Some jet bundle structure

Andrew D. Lewis*

26/10/2005

Abstract

Two results concerning the structure of jet bundles are presented and explained.

Introduction. We present, discuss, and hopefully make clear, two results stated in [Pommaret 1978] as Lemma 9.12 and Proposition 9.13 in Chapter 1.

The setup is a fibred manifold $\pi\colon \mathsf{Y} \to \mathsf{X}$. By $\mathsf{V}\pi = \ker(T\pi)$ we denote the vertical subbundle of TY. Let us denote $\nu = \pi \circ (\pi_{\mathsf{TY}}|\mathsf{V}\pi)$ the natural projection from $\mathsf{V}\pi$ to X. (By $\pi_{\mathsf{TM}}\colon \mathsf{TM} \to \mathsf{M}$ we denote the tangent bundle projection.) We denote by $\mathsf{J}_k\pi$ the bundle of k-jets of sections of Y, and we denote by $\pi_k\colon \mathsf{J}_k\pi \to \mathsf{X}$ and $\pi_l^k\colon \mathsf{J}_k\pi \to \mathsf{J}_l\pi$ $(k \geq l)$ the natural projections. If $(\xi, \mathcal{U})$ is a local section, then $j_k\xi\colon \mathcal{U} \to \mathsf{J}_k\pi$ denotes the corresponding local section of $\mathsf{J}_k\pi$.

First result. We consider the fibred manifold $\nu\colon \mathsf{V}\pi \to \mathsf{X}$ and denote by $\mathsf{J}_k\nu$ the bundle of k-jets of sections. We first make an observation about this manifold being, in fact, the total space of a vector bundle over $\mathsf{J}_k\pi$. Let us define the projection, which we denote by $\alpha_k\colon \mathsf{J}_k\nu \to \mathsf{J}_k\pi$. For a local section $(\eta, \mathcal{U})$ of $\nu\colon \mathsf{V}\pi \to \mathsf{X}$, denote by $(\xi_\eta, \mathcal{U})$ the local section of $\pi\colon \mathsf{Y} \to \mathsf{X}$ defined by $\xi_\eta(x) = \pi_{\mathsf{TY}}(\eta(x))$. Let us define $\alpha_k(j_k\eta(x)) = j_k\xi_\eta(x)$.

We then have the following result.

1 Lemma: (The vector bundle structure of $\mathsf{J}_k\nu$) *The map $\alpha_k\colon \mathsf{J}_k\nu \to \mathsf{J}_k\pi$ is a surjective submersion, and resulting fibred manifold has a vector bundle structure.*

Proof: That α_k is a surjective submersion is easily checked in local coordinates. We exhibit the vector bundle structure of $\alpha_k\colon \mathsf{J}_k\nu \to \mathsf{J}_k\pi$ by indicating how one defines vector addition and scalar multiplication in the fibres of α_k. Let $(\eta_1, \mathcal{U})$ and $(\eta_2, \mathcal{U})$ be local sections of $\nu\colon \mathsf{Y} \to \mathsf{X}$ such that $j_k\xi_{\eta_1}(x) = j_k\xi_{\eta_2}(x)$ for all $x \in \mathcal{U}$. Thus $j_k\eta_1(x)$ and $j_k\eta_2(x)$ are in the same fiber of α_k. Moreover, $\eta_1(x)$ and $\eta_2(x)$ are in the same fibre of $\pi_{\mathsf{TY}}|\mathsf{V}\pi$ for each $x \in \mathcal{U}$. Thus we may define the local section $(\eta_1+\eta_2, \mathcal{U})$ by asking that $(\eta_1+\eta_2)(x) = \eta_1(x)+\eta_2(x)$, where addition on the right takes place in $\mathsf{V}_{\xi_{\eta_1}(x)}\pi$. We then define

$$(j_k\eta_1 + j_k\eta_2)(x) = j_k(\eta_1 + \eta_2)(x),$$

so defining addition in the fibres of α_k. Scalar multiplication is defined, for $a \in \mathbb{R}$, by $a(j_k\eta(x)) = j_k(a\eta)(x)$, where $(a\eta, \mathcal{U})$ is the local section defined by $a\eta(x) = a(\eta(x))$ with scalar multiplication on the right being that in $\mathsf{V}_{\xi_\eta(x)}\pi$.

