Geometric partial differential equations: Definitions and properties

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Abstract

Nonlinear partial differential equations are defined as fibred submanifolds of a jet bundle. The definitions of prolongation and symbol are given and given interpretations. Projections, formal properties, and linearisation are also discussed.

Almost no attempt is made to maintain notational consistency with the standard literature.

Contents

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1. Introduction

A partial differential equation is, roughly and somewhat abstractly, a collection of relations between the dependent variables and their derivatives with respect to the independent variables. In this sort of way of thinking about partial differential equations, it is intuitive to think of them as submanifolds of jet bundles. The relations are the equations. However, in the geometric formulation, it is often convenient to think not of equations, but of solutions to these equations. Locally, the two situations are equivalent. This is rather like the fact that a submanifold is locally the level set of a function.

With this geometric idea of partial differential equations at hand, the geometry of jet bundles becomes a useful abstract tool for studying these equations. Here the main ideas are the prolongations of a partial differential equation and the symbol of a partial differentiation. The prolongation of a partial differential is the new partial differential equation obtained by differentiating the original one. The symbol of a partial differential equation is the highest order component in the linearisation of the equation. All of this is made precise and hopefully clear.

2. Fibre bundle and jet bundle notation

I am officially parting ways with the notation of Pommaret [1978], and also of Goldschmidt [1967] and many others. The many (convenient and justifiable) notational abuses in the standard literature can make it difficult for newcomers to capture some of the nuances of the subject.

The review in this section is hasty, unsystematic, disorganised, and incomplete.

All objects will be assumed to be of at least class C^∞.

For a surjective submersion $\pi\colon \mathsf{Y} \to \mathsf{X}$, we denote by $\mathsf{V}\pi \triangleq \ker(T\pi)$ the vertical bundle of π. We denote by 0_{X} the fibred manifold over X with trivial fibre (as manifolds we have $0_{\mathsf{X}} = \mathsf{X}$). Denote the point in the fibre over $x \in \mathsf{X}$ by 0_x. The fibred manifold 0_{X} allows us to be precise in saying that the exactness of certain sequences corresponds to injectivity and surjectivity.

2.1 Lemma: (Exact sequences for fibred monomorphisms and epimorphisms) *Let* $\pi\colon \mathsf{Y} \to \mathsf{X}$ *and* $\tau\colon \mathsf{Z} \to \mathsf{X}$ *be fibred manifolds and let* $\phi\colon \mathsf{Y} \to \mathsf{Z}$ *be fibred morphisms over* id_{X}. *Then* ϕ *is a monomorphism or an epimorphism if and only if the respective sequences*

$$0_{\mathsf{X}} \xrightarrow{\iota} \mathsf{Y} \xrightarrow{\phi} \mathsf{Z} \qquad \mathsf{Y} \xrightarrow{\phi} \mathsf{Z} \xrightarrow{p} 0_{\mathsf{X}}$$

are exact for any fibred morphisms ι *and* p.

Proof: Suppose that the left sequence in the lemma is exact for any fibred morphism ι and suppose that $\phi(y_1) = \phi(y_2)$ for $y_1, y_2 \in \mathsf{Y}$. Clearly we must have $\pi(y_1) = \pi(y_2)$ since ϕ is a fibred morphism. Choose ι such that $\iota(0_{\pi(y_1)}) = y_1$. Then there exists a section η of $\tau\colon \mathsf{Z} \to \mathsf{X}$ such that $\mathrm{image}(\iota) = \ker_\eta(\phi)$. Thus

$$\{\iota(0_{\pi(y)})\} = \{y' \in \pi^{-1}(\pi(y)) \mid \phi(y') = \eta(\pi(y))\}$$

for every $y \in \mathsf{Y}$. In particular, taking y to be y_1 gives

$$\{y_1\} = \{y \in \pi^{-1}(\pi(y_1)) \mid \phi(y) = \eta(\pi(y_1))\}.$$

Note that $\phi(y_2) = \phi(y_1) = \eta(\pi(y_1))$. Therefore,

$$y_2 \in \{y \in \pi^{-1}(\pi(y_1)) \mid \phi(y) = \eta(\pi(y_1))\}$$

and so we must have $y_2 = y_1$. Thus ϕ is injective. Exactness directly implies that $V\phi$ is injective, so ϕ is a monomorphism.

Next suppose that ϕ is injective and let $\iota\colon 0_{\mathsf{X}} \to \mathsf{Y}$ be a fibred morphism. Define a section η of $\tau\colon \mathsf{Z} \to \mathsf{X}$ by $\eta(x) = \phi \circ \iota(x)$. Let $x \in \mathsf{X}$. Since ϕ is injective there exists at most one $y \in \pi^{-1}(x)$ such that $\phi(y) = \eta(\pi(y))$. By definition of η we have $\phi(\iota(x)) = \eta(x)$. Therefore, we then have

$$\ker_\eta(\phi) = \{\, y \in \mathsf{Y} \mid \ \phi(y) = \eta(\pi(y))\} = \{\, \iota(x) \mid \ x \in \mathsf{X}\} = \mathrm{image}(\iota).$$

Since ϕ is a monomorphism, $V\phi$ is injective. Thus the left sequence is exact for any fibred morphism ι.

For the right sequence in the statement of the lemma we make the following observations which are independent of ϕ:

1. there is only one fibred morphism from Z to 0_{X} and it is the one defined by $p(z) = 0_{\tau(z)}$;
2. there is only one section of 0_{X} and it is given by $\zeta\colon x \mapsto 0_x$.

From these observations it follows that

$$\mathsf{Z} = \{\, z \in \mathsf{Z} \mid \ p(z) = \zeta(\tau(z))\} = \ker_\zeta(p).$$

Now suppose that the right sequence in the statement of the lemma is exact. Then

$$\mathrm{image}(\phi) = \ker_\zeta(p) = \mathsf{Z},$$

i.e., ϕ is surjective. It also holds that $V\phi$ is surjective by exactness. Thus ϕ is an epimorphism.

Now suppose that ϕ is surjective. Then

$$\mathrm{image}(\phi) = \mathsf{Z} = \ker_\zeta(p)$$

and $V\phi$ is also immediately surjective, i.e., the right sequence is exact. ■

2.2 Notation: (Fibred monomorphisms and epimorphisms) We shall often simply write

$$0_{\mathsf{X}} \longrightarrow \mathsf{Y} \xrightarrow{\phi} \mathsf{Z} \qquad \mathsf{Y} \xrightarrow{\phi} \mathsf{Z} \longrightarrow 0_{\mathsf{X}}$$

to denote a fibred monomorphism or epimorphism, respectively, without denoting explicitly the injection ι or the projection p as in the lemma. But one should be sure to understand what this means. In the case of the left sequence it means that, for any fibred morphism $\iota\colon 0_{\mathsf{X}} \to \mathsf{Y}$ there exists a section η of $\tau\colon \mathsf{Z} \to \mathsf{X}$ such that $\phi(\iota(0_x)) = \eta(x)$. In the case of the right sequence it directly means that $\mathrm{image}(\phi) = \mathsf{Z}$ since the morphism from Z to 0_{X} is unique. •

For a vector bundle $\pi\colon \mathsf{E} \to \mathsf{X}$ the set smooth of sections is denoted by $\Gamma^\infty(\pi)$.

For a fibred manifold $\pi\colon \mathsf{Y} \to \mathsf{X}$, we denote by $\mathscr{G}_{\mathsf{X}}(\mathsf{Y})$ the sheaf of germs of local sections of $\pi\colon \mathsf{Y} \to \mathsf{X}$. For $k \in \mathbb{Z}_{\geq 0}$ we denote by $\mathsf{J}_k\pi$ the bundle of k-jets of sections of π. We denote $\mathsf{J}_0\pi = \mathsf{Y}$. If $(\xi, \mathcal{U})$ is a local section of π, its k-jet is denoted by $j_k\xi$. An element of $\mathsf{J}_k\pi$ we typically denote by $j_k\xi(x)$, or by p_k if compactness of notation is important. We let $\pi_k\colon \mathsf{J}_k\pi \to \mathsf{X}$ and $\pi^k_l\colon \mathsf{J}_k\pi \to \mathsf{J}_l\pi$, $l \leq k$, be the canonical projections. We can and will think of j_k as being a morphism of the sheaves $\mathscr{G}_{\mathsf{X}}(\mathsf{Y})$ and $\mathscr{G}_{\mathsf{X}}(\mathsf{J}_k\pi)$.

Recall that $\pi^k_{k-1}\colon \mathsf{J}_k\pi \to \mathsf{J}_{k-1}\pi$ is an affine bundle for which the fibre $(\pi^k_{k-1})^{-1}(p_{k-1})$ is an affine space modelled on $S_k(\mathsf{T}^*_{\pi_{k-1}(p_{k-1})}\mathsf{X}) \otimes \mathsf{V}_{\pi_0^{k-1}(p_{k-1})}\pi$.

If $\phi\colon \mathsf{Y} \to \mathsf{Y}'$ is a morphism of fibred manifolds $\pi\colon \mathsf{Y} \to \mathsf{X}$ and $\pi'\colon \mathsf{Y}' \to \mathsf{X}'$ over $\phi_0\colon \mathsf{X} \to \mathsf{X}'$, then we have the induced vector bundle map between the vertical bundles, denoted by $V\phi\colon \mathsf{V}\pi \to \mathsf{V}\pi'$ and defined by

$$V\phi(v_x) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\phi(\gamma(t)),$$

where $\gamma\colon I \to \pi^{-1}(x)$ is a curve such that $\gamma'(0) = v_x$. Note that $V\phi = T\phi|\mathsf{V}\pi$.[1] If $\mathsf{X}' = \mathsf{X}$ and if $\phi_0 = \mathrm{id}_{\mathsf{X}}$, we also have an induced morphism $J_k\phi\colon \mathsf{J}_k\pi \to \mathsf{J}_k\pi'$ defined by $J_k\phi(j_k\xi(x)) = j_k(\xi_\phi(x))$, where $(\xi, \mathcal{U})$ is local section of $\pi\colon \mathsf{Y} \to \mathsf{X}$ and $(\xi_\phi, \mathcal{U})$ is the section of $\pi'\colon \mathsf{Y}' \to \mathsf{X}$ defined by $\xi_\phi(x) = \phi \circ \xi(x)$.

If $\pi\colon \mathsf{E} \to \mathsf{X}$ is a vector bundle then $\pi_k\colon \mathsf{J}_k\pi \to \mathsf{X}$ is itself a vector bundle over X with the operations of addition and scalar multiplication defined by

$$(j_k\xi_1 + j_k\xi_2)(x) = j_k\xi_1(x) + j_k\xi_2(x), \quad (a j_k\xi)(x) = a(j_k\xi(x)).$$

The zero section of a vector bundle $\pi\colon \mathsf{E} \to \mathsf{X}$ we denote by $Z(\mathsf{E})$.

We recall that there is a canonical injection $\lambda_{k,l}\colon \mathsf{J}_{k+l}\pi \to \mathsf{J}_l\pi_k$ given by

$$\lambda_{k,l}(j_{k+l}\xi(x)) = j_l(j_k\xi)(x).$$

3. Partial differential equations and their solutions

In this section we define what we mean by a partial differential equation. What we actually define as a partial differential equation is not actually an equation; it is better thought of as a collection of solutions, or more precisely, formal solutions (see Section 7).

[1]The reader will be annoyed by the fact that $\mathsf{V}f$ and Vf are quite different, and are distinguished only by a difference of font.

This "equation" versus "solution" formulation should be kept in mind. The former is more general, although locally the two ways of thinking about partial differential equations are identical (this is one way to view Proposition 3.3).

3.1. Definitions and local representations. We let $\pi\colon \mathsf{Y} \to \mathsf{X}$ be a fibred manifold, i.e., π is a surjective submersion. The manifold X is to be thought of as the set of independent variables, and the manifold Y is to be thought of as the set of both independent and dependent variables. For $x \in \mathsf{X}$ one might think of the fibre $\pi^{-1}(x)$ as being the set of dependent variables associated to the value x of the independent variable.

3.1 DEFINITION: (Partial differential equation) Let $\pi\colon \mathsf{Y} \to \mathsf{X}$ be a fibred manifold and let $k \in \mathbb{Z}_{\geq 0}$. A ***partial differential equation*** of order k is a fibred submanifold $\mathsf{R}_k \subset \mathsf{J}_k\pi$ of $\pi_k\colon \mathsf{J}_k\pi \to \mathsf{X}$. The partial differential equation R_k is of ***homogeneous degree k*** if $\pi^k_l(\mathsf{R}_k) = \mathsf{J}_l\pi$ for each $l \in \{0, 1, \dots, k-1\}$. •

The idea of a partial differential equation of homogeneous degree is that only the derivatives of highest-order are restricted by the requirement that they lie in R_k.

We denote by $\hat{\pi}_k\colon \mathsf{R}_k \to \mathsf{X}$ the restriction of π_k to R_k. We also write $\mathsf{R}_{k,x} = \mathsf{R}_k \cap \pi_k^{-1}(x)$ for $x \in \mathsf{X}$.

Let us give some simple examples of partial differential equations.

3.2 EXAMPLES: (Partial differential equations)

1. Let $\mathsf{X} = \mathbb{R}^2$ and let $\mathsf{Y} = \mathbb{R}^2 \times \mathbb{R}^1$ with $\pi((x, y), u) = (x, y)$. Let us denote coordinates for $\mathsf{J}_2\pi$ by $(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy})$. We define a second-order partial differential equation by

$$\mathsf{R}_{\mathrm{lap}} = \{\, (x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) \in \mathsf{J}_2\pi \mid\ u_{xx} + u_{yy} = 0 \} .$$

   We note that this is indeed a fibred submanifold. The subscript "lap" stands for "Laplace's equation," as we shall see that $\mathsf{R}_{\mathrm{lap}}$ is the geometric form for Laplace's equation.

2. We again take $\mathsf{X} = \mathbb{R}^2$, $\mathsf{Y} = \mathbb{R}^2 \times \mathbb{R}$, and $\pi((x, y), f) = (x, y)$. We also let $\alpha, \beta\colon \mathbb{R}^2 \to \mathbb{R}$ be smooth functions. Now define a first-order partial differential equation by

$$\mathsf{R}_{\mathrm{der}} = \{\, (x, y, f, f_x, f_y) \in \mathsf{J}_1\pi \mid\ f_x = \alpha(x, y),\ f_y = \beta(x, y) \} .$$

   Note that $\mathsf{R}_{\mathrm{der}}$ is a fibred submanifold, and so is indeed a partial differential equation. The subscript "der" represents the fact that this partial differential equation is that which comes out of asking whether there exists a function $f\colon \mathbb{R}^2 \to \mathbb{R}$ such that

$$\boldsymbol{d}f(x, y) = \alpha(x, y)\mathrm{d}x + \beta(x, y)\mathrm{d}y.$$

   This amounts to asking whether the differential one-form

$$(x, y) \mapsto \alpha(x, y)\mathrm{d}x + \beta(x, y)\mathrm{d}y$$

   is exact.

