders  $\mathbf{cAb} = 0$ .

**6.2. Stabilisation using linearisation.** In this section we illustrate an implementation of the ideas behind Theorem 5.3 when applied to the inverted pendulum example. Thus we work with the linearisation of the system, doing pole placement on the sliding subspace via Theorem 3.1. To implement the linear controller for the nonlinear system, we use the near identity diffeomorphism of the first part of Lemma 5.2 corresponding to the coordinate chart for  $T(\mathbb{R} \times \mathbb{S}^1)$  chosen above (thus we implement the linear controller for the nonlinear system in the “obvious” way). To make the resulting simulations interesting, we choose the sliding subspace for the linearisation by first selecting poles using LQR. These poles turn out to have the form  $\{-\lambda_1, -\lambda_2, -\sigma \pm i\omega\}$  for  $\lambda_1, \lambda_2, \sigma, \omega > 0$  and  $\lambda_1 > \lambda_2$ . The sliding subspace for the linearised system is then chosen to be the eigenspace for the closed-loop system corresponding to the eigenvalues  $\{-\lambda_2, -\sigma \pm i\omega\}$ . This allows us to make a meaningful comparison with the LQR controller for the system. In Figures 8 and 9 we show the resulting simulations for two different initial conditions and for the parameters  $M = 2$ ,  $m = 1$ ,  $\ell = \frac{1}{2}$ , and  $g = 9.81$ . The sliding mode controller uses the local control law of Section 3.2 with  $K = 50$ . This value of  $K$  was chosen so that the maximum value of the sliding mode control law and the LQR control were roughly the same for the set of initial conditions simulated, cf. Figure 9. Note that there is not a radical difference in performance of the two systems. The sliding mode controller works better for the initial condition closer to the equilibrium point. This is not surprising since it uses a larger maximum control effort than the LQR controller is allowed.

## 7. Summary

In this paper we have provided a viewpoint for sliding mode control that emphasises structural aspects of the method different from those commonly emphasised. We saw in Section 2 that this leads to a simplification of the notion of what is accomplished by the equivalent control. Also, we saw in Section 2.4 that our way of looking at sliding mode control allowed a simple characterisation of all sliding mode controllers for a simple example. In Section 3 we considered the linear case, providing a characterisation of known results in our geometric setting. The matter of locally extending the linearisation methods

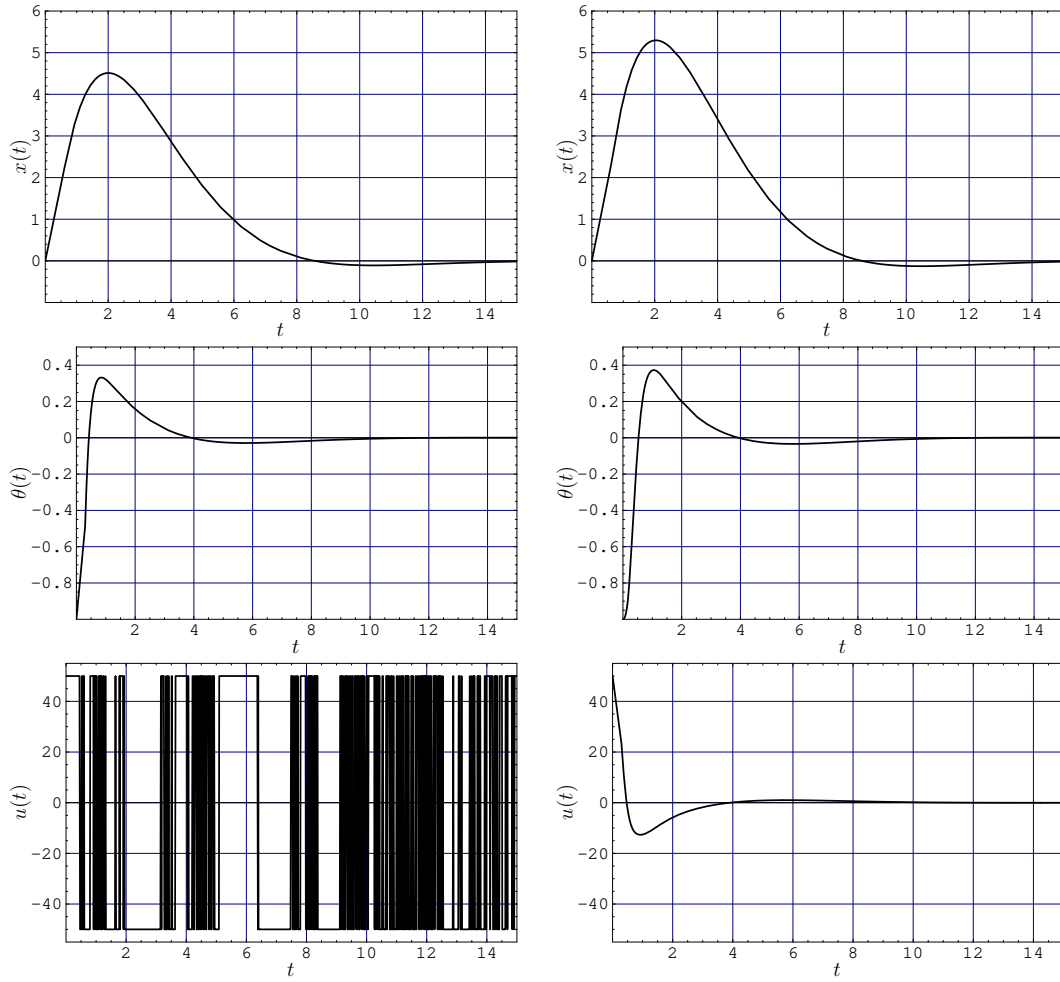


**Figure 8.** Sliding mode control (left) versus LQR (right). The top plot is  $x$ , the middle plot is  $\theta$ , and the bottom plot is  $u$ . The initial state is  $(x, \theta, \dot{x}, \dot{\theta}) = (0, -\frac{1}{2}, 0, 0)$ .

to a neighbourhood of a linearly controllable equilibrium were systematically explored in Section 5. In future work, we will further illustrate the utility of our geometric approach by investigating approaches to extend local sliding submanifolds obtained by linearisation as in Section 5 to more global sliding submanifolds. We will also look at systems for which it is not possible to design a sliding submanifold using the linearisation as a starting point.

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**Figure 9.** Sliding mode control (left) versus LQR (right). The top plot is  $x$ , the middle plot is  $\theta$ , and the bottom plot is  $u$ . The initial state is  $(x, \theta, \dot{x}, \dot{\theta}) = (0, -1, 0, 0)$ .

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