Now one should really verify that there is a vector bundle structure by defining a vector bundle atlas. This is done easily in natural coordinates, however. We leave the working out of the details of this to the reader. ■

*Associate Professor, Department of Mathematics and Statistics, Queen's University, Kingston, ON K7L 3N6, Canada
Email: andrew@mast.queensu.ca, URL: http://penelope.mast.queensu.ca/~andrew/

To get an understanding of the vector bundle structure, let us consider in coordinates the case when $k = 1$. We denote adapted coordinates for Y by $(\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{U} \times \mathcal{V} \subset \mathbb{R}^n \times \mathbb{R}^m$. Coordinates for $\mathsf{V}\pi$ are denoted by $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{v})$ where $\boldsymbol{v} \in \mathbb{R}^m$. Coordinates for $\mathsf{J}_1\nu$ are then denoted by $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{v}, \boldsymbol{p}_1, \boldsymbol{p}_2)$ where $\boldsymbol{p}_a \in L(\mathbb{R}^n; \mathbb{R}^m)$ for $a \in \{1, 2\}$. Thus $\boldsymbol{p}_1$ is to be thought of as the partial derivatives of $\boldsymbol{y}$ with respect to $\boldsymbol{x}$, and $\boldsymbol{p}_2$ is to be thought of as the partial derivatives of $\boldsymbol{v}$ with respect to $\boldsymbol{x}$. The local representative of α_1 is then easily seen to be

$$(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{v}, \boldsymbol{p}_1, \boldsymbol{p}_2) \mapsto (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{p}_1).$$

Thus, imagining how this looks for arbitrary k, the fibres of α_k are formed by $\boldsymbol{v}$ and its partial derivatives with respect to $\boldsymbol{x}$. The vector bundle structure is obtained from the fact that $\boldsymbol{v}$ is to be thought of as a vertical vector, and so is an element of a vector space. Addition and scalar multiplication are then just addition and scalar multiplication of these vectors and their derivatives.

Now we consider another vector bundle over $\mathsf{J}_k\pi$, namely the vertical bundle $\mathsf{V}\pi_k$ of the fibred manifold $\pi_k \colon \mathsf{J}_k\pi \to \mathsf{X}$. This is obviously a vector bundle over $\mathsf{J}_k\pi$.

The final construction preliminary to the statement of the result is the following. Let $\eta \colon \mathsf{X} \to \mathsf{V}\pi$ be a section of $\nu \colon \mathsf{V}\pi \to \mathsf{X}$ with ξ_η the corresponding section of $\pi \colon \mathsf{Y} \to \mathsf{X}$. Let $x_0 \in \mathsf{X}$, let $\mathcal{U}$ be a neighbourhood of x_0, and define $\sigma_{\eta,x_0} \colon \mathcal{U} \times I \to \mathsf{Y}$ such that

1. $I \subset \mathbb{R}$ is an interval for which $0 \in \operatorname{int}(I)$,
2. $\sigma_{\eta,x_0}(x, 0) = \xi_\eta(x)$ for all $x \in \mathcal{U}$,
3. $\pi \circ \sigma_{\eta,x_0}(x, t) = x$ for all $(x, t) \in \mathcal{U} \times I$, and
4. $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \sigma_{\eta,x_0}(x_0, t) = \eta(x_0) \in \mathsf{V}_{\xi_\eta(x_0)}\pi$.

Thus σ_{η,x_0} is a variation of ξ_η at x_0 in the direction of $\eta(x_0)$. Define $j_k\sigma_{\eta,x_0} \colon \mathcal{U} \times I \to \mathsf{J}_k\pi$ to be such that $x \mapsto j_k\sigma_{\eta,x_0}(x, t)$ is the k-jet of the local section $x \mapsto \sigma_{\eta,x_0}(x, t)$. Thus $t \mapsto j_k\sigma_{\eta,x_0}(x, t)$ is a curve in $\pi_k^{-1}(x)$ for each $x \in \mathcal{U}$, and so has a tangent vector field that is vertical with respect to the projection π_k.