3. We take $(\mathsf{X}, \mathbb{G})$ to be a Riemannian manifold with $\overset{\mathbb{G}}{\nabla}$ the Levi-Civita affine connection. If $X \in \Gamma^\infty(\pi_{\mathsf{TX}})$ is a vector field then $\overset{\mathbb{G}}{\nabla} X$ is a $(1,1)$-tensor field. Moreover, $\overset{\mathbb{G}}{\nabla} X$ depends on the 1-jet of X. Thus the map $X \mapsto \overset{\mathbb{G}}{\nabla} X$ can be thought of as one from $\mathsf{J}_1\pi_{\mathsf{TX}}$ to $T^1_1(\mathsf{TX})$, where $\pi_{\mathsf{TX}} \colon \mathsf{TX} \to \mathsf{X}$ denotes the tangent bundle projection. We let $B \in \Gamma^\infty(\pi_{\mathsf{TX}})$ be a vector field and $P \in C^\infty(\mathsf{X}; \mathbb{R})$ be a function with $\operatorname{grad} P$ the gradient vector field of P. We let $\mathsf{Y} = \mathsf{TX}$ and define a first-order partial differential equation by
$$\mathsf{R}_{\mathrm{sse}} = \{j_1 X(x) \in \mathsf{J}_1\pi_{\mathsf{TX}} \mid \overset{\mathbb{G}}{\nabla} X(X(x)) = -\operatorname{grad} P(x) + B(x)\}.$$
This partial differential equation is the incompressible steady-state Euler equation describing the steady motion of an incompressible inviscid fluid in the manifold X. The function P is the pressure and the vector field B is the body force on the fluid.
Don't get too excited about this geometric formulation of fluid mechanics. Apart from the simplifications of no viscosity, incompressibility, and steady-state, the important matter of initial conditions is completely missing from the discussion at this point.
4. We let $\mathsf{X} = \mathbb{R}$, $\mathsf{Y} = \mathbb{R} \times \mathbb{R}^2$, and $\pi(t, (x, y)) = t$. We let $f, g \colon \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ be smooth functions, let $\alpha \in \mathbb{R}$, and we define a first-order partial differential equation $\mathsf{R}_{\mathrm{ode}} \subset \mathsf{J}_1\pi$ by
$$\mathsf{R}_{\mathrm{ode}} = \{ (t, x, y, x_t, y_t) \mid x_t = f(x, y, t),\ \alpha y_t = g(t, x, y) \}.$$
The subscript "ode" reflects the fact that this partial differential equation is, in fact, an ordinary differential equation, its independent variable being $t \in \mathbb{R}$.
5. Let us next consider $\mathsf{X} = \mathbb{R}$, $\mathsf{Y} = \mathbb{R} \times \mathbb{R}^n$, $\pi(t, \boldsymbol{x}) = t$, and consider a function $L \colon \mathsf{T}\mathbb{R}^n \to \mathbb{R}$ which is to be thought of as a Lagrangian (we do not work on a general manifold so that we can keep things as simple as possible by using the canonical coordinate system on $\mathbb{R}^n$). We let $\boldsymbol{D}_1 L$ and $\boldsymbol{D}_2 L$ denote the partial derivatives which we think of as being of $\mathbb{R}^n$-valued using the standard identification of $\mathbb{R}^n$ with its dual. We consider the matrices of second partial derivatives $\boldsymbol{D}_1\boldsymbol{D}_2 L$ and $\boldsymbol{D}_2^2 L$ as taking values in $\operatorname{End}(\mathbb{R}^n)$. With these identifications we define a second-order partial differential equation R_{el} by
$$\mathsf{R}_{\mathrm{el}} = \{(t, \boldsymbol{x}, \boldsymbol{v}, \boldsymbol{a}) \in \mathsf{J}_2\pi \mid \boldsymbol{D}_2^2 L(\boldsymbol{x}, \boldsymbol{v})\boldsymbol{a} + \boldsymbol{D}_1\boldsymbol{D}_2 L(\boldsymbol{x}, \boldsymbol{v})\boldsymbol{v} - \boldsymbol{D}_1 L(\boldsymbol{x}, \boldsymbol{v}) = \boldsymbol{0}\}.$$
The subscript "el" refers to "Euler–Lagrange" since these are the Euler–Lagrange equations for the Lagrangian L.
(For those who want to think about this geometrically, we would take $\mathsf{X} = \mathbb{R}$ and $\mathsf{Y} = \mathscr{Q}$ with $\pi \colon \mathscr{Q} \to \mathbb{R}$ being an arbitrary surjective submersion. Then the Lagrangian would be a function on $\mathsf{J}_1\pi$, and so would be naturally time-dependent (we have only considered time-independent Lagrangians). The differential equation would still be thought of as a subset of $\mathsf{J}_2\pi$ which in local jet bundle coordinates has the form
$$\begin{aligned}\mathsf{R}_{\mathrm{el}} = \{(t, \boldsymbol{x}, \boldsymbol{v}, \boldsymbol{a}) \in \mathsf{J}_2\pi \mid\ & \boldsymbol{D}_3^2 L(t, \boldsymbol{x}, \boldsymbol{v})\boldsymbol{a} + \boldsymbol{D}_2\boldsymbol{D}_3 L(t, \boldsymbol{x}, \boldsymbol{v})\boldsymbol{v} \\ & + \boldsymbol{D}_1\boldsymbol{D}_3 L(t, \boldsymbol{x}, \boldsymbol{x}) - \boldsymbol{D}_1 L(t, \boldsymbol{x}, \boldsymbol{v}) = \boldsymbol{0}\},\end{aligned}$$
noting now that partial derivatives are taken with respect to the coordinates $(t, \boldsymbol{x}, \boldsymbol{v})$ and not $(\boldsymbol{x}, \boldsymbol{v})$.) •

The following local characterisation of a partial differential equation is often useful.

3.3 Proposition: (Local defining equations) *Let* $\pi\colon \mathsf{Y} \to \mathsf{X}$ *be a fibred manifold. If* $\mathsf{R}_k \subset \mathsf{J}_k\pi$ *is a kth-order partial differential equation then, for each* $x \in \mathsf{X}$*, there exists*

(i) a neighbourhood $\mathcal{U}$ *of* x*,*

(ii) a fibred manifold $\tau\colon \mathsf{Z} \to \mathcal{U}$*,*

(iii) a morphism $\Phi\colon \pi_k^{-1}(\mathcal{U}) \to \mathsf{Z}$ *of constant rank, and*

(iv) and a smooth section η *of* $\tau\colon \mathsf{Z} \to \mathcal{U}$

such that

$$\pi_k^{-1}(\mathcal{U}) \cap \mathsf{R}_k = \ker_\eta \Phi \triangleq \{p_k \in \pi_k^{-1}(\mathcal{U}) \mid \Phi(p_k) = \eta(\pi_k(p_k))\}.$$

Proof: Since R_k is a fibred submanifold there exists an adapted chart $(\mathcal{V}, \psi)$ for $\mathsf{J}_k\pi$ with $(\mathcal{U}, \phi)$ the induced chart for X such that

$$\psi(\mathcal{V}) \subset \phi(\mathcal{U}) \times \mathcal{W}_1 \times \mathcal{W}_2 \subset \mathbb{R}^n \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$$

and such that

$$\pi_k^{-1}(\mathcal{U}) \cap \mathsf{R}_k = \{\, (\boldsymbol{x}, \boldsymbol{y}_1, \boldsymbol{0}) \mid \ \boldsymbol{x} \in \phi(\mathcal{U}),\ \boldsymbol{y}_1 \in \mathcal{W}_1\}\,.$$

We then take

1. $\mathsf{Z} = \mathcal{U} \times \mathcal{W}_1$ and $\tau(x, \boldsymbol{y}_1) = x$,
2. $\Phi(y) = (x, \boldsymbol{y}_1)$ where $(\phi(x), \boldsymbol{y}_1, \boldsymbol{y}_2) = \psi(y)$, and
3. $\eta(x) = (x, \boldsymbol{0})$

to give the result. ■

It will be useful to be able to refer precisely to the constructions of the preceding result.

3.4 Definition: (Local defining equation) Let $\pi\colon \mathsf{Y} \to \mathsf{X}$ be a fibred manifold and let $\mathsf{R}_k \subset \mathsf{J}_k\pi$ be a partial differential equation of order k. A ***local defining equation*** for R_k is a quintuple $(\mathcal{U}, \mathsf{Z}, \tau, \Phi, \eta)$ satisfying the conditions of Proposition 3.3:

(i) $\mathcal{U}$ is an open subset of X,

(ii) $\tau\colon \mathsf{Z} \to \mathcal{U}$ is a fibred manifold,

(iii) $\Phi\colon \pi_k^{-1}(\mathcal{U}) \to \mathsf{Z}$ is a constant rank morphism of fibred manifolds,

(iv) $\eta\colon \mathcal{U} \to \mathsf{Z}$ is a section, and

(v) $\pi_k^{-1}(\mathcal{U}) \cap \mathsf{R}_k = \ker_\eta(\Phi)$. •

Our examples make transparent the definition and the result preceding it.

3.5 Examples: (Local characterisation of partial differential equations)

1. For $\mathsf{X} = \mathbb{R}^2$, $\mathsf{Y} = \mathbb{R}^2 \times \mathbb{R}$, and $\pi((x, y), u) = (x, y)$, define

   (a) $\mathcal{U} = \mathbb{R}^2$,

   (b) $\mathsf{Z}_{\text{lap}} = \mathbb{R}^2 \times \mathbb{R}$ with $\tau((x, y), z) = (x, y)$,

   (c) $\Phi_{\text{lap}}(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = ((x, y), u_{xx} + u_{yy})$, and

   (d) $\eta_{\text{lap}}(x, y) = ((x, y), 0)$.

   We then see that $\mathsf{R}_{\text{lap}} = \ker_{\eta_{\text{lap}}} \Phi_{\text{lap}}$.

2. For $\mathsf{X} = \mathbb{R}^2$, $\mathsf{Y} = \mathbb{R}^2 \times \mathbb{R}$, and $\pi((x,y),f) = (x,y)$, define
   (a) $\mathcal{U} = \mathbb{R}^2$,
   (b) $\mathsf{Z}_{\mathrm{der}} = \mathbb{R}^2 \times \mathbb{R}^2$ with $\tau((x,y),(w,z)) = (x,y)$,
   (c) $\Phi_{\mathrm{der}}(x,y,f,f_x,f_y) = ((x,y),(f_x,f_y))$, and
   (d) $\eta_{\mathrm{der}}(x,y) = ((x,y),(\alpha(x,y),\beta(x,y))$.

   Then $\mathsf{R}_{\mathrm{der}} = \ker_{\eta_{\mathrm{der}}} \Phi_{\mathrm{der}}$.
3. For the partial differential equation $\mathsf{R}_{\mathrm{sse}}$ we take
   (a) $\mathcal{U} = \mathsf{X}$,
   (b) $\mathsf{Z}_{\mathrm{sse}} = \mathsf{TX}$ with $\tau = \pi_{\mathsf{TX}}$,
   (c) $\Phi_{\mathrm{sse}}(j_1X(x)) = \overset{\mathbb{G}}{\nabla} X(X(x))$, and
   (d) $\eta_{\mathrm{sse}}(x) = -\operatorname{grad} P(x) + B(x)$.

   Then $\mathsf{R}_{\mathrm{sse}} = \ker_{\eta_{\mathrm{sse}}} \Phi_{\mathrm{sse}}$.
4. For $\mathsf{X} = \mathbb{R}$, $\mathsf{Y} = \mathbb{R} \times \mathbb{R}^2$, and $\pi(t,(x,y)) = t$, define
   (a) $\mathcal{U} = \mathbb{R}$,
   (b) $\mathsf{Z}_{\mathrm{ode}} = \mathbb{R} \times \mathbb{R}^2$ with $\tau(t,u,v) = t$,
   (c) $\Phi_{\mathrm{ode}}(t,x,y,x_t,y_t) = (t,(x_t - f(t,x,y), y_t - \alpha g(t,x,y)))$, and
   (d) $\eta_{\mathrm{ode}}(t) = (t,(0,0))$.

   Then $\mathsf{R}_{\mathrm{ode}} = \ker_{\eta_{\mathrm{ode}}} \Phi_{\mathrm{ode}}$.
5. For $\mathsf{X} = \mathbb{R}$, $\mathsf{Y} = \mathbb{R} \times \mathbb{R}^n$, and $\pi(t,\boldsymbol{x}) = t$, define
   (a) $\mathcal{U} = \mathbb{R}$,
   (b) $\mathsf{Z}_{\mathrm{el}} = \mathbb{R} \times \mathbb{R}^n$ with $\tau(t,\boldsymbol{y}) = \boldsymbol{y})$,
   (c) $\Phi_{\mathrm{el}}(t,\boldsymbol{x},\boldsymbol{v},\boldsymbol{a}) = \boldsymbol{D}_2^2 L(\boldsymbol{x},\boldsymbol{v})\boldsymbol{a} + \boldsymbol{D}_1\boldsymbol{D}_2 L(\boldsymbol{x},\boldsymbol{v})\boldsymbol{v} - \boldsymbol{D}_1 L(\boldsymbol{x},\boldsymbol{v})$, and
   (d) $\eta_{\mathrm{el}}(t) = (t,\boldsymbol{0})$.

   Then $\mathsf{R}_{\mathrm{el}} = \ker_{\eta_{\mathrm{el}}}(\Phi_{\mathrm{eta}})$.
   (Again for the benefit of those liking the geometry, we would take Z_{el} to be the subbundle of $\mathsf{T}^*\mathsf{J}_1\pi$ given by the first contact system. It turns out that this is the natural domain of the map which in coordinates looks like

$$(t,\boldsymbol{x},\boldsymbol{v},\boldsymbol{a}) \mapsto \boldsymbol{D}_3^2 L(t,\boldsymbol{x},\boldsymbol{v})\boldsymbol{a} + \boldsymbol{D}_2\boldsymbol{D}_3 L(t,\boldsymbol{x},\boldsymbol{v})\boldsymbol{v} + \boldsymbol{D}_1\boldsymbol{D}_3 L(t,\boldsymbol{x},\boldsymbol{v}) - \boldsymbol{D}_2 L(t,\boldsymbol{x},\boldsymbol{v}),$$

   again keeping in mind that partial derivatives are now taken with respect to the coordinates $(t,\boldsymbol{x},\boldsymbol{v})$ and not $(\boldsymbol{x},\boldsymbol{v})$ as originally.) •

3.2. Solutions. Next we introduce the notion of a solution. We see here why we should think of a partial differential equation R_k as being a set of solutions rather than a set of equations.

3.6 Definition: (Solution to a partial differential equation) Let $\pi\colon \mathsf{Y} \to \mathsf{X}$ be a fibred manifold and let $\mathsf{R}_k \subset \mathsf{J}_k\pi$ be a kth-order partial differential equation. A ***local solution*** of R_k is a local section $(\xi,\mathcal{U})$ with the property that $j_k\xi(x) \in \mathsf{R}_k$ for every $x \in \mathcal{U}$. •

Let us examine the matter of solutions for our examples.

3.7 Examples: (Solutions of partial differential equations)

1. We consider the second-order partial differential equation $\mathsf{R}_{\mathrm{lap}}$. A solution is then a section $(x,y)\mapsto((x,y),u(x,y))$ that satisfies

$$\frac{\partial^2 u}{\partial x^2}(x,y)+\frac{\partial^2 u}{\partial y^2}(x,y)=0.$$

This is Laplace's equation in $\mathbb{R}^2$. This equation has many solutions; let us merely list a bunch of them.

(a) Any affine function of (x,y), $u\colon(x,y)\mapsto ax+by+c+d$, gives rise to a solution.

(b) The function $u\colon(x,y)\mapsto x^2-y^2$ defines a solution.

(c) The function $u\colon(x,y)\mapsto xy$ defines a solution. •

2. Next we consider the first-order partial differential equation $\mathsf{R}_{\mathrm{der}}$. Solutions are sections $(x,y)\mapsto f(x,y)$ that satisfy

$$\frac{\partial f}{\partial x}(x,y)=\alpha(x,y),\quad \frac{\partial f}{\partial y}=\beta(x,y).$$

Thus we see that solutions are in 1–1 correspondence with functions f on $\mathbb{R}^2$ that satisfy

$$\boldsymbol{d}f(x,y)=\alpha(x,y)\mathrm{d}x+\beta(x,y)\mathrm{d}y.$$

Since f is assumed to be infinitely differentiable we must have $\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial^2 f}{\partial y\partial x}$, which gives the necessary conditions

$$\frac{\partial\alpha}{\partial y}(x,y)=\frac{\partial\beta}{\partial x}(x,y),\qquad (x,y)\in\mathcal{U},\tag{3.1}$$

for the existence of solutions on any open subset $\mathcal{U}$ of $\mathbb{R}^2$. It turns out that this condition is also sufficient if $\mathcal{U}$ is simply connected. We shall not concern ourselves in our development with global topological issues such as simple connectedness, but rather will be interested in conditions on α and β which ensure that, for $(x,y)\in\mathbb{R}^2$, there exists a neighbourhood $\mathcal{U}$ of (x,y) such that local solutions $(\xi,\mathcal{U})$ exist.

3. Of course, for general pressure P and body force B it will be impossible to describe solutions to the partial differential equation $\mathsf{R}_{\mathrm{sse}}$ in a useful way. A trivial special case is when the pressure is constant and the body force is zero. In this case the steady-state motions are vector fields whose integral curves are geodesics for $\overset{\mathbb{G}}{\nabla}$. For the case where $\mathsf{X}=\mathbb{R}^n$ and $\mathbb{G}$ is the standard Riemannian metric on $\mathbb{R}^n$, the solutions to $\mathsf{R}_{\mathrm{sse}}$ are thus the constant vector fields, and the motion of the fluid is just straight line motion; Newton's First Law manifested for fluids.

4. Now consider the ordinary differential equation $\mathsf{R}_{\mathrm{ode}}$. Solutions to this equation are sections $t\mapsto(x(t),y(t)$ satisfying

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t)=f(t,x(t),y(t)),\quad \alpha\frac{\mathrm{d}y}{\mathrm{d}t}(t)=g(t,x(t),y(t)).$$

When $\alpha \neq 0$ these equations are just the standard ordinary differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t) = f(t, x(t), y(t)), \quad \frac{\mathrm{d}y}{\mathrm{d}t}(t) = \alpha^{-1} g(t, x(t), y(t)).$$

When $\alpha = 0$ these equations will have solutions only at points where $g(t, x, y) = 0$. If $g(t, x, y) = 0$ for all $(t, x, y) \in \mathbb{R} \times \mathbb{R}^2$, there is no equation governing y, and the solutions are those for the scalar ordinary differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t) = f(t, x(t), y(t))$$

where $t \mapsto y(t)$ is an arbitrary smooth function. Thus we have two things happening here: (1) there is a compatibility condition and (2) the set of solutions is parameterised by an arbitrary function.

5. Solutions to the Euler–Lagrange equations R_{el} are sections $t \mapsto (t, \boldsymbol{x}(t))$ that satisfy

$$\frac{\partial^2 L}{\partial v^i \partial v^k} \ddot{x}^i(t) + \frac{\partial^2 L}{\partial v^i \partial x^j} \dot{x}^i(t) - \frac{\partial L}{\partial x^i} = 0, \qquad i \in \{1, \dots, n\}.$$

These can be written in mnemonic form as

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\partial L}{\partial v^i}\Big) - \frac{\partial L}{\partial x^i} = 0, \qquad i \in \{1, \dots, n\},$$

giving the Euler-Lagrange equations in their familiar representation. Note that, depending on the character of L, these implicit differential equations may have (a) unique solutions, (b) solutions that are not unique, or (c) no solutions. We leave it to the reader to come up with Lagrangians giving rise to equations in each of these categories. •

The next result further clarifies the "solution" versus "equation" point of view of partial differential equations.

3.8 Proposition: (Solutions in terms of defining equations) *Let $\pi\colon \mathsf{Y} \to \mathsf{X}$ be a fibred manifold and let R_k be a kth-order partial differential equation. If $(\mathcal{U}, \mathsf{Z}, \tau, \Phi, \eta)$ is a local defining equation for R_k then $(\xi, \mathcal{U})$ is a local solution of R_k if and only if $\Phi(j_k\xi(x)) = \eta(x)$ for all $x \in \mathcal{U}$.*

Proof: This is clear from the definition of a local defining equation. ■

3.3. Linear and quasilinear partial differential equations. For linear partial differential equations we simply take into account the additional structure in the problem.

3.9 Definition: (Linear partial differential equation) Let $\pi\colon \mathsf{E} \to \mathsf{X}$ be a vector bundle and let $k \in \mathbb{Z}_{\geq 0}$.

(i) A ***homogeneous linear partial differential equation*** of order k is a vector subbundle $\mathsf{R}_k \subset \mathsf{J}_k\pi$ of $\pi_k\colon \mathsf{J}_k\pi \to \mathsf{X}$.

(ii) An ***inhomogeneous linear partial differential equation*** of order k is an affine subbundle of $\mathsf{R}_k \subset \mathsf{J}_k\pi$ of $\pi_k\colon \mathsf{J}_k\pi \to \mathsf{X}$.

Let $\pi\colon \mathsf{Y} \to \mathsf{X}$ be a fibred manifold and let $k \in \mathbb{Z}_{>0}$.

(iii) A ***quasilinear partial differential equation*** of order k is a partial differential equation $\mathsf{R}_k \subset \mathsf{J}_k\pi$ such that $(\pi^k_{k-1})^{-1}(\pi^k_{k-1}(p_k)) \cap \mathsf{R}_k$ is an affine subspace of $(\pi^k_{k-1})^{-1}(\pi^k_{k-1}(p_k))$ for each $p_k \in \mathsf{R}_k$. •

Often one simply uses the words "linear partial differential equation" without explicit reference to whether it is homogeneous or inhomogeneous. In such cases it is hopefully clear from context which situation is intended. However, some authors explicitly mean "homogeneous linear" when they say "linear." Of course, homogeneous linear partial differential equations are a special case of inhomogeneous linear partial differential equations.

For linear partial differential equations, the local representation of Proposition 3.3 is, in fact, global. Thus, for linear partial differential equations, the "solution" and "equation" points of view are equivalent. For general partial differential equations, this is not the case; the "solution" point of view is more general.

3.10 Proposition: (Linear partial differential equations have global defining equations) *Let $\pi\colon \mathsf{E} \to \mathsf{X}$ be a vector bundle and let $\mathsf{R}_k \subset \mathsf{J}_k\pi$ be an inhomogeneous linear partial differential equation. Then there exists a vector bundle $\tau\colon \mathsf{F} \to \mathsf{X}$, a vector bundle map $\Phi\colon \mathsf{J}_k\pi \to \mathsf{F}$, and a section η of $\tau\colon \mathsf{F} \to \mathsf{X}$ such that $\mathsf{R}_k = \ker_\eta(\Phi)$. Moreover, if R_k is a homogeneous linear partial differential equation, then we may take η to be the zero section.*

Proof: Let $L(\mathsf{R}_k) \subset \mathsf{J}_k\pi$ be the vector subbundle defined by the affine subbundle R_k. Take $\mathsf{F} = \operatorname{coker}(L(\mathsf{R}_k))$, let Φ be the canonical projection, and let $\eta(x)$ be the equivalence class of $\mathsf{R}_{k,x}$ in $\operatorname{coker}(L(\mathsf{R}_k))_x$. This is easily seen to give the first assertion. The second follows since, if R_k is a homogeneous linear partial differential equation, then $L(\mathsf{R}_k) = \mathsf{R}_k$. ■

Let us see how our running examples fit into the various classifications of Definition 3.9.

3.11 Examples: (Flavours of linear partial differential equations)

1. $\mathsf{R}_{\mathrm{lap}}$ is a homogeneous linear partial differential equation, evidenced by the fact that it has a global linear defining equation from $\mathsf{J}_2\pi$ to $\mathsf{F} = \mathsf{X} \times \mathbb{R}$ given by

$$\Phi(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = (x, y, u_{xx} + u_{yy}).$$

   That is to say, Φ is a vector bundle map and $\mathsf{R}_{\mathrm{lap}} = \Phi^{-1}(Z(\mathsf{F}))$.

2. The partial differential equation $\mathsf{R}_{\mathrm{der}}$ is an inhomogeneous linear partial differential equation since $\mathsf{R}_{\mathrm{der}} = \ker_\eta \Phi$ where $\Phi\colon \mathsf{J}_1\pi \to \mathsf{F} = \mathbb{R}^2 \times \mathbb{R}^2$ is the vector bundle map

$$\Phi(x, y, f, f_x, f_y) = (x, y, f_x, f_y)$$

   and η is the section of F given by $\eta(x, y) = (x, y, \alpha(x, y), \beta(x, y))$.

3. The partial differential equation $\mathsf{R}_{\mathrm{sse}}$ describing the steady motion of an incompressible inviscid fluid in a Riemannian manifold $(\mathsf{X}, \mathbb{G})$ is a quasilinear first-order partial differential equation. To see that it is quasilinear we observe that, for a vector field $X \in \Gamma^\infty(\pi_{\mathsf{TX}})$, the map from $\mathsf{J}_1\pi_{\mathsf{TX}}$ to $\pi^*_{\mathsf{TX}}\mathsf{TX}$ given by $j_1X(x) \mapsto \overset{\mathbb{G}}{\nabla} X(X(x))$ is a vector bundle map over the identity on TX. By our characterisation of $\mathsf{R}_{\mathrm{sse}}$ in Example 3.5–3 it follows that the equation is indeed quasilinear.

4. The partial differential equation $\mathsf{R}_{\mathrm{ode}}$ is not generally a linear partial differential equation, either homogeneous or inhomogeneous. If the functions f and g are linear in x and y, i.e., if

$$\begin{bmatrix} f(t,x,y) \\ \alpha g(t,x,y) \end{bmatrix} = \boldsymbol{A}(t) \begin{bmatrix} x \\ y \end{bmatrix}, \qquad \boldsymbol{A} \colon \mathbb{R} \to \mathbb{R}^{2\times 2},$$

then the $\mathsf{R}_{\mathrm{ode}}$ is linear and homogeneous.

5. The partial differential equation R_{el} is quasilinear, essentially because it is linear in $\boldsymbol{a}$. The reader can see for themselves that it is this that implies quasilinearity of the partial differential equation, cf. Proposition 3.18. •

3.4. Partial differential equations with constant coefficients. In certain applications partial differential equations are defined, not on manifolds, but on vector spaces. In such cases it is possible to make sense of the notion of a partial differential equation with "constant coefficients." In this section we shall say precisely what this means, and cast such partial differential equations in our more general framework.

We let V and U be finite-dimensional $\mathbb{R}$-vector spaces. We then define a fibred manifold, indeed a vector bundle, $\pi \colon \mathsf{V} \times \mathsf{U} \to \mathsf{V}$ by $\pi(v,u) = v$. Thus V is where the "independent variables" live and U is where the "dependent variables" live. The bundle of k-jets is naturally isomorphic to the vector bundle

$$\mathsf{V} \times \left(\oplus_{j=0}^{k} (S_j(\mathsf{V}^*) \otimes \mathsf{U})\right),$$

and we shall simply suppose that $\mathsf{J}_k\pi$ is *equal* to this vector bundle.

We now make the following definition.

3.12 Definition: (Partial differential equation with constant coefficients) Let V and U be finite-dimensional $\mathbb{R}$-vector spaces. A ***kth-order partial differential equation with constant coefficients*** is a subspace S_k of $\oplus_{j=0}^{k}(S_j(\mathsf{V}^*) \otimes \mathsf{U})$. A ***kth-order tableau*** is a subspace of $S_k(\mathsf{V}^*) \otimes \mathsf{U}$. •

3.13 Remark: (Characterisation of first-order tableaux) By definition, a first-order tableau is a subspace S_k of $\mathsf{V}^* \otimes \mathsf{U} \simeq \mathrm{Hom}_{\mathbb{R}}(\mathsf{V};\mathsf{U})$. Choosing bases $\{v_1,\dots,v_n\}$ and $\{u_1,\dots,u_m\}$ for V and U, respectively, S_k is to be thought of as a subspace of the $m \times n$-matrices. It will occasionally be useful to use this as a means of representing such partial differential equations. •

Let us see how one associates to a partial differential equation S_k with constant coefficients a partial differential equation in the usual sense.

3.14 Proposition: (Constant coefficient partial differential equations are partial differential equations) *Let V and U be finite-dimensional $\mathbb{R}$-vector spaces and let $\mathsf{S}_k \subset \oplus_{j=0}^{k}(S_j(\mathsf{V}^*) \otimes \mathsf{U})$ be a partial differential equation with constant coefficients. Then*

$$\mathsf{R}(\mathsf{S}_k) = \{(x,A) \in \mathsf{V} \times \oplus_{j=0}^{k}(S_j(\mathsf{V}^*) \otimes \mathsf{U}) \mid A \in \mathsf{S}_k\}$$

is a linear homogeneous partial differential equation. If S_k is a kth-order tableau, then $\mathsf{R}(\mathsf{S}_k)$ is of homogeneous degree k.

Proof: It is obvious that $\mathsf{R}(\mathsf{S}_k)$ is a vector subbundle of $\mathsf{J}_k\pi$. The final assertion is also immediate. ■

We also have the notion of the solution of a partial differential equation with constant coefficients.

3.15 DEFINITION: (Solution of partial differential equation with constant coefficients) Let V and U be finite-dimensional $\mathbb{R}$-vector spaces and let $\mathsf{S}_k \subset \oplus_{j=0}^k (S_j(\mathsf{V}^*) \otimes \mathsf{U})$ be a partial differential equation with constant coefficients. A ***solution*** to S_k is a map $f\colon \mathsf{V} \to \mathsf{U}$ such that $\oplus_{j=0}^k \boldsymbol{D} f(x) \in \mathsf{S}_k$ for every $x \in \mathsf{V}$. •

Of course, there is a natural correspondence between this notion of solution and the usual notion of solution.

3.16 PROPOSITION: (Solutions of S_k give rise to solutions of $\mathsf{R}(\mathsf{S}_k)$) *Let V and U be finite-dimensional $\mathbb{R}$-vector spaces and let $\mathsf{S}_k \subset \oplus_{j=0}^k (S_j(\mathsf{V}^*)\otimes\mathsf{U})$ be a partial differential equation with constant coefficients. A map $f\colon \mathsf{V} \to \mathsf{U}$ is solution to S_k if and only if $\mathsf{V} \ni x \mapsto (x, f(x)) \in \mathsf{V} \times \mathsf{U}$ is a solution to $\mathsf{R}(\mathsf{S}_k)$.*

Let us consider some examples of partial differential equations with constant coefficients.

3.17 EXAMPLES: (Partial differential equations with constant coefficients)

1. Of our list of running examples, the only one that is generally a partial differential equations is $\mathsf{R}_{\mathrm{lap}}$. In this case we take $\mathsf{V} = \mathbb{R}^2$, $\mathsf{U} = \mathbb{R}$, and $k = 2$. Then this partial differential equation is given by the subspace

$$\mathsf{S}_{\mathrm{lap}} = \{(u, u_x.u_y, u_{xx}, u_{xy}, u_{yy}) \in \oplus_{j=0}^k (S_j(\mathsf{V}^*) \otimes \mathsf{U}) \mid u_{xx} + u_{yy} = 0\},$$

   where we adopt convenient, but rather unalgebraic looking, notation, u, (u_x, u_y), and (u_{xx}, u_{xy}, u_{yy}) for coordinates in $S_j(\mathsf{V}^*)$, $j \in \{0, 1, 2\}$. Of course, solutions are functions $(x, y) \mapsto u(x, y)$ satisfying

$$\frac{\partial^2 u}{\partial x^2}(x, y) + \frac{\partial^2 u}{\partial y^2}(x, y) = 0.$$

2. Let us take $\mathsf{V} = \mathbb{R}$, $\mathsf{U} = \mathbb{R}^n$, $k = 1$, and define

$$\mathsf{S}_{\mathrm{ode}} = \{\, (\boldsymbol{x}, \boldsymbol{x}_t) \in \mathbb{R}^n \oplus \mathbb{R}^n \mid \ \boldsymbol{E}\boldsymbol{x}_t = \boldsymbol{A}\boldsymbol{x} \}\,.$$

   Here we use the natural identifications of $\mathbb{R}^* \otimes \mathbb{R}^n$ and $S_1(\mathbb{R}^*) \otimes \mathbb{R}^n$ with $\mathbb{R}^n$. Solutions of this system of partial differential equations are functions $t \mapsto \boldsymbol{x}(t)$ that satisfy

$$\boldsymbol{E}\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t).$$

   When $\boldsymbol{E}$ is invertible, solutions are given by $t \mapsto \exp(\boldsymbol{E}^{-1}\boldsymbol{A}t)\boldsymbol{x}(0)$.

3. We next consider what appears to be a rather particular first-order partial differential equation with constant coefficients. However, it is actually representative of *any* constant coefficient partial differential equation, after some operations (order reduction and prolongation) have been performed.