We now have the following result.

2 Proposition: ($\mathsf{V}\pi_k \simeq \mathsf{J}_k\nu$) *The map* $j_k\eta(x) \mapsto \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} j_k\sigma_{\eta,x}(x)$ *is an isomorphism of vector bundles over* $\mathsf{J}_k\pi$.

Proof: This is easily, but tediously, checked in local coordinates. ■

Let us illustrate the result in the case where $k = 1$. We use, as above, local coordinates for $\mathsf{J}_1\nu$ denoted by

$$(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{v}, \boldsymbol{p}_1, \boldsymbol{p}_2) \in \mathcal{U} \times \mathcal{V} \times \mathbb{R}^m \times L(\mathbb{R}^n; \mathbb{R}^m) \times L(\mathbb{R}^n; \mathbb{R}^m).$$

Local coordinates for $\mathsf{V}\pi_1$ are denoted by

$$(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{p}_1, \boldsymbol{v}, \boldsymbol{A}) \in \mathcal{U} \times \mathcal{V} \times L(\mathbb{R}^n; \mathbb{R}^m) \times \mathbb{R}^m \times L(\mathbb{R}^n; \mathbb{R}^m).$$

The vector bundle isomorphism of the proposition looks like

$$(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{v}, \boldsymbol{p}_1, \boldsymbol{p}_2) \mapsto (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{p}_1, \boldsymbol{v}, \boldsymbol{p}_2).$$

The idea, roughly speaking, is that $j_k\eta(x)$ is mapped to the vertical vector consisting of the derivatives of the vertical part of η.

Second result. The second result we state has essentially to do with fixed point sets for the symmetric group. So let us first consider some seemingly pointless constructions along these lines.

For $m \in \mathbb{N}$ denote by S_m the group of bijections of $\{1, \dots, m\}$, i.e., the symmetric group of order m. For $k, l \in \mathbb{N}$, let us define group monomorphisms $\iota_k^L \colon S_k \to S_{k+l}$ and $\iota_l^R \colon S_l \to S_{k+l}$ by

$$\iota_k^L(\sigma)(j) = \begin{cases} \sigma(j), & j \in \{1, \dots, k\}, \\ j, & j \in \{k+1, \dots, k+l\}, \end{cases}$$

and

$$\iota_l^R(\sigma)(j) = \begin{cases} j, & j \in \{1, \dots, k\}, \\ \sigma(j-k)+k, & j \in \{k+1, \dots, k+l\}. \end{cases}$$

Thus $\text{image}(\iota_k^L)$ consists of the permutations of only the first k elements and $\text{image}(\iota_l^R)$ consists of the permutations of only the last l elements. Let us denote $S_k^L = \text{image}(\iota_k^L)$ and $S_l^R = \text{image}(\iota_l^R)$. Note that $S_k^L \times S_l^R$ then consists of those permutations that independently permute the first k terms and the last l terms.

If $\Phi \colon G \times A \to A$ is an action of a group G on a set A, let

$$\text{Fix}(G, A) = \{ x \in A \mid \ \Phi(g, x) = x, \ g \in G \}$$

be the fixed point set.

We now have the following result.

3 LEMMA: (Fixed point sets for actions of the symmetric group) *Let $k, l \in \mathbb{N}$ and let $\Phi \colon S_{k+l} \times A \to A$ be an action of the symmetric group on the set A. Then*

$$\text{Fix}(S_{k+l}, A) = \text{Fix}(S_k^L \times S_l^R, A) \cap \text{Fix}(S_{k+1}^L \times S_{l-1}^R, A).$$