We let $\mathsf{V} = \mathbb{R}^n$ and $\mathsf{U} = \mathbb{R}^m$. We let $k \in \{1, \dots, n\}$, let $s_1, \dots, s_k \in \mathbb{Z}_{>0}$ satisfy

$$s_1 > s_2 > \cdots > s_k,$$

and let $\mathsf{S}_{\mathrm{inv}}$ be the subspace of $\mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^n; \mathbb{R}^m)$ spanned by matrices of the form

$$\boldsymbol{A} = \begin{bmatrix} a_1^1 & a_2^1 & \cdots & a_k^1 & 0 & \cdots & 0 \\ a_1^2 & a_2^2 & \cdots & a_k^2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_1^{s_k} & a_2^{s_k} & \cdots & a_k^{s_k} & 0 & \cdots & 0 \\ a_1^{s_k+1} & a_2^{s_k+1} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_1^{s_2} & a_2^{s_2} & \cdots & 0 & 0 & \cdots & 0 \\ a_1^{s_2+1} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_1^{s_1} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}$$

for entries a_j^i take arbitrary $\mathbb{R}$-values. This subspace thus has dimension $nm - s_1 + \cdots + s_k$. The easiest way to think about $\mathsf{S}_{\mathrm{inv}}$ is to consider the equations which must be satisfied by a solution $\mathbb{R}^n \ni \boldsymbol{x} \mapsto \boldsymbol{u}(\boldsymbol{x}) \in \mathbb{R}^m$:

$$\begin{aligned} \frac{\partial u^a}{\partial x^j} &= 0, \qquad a \in \{s_1 + 1, \dots, m\},\ j \in \{1, \dots, n\}; \\ \frac{\partial u^a}{\partial x^j} &= 0, \qquad a \in \{s_2 + 1, \dots, s_1\},\ j \in \{2, \dots, n\}; \\ &\vdots \\ \frac{\partial u^a}{\partial x^j} &= 0, \qquad a \in \{1, \dots, s_k\},\ j \in \{k + 1, \dots, n\}. \end{aligned} \tag{3.2}$$

It is possible to explicitly define a collection of C^∞ solutions to these equations by simply integrating the equations group-by-group as they appear in (3.2). The first $m - s_1$ equations give $u^{s_1+1}, \dots, u^m$ being independent of $x^1, \dots, x^n$. Thus these functions are constant. The second $s_1 - s_2$ equations give $u^{s_2+1}, \dots, u^{s_1}$ being independent of $x^2, \dots, x^n$. We carry on this way to the last s_k equations which give $u^1, \dots, u^{s_k}$ being independent of $x^{k+1}, \dots, x^n$. To summarise, this method of solution gives the following class of solutions:

(a) $u^a =$ function of $x^1, \dots, x^k$, $a \in \{1, \dots, s_k\}$;

(b) $u^a =$ function of $x^1, \dots, x^{k-1}$, $a \in \{s_k + 1, \dots, s_{k-1}\}$;

(c) $\vdots$

(d) $u^a =$ function of x^1, $a \in \{s_2 + 1, \dots, s_1\}$;

(e) $u^a =$ constant, $a \in \{s_1 + 1, \dots, m\}$.

Of course, we have seen that partial differential equations may have incompatibilities, and there are no guarantees that our method of solution has produced actual solutions. However, one can check that the set of functions defined above *are* solutions simply by substituting them into the defining equations. Indeed, the special thing about this class of equations is that this "block-by-block" method of solution works. But this is getting into the idea of involutivity of symbols (this motivating, by the way, the subscript "inv"). •

3.5. Coordinate formulae. Apart from indicating how one might do computations in examples, it is sometimes helpful in terms of understanding concepts to provide coordinate formulae for some of the objects associated to partial differential equations. In this section we provide the notation we will use to give our coordinate representations. We will not use indicial notation here since it is sometimes difficult to manage the multi-indices in a way that is both useful and insightful.

We let $\pi\colon \mathsf{Y} \to \mathsf{X}$ be a fibred manifold, as usual. We denote local coordinates for X by $\boldsymbol{x} \in \mathbb{R}^n$ and fibred coordinates for Y by $(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^n \times \mathbb{R}^m$. Thus the local representative of π is $(\boldsymbol{x}, \boldsymbol{y}) \mapsto \boldsymbol{x}$. As coordinates for $\mathsf{J}_k\pi$ we use

$$(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{p}_1, \dots, \boldsymbol{p}_k) \in \mathbb{R}^n \times \mathbb{R}^m \times L^1_{\mathrm{sym}}(\mathbb{R}^n; \mathbb{R}^m) \times \cdots \times L^k_{\mathrm{sym}}(\mathbb{R}^n; \mathbb{R}^m).$$

Now let R_k be a kth-order partial differential equation. A solution then is locally represented by a map $\boldsymbol{x} \mapsto (\boldsymbol{x}, \boldsymbol{\xi}(\boldsymbol{x}))$ that has the property that the map

$$\boldsymbol{x} \mapsto (\boldsymbol{x}, \boldsymbol{\xi}(\boldsymbol{x}), \boldsymbol{D}\boldsymbol{\xi}(\boldsymbol{x}), \dots, \boldsymbol{D}^k\boldsymbol{\xi}(\boldsymbol{x}))$$

takes its values in R_k.

Let $(\mathcal{U}, \mathsf{Z}, \tau, \Phi, \eta)$ be a local defining equation and suppose that $\mathcal{U}$ is a coordinate chart for X with coordinates $(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^n \times \mathbb{R}^m$ for Y and $(\boldsymbol{x}, \boldsymbol{z}) \in \mathbb{R}^n \times \mathbb{R}^r$ for Z. Then Φ has the local representative

$$(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{p}_1, \dots, \boldsymbol{p}_k) \mapsto (\boldsymbol{x}, \boldsymbol{\Phi}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{p}_1, \dots, \boldsymbol{p}_k)),$$

so defining the map $\boldsymbol{\Phi}$. If the local representative of η is $\boldsymbol{x} \mapsto (\boldsymbol{x}, \boldsymbol{\eta}(\boldsymbol{x}))$, then $\mathsf{R}_k \cap \pi_k^{-1}(\mathcal{U})$ is given by

$$\{(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{p}_1, \dots, \boldsymbol{p}_k) \mid \ \boldsymbol{\Phi}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{p}_1, \dots, \boldsymbol{p}_k) = \boldsymbol{\eta}(\boldsymbol{x})\}.$$

It is possible to characterise the various sorts of partial differential equations by properties of the map $\boldsymbol{\Phi}$. We leave it for the reader to verify the following facts.

3.18 Proposition: (Coordinate forms for various classes of partial differential equations) *Let $\pi\colon \mathsf{E} \to \mathsf{X}$ be a vector bundle and let $\mathsf{R}_k \subset \mathsf{J}_k\pi$ be a kth-order partial differential equation. Then the following statements hold:*

(i) R_k is a homogeneous linear partial differential equation if and only if, about each $x \in \mathsf{X}$, there exists a local defining equation $(\mathcal{U}, \mathsf{F}, \tau, \Phi, \eta)$ for which

(a) $\tau\colon \mathsf{F} \to \mathcal{U}$ is a vector bundle,

(b) the map $\boldsymbol{\Phi}$ above is linear in last $k+1$ entries, and

(c) the section η has the local representative $\boldsymbol{x} \mapsto (\boldsymbol{x}, \boldsymbol{0})$;

(ii) R_k *is an inhomogeneous linear partial differential equation if and only if, about each* $x \in \mathsf{X}$*, there exists a local defining equation* $(\mathcal{U}, \mathsf{F}, \tau, \Phi, \eta)$ *for which*

(a) $\tau\colon \mathsf{F} \to \mathcal{U}$ *is a vector bundle and*

(b) the map $\boldsymbol{\Phi}$ *above is linear in last* $k+1$ *entries.*

Suppose that $\pi\colon \mathsf{Y} \to \mathsf{X}$ *is a fibred manifold and let* $\mathsf{R}_k \subset \mathsf{J}_k\pi$ *be a* k*th-order partial differential equation. Then*

(iii) R_k *is a quasilinear partial differential equation if and only if, about each* $x \in \mathsf{X}$*, there exists a local defining equation* $(\mathcal{U}, \mathsf{F}, \tau, \Phi, \eta)$ *for which*

(a) $\tau\colon \mathsf{F} \to \mathcal{U}$ *is a vector bundle and*

(b) the map $\boldsymbol{\Phi}$ *above is linear in last entry;*

(iv) R_k *is a partial differential equation that is of homogeneous degree* k *if and only if, about each* $x \in \mathsf{X}$*, there exists a local defining equation* $(\mathcal{U}, \mathsf{Z}, \tau, \Phi, \eta)$ *for which the map* $\boldsymbol{\Phi}$ *above is independent of all except the first and last components.*

4. Prolongation

A key to the formal theory of partial differential equations is the notion of differentiating a partial differential equation to arrive at a partial differential equation of higher order. This process is called prolongation.

4.1. Definition and properties.

We begin with the definition.

4.1 Definition: (Prolongation of a partial differential equation) Let $\pi\colon \mathsf{Y} \to \mathsf{X}$ be a fibred manifold and let $\mathsf{R}_k \subset \mathsf{J}_k\pi$ be a partial differential equation of order k. For $l \in \mathbb{Z}_{\geq 0}$ the ***lth prolongation*** of R_k is the subset

$$\rho_l(\mathsf{R}_k) = \mathsf{J}_l\hat{\pi}_k \cap \mathsf{J}_{k+l}\pi$$

of $\mathsf{J}_{k+l}\pi$ (where we use the natural inclusion of $\mathsf{J}_{k+l}\pi$ in $\mathsf{J}_l\pi_k$). The partial differential equation R_k is ***regular*** if $\rho_l(\mathsf{R}_k)$ is a fibred submanifold of $\pi_{k+l}\colon \mathsf{J}_{k+l}\pi \to \mathsf{X}$ for each $l \in \mathbb{Z}_{\geq 0}$. •

4.2 Remark: (Empty prolongations) It is possible that $\rho_l(\mathsf{R}_k)$ will be empty for $l \in \mathbb{Z}_{>0}$; we shall see examples of this. We adopt the convention that the prolongation of the empty set is again the empty set, in order to be able to define all constructions related to prolongation without having to constantly add the proviso that a prolongation be nonempty. •

We denote $\rho_l(\mathsf{R}_k)_x = \rho_l(\mathsf{R}_k) \cap \pi_{k+l}^{-1}(x)$ for $x \in \mathsf{X}$. We also adopt the convention that $\rho_{-l}(\mathsf{R}_k) = \mathsf{J}_{k-l}\pi$ for $l \in \{1, \dots, k\}$.

The following result gives a fairly obvious but essential observation about the successive prolongations.

4.3 Proposition: (Prolongations project to subsets of prolongations) *Let* $\pi\colon \mathsf{Y} \to \mathsf{X}$ *be a fibred manifold and let* $\mathsf{R}_k \subset \mathsf{J}_k\pi$ *be a* k*th-order partial differential equation. Then, for* $l, j \in \mathbb{Z}_{\geq 0}$ *with* $j \leq l$*,* $\pi_{k+j}^{k+l}(\rho_l(\mathsf{R}_k)) \subset \rho_j(\mathsf{R}_k)$*.*

Proof: We think of $\rho_l(\mathsf{R}_k)$ and $\rho_j(\mathsf{R}_k)$ as being subsets of $\mathsf{J}_l\pi_k$ and $\mathsf{J}_j\pi_k$, respectively. Certainly $\pi^{k+l}_{k+j}(\mathsf{J}_l\hat{\pi}_k) \subset \mathsf{J}_j\hat{\pi}_k$ and $\pi^{k+l}_{k+j}(\mathsf{J}_{k+l}\pi) \subset \mathsf{J}_{k+j}\pi$. It therefore follows that

$$\pi^{k+l}_{k+j}(\mathsf{J}_l\hat{\pi}_k \cap \mathsf{J}_{k+l}\pi) \subset \mathsf{J}_j\hat{\pi}_k \cap \mathsf{J}_{k+j}\pi,$$

which is the result. ■

We denote by $\hat{\pi}^{k+l}_{k+j}\colon \rho_l(\mathsf{R}_k) \to \rho_j(\mathsf{R}_k)$, $j \leq l$, and $\hat{\pi}_{k+l}\colon \rho_l(\mathsf{R}_k) \to \mathsf{X}$ the canonical projections, these making sense by the preceding result. Note that these maps are *not* necessarily surjective submersions. If R_k is regular, then by definition the maps $\hat{\pi}_{k+l}$ are surjective submersions for $l \in \mathbb{Z}_{\geq 0}$. However, even if R_k is regular, the maps $\hat{\pi}^{k+l}_{k+j}$, $j \leq l$, may fail to be surjective submersions, by failing to be surjective and/or by failing to be submersions. Indeed, as we shall see, surjectivity of the projections $\hat{\pi}^{k+l}_{k+j}$, $j \leq l$, is related to the important notion of formal integrability.

We are also interested in infinite prolongations. The definition is analogous to the definition of the set of infinite jets.

4.4 Definition: (Infinite prolongation) Let $\pi\colon \mathsf{Y} \to \mathsf{X}$ be a fibred manifold and let $\mathsf{R}_k \subset \mathsf{J}_k\pi$ be a kth-order partial differential equation. An ***infinite prolongation*** of R_k at x is a map $p_\infty\colon \mathbb{Z}_{\geq 0} \to \cup_{l\in\mathbb{Z}_{\geq 0}}\rho_l(\mathsf{R}_k)_x$ with the following properties:

(i) $p_\infty(l) \in \rho_l(\mathsf{R}_k)_x$ for each $l \in \mathbb{Z}_{\geq 0}$;

(ii) $\hat{\pi}^{k+l}_{k+j}(p_\infty(l)) = p_\infty(j)$ for all $j, l \in \mathbb{Z}_{\geq 0}$ such that $j \leq l$.

The set of infinite prolongations of R_k at x is denoted by $\rho_\infty(\mathsf{R}_k)_x$, and the ***infinite prolongation*** of R_k is $\rho_\infty(\mathsf{R}_k) = \cup_{x\in\mathsf{X}}\rho_\infty(\mathsf{R}_k)_x$. •

We denote by $\hat{\pi}_\infty\colon \rho_\infty(\mathsf{R}_k) \to \mathsf{X}$ and $\hat{\pi}^\infty_{k+l}\colon \rho_\infty(\mathsf{R}_k) \to \rho_l(\mathsf{R}_k)$ the canonical projections.

A useful way to see prolongation is via the following characterisation.

4.5 Proposition: (Characterisation of prolongation) *Let $\pi\colon \mathsf{Y} \to \mathsf{X}$ be a fibred manifold, let $\mathsf{R}_k \subset \mathsf{J}_k\pi$ be a kth-order partial differential equation, and let $l \in \mathbb{Z}_{>0}$. If $\rho_l(\mathsf{R}_k)$ is a fibred submanifold of $\pi_{k+l}\colon \mathsf{J}_{k+l}\pi \to \mathsf{X}$ then the following diagram is commutative and has exact rows and columns:*

$$\begin{tikzcd}
 & 0_{\mathsf{X}} \arrow[d] & 0_{\mathsf{X}} \arrow[d] \\
0_{\mathsf{X}} \arrow[r] & \rho_l(\mathsf{R}_k) \arrow[r] \arrow[d] & \mathsf{J}_l\hat{\pi}_k \arrow[d] \\
0_{\mathsf{X}} \arrow[r] & \mathsf{J}_{k+l}\pi \arrow[r] & \mathsf{J}_l\pi_k
\end{tikzcd}$$

Proof: Commutativity of the diagram

$$\begin{tikzcd}
\rho_l(\mathsf{R}_k) \arrow[r] \arrow[d] & \mathsf{J}_l\hat{\pi}_k \arrow[d] \\
\mathsf{J}_{k+l}\pi \arrow[r] & \mathsf{J}_l\pi_k
\end{tikzcd}$$

does not rely on $\rho_l(\mathsf{R}_k)$ being a fibred submanifold, and follows simply by the definition of prolongation as being the intersection of $\mathsf{J}_l\hat{\pi}_k$ and $\mathsf{J}_{k+l}\pi$ in $\mathsf{J}_l\pi_k$ (i.e., commutativity is simply reliant on general set-type arguments). Exactness of the rows and columns follows since all maps are injective. ■

Diagrams of this sort arise when one defines an object by intersection with two other objects, cf. Lemma 1 in the proof of Proposition 5.2. The way to think of the reformulation given in the preceding proposition of the definition of the prolongation is like this: $\rho_l(\mathsf{R}_k)$ is to $\mathsf{J}_l\hat{\pi}_k$ as $\mathsf{J}_{k+l}\pi$ is to $\mathsf{J}_l\pi_k$. The inclusion of $\mathsf{J}_{k+l}\pi$ in $\mathsf{J}_l\pi_k$ is easy to understand (we have done this previously), and so $\rho_l(\mathsf{R}_k)$ ought to be thought of in the same way. It is simply the lth-jet bundle of $\pi_k\colon \mathsf{R}_k \to \mathsf{X}$, taking into account some canonical jet bundle geometry.

Let us give some prolongations for the examples we have introduced.

4.6 Examples: (Prolongation)

1. We consider the second-order partial differential equation R_{lap} defined by $\mathsf{X} = \mathbb{R}^2$, $\mathsf{Y} = \mathbb{R}^2 \times \mathbb{R}$, and

$$\mathsf{R}_{\text{lap}} = \{\, (x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) \mid \; u_{xx} + u_{yy} = 0\}\,.$$

Let us use coordinates $(x, y, u, u_x, u_y, u_{xx}, u_{xy})$ for R_{lap}, noting that the inclusion of R_{lap} in $\mathsf{J}_2\pi$ is then

$$(x, y, u, u_x, u_y, u_{xx}, u_{xy}) \mapsto (x, y, u, u_x, u_y, u_{xx}, u_{xy}, -u_{xx}). \tag{4.1}$$

Let us denote by $\rho_1(\mathsf{R}_{\text{lap}})$ the first prolongation of R_{lap}. To determine this, we note that coordinates for $\mathsf{J}_1\hat{\pi}_2$ are denoted by

$$(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{,x}, u_{,y}, u_{x,x}, u_{x,y}, u_{y,x}, u_{y,y}, u_{xx,x}, u_{xx,y}, u_{xy,x}, u_{xy,y}),$$

where the terms to the right of the commas mean partial differentiation of the fibre variables of $\mathsf{J}_1\hat{\pi}_2$. If we think of $\mathsf{J}_1\hat{\pi}_2$ as a subset of $\mathsf{J}_1\pi_2$ then, using the inclusion (4.1), this subset is given by

$$\begin{aligned}&\{(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, u_{,x}, u_{,y}, u_{x,x}, u_{x,y}, u_{y,x}, u_{y,y},\\ &u_{xx,x}, u_{xx,y}, u_{xy,x}, u_{xy,y}, u_{yy,x}, u_{yy,y})\mid\; u_{yy} = -u_{xx},\; u_{yy,x} = -u_{xx,x},\; u_{yy,y} = -u_{xx,y}\}.\end{aligned} \tag{4.2}$$

Now the inclusion of $\mathsf{J}_3\pi$ in $\mathsf{J}_1\pi_2$ is given by

$$\begin{aligned}&(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, u_{xxx}, u_{xxy}, u_{xyy}, u_{yyy}) \mapsto\\ &(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, u_x, u_y, u_{xx}, u_{xy}, u_{xy}, u_{yy}, u_{xxx}, u_{xxy}, u_{xxy}, u_{xyy}, u_{xyy}, u_{yyy}).\end{aligned}$$

Thus $\mathsf{J}_3\pi$ is the subset of $\mathsf{J}_1\pi_2$ given by

$$\begin{aligned}&\{(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, u_{,x}, u_{,y}, u_{x,x}, u_{x,y}, u_{y,x}, u_{y,y},\\ &\qquad u_{xx,x}, u_{xx,y}, u_{xy,x}, u_{xy,y}, u_{yy,x}, u_{yy,y})\mid\; u_{,x} = u_x,\; u_{,y} = u_y,\\ &\qquad u_{x,x} = u_{xx},\; u_{x,y} = u_{y,x} = u_{xy},\; u_{y,y} = u_{yy},\; u_{xy,x} = u_{xx,y},\; u_{xy,y} = u_{yy,x}\}.\end{aligned} \tag{4.3}$$

Thus $\mathsf{J}_1\hat{\pi}_2 \cap \mathsf{J}_3\pi$ is the subset $\mathsf{J}_1\pi_2$ given by combining the relations given by equations (4.2) and (4.3):

$$\begin{aligned}\rho_1(\mathsf{R}_{\text{lap}}) = \{(&x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, u_{,x}, u_{,y}, u_{x,x}, u_{x,y}, u_{y,x}, u_{y,y},\\ &u_{xx,x}, u_{xx,y}, u_{xy,x}, u_{xy,y}, u_{yy,x}, u_{yy,y})|\ u_{,x} = u_x,\ u_{,y} = u_y,\\ &u_{x,x} = u_{xx},\ u_{x,y} = u_{y,x} = u_{xy},\ u_{y,y} = u_{yy} = -u_{xx},\ u_{xy,x} = u_{xx,y},\\ &u_{xy,y} = u_{yy,x} = -u_{xx,x},\ u_{yy,y} = -u_{xx,y}\}.\end{aligned}$$

This is the first prolongation of R_{lap} thought of as a subset of $\mathsf{J}_1\pi_2$. It is most revealing to think of this as a subset of $\mathsf{J}_3\pi$ via (4.3):

$$\begin{aligned}\rho_1(\mathsf{R}_{\text{lap}}) = \{(&x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, u_{xxx}, u_{xxy}, u_{xyx}, u_{xyy}, u_{yyy})|\\ &u_{xx} + u_{yy} = 0,\ u_{xxx} + u_{yyx} = 0,\ u_{xxy} + u_{yyy} = 0\}.\end{aligned}$$

2. Now let us consider the first-order partial differential equation R_{der} defined by $\mathsf{X} = \mathbb{R}^2$, $\mathsf{Y} = \mathbb{R}^2 \times \mathbb{R}$, and

$$\mathsf{R}_{\text{der}} = \{ (x, y, f, f_x, f_y) \mid\ f_x = \alpha(x, y),\ f_y = \beta(x, y)\} .$$

We wish to compute the first prolongation. Since presumably the reader can gather the details of the process from the proceeding example, let us merely give the answer:

$$\begin{aligned}\rho_1(\mathsf{R}_{\text{der}}) = \{(&x, y, f_x, f_y, f_{xx}, f_{xy}, f_{yy})|\ f_x = \alpha(x, y),\ f_y = \beta(x, y),\\ &f_{xx} = \tfrac{\partial\alpha}{\partial x}(x, y),\ f_{xy} = \tfrac{\partial\alpha}{\partial y}(x, y),\ f_{xy} = \tfrac{\partial\beta}{\partial x}(x, y),\ f_{yy} = \tfrac{\partial\beta}{\partial y}(x, y)\}.\end{aligned}$$

In particular, note that if $\frac{\partial\alpha}{\partial y}(x, y) \neq \frac{\partial\beta}{\partial x}(x, y)$ for every $(x, y) \in \mathbb{R}^2$ then $\rho_1(\mathsf{R}_{\text{der}})_{(x,y)} = \emptyset$.

3. Let us forgo the chore of writing the first prolongation of the steady-state Euler equations.
4. For the partial differential equation R_{ode} with $\mathsf{X} = \mathbb{R}$ and $\mathsf{Y} = \mathbb{R} \times \mathbb{R}^2$ we determine the first prolongation to be

$$\begin{aligned}\rho_1(\mathsf{R}_{\text{ode}}) = \{(&t, x, y, x_t, y_t, x_{tt}, y_{tt})|\ x_t = f(t, x, y),\ \alpha y_t = g(t, x, y),\\ &x_{tt} = \tfrac{\partial f}{\partial t}(t, x, y) + \tfrac{\partial f}{\partial x}(t, x, y)x_t + \tfrac{\partial f}{\partial y}(t, x, y)y_t,\\ &\alpha y_{tt} = \tfrac{\partial g}{\partial t}(t, x, y) + \tfrac{\partial g}{\partial x}(t, x, y)x_t + \tfrac{\partial g}{\partial y}(t, x, y)y_t\}.\end{aligned}$$

If $\alpha = 0$ and if $g(t, x, y) \neq 0$ for every $(t, x, y) \in \mathbb{R} \times \mathbb{R}^2$ then $\mathsf{R}_{\text{ode},t} = \emptyset$ and also $\rho_1(\mathsf{R}_{\text{ode}})_t = \emptyset$ as a consequence.
5. Let us not bother to prolong the Euler–Lagrange equations. •

The next result shows that the order in which one constructs prolongations is immaterial.

4.7 PROPOSITION: (The mth prolongation of the lth prolongation is the $(l + m)$th prolongation) *Let $\pi\colon \mathsf{Y} \to \mathsf{X}$ be a fibred manifold and let $\mathsf{R}_k \subset \mathsf{J}_k\pi$ be a kth-order partial differential equation. For $l, m \in \mathbb{Z}_{>0}$, if $\rho_l(\mathsf{R}_k)$ is a fibred submanifold of $\pi_{k+l}\colon \mathsf{J}_{k+l}\pi \to \mathsf{X}$ then $\rho_m(\rho_l(\mathsf{R}_k)) = \rho_{l+m}(\mathsf{R}_k)$.*

Proof: It suffices to work locally in a neighbourhood $\mathcal{U}$ in X. Therefore, to simplify notation, and without loss of generality, we suppose that there exists a global defining equation $(\mathsf{X}, \mathsf{Z}, \tau, \Phi, \eta)$ for R_k. That is, we suppose that $\mathsf{R}_k = \ker_\eta(\Phi)$. (Jumping ahead) we will then use the local defining equation for the prolongation defined in Proposition 4.13. From Proposition 4.13 we then have

$$\rho_{l+m}(\mathsf{R}_k) = \ker_{j_{l+m}\eta}(\rho_{l+m}(\Phi)), \quad \rho_m(\rho_l(\mathsf{R}_k)) = \ker_{j_m j_l \eta}(\rho_m(\rho_l(\Phi))).$$

This gives the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & & 0_{\mathsf{X}} \arrow[d] & 0_{\mathsf{X}} \arrow[d] & \\
0_{\mathsf{X}} \arrow[r] & \rho_{l+m}(\mathsf{R}_k) \arrow[r] \arrow[dd, dashed, equal] & \mathsf{J}_{k+l+m}\pi \arrow[r, "\rho_{l+m}(\Phi)"] \arrow[dd, equal] & \mathsf{J}_{l+m}\tau \arrow[dd, "\lambda_{l,m}"'] \arrow[dr] & \\
 & & & & \mathsf{X} \arrow[ul, bend right, "j_{l+m}\eta"'] \arrow[dl, bend left, "j_m j_l \eta"] \\
0_{\mathsf{X}} \arrow[r, "\iota_{k,l,m}"'] & \rho_m(\rho_l(\mathsf{R}_k)) \arrow[r] & \mathsf{J}_{k+l+m}\pi \arrow[r, "\rho_m(\rho_l(\Phi))"'] & \mathsf{J}_m\tau_l \arrow[ur] &
\end{tikzcd}$$

We wish to assert the existence of the map indicated by the dashed arrow, and show that it is an isomorphism of fibred manifolds. For this it suffices to show that $\rho_{l+m}(\mathsf{R}_k)$ and $\rho_m(\rho_l(\mathsf{R}_k))$ agree as subsets of $\mathsf{J}_{k+l+m}\pi$.

Let $p_{k,l+m} \in \rho_{l+m}(\mathsf{R}_k) \subset \mathsf{J}_{k+l+m}\pi$, supposing that $\pi_{k+l+m}(p_{k,l+m}) = x$. Then $p_{k,l+m} \in \ker_{j_{l+m}\eta}(\rho_{l+m}(\Phi))$; in other words $\rho_{l+m}(\Phi)(p_{k,l+m}) = j_{l+m}\eta(x)$. Then

$$\lambda_{l,m} \circ \rho_{l+m}(\Phi)(p_{k,l+m}) = \lambda_{l,m}(j_{l+m}\eta(x)) \quad\Longrightarrow\quad \rho_m(\rho_l(\Phi))(p_{k,l+m}) = j_m j_l \eta(x)$$

by commutativity of the above diagram and by definition of $\lambda_{l,m}$. Thus $p_{k,l+m} \in \ker_{j_m j_l \eta}(\rho_m(\rho_l(\Phi)))$ implying that $p_{k,l+m} \in \rho_m(\rho_l(\mathsf{R}_k))$. Thus $\rho_{l+m}(\mathsf{R}_k) \subset \rho_m(\rho_l(\mathsf{R}_k))$.

Now suppose that $p_{k,l,m} \in \rho_m(\rho_l(\mathsf{R}_k)) \subset \mathsf{J}_{k+l+m}(\pi)$. Then $p_{k,l,m} \in \ker_{j_m j_l \eta}(\rho_m(\rho_l(\Phi)))$ which gives

$$\rho_m(\rho_l(\Phi))(p_{k,l,m}) = j_m j_l \eta(x) \quad\Longrightarrow\quad \lambda_{l,m} \circ \rho_{l+m}(\Phi)(p_{k,l,m}) = j_m j_l \eta(x)$$

by commutativity of the above diagram. Thus $j_m j_l \eta(x) \in \mathrm{image}(\lambda_{l,m})$ and so we have

$$\lambda_{l,m} \circ \rho_{l+m}(\Phi)(p_{k,l,m}) = \lambda_{l+m}(j_{l+m}\eta(x)).$$

By injectivity of $\lambda_{l,m}$ this gives $\rho_{l+m}(\Phi)(p_{k,l,m}) = j_{l+m}\eta(x)$ so $p_{k,l,m} \in \ker_{j_{l+m}\eta}(\rho_{l+m}(\Phi))$. Thus $p_{k,l,m} \in \rho_{l+m}(\mathsf{R}_k)$, giving the result. ■

4.8 Notation: (Some convenient notational abuses for prolongation that we will not make) If R_k is regular, then it matters not the order in which we prolong. In such cases we could denote $\mathsf{R}_{k+l+m} = \rho_m(\rho_l(\mathsf{R}_k)) = \rho_{l+m}(\mathsf{R}_k)$ for any $l, m \in \mathbb{Z}_{\geq 0}$. However, when R_k is not regular, this notation can be potentially confusing. Nonetheless, it is common practice to write R_{k+l} for the lth prolongation. However, it is generally only the case that "$\mathsf{R}_{k+(l+m)} = \mathsf{R}_{(k+l)+m}$" when R_{k+l} is a fibred submanifold. Moreover, in order to understand the meaning of the symbols "$\mathsf{R}_{k+(l+m)}$" and "$\mathsf{R}_{(k+l)+m}$" one should, for example, discard thinking of $k + (l + m)$ as an integer, and think of it as a symbol. Thus $4 + (3 + 2) \neq 9$. Since this is a little disruptive, we will not use this common and sometimes convenient notation. •

Let us next consider the prolongation of linear partial differential equations.

4.9 PROPOSITION: (Prolongation of linear partial differential equations) *Let* $\pi\colon \mathsf{E} \to \mathsf{X}$ *be a vector bundle and let* $\mathsf{R}_k \subset \mathsf{J}_k\pi$ *be an inhomogeneous linear partial differential equation. Then, for* $l \in \mathbb{Z}_{\geq 0}$, $\rho_l(\mathsf{R}_k)$ *is a family of affine subspaces over* X. *Moreover, if* R_k *is homogeneous then* $\rho_l(\mathsf{R}_k)$ *is a family of vector subspaces over* X.

Proof: We claim that since $\hat{\pi}_k\colon \mathsf{R}_k \to \mathsf{X}$ is an affine subbundle of the vector bundle $\pi_k\colon \mathsf{J}_k\pi \to \mathsf{X}$, $\mathsf{J}_l\hat{\pi}_k$ is an affine subbundle of the vector bundle $(\pi_k)_l\colon \mathsf{J}_l\pi_k \to \mathsf{X}$. Indeed, by Proposition 3.10 there exists a vector bundle $\tau\colon \mathsf{F} \to \mathsf{X}$, a section η of this vector bundle, and a vector bundle morphism $\Phi\colon \mathsf{J}_k\pi \to \mathsf{F}$ such that such that $\mathsf{R}_k = \ker_\eta(\Phi)$. That is, we have the following exact sequence of fibred manifolds:

$$\begin{tikzcd} 0_{\mathsf{X}} \arrow[r] & \mathsf{R}_k \arrow[r] & \mathsf{J}_k\pi \arrow[rr, "\Phi"] \arrow[dr, "\pi_k"'] & & \mathsf{F} \arrow[dl, "\tau"'] \\ & & & \mathsf{X} \arrow[ur, bend right, "\eta"'] & \end{tikzcd}$$

By Proposition 2.1.11 of [Pommaret 1978] this gives rise to the exact sequence

$$\begin{tikzcd} 0_{\mathsf{X}} \arrow[r] & \mathsf{J}_l\hat{\pi}_k \arrow[r] & \mathsf{J}_l\pi_k \arrow[rr, "J_l\Phi"] \arrow[dr, "\pi_k"'] & & \mathsf{J}_l\mathsf{F} \arrow[dl, "\tau_l"'] \\ & & & \mathsf{X} \arrow[ur, bend right, "j_l\eta"'] & \end{tikzcd}$$

of fibred manifolds, and our claim follows since $J_l\Phi$ is a vector bundle morphism.

Since $\rho_l(\mathsf{R}_k) = \mathsf{J}_l\hat{\pi}_k \cap \mathsf{J}_{k+l}\pi$ and since the intersection of two affine subspaces is again an affine subspace, this gives the first part of the proposition, and the second part follows since, if R_k is homogeneous, the above argument can be made with η being the zero section, and so the sequences above become exact sequences of vector bundles. ■

Finally, let us note some special structure of the fibres of the projections $\hat{\pi}^{k+l+1}_{k+l}\colon \rho_{l+1}(\mathsf{R}_k) \to \rho_l(\mathsf{R}_k)$.