Proof: If $x \in \text{Fix}(S_{k+l}, A)$ then $\Phi(\sigma_1, x) = x$ and $\Phi(\sigma_2, x) = x$ for all $\sigma_1 \in S_k^L \times S_l^R$ and $\sigma_2 \in S_{k+1}^L \times S_{l-1}^R$. Thus

$$x \in \text{Fix}(S_k^L \times S_l^R, A) \cap \text{Fix}(S_{k+1}^L \times S_{l-1}^R, A)$$

and so

$$\text{Fix}(S_{k+l}, A) \subset \text{Fix}(S_k^L \times S_l^R, A) \cap \text{Fix}(S_{k+1}^L \times S_{l-1}^R, A).$$

To prove the opposite inclusion we note that, since every element of S_{k+l} is the product of a finite number of transpositions, it suffices to prove that if

$$x \in \text{Fix}(S_k^L \times S_l^R, A) \cap \text{Fix}(S_{k+1}^L \times S_{l-1}^R, A)$$

then $\Phi(\sigma, x) = x$ for every transposition $\sigma \in S_{k+l}$. Indeed, if $\sigma \in S_{k+l}$ is arbitrary, then write $\sigma = \sigma_1 \circ \cdots \circ \sigma_r$ for transpositions $\sigma_1, \dots, \sigma_r$. Then, if $\Phi(\sigma_a, x) = x$ for each $a \in \{1, \dots, r\}$, we have

$$\Phi(\sigma, x) = \Phi(\sigma_1 \circ \cdots \circ \sigma_r, x) = \Phi(\sigma_1, \Phi(\sigma_2, \dots, \Phi(\sigma_r, x))) = x.$$

Thus we let

$$x \in \text{Fix}(S_k^L \times S_l^R, A) \cap \text{Fix}(S_{k+1}^L \times S_{l-1}^R, A)$$

and we let $\sigma \in S_{k+l}$ be a transposition. If $\sigma \in S^L_k \times S^R_l$ or if $\sigma \in S^L_{k+1} \times S^R_{l-1}$ then we immediately have $\Phi(\sigma, x) = x$. Thus we may as well suppose that

$$\sigma \in S_{k+l} \setminus ((S^L_k \times S^R_l) \cup (S^L_{k+1} \times S^R_{l-1})).$$

If σ is a transposition of the elements $j_1, j_2 \in \{1, \dots, k+l\}$, this means that we may assume that $j_1 \in \{1, \dots, k+1\}$ and $j_2 \in \{k+2, \dots, k+l\}$. Thus

$$\sigma = \begin{pmatrix} 1 & \cdots & j_1 & \cdots & k+1 & \cdots & j_2 & \cdots & k+l \\ 1 & \cdots & j_2 & \cdots & k+1 & \cdots & j_1 & \cdots & k+l \end{pmatrix}.$$

Let us write $\sigma = \sigma_1 \circ \sigma_2 \circ \sigma_1$ where

$$\sigma_1 = \begin{pmatrix} 1 & \cdots & j_1 & \cdots & k+1 & \cdots & j_2 & \cdots & k+l \\ 1 & \cdots & k+1 & \cdots & j_1 & \cdots & j_2 & \cdots & k+l \end{pmatrix}$$
$$\sigma_2 = \begin{pmatrix} 1 & \cdots & j_1 & \cdots & k+1 & \cdots & j_2 & \cdots & k+l \\ 1 & \cdots & j_1 & \cdots & j_2 & \cdots & k+1 & \cdots & k+l \end{pmatrix},$$

noting that $\sigma_1 \in S^L_{k+1}$ and $\sigma_2 \in S^R_l$. Thus $\Phi(\sigma_1, x) = x$ and $\Phi(\sigma_2, x) = x$ and so

$$\Phi(\sigma, x) = \Phi(\sigma_1, \Phi(\sigma_2, \Phi(\sigma_1, x))) = x,$$

and so $x \in \mathrm{Fix}(S_{k+l}, A)$ since we have shown that x is fixed by every transposition. ■

Now let us proceed with what we are really interested in here. We first will find it useful to have the following characterisation of $\mathsf{J}_k\pi$ as a submanifold of an iterated collection of first-order jet bundles. For $k \in \mathbb{N}$, define $\mathsf{J}^k_1\pi$ inductively as follows. Take $\mathsf{J}^1_1\pi = \mathsf{J}_1\pi$, thinking of this as a fibred manifold over X. Then define $\mathsf{J}^k_1\pi$ to be the bundle of 1-jets of sections of $\mathsf{J}^{k-1}_1\pi$ over X.