4.10 PROPOSITION: (Fibres of prolongations) *Let* $\pi\colon \mathsf{Y} \to \mathsf{X}$ *be a fibred manifold, let* $\mathsf{R}_k \subset \mathsf{J}_k\pi$ *be a* k*th-order partial differential equation, and let* $l \in \mathbb{Z}_{\geq 0}$. *Then, for each* $p_{k+l} \in \rho_l(\mathsf{R}_k)$, $(\hat{\pi}^{k+l+1}_{k+l})^{-1}(p_{k+l})$ *is an affine subspace of* $(\pi^{k+l+1}_{k+l})^{-1}(p_{k+l})$.

Proof: We think of $\rho_l(\mathsf{R}_k) = \mathsf{J}_l\hat{\pi}_k \cap \mathsf{J}_{k+l}\pi$ as being a subset of $\mathsf{J}_l\pi_k$ and we think of $\rho_{l+1}(\mathsf{R}_k) = \mathsf{J}_{l+1}\hat{\pi}_k \cap \mathsf{J}_{k+l+1}\pi$ as being a subset of $\mathsf{J}_{l+1}\pi_k$. Since $\mathsf{J}_{l+1}\hat{\pi}_k$ is an affine bundle over $\mathsf{J}_l\hat{\pi}_k$ it follows that the fibres of $(\hat{\pi}_k)^{l+1}_l$ are affine subspaces of the fibres of $(\pi_k)^{l+1}_l$. Similarly, the fibres of π^{k+l+1}_{k+l} are affine subspaces of the fibres of $(\pi_k)^{l+1}_l$. Since the intersection of affine subspaces is an affine subspace, the result follows. ■

4.2. Local description of prolongation. Another insightful way to view the prolongation involves the defining equation point of view. In order to present this, we first define the prolongation of a morphism from a jet bundle.

4.11 Definition: (Prolongation of a morphism) Let $\pi\colon \mathsf{Y}\to\mathsf{X}$ and $\tau\colon \mathsf{Z}\to\mathsf{X}$ be fibred manifolds and let $\Phi\colon \mathsf{J}_k\pi\to\mathsf{Z}$ be a morphism of fibred manifolds over X. The ***lth prolongation*** of Φ is the unique morphism $\rho_l(\Phi)\colon \mathsf{J}_{k+l}\pi\to\mathsf{J}_l\tau$ such that the following diagram commutes:

$$\begin{tikzcd} \mathscr{G}_{\mathsf{X}}(\mathsf{J}_{k+l}\pi) \arrow[r, "\rho_l(\Phi)"] & \mathscr{G}_{\mathsf{X}}(\mathsf{J}_l\tau) \\ \mathscr{G}_{\mathsf{X}}(\mathsf{Y}) \arrow[u, "j_{k+l}"] \arrow[r, "\Phi"'] & \mathscr{G}_{\mathsf{X}}(\mathsf{Z}) \arrow[u, "j_l"'] \end{tikzcd}$$

•

The prolongation of a morphism has a simple relationship with the jet of a morphism.

4.12 Proposition: (The prolongation of a morphism and the jet of a morphism) *Let $\pi\colon \mathsf{Y}\to\mathsf{X}$ and $\tau\colon \mathsf{Z}\to\mathsf{X}$ be fibred manifolds and let $\Phi\colon \mathsf{J}_k\pi\to\mathsf{Z}$ be a morphism of fibred manifolds over X. Then the following diagram is commutative:*

$$\begin{tikzcd} \mathsf{J}_{k+l}\pi \arrow[r, "\rho_l(\Phi)"] \arrow[d, "\lambda_{k,l}"'] & \mathsf{J}_l\tau \\ \mathsf{J}_l\pi_k \arrow[ur, "J_l\Phi"'] & \end{tikzcd}$$

Proof: This is Application 2.1.17 of [Pommaret 1978]. ■

The point of the proposition is that the prolongation of $\Phi\colon \mathsf{J}_k\pi\to\mathsf{Z}$ is the same as the jet of Φ, up to some canonical jet bundle geometry.

Let us see how this is useful to characterise the process of prolongation locally.

4.13 Proposition: (Prolongation is differentiation of equations) *Let $\pi\colon \mathsf{Y}\to\mathsf{X}$ be a fibred manifold, let $\mathsf{R}_k\subset\mathsf{J}_k\pi$ be a kth-order partial differential equation, and let $(\mathcal{U},\mathsf{Z},\tau,\Phi,\eta)$ be a local defining equation for R_k. Then, for $l\in\mathbb{Z}_{\geq 0}$, $(\mathcal{U},\mathsf{J}_l\tau,\tau_l,\rho_l(\Phi),j_l\eta)$ is a local defining equation for $\rho_l(\mathsf{R}_k)$.*

Proof: To simplify notation, let us suppose that $\mathsf{X}=\mathcal{U}$. We then have the following diagram

$$\begin{tikzcd} 0_{\mathsf{X}} \arrow[r] & \mathsf{R}_k \arrow[r] & \mathsf{J}_k\pi \arrow[rr, "\Phi"] \arrow[dr, "\pi_k"'] & & \mathsf{Z} \arrow[dl, "\tau"'] \\ & & & \mathsf{X} \arrow[ur, bend right, "\eta"'] & \end{tikzcd}$$

which commutes and is exact. This then gives the diagram

$$\begin{tikzcd} 0_{\mathsf{X}} \arrow[r] & \mathsf{J}_l\hat{\pi}_k \arrow[r] & \mathsf{J}_l\pi_k \arrow[rr, "J_l\Phi"] \arrow[dr, "(\pi_k)_l"'] & & \mathsf{J}_l\tau \arrow[dl, "\tau_l"'] \\ & & & \mathsf{X} \arrow[ur, bend right, "j_l\eta"'] & \end{tikzcd}$$

which is also commutative and exact by Proposition 2.1.11 of [Pommaret 1978]. From Propositions 4.5 and 4.12 this leads to the commutative and exact diagram

$$\begin{tikzcd}
 & 0_{\mathsf{X}} \arrow[d] & 0_{\mathsf{X}} \arrow[d] & \\
0_{\mathsf{X}} \arrow[r] & \rho_l(\mathsf{R}_k) \arrow[r] \arrow[d] & \mathsf{J}_{k+l}\pi \arrow[r, "\rho_l(\Phi)"] \arrow[d] & \mathsf{J}_l\tau \arrow[d, equal] \\
0_{\mathsf{X}} \arrow[r] & \mathsf{J}_l\hat{\pi}_k \arrow[r] & \mathsf{J}_l\pi_k \arrow[r, "J_l\Phi"] \arrow[dr, "(\pi_k)_l"'] & \mathsf{J}_l\tau \arrow[d, "\tau_l"'] \\
 & & & \mathsf{X} \arrow[u, bend right, "j_l\eta"']
\end{tikzcd}$$

From this diagram we can extract the following diagram

$$\begin{tikzcd}
0_{\mathsf{X}} \arrow[r] & \rho_l(\mathsf{R}_k) \arrow[r] & \mathsf{J}_{k+l}\pi \arrow[r, "\rho_l(\Phi)"] \arrow[dr, "\pi_{k+l}"'] & \mathsf{J}_l\tau \arrow[d, "\tau_l"'] \\
 & & & \mathsf{X} \arrow[u, bend right, "j_l\eta"']
\end{tikzcd}$$

which is commutative and exact. This is the result. ■

It will also be useful to know the relationship of prolongations to those of lower-order. The following result gives one such relationship.

4.14 PROPOSITION: (Prolongations fibre over lower-order prolongations) *Let* $\pi\colon \mathsf{Y} \to \mathsf{X}$ *and* $\tau\colon \mathsf{Z} \to \mathsf{X}$ *be a fibred manifolds, let* $\Phi\colon \mathsf{J}_k\pi \to \mathsf{Z}$ *be a morphism over* id_{X}*, and let* $l \in \mathbb{Z}_{>0}$*. Then the following diagram commutes:*

$$\begin{tikzcd}
\mathsf{J}_{k+l}\pi \arrow[r, "\rho_l(\Phi)"] \arrow[d, "\pi^{k+l}_{k+l-1}"'] & \mathsf{J}_l\tau \arrow[d, "\tau^l_{l-1}"] \\
\mathsf{J}_{k+l-1}\pi \arrow[r, "\rho_{l-1}(\Phi)"'] & \mathsf{J}_{l-1}\tau
\end{tikzcd}$$

Moreover, $\rho_l(\Phi)$ *is an affine bundle morphism over* $\rho_{l-1}(\Phi)$*.*

Proof: This is Application 2.1.18 of [Pommaret 1978]. ■

Let us see how this local characterisation of prolongation comes up in our examples. As we shall see, this local characterisation is perhaps a more natural one when it comes to actually computing a prolongation.

4.15 EXAMPLES: (Prolongation via local defining equations) We refer to Example 3.5 for the definitions of the local defining equations for the examples we consider here.

1. We consider the Laplacian partial differential equation $\mathsf{R}_{\mathrm{lap}}$ which is a linear partial differential equation with defining equation $\Phi_{\mathrm{lap}}\colon \mathsf{J}_2\pi \to \mathsf{Z}_{\mathrm{lap}} = \mathbb{R}^2 \times \mathbb{R}$ given by

$$\Phi_{\mathrm{lap}}(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = (x, y, u_{xx} + u_{yy}).$$

We compute that $J_1\Phi_{\mathrm{lap}}\colon \mathsf{J}_1\pi_2 \to \mathsf{J}_1\tau$ (where $\tau\colon \mathsf{Z}_{\mathrm{lap}} \to \mathsf{X}$ is the vector bundle projection) is given by

$$\begin{aligned}&J_1\Phi_{\mathrm{scr}}(x,y,u,u_x,u_y,u_{xx},u_{xy},u_{yy},u_{,x},u_{,y},u_{x,x}u_{x,y},u_{y,x},u_{y,y},\\&\quad u_{xx,x},u_{xx,y},u_{xy,x},u_{xy,y},u_{yy,x},u_{yy,y}) = (x,y,u_{xx}+u_{yy},u_{xx,x}+u_{yy,x},u_{xx,y}+u_{yy,y}),\end{aligned}$$

from which we deduce that $\rho_1(\Phi_{\mathrm{lap}})\colon \mathsf{J}_3\Phi \to \mathsf{J}_1\tau$ is given by

$$\begin{aligned}&\rho_1(\Phi_{\mathrm{lap}})(x,y,u,u_x,u_y,u_{xx},u_{xy},u_{yy},u_{xxx},u_{xxy},u_{xyx},u_{xyy},u_{yyx},u_{yyy})\\&\qquad\qquad = (x,y,u_{xx}+u_{yy},u_{xxx}+u_{xxy},u_{xxy}+u_{yyy}),\end{aligned}$$

using the inclusion of $\mathsf{J}_3\pi$ in $\mathsf{J}_1\pi_2$ given in (4.2). Since this equation is linear, this is all we need to do to determine the local defining equation for the first prolongation, i.e., $\rho_1(\mathsf{R}_{\mathrm{lap}}) = \ker(\rho(\Phi_{\mathrm{lap}}))$.

Let us make this even more transparent. Writing the partial differential equation as an actual…er…partial differential equation, we have

$$\frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) = 0.$$

Simply differentiating these equations with respect to x then y gives the additional equations

$$\frac{\partial^3 u}{\partial x^3}(x,y) + \frac{\partial^3 u}{\partial y^2\partial x}(x,y) = 0, \quad \frac{\partial^3 u}{\partial x^2\partial y}(x,y) + \frac{\partial^3 u}{\partial y^3}(x,y) = 0$$

which, together with the original equations, define the partial differential equations which are the first prolongation. This is how one would compute the prolongation in practice, although it is still instructive to simply apply the definition as we did above, and as we shall do in the examples below.

2. The exterior derivative partial differential equation $\mathsf{R}_{\mathrm{der}}$ has the local defining equation defined with $\Phi_{\mathrm{der}}\colon \mathsf{J}_1\pi \to \mathsf{Z}_{\mathrm{der}} = \mathbb{R}^2\times\mathbb{R}$ and η_{der} given by

$$\Phi_{\mathrm{der}}(x,y,f,f_x,f_y) = (x,y,f_x,f_y), \quad \eta_{\mathrm{der}}(x,y) = (x,y,\alpha(x,y),\beta(x,y)).$$

We compute

$$\begin{aligned}&\rho_1(\Phi_{\mathrm{der}})(x,y,f,f_x,f_y,f_{xx},f_{xy},f_{yy}) = (x,y,f_x,f_y,f_{xx},f_{xy},f_{xy},f_{yy}),\\&j_1\eta_{\mathrm{der}}(x,y) = (x,y,\alpha(x,y),\beta(x,y),\tfrac{\partial\alpha}{\partial x}(x,y),\tfrac{\partial\alpha}{\partial y}(x,y),\tfrac{\partial\beta}{\partial x}(x,y),\tfrac{\partial\beta}{\partial y}(x,y)).\end{aligned}$$

We then have $\rho_1(\mathsf{R}_{\mathrm{der}}) = \ker_{j_1\eta_{\mathrm{der}}}(\rho_1(\Phi_{\mathrm{der}}))$.

3. Again we forgo the pleasure of prolonging $\mathsf{R}_{\mathrm{sse}}$.
4. For $\mathsf{R}_{\mathrm{ode}}$ we compute

$$\begin{aligned}\rho_1(\Phi_{\mathrm{ode}})(t,x,y,x_t,y_t,x_{tt},y_{tt}) = (x,y,x_t &- f(t,x,y),\alpha y_t - g(t,x,y),\\ &x_{tt} - \tfrac{\partial f}{\partial t}(t,x,y) - \tfrac{\partial f}{\partial x}(t,x,y)x_t - \tfrac{\partial f}{\partial y}(t,x,y)y_t,\\ &\qquad \alpha y_{tt} - \tfrac{\partial g}{\partial t}(t,x,y) - \tfrac{\partial g}{\partial x}(t,x,y)x_t - \tfrac{\partial g}{\partial y}(t,x,y)y_t),\end{aligned}$$

then observe that $\rho_1(\mathsf{R}_{\mathrm{ode}}) = \ker_{j_1\eta_{\mathrm{ode}}}(\rho_1(\Phi_{\mathrm{ode}}))$, where η is the zero section.

5. Again, we do not bother to prolong the Euler–Lagrange equations. •

4.3. Coordinate formulae. Let us give the coordinate form for the prolongation of a morphism, and from this determine the coordinate form for the prolongation of a partial differential equation in terms of a local defining equation.

We let $\pi\colon \mathsf{Y} \to \mathsf{X}$ and we suppose that we have a local defining equation $(\mathcal{U}, \mathsf{Z}, \tau, \Phi, \eta)$ for a partial differential equation $\mathsf{R}_k \subset \mathsf{J}_k\pi$. We also suppose that $\mathcal{U}$ is a coordinate chart and that Φ is given locally by

$$(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{p}_1, \dots, \boldsymbol{p}_k) \mapsto (\boldsymbol{x}, \boldsymbol{\Phi}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{p}_1, \dots, \boldsymbol{p}_k)).$$

Let us denote the local form for lth prolongation of Φ by

$$\begin{aligned}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{p}_1, \dots, \boldsymbol{p}_{k+l}) \mapsto (\boldsymbol{x}, &\boldsymbol{\Phi}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{p}_1, \dots, \boldsymbol{p}_k),\\ &\rho_1(\boldsymbol{\Phi})((\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{p}_1, \dots, \boldsymbol{p}_{k+1})), \dots, \rho_l(\boldsymbol{\Phi})(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{p}_1, \dots, \boldsymbol{p}_{k+l})),\end{aligned}$$

so defining the maps $\rho_j(\boldsymbol{\Phi})$, $j \in \{1, \dots, l\}$. Note that

$$\rho_j(\boldsymbol{\Phi})(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{p}_1, \dots, \boldsymbol{p}_{k+j}) \in L^j_{\mathrm{sym}}(\mathbb{R}^n; \mathbb{R}^m).$$

To obtain explicit expressions for these maps requires some notation and effort. We work with multi-indices in $\{1, 2, \dots, k\}$ and we denote the set of all such multi-indices by $\mathscr{I}_k$. Thus an element of $\mathscr{I}_k$ is a map from $\{1, \dots, k\}$ into $\mathbb{Z}_{\geq 0}$, and we denote a typical such multi-index by $(\alpha(1), \alpha(2), \dots, \alpha(k))$. For $\alpha \in \mathscr{I}_k$ we denote

$$|\alpha| = \sum_{r=1}^{k} \alpha(r),$$

which is the ***order*** of α. We also denote by $\#\alpha$ the number of nonzero elements in the list $(\alpha(1), \dots, \alpha(k))$, and we write these nonzero elements as $\{\alpha(r^\alpha_1), \dots, \alpha(r^\alpha_{\#\alpha})\}$ with the convention that

$$r_{\alpha,1} < \dots < r_{\alpha,\#\alpha}.$$

For $s \in \{1, \dots, \#\alpha\}$ let us write

$$\beta_\alpha(s) = \alpha(r^\alpha_1) + \dots + \alpha(r^\alpha_s).$$

Thus $\beta_\alpha(s)$ gives the order of the first s nonzero elements in α. In particular, $\beta_\alpha(\#\alpha) = |\alpha|$. Given $\alpha \in \mathscr{I}_k$, let $S(\alpha)$ denote the $|\alpha|$-tuples $(a_1, \dots, a_{|\alpha|}) \subset \{0, 1, \dots, k\}^{|\alpha|}$ such that

1. $a_1 \leq \dots \leq a_{|\alpha|}$ and
2. $\displaystyle\sum_{s=1}^{|\alpha|} a_s = \sum_{l=1}^{k} l\alpha(l) - (2\,|\alpha| - j)$.

Let us consider some special cases to illustrate what $S(\alpha)$ is.

1. $k = 1$, $j = 2$, $\alpha = (2, 0, 0, 0)$: We have $|\alpha| = 2$. Thus we want pairs from $\{0, 1, 2\}$ which sum to $1 \cdot 2 - (2 \cdot 2 - 2) = 0$. Thus $S(\alpha) = \{(0, 0)\}$.
2. $k = 1$, $j = 3$, $\alpha = (0, 1, 1, 0, 0, 0)$: We have $|\alpha| = 2$. Thus we want pairs from $\{1, 2, 3\}$ which sum to $(2 \cdot 1 + 3 \cdot 1) - (2 \cdot 2 - 3) = 4$. Thus $S(\alpha) = \{(1, 3), (2, 2)\}$.

Given $\alpha \in \mathscr{I}_k$ we define an index by

$$\underbrace{r_1^\alpha \cdots r_1^\alpha}_{\alpha(r_1^\alpha)\ \text{times}} \cdots \underbrace{r_{\#\alpha}^\alpha \cdots r_{\#\alpha}^\alpha}_{\alpha(r_{\#\alpha}^\alpha)\ \text{times}}.$$

Let us denote the lth element in this index by $f_\alpha(l)$, $l \in \{1, \dots, |\alpha|\}$. If $\alpha \in \mathscr{I}_k$ and if $(a_1, \dots, a_{|\alpha|}) \in S(\alpha)$ then, for $l \in \{1, \dots, |\alpha|\}$ define $q_l^\alpha = a_l - r_l^\alpha + 2$. Given $\alpha \in \mathscr{I}_k$ and $(a_1, \dots, a_{|\alpha|}) \in S(\alpha)$ we can now make a table that summarises the constructions made thus far.

l	1	$\cdots$	$\alpha(r_1^\alpha)$	$\cdots$	$\beta_\alpha(\#\alpha - 1) + 1$	$\cdots$	$\|\alpha\|$
$f_\alpha(l)$	r_1^α	$\cdots$	r_1^α	$\cdots$	$r_{\#\alpha}^\alpha$	$\cdots$	$r_{\#\alpha}^\alpha$
a_l	a_1	$\cdots$	$a_{\alpha(r_1^\alpha)}$	$\cdots$	$a_{\beta_\alpha(\#\alpha-1)+1}$	$\cdots$	$a_{\|\alpha\|}$
q_l^α	$a_1 - r_1^\alpha + 2$	$\cdots$	$a_{\alpha(r_1^\alpha)} - r_1^\alpha + 2$	$\cdots$	$a_{\beta_\alpha(\#\alpha-1)+1} - r_{\#\alpha}^\alpha + 2$	$\cdots$	$a_{\|\alpha\|} + r_{\#\alpha}^\alpha + 2$

Now let $s \in \{1, \dots, \#\alpha\}$. Corresponding to s are the columns $\beta_\alpha(s-1)+1, \dots, \beta_\alpha(s)$ in the table above (the first column in the table does not count, it being a title). In the second row of the table the entries in these columns will all be equal to r_s^α. In the third row in these columns, the entries may be repeated or distinct. They will, however, by definition of $S(\alpha)$ be ordered. Thus we may write the entries in these columns in the third row as

$$\underbrace{\tilde{a}_1 \cdots \tilde{a}_1}_{b_{s,1}\ \text{terms}} \cdots \underbrace{\tilde{a}_{l_s} \cdots \tilde{a}_{l_s}}_{b_{s,\alpha(r_s^\alpha)}\ \text{terms}},$$

where

$$\tilde{a}_1 < \cdots < \tilde{a}_{l_s},$$

so defining numbers $b_{s,1}, \dots, b_{s,\alpha(r_s^\alpha)}$. Thus, given $\alpha \in \mathscr{I}_k$, $P = (a_1, \dots, a_{|\alpha|}) \in S(\alpha)$, and $s \in \{1, \dots, r_{\#\alpha}^\alpha\}$ we define

$$N(\alpha, P, s) = b_{s,1}! \cdots b_{s,\alpha(r_s^\alpha)}!.$$

If $\boldsymbol{A} \in L^s_{\mathrm{sym}}(\mathbb{R}^n; \mathbb{R}^m)$ and of $\boldsymbol{v}_1, \dots, \boldsymbol{v}_l \in \mathbb{R}^n$, $l \leq s$, then we denote

$$(\boldsymbol{v}_1, \dots, \boldsymbol{v}_l) \lrcorner \boldsymbol{A} = \boldsymbol{v}_1 \lrcorner \cdots \lrcorner \boldsymbol{v}_l \lrcorner \boldsymbol{A}.$$

Now we are in a position to give an explicit expression for $\rho_j(\boldsymbol{\Phi})$:

$$\begin{aligned}
\rho_j(\boldsymbol{\Phi})(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{p}_1, \dots, \boldsymbol{p}_{k+j}) \cdot (\boldsymbol{v}_1, \dots, \boldsymbol{v}_j) \\
= \sum_{\sigma \in S_j} \sum_{\substack{\alpha \in \mathscr{I}_k \\ |\alpha| \leq j}} \sum_{\substack{P = (a_1, \dots, a_{|\alpha|}) \\ \in S(\alpha)}} \frac{1}{q_1^\alpha! \cdots q_{|\alpha|}^\alpha! N(\alpha, P, 1)! \cdots N(\alpha, P, \#\alpha)!} \\
\boldsymbol{D}_{r_1^\alpha}^{\alpha(r_1^\alpha)} \cdots \boldsymbol{D}_{r_{\#\alpha}^\alpha}^{\alpha(r_{\#\alpha}^\alpha)} \boldsymbol{\Phi}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{p}_1, \dots, \boldsymbol{p}_k) \\
\cdot ((\boldsymbol{v}_{\sigma(1)}, \dots, \boldsymbol{v}_{\sigma(q_1^\alpha)}) \lrcorner \boldsymbol{p}_{a_1}, \dots, (\boldsymbol{v}_{\sigma(|\alpha| - q_{|\alpha|}^\alpha + 1)}, \dots, \boldsymbol{v}_{\sigma(j)}) \lrcorner \boldsymbol{p}_{a_{|\alpha|}}).
\end{aligned}$$

In writing this expression it has been convenient to adopt the convention that $\boldsymbol{v}_l \lrcorner \boldsymbol{p}_0 = \boldsymbol{v}_l$, $l \in \{1, \dots, j\}$. It will likely take a moment or two to understand where this expression comes from. Thankfully, it will not be all that important for us to have this formula at hand.

In order to aid in understanding the general formula, and also perhaps get the interested reader pointed in the direction of understanding the formula, let us explicitly give the expressions for the case when $k = 1$ and $j \in \{1, 2\}$:

$$\begin{aligned}
\rho_1(\boldsymbol{\Phi})(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{p}_1, \boldsymbol{p}_2) \cdot \boldsymbol{v}_1 &= \boldsymbol{D}_1\boldsymbol{\Phi} \cdot \boldsymbol{v}_1 + \boldsymbol{D}_2\boldsymbol{\Phi} \cdot (\boldsymbol{v}_1 \lrcorner\, \boldsymbol{p}_1) + \boldsymbol{D}_3\boldsymbol{\Phi} \cdot (\boldsymbol{v}_1 \lrcorner\, \boldsymbol{p}_2), \\
\rho_2(\boldsymbol{\Phi})(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3) \cdot (\boldsymbol{v}_1, \boldsymbol{v}_2) &= \boldsymbol{D}_1^2\boldsymbol{\Phi} \cdot (\boldsymbol{v}_1, \boldsymbol{v}_2) + \boldsymbol{D}_1\boldsymbol{D}_2\boldsymbol{\Phi} \cdot (\boldsymbol{v}_1, \boldsymbol{v}_2 \lrcorner\, \boldsymbol{p}_1) \\
&\quad + \boldsymbol{D}_1\boldsymbol{D}_2\boldsymbol{\Phi} \cdot (\boldsymbol{v}_2, \boldsymbol{v}_1 \lrcorner\, \boldsymbol{p}_1) + \boldsymbol{D}_1\boldsymbol{D}_3\boldsymbol{\Phi} \cdot (\boldsymbol{v}_1, \boldsymbol{v}_2 \lrcorner\, \boldsymbol{p}_2) \\
&\quad + \boldsymbol{D}_1\boldsymbol{D}_3\boldsymbol{\Phi} \cdot (\boldsymbol{v}_2, \boldsymbol{v}_1 \lrcorner\, \boldsymbol{p}_2) + \boldsymbol{D}_2^2\boldsymbol{\Phi} \cdot (\boldsymbol{v}_1 \lrcorner\, \boldsymbol{p}_1, \boldsymbol{v}_2 \lrcorner\, \boldsymbol{p}_1) \\
&\quad + \boldsymbol{D}_2\boldsymbol{D}_3\boldsymbol{\Phi} \cdot (\boldsymbol{v}_1 \lrcorner\, \boldsymbol{p}_1, \boldsymbol{v}_2 \lrcorner\, \boldsymbol{p}_2) + \boldsymbol{D}_2\boldsymbol{D}_3\boldsymbol{\Phi} \cdot (\boldsymbol{v}_2 \lrcorner\, \boldsymbol{p}_1, \boldsymbol{v}_1 \lrcorner\, \boldsymbol{p}_2) \\
&\quad + \boldsymbol{D}_3^2\boldsymbol{\Phi} \cdot (\boldsymbol{v}_1 \lrcorner\, \boldsymbol{p}_2, \boldsymbol{v}_2 \lrcorner\, \boldsymbol{p}_2) + \boldsymbol{D}_2\boldsymbol{\Phi} \cdot (\boldsymbol{v}_1 \lrcorner\, \boldsymbol{v}_2 \lrcorner\, \boldsymbol{p}_2) \\
&\quad + \boldsymbol{D}_3\boldsymbol{\Phi} \cdot (\boldsymbol{v}_1 \lrcorner\, \boldsymbol{v}_2 \lrcorner\, \boldsymbol{p}_3),
\end{aligned}$$

where all partial derivatives are evaluated at $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{p}_1)$.

As a final aid in understanding the general formula for the prolongation, let us make some comments on the character of the various components of the formula. These comments will be pretty much meaningless until the reader has looked in detail at the two special cases above, and thought about how these special cases can be generalised.

1. When computing the jth prolongation we have to derive $\boldsymbol{\Phi}$ j times using the Chain Rule. In doing this, derivatives will arise in two ways: (a) directly in terms of derivatives of $\boldsymbol{\Phi}$ and (b) by differentiation of the partial derivatives $\boldsymbol{p}_1, \dots, \boldsymbol{p}_k$. One can get some insight into this process by looking at the case of $k = 1$ and $j = 1$ above. Because of the fact that differentiation arises by the second of these processes, the expression for the jth prolongation will involve derivatives of $\boldsymbol{\Phi}$ of order, not just j, but order up to j. This explains the sum over multi-indices of order at most j.
2. When partially differentiating $\boldsymbol{\Phi}$ directly with respect to the lth component, the derivative is evaluated on something in $\mathbb{R}^n$ if $l = 1$ and something in $L^{l-1}_{\mathrm{sym}}(\mathbb{R}^n; \mathbb{R}^m)$ if $l > 1$. This argument arises in the process by taking one of the partial derivatives, say $\boldsymbol{p}_s$, and contracting it with an appropriate number of vectors in $\mathbb{R}^n$. These will be expressions of the sort
$$(\boldsymbol{v}_{j_1}, \dots, \boldsymbol{v}_{j_q}) \lrcorner\, \boldsymbol{p}_s \tag{4.4}$$
where q and s have the property that the expression (4.4) is of the proper form, i.e., $s - q = l - 1$. For a given multi-index α with the corresponding partial derivative of $\boldsymbol{\Phi}$, there will be a collection of possible $\boldsymbol{p}_s$'s, $s \in \{1, \dots, k + j\}$, that will appear in the argument of the partial derivative. This explains where $S(\alpha)$ comes from. It also explains where the numbers q^α_l come from. In fact, in the table above, the second row represents the component of $\boldsymbol{\Phi}$ with respect to which differentiation is being done, the third row represents which of the partial derivatives $\boldsymbol{p}_1, \dots, \boldsymbol{p}_{k+j}$ appears in the argument of the partial derivative, and the fourth row gives the number q in the expression (4.4) in order to ensure that the arguments work out.
3. There is a sum over permutations in the general formula for the jth prolongation. This achieves symmetry of the resulting expression. However, some of the terms in the sum are already symmetric. For example, the expression (4.4) is symmetric in $\boldsymbol{v}_{j_1}, \dots, \boldsymbol{v}_{j_q}$.

Also, repeated partial derivatives of $\mathbf{\Phi}$ with respect to one of its components also gives something symmetric. One needs to account for these preexisting symmetries since, if left unaccounted for, one will get repeated terms in the sum that ought to not be there. This is where the factor

$$\frac{1}{q_1^{\alpha}! \cdots q_{|\alpha|}^{\alpha}! N(\alpha, P, 1)! \cdots N(\alpha, P, \#\alpha)!}$$

in the sum arises from.

By keeping in mind these comments, and by considering the above special cases, we hope that the reader can begin to see where the general formula for the jth prolongation comes from.

5. Symbols and their prolongations

The symbol, roughly, captures the highest order term in the linearisation of a partial differential equation, cf. Proposition 8.7. Just why this is an important thing to consider is not so obvious at first glance. But it in nonetheless the case that much of the formal study of partial differential equations rests on properties of the symbol. There are several ways of thinking of the symbol (and its prolongations), and each has value depending on what one wishes to accomplish. In terms of understanding the symbol (and its prolongations) as it relates (they relate) to formal integrability, it is useful to understand the symbol (and its prolongations) as it relates to the partial differential equation (and its prolongations). To understand the important algebraic properties of the symbol (and its prolongations), it is useful to think of the symbol as a pointwise vector space construction, in many ways completely independent of the partial differential equation from which it is derived. We shall consider both points of view, and the relationships between them.

5.1. The various characterisations of the symbol. We begin by assembling the machinery needed to provide our multiple equivalent characterisations of the symbol.

First recall that $\pi^k_{k-1} \colon \mathsf{J}_k\pi \to \mathsf{J}_{k-1}\pi$ is an affine bundle whose fibre at $p_{k-1} \in \mathsf{J}_{k-1}\pi$ is an affine space modelled on $S_k(\mathsf{T}^*_x\mathsf{X}) \otimes \mathsf{V}_y\pi$ where $x = \pi_{k-1}(p_{k-1})$ and where $y = \pi^{k-1}_0(p_{k-1})$. Therefore, $\mathsf{V}_{p_k}\pi^k_{k-1} \simeq S_k(\mathsf{T}^*_x\mathsf{X}) \otimes \mathsf{V}_y\pi$ for each $p_k \in \mathsf{J}_k\pi$. Let us write

$$\phi_k \colon \pi_k^* S_k(\mathsf{T}^*\mathsf{X}) \otimes (\pi^k_0)^*\mathsf{V}\pi \to \mathsf{V}\pi^k_{k-1}$$

as the isomorphism of vector bundles, and denote by ϕ_{k,p_k} its restriction to the fibre over $p_k \in \mathsf{J}_k\pi$. (Very often in the literature this isomorphism is suppressed, and it is taken that the two vector bundles are equal.)