4 Lemma: ($\mathsf{J}_k\pi$ is a fibred submanifold of $\mathsf{J}^k_1\pi$) *The map*

$$j_k\xi(x) \mapsto \underbrace{j_1 \cdots j_1}_{k \text{ times}} \xi(x)$$

is an embedding of $\mathsf{J}_k\pi$ *in* $\mathsf{J}^k_1\pi$ *as a fibred submanifold over* X.

Proof: This is easily, if a little tediously, verified in natural coordinates. ■

Let us understand the lemma in coordinates in the case when $k = 2$. We denote adapted coordinates for Y by $(\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{U} \times \mathcal{V} \subset \mathbb{R}^n \times \mathbb{R}^m$. Coordinates for $\mathsf{J}_1\pi$ are then denoted by

$$(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{p}_1) \in \mathcal{U} \times \mathcal{V} \times L(\mathbb{R}^n; \mathbb{R}^m),$$

coordinates for $\mathsf{J}_2\pi$ are denoted by

$$(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{p}_1, \boldsymbol{p}_2) \in \mathcal{U} \times \mathcal{V} \times L(\mathbb{R}^n; \mathbb{R}^m) \times L^2_{\mathrm{sym}}(\mathbb{R}^n; \mathbb{R}^m),$$

and coordinates for J^2_1 are denoted by

$$(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{p}_1, \boldsymbol{q}_1, \boldsymbol{q}_2) \in \mathcal{U} \times \mathcal{V} \times L(\mathbb{R}^n; \mathbb{R}^m) \times L(\mathbb{R}^n; \mathbb{R}^m) \times L^2(\mathbb{R}^n; \mathbb{R}^m),$$

recalling that $L(\mathbb{R}^n; L(\mathbb{R}^n; \mathbb{R}^m)) \simeq L^2(\mathbb{R}^n; \mathbb{R}^m)$ via the isomorphism ι_2 given by

$$\iota_2(L)(\boldsymbol{u}_1, \boldsymbol{u}_2) = L(\boldsymbol{u}_1)(\boldsymbol{u}_2).$$

A working through in local coordinates of the map in the lemma gives the local model for the submanifold $\mathsf{J}_2\pi$ of $\mathsf{J}^2_1\pi$ as

$$\left\{ (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{p}_1, \boldsymbol{q}_1, \boldsymbol{q}_2) \mid \ \boldsymbol{q}_1 = \boldsymbol{p}_1, \ \boldsymbol{q}_2 \in L^2_{\text{sym}}(\mathbb{R}^n; \mathbb{R}^m) \right\}.$$

Now, using this inclusion of $\mathsf{J}_k\pi$ in $\mathsf{J}^k_1\pi$, we wish to characterise $\mathsf{J}_{k+l}\pi$ for $k, l \in \mathbb{N}$. First note that we have $\mathsf{J}_{k+l}\pi$ as a submanifold of $\mathsf{J}^{k+l}_1\pi$. Now consider the fibred manifold $\pi_l \colon \mathsf{J}_l\pi \to \mathsf{X}$. Denote by $\mathsf{J}_k\pi_l$ the bundle of k-jets of this fibred manifold. Note that $\mathsf{J}_l\pi$ is a submanifold of $\mathsf{J}^l_1\pi$ with inclusion given by

$$j_l\xi(x) \mapsto \underbrace{j_1 \cdots j_1}_{l \text{ times}}\xi(x).$$

Also, $\mathsf{J}_k\pi_l$ is a submanifold of $\mathsf{J}^k_1\pi_l$ with inclusion

$$j_k j_l\xi(x) = \underbrace{j_1 \cdots j_1}_{k \text{ times}} j_l\xi(x).$$

Since $\mathsf{J}_l\pi$ is a fibred submanifold over X of $\mathsf{J}_k\pi_l$, this gives $\mathsf{J}_k\pi_l$ as a fibred submanifold of $\mathsf{J}^{k+l}_1\pi$.[1]

With these inclusions in mind, we state the following result.

5 PROPOSITION: (Characterisation of $\mathsf{J}_{k+l}\pi$) *For $k, l \in \mathbb{N}$, $\mathsf{J}_{k+l}\pi = (\mathsf{J}_k\pi_l) \cap (\mathsf{J}_{k+1}\pi_{l-1})$.*

Idea of Proof: We work locally, and so let $\mathsf{Y} = \mathcal{U} \times \mathcal{V} \subset \mathbb{R}^n \times \mathbb{R}^m$, taking $\mathsf{X} = \mathcal{U}$ and $\pi(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{x}$. For $r \in \mathbb{N}$ let $\mathcal{U}$ be the r-fold Cartesian product of $\mathcal{U}$ with itself. Denote by $C^\infty(\mathcal{U}^r, \mathcal{V})$ the set of smooth maps from $\mathcal{U}^r$ to $\mathcal{V}$. Define an equivalence relation in $C^\infty(\mathcal{U}^r, \mathcal{V})$ by saying that $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$ are ***equivalent at $\boldsymbol{x}_0 \in \mathcal{U}$*** if

1. $\boldsymbol{f}_1(\boldsymbol{x}_0, \dots, \boldsymbol{x}_0) = \boldsymbol{f}_2(\boldsymbol{x}_0, \dots, \boldsymbol{x}_0)$ and if,
2. for each $r \in \{1, \dots, r\}$ and for each $j_1, \dots, j_r \in \{1, \dots, r\}$,

$$\boldsymbol{D}_{j_1} \cdots \boldsymbol{D}_{j_r}\boldsymbol{f}_1(\boldsymbol{x}_0, \dots, \boldsymbol{x}_0) = \boldsymbol{D}_{j_1} \cdots \boldsymbol{D}_{j_r}\boldsymbol{f}_2(\boldsymbol{x}_0, \dots, \boldsymbol{x}_0).$$

With a little effort, one can convince oneself that the union over $\boldsymbol{x}_0$ of the sets of equivalence classes at $\boldsymbol{x}_0$ is in 1–1 correspondence with the local model for $\mathsf{J}^r_1\pi$.

Now define an action $\tilde{\Phi}$ of S_r on $C^\infty(\mathcal{U}^r, \mathcal{V})$ by

$$\tilde{\Phi}(\sigma, \boldsymbol{f})(\boldsymbol{x}_1, \dots, \boldsymbol{x}_r) = \boldsymbol{f}(\boldsymbol{x}_{\sigma(1)}, \dots, \boldsymbol{x}_{\sigma(r)})$$

and define an action Φ on the local model for $\mathsf{J}^r_1\pi$ by

$$\Phi(\sigma, [\boldsymbol{f}]) = [\tilde{\Phi}(\sigma, \boldsymbol{f})].$$

One can then verify that $\mathsf{J}_k\pi_l \subset \mathsf{J}^{k+l}_1\pi$ is exactly given by $\text{Fix}(S^L_k \times S^R_l, \mathsf{J}^{k+1}_1\pi)$. After this verification, the result follows immediately from Lemma 3. ■

[1] We use here the fact that, if $\mathsf{Z} \subset \mathsf{Y}$ is a fibred submanifold of a fibred manifold $\pi \colon \mathsf{Y} \to \mathsf{X}$, then $\mathsf{J}_k(\pi|\mathsf{Z})$ is a fibred submanifold of $\mathsf{J}_k\pi$ over X for each $k \in \mathbb{N}$.

References

Pommaret, J.-F. [1978] *Systems of Nonlinear Partial Differential Equations and Lie Pseudogroups*, number 14 in Mathematics and its Applications, Gordon & Breach Science Publishers, New York, ISBN 0-677-00270-X.