Now suppose that $\mathsf{R}_k \subset \mathsf{J}_k\pi$ is a kth-order partial differential equation. Let us denote by $\iota_k \colon \mathsf{R}_k \to \mathsf{J}_k\pi$ the inclusion, noting that this gives rise to, for each $p_k \in \mathsf{R}_k$, the exact sequence

$$0_{\mathsf{X}} \longrightarrow \mathsf{V}\hat{\pi}_k \xrightarrow{V\iota_k} \mathsf{V}\pi_k \longrightarrow \operatorname{coker}(V\iota_k) \longrightarrow 0_{\mathsf{X}}$$

of vector bundles over X. Let us denote by θ_{k,p_k} the restriction to $V_{p_k}\pi^k_{k-1}$ of the canonical projection from $\mathsf{V}_{p_k}\pi_k$ to $\operatorname{coker}(V_{p_k}\iota_k)$.

Next let us turn to providing the setup for a local characterisation of the symbol. To do this we recall that there is a morphism $\epsilon_k\colon \pi_k^* S_k(\mathsf{T}^*\mathsf{X}) \otimes (\pi_0^k)^*\mathsf{V}\pi \to \mathsf{V}\pi_k$ of vector bundles over $\mathsf{J}_k\pi$ such that the sequence

$$0_{\mathsf{X}} \longrightarrow \mathsf{V}\pi^k_{k-1} \xrightarrow{\epsilon_k \circ \phi_k^{-1}} \mathsf{V}\pi_k \xrightarrow{V\pi^k_{k-1}} (\pi^k_{k-1})^*\mathsf{V}\pi_{k-1} \longrightarrow 0_{\mathsf{X}}$$

is exact. With this, we make the following definition.

5.1 DEFINITION: (Symbol of a morphism) Let $\pi\colon \mathsf{Y} \to \mathsf{X}$ and $\tau\colon \mathsf{Z} \to \mathsf{X}$ be fibred manifolds and let $\Phi\colon \mathsf{J}_k\pi \to \mathsf{Z}$ be a morphism over id_{X}. The morphism $\sigma(\Phi) = V\Phi \circ \epsilon_k$ of the vector bundles $\pi_k^* S_k(\mathsf{T}^*\mathsf{X}) \otimes (\pi_0^k)^*\mathsf{V}\pi$ and $\mathsf{V}\tau$ is the ***symbol*** of Φ. •

We denote by $\sigma(\Phi)_{p_k}$ the restriction of $\sigma(\Phi)$ to the fibre over $p_k \in \mathsf{J}_k\pi$.

With the above notation, we are ready to state equivalent characterisations of the symbol.

5.2 PROPOSITION: (Equivalent characterisations of the symbol) *Let $\pi\colon \mathsf{Y} \to \mathsf{X}$ be a fibred manifold and let $\mathsf{R}_k \subset \mathsf{J}_k\pi$ be a kth-order partial differential equation. Let $p_k \in \mathsf{R}_k$, let $y = \pi_0^k(p_k)$, and let $x = \pi(y)$. For a subspace $\mathsf{G}(\mathsf{R}_k, p_k)$ of $S_k(\mathsf{T}_x^*\mathsf{X}) \otimes \mathsf{V}_y\pi$ the following statements are equivalent:*

(i) $\phi_{k,p_k}(\mathsf{G}(\mathsf{R}_k, p_k)) = \mathsf{V}_{p_k}\hat{\pi}_k \cap \mathsf{V}_{p_k}\pi^k_{k-1}$*;*

(ii) the sequence

$$0 \longrightarrow \phi_{k,p_k}(\mathsf{G}(\mathsf{R}_k, p_k)) \longrightarrow \mathsf{V}_{p_k}\hat{\pi}_k \xrightarrow{V_{p_k}\hat{\pi}^k_{k-1}} \mathsf{V}_{p_{k-1}}\pi_{k-1}$$

is exact (where $p_{k-1} = \hat{\pi}^k_{k-1}(p_k)$);

(iii) the sequence

$$0 \longrightarrow \phi_{k,p_k}(\mathsf{G}(\mathsf{R}_k, p_k)) \longrightarrow \mathsf{V}_{p_k}\pi^k_{k-1} \xrightarrow{\theta_{k,p_k}} \mathrm{coker}(V_{p_k}\iota_k)$$

is exact.

If $(\mathcal{U}, \mathsf{Z}, \tau, \Phi, \eta)$ is a local defining equation for R_k about $x \in \mathsf{X}$, then the preceding characterisations are equivalent to:

(iv) the sequence

$$0 \longrightarrow \mathsf{G}(\mathsf{R}_k, p_k) \longrightarrow S_k(\mathsf{T}_x^*\mathsf{X}) \otimes \mathsf{V}_y\pi \xrightarrow{\sigma(\Phi)_{p_k}} \mathsf{V}_{\Phi(p_k)}\tau$$

is exact.

Proof: The following lemma is useful in the proof.

1 LEMMA: *Let V be a vector space with U_1 and U_2 subspaces of V, and let $\mathsf{W} = \mathsf{U}_1 \cap \mathsf{U}_2$. Suppose that U is a vector space such that the sequence*

$$0 \longrightarrow \mathsf{U}_2 \xrightarrow[\iota_{\mathsf{U}_2}]{} \mathsf{V} \xrightarrow[\pi]{} \mathsf{U}$$

is exact for some $\pi \in \mathrm{Hom}_{\mathbb{R}}(\mathsf{V};\mathsf{U})$*. Then the following diagram is commutative and has exact rows and columns:*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathsf{W} \arrow[r, "\iota_{\mathsf{W},1}"] \arrow[d, "\iota_{\mathsf{W},2}"'] & \mathsf{U}_1 \arrow[r, "\pi|\mathsf{U}_1"] \arrow[d, "\iota_{\mathsf{U}_1}"'] & \mathsf{U} \arrow[d, equal] \\
0 \arrow[r] & \mathsf{U}_2 \arrow[r, "\iota_{\mathsf{U}_2}"'] & \mathsf{V} \arrow[r, "\pi"'] & \mathsf{U}
\end{tikzcd}$$

where $\iota_{\mathsf{W},1}$*,* $\iota_{\mathsf{W},2}$*,* ι_{U_1}*, and* ι_{U_2} *are the inclusions.*

Proof: Only the exactness of the top row is not obvious. Let $w \in \mathsf{W}$. Since $\mathsf{W} \subset \mathsf{U}_2$ it follows that $\pi(w) = 0$. Thus $\mathrm{image}(\iota_{\mathsf{W},1}) \subset \ker(\pi|\mathsf{U}_1)$. Now let $u_1 \in \ker(\pi|\mathsf{U}_1)$. Thus $\pi \circ \iota_{\mathsf{U}_1}(u_1) = 0$ and so there exists $u_2 \in \mathsf{U}_2$ such that $\iota_{\mathsf{U}_2}(u_2) = \iota_{\mathsf{U}_1}(u_1)$ by exactness of the bottom row. Thus $u_1 = u_2$. Thus $u_1 \in \mathsf{U}_1 \cap \mathsf{U}_2 = \mathsf{W}$ has the property that $\iota_{\mathsf{W},1}(u_1) = u_1$, giving exactness of the top row. ▼

(i) $\Longleftrightarrow$ (ii) We have the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \mathsf{V}_{p_k}\hat{\pi}_k \cap \mathsf{V}_{p_k}\pi^k_{k-1} \arrow[r] \arrow[d] & \mathsf{V}_{p_k}\hat{\pi}_k \arrow[r, "V\pi^k_{k-1}|\mathsf{V}\hat{\pi}_k"] \arrow[d] & \mathsf{V}_{p_{k-1}}\pi_{k-1} \arrow[d, equal] & \\
0 \arrow[r] & \mathsf{V}_{p_k}\pi^k_{k-1} \arrow[r] & \mathsf{V}_{p_k}\pi_k \arrow[r, "V\pi^k_{k-1}"'] & \mathsf{V}_{p_{k-1}}\pi_{k-1} \arrow[r] & 0
\end{tikzcd}$$

The last row is exact because $V\pi^k_{k-1}$ is an epimorphism of vector bundles by virtue of π^k_{k-1} being an epimorphism of fibred manifolds. We may then apply the lemma to give

$$\mathsf{V}_{p_k}\hat{\pi}_k \cap \mathsf{V}_{p_k}\pi^k_{k-1} = \ker(V_{p_k}\pi^k_{k-1}|\mathsf{V}_{p_k}\hat{\pi}_k).$$

This part of the result will follow if we can show that $V_{p_k}\pi^k_{k-1}|\mathsf{V}_{p_k}\hat{\pi}_k = V_{p_k}\hat{\pi}^k_{k-1}$. We have $\hat{\pi}^k_{k-1} = \pi^k_{k-1} \circ \iota_k$ which gives $T_{p_k}\hat{\pi}^k_{k-1} = T_{p_k}\pi^k_{k-1} \circ T_{p_k}\iota_k$. Thus $T_{p_k}\hat{\pi}^k_{k-1}$ is the restriction of $T_{p_k}\pi^k_{k-1}$ to $\mathsf{T}_{p_k}\mathsf{R}_k$. Since $V_{p_k}\pi^k_{k-1}$ is the restriction of $T_{p_k}\pi^k_{k-1}$ to $\mathsf{V}_{p_k}\pi_k$ and since $\mathsf{V}_{p_k}\hat{\pi}_k \subset \mathsf{V}_{p_k}\pi_k$, it follows that $V_{p_k}\pi^k_{k-1}|\mathsf{V}_{p_k}\hat{\pi}_k = V_{p_k}\hat{\pi}^k_{k-1}$, as desired.

(i) $\Longleftrightarrow$ (iii) We have the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \mathsf{V}_{p_k}\hat{\pi}_k \cap \mathsf{V}_{p_k}\pi^k_{k-1} \arrow[r] \arrow[d] & \mathsf{V}_{p_k}\pi^k_{k-1} \arrow[r, "\theta_{k,p_k}"] \arrow[d] & \mathsf{V}_{p_k}\pi_k/\mathsf{V}_{p_k}\hat{\pi}_k \arrow[d, equal] & \\
0 \arrow[r] & \mathsf{V}_{p_k}\hat{\pi}_k \arrow[r] & \mathsf{V}_{p_k}\pi_k \arrow[r] & \mathsf{V}_{p_k}\pi_k/\mathsf{V}_{p_k}\hat{\pi}_k \arrow[r] & 0
\end{tikzcd}$$

Since the bottom row is exact, one need only apply Lemma 1.

(ii) $\iff$ (iv) We have the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathsf{V}_{p_k}\hat{\pi}_k \cap \mathsf{V}_{p_k}\pi^k_{k-1} \arrow[r] \arrow[d] & \mathsf{V}_{p_k}\pi^k_{k-1} \arrow[r, "V\Phi|\mathsf{V}\pi^k_{k-1}"] \arrow[d] & \mathsf{V}_{\Phi(p_k)}\tau \arrow[d, equal] \\
0 \arrow[r] & \mathsf{V}_{p_k}\hat{\pi}_k \arrow[r] & \mathsf{V}_{p_k}\pi_k \arrow[r, "V\Phi"'] & \mathsf{V}_{\Phi(p_k)}\tau
\end{tikzcd}$$

The bottom row is exact since the sequence

$$\begin{tikzcd}
0_{\mathsf{X}} \arrow[r] & \mathsf{R}_k \arrow[r] & \mathsf{J}_k\pi \arrow[rr, "\Phi"] \arrow[dr] & & \mathsf{Z} \arrow[dl] \\
 & & & \mathsf{X} \arrow[ur, bend right, "\eta"'] &
\end{tikzcd}$$

is exact. By the lemma above this then gives

$$\mathsf{V}_{p_k}\hat{\pi}_k \cap \mathsf{V}_{p_k}\pi^k_{k-1} = \ker(V_{p_k}\Phi|\mathsf{V}_{p_k}\pi^k_{k-1}).$$

This part of the result will follow if we can show that $V_{p_k}\Phi \circ \epsilon_k$ agrees with $V_{p_k}\Phi|\mathsf{V}_{p_k}\pi^k_{k-1}$ under the identification of $\mathsf{V}_{p_k}\pi^k_{k-1}$ with $S_k(\mathsf{T}^*_x\mathsf{X}) \otimes \mathsf{V}_y\pi$. This is obvious, however, since ϵ_k agree with the inclusion of $\mathsf{V}_{p_k}\pi^k_{k-1}$ in $\mathsf{V}_{p_k}\pi_k$ under this identification. ■

It is the subspace of the proposition that is of interest to us.

5.3 Definition: (Symbol of a partial differential equation) Let $\pi\colon \mathsf{Y} \to \mathsf{X}$ be a fibred manifold and let $\mathsf{R}_k \subset \mathsf{J}_k\pi$ be a partial differential equation. For $p_k \in \mathsf{R}_k$ with $x = \pi_k(p_k)$ and $y = \pi^k_0(p_k)$, the subspace $\mathsf{G}(\mathsf{R}_k, p_k)$ of $S_k(\mathsf{T}^*_x\mathsf{X}) \otimes \mathsf{V}_y\pi$ satisfying any one of the equivalent characterisations of Proposition 5.2 is the ***symbol*** of R_k at p_k. The ***symbol bundle*** of R_k is the family $\mathsf{G}(\mathsf{R}_k)$ of subspaces of the vector bundle $\pi^*_k S_k(\mathsf{T}^*\mathsf{X}) \otimes (\pi^k_0)^*\mathsf{V}\pi$ over R_k defined by $\mathsf{G}(\mathsf{R}_k)_{p_k} = \mathsf{G}(\mathsf{R}_k, p_k)$. •

The notational distinction between the symbol at a point in R_k as a subspace of $S_k(\mathsf{T}^*_x\mathsf{X}) \otimes \mathsf{V}_y\pi$ and the symbol as being a collection of subspaces of the vector bundle $\mathsf{V}\pi^k_{k-1}$ is subtle, but does exist. This is because we wish to maintain some separation between the algebraic object, $\mathsf{G}(\mathsf{R}_k, p_k)$, and the geometric object, $\mathsf{G}(\mathsf{R}_k)_{p_k}$. In Propositions 6.4 and 8.7 we shall give some further properties of the symbol which will be useful in understanding its importance. At the moment, it merely is what it is.

It is useful to record the manner in which the dimension of the symbol varies on R_k.

5.4 Proposition: (Semicontinuity of dimension of symbol) *Let $\pi\colon \mathsf{Y} \to \mathsf{X}$ be a fibred manifold and let R_k be a kth-order partial differential equation. Then the function $\mathsf{R}_k \ni p_k \mapsto \dim(\mathsf{G}(\mathsf{R}_k)_{p_k})$ is upper semicontinuous.*

Proof: This follows since the subspace $\mathsf{G}(\mathsf{R}_k)_{p_k}$ is the kernel of a C^∞-vector bundle map. Indeed, Proposition 5.2 gives three different such characterisations of the symbol. Since the nullity of a continuous vector bundle map is upper semicontinuous, the result follows. ■

Let us give some examples of symbols. We shall not go through the details of deriving the symbol in the examples, but merely produce the answer.

5.5 Examples: (Symbols)

1. The symbol for the partial differential equation $\mathsf{R}_{\mathrm{lap}}$ is

$$\mathsf{G}(\mathsf{R}_{\mathrm{lap}},(x,y,u,u_x,u_y,u_{xx},u_{xy},u_{yy})) = \{\,(v_{xx},v_{xy},v_{yy}) \mid\ v_{xx}+v_{yy}=0\}\,.$$

   Here we identify $S_2(\mathbb{R}^2)\otimes\mathbb{R}$ with $S_2(\mathbb{R}^2)$.

2. The symbol for $\mathsf{R}_{\mathrm{der}}$ is

$$\mathsf{G}(\mathsf{R}_{\mathrm{der}},(x,y,f,f_x,f_y)) = \{\,(v_x,v_y) \mid\ v_x=0,\ v_y=0\}\,.$$

   We identify $S_1(\mathbb{R}^2)\otimes\mathbb{R}$ and $\mathbb{R}^2$.

3. The symbol for $\mathsf{R}_{\mathrm{sse}}$ is

$$\mathsf{G}(\mathsf{R}_{\mathrm{sse}},j_1X(x)) = \{\,A\in\mathrm{End}(\mathsf{T}_x\mathsf{X}) \mid\ A(X(x))=0_x\}\,.$$

   We are identifying $S_1(\mathsf{T}_x^*\mathsf{X})\otimes\mathsf{T}_x\mathsf{X}$ and $\mathrm{End}(\mathsf{T}_x\mathsf{X})$.

4. The symbol for $\mathsf{R}_{\mathrm{ode}}$ is

$$\mathsf{G}(\mathsf{R}_{\mathrm{ode}},(t,x,y,x_t,y_t)) = \{\,(u_t,v_t) \mid\ u_t=0,\ \alpha v_t=0\}\,.$$

   We identify $S_1(\mathbb{R})\otimes\mathbb{R}^2$ with $\mathbb{R}^2$.

5. For R_{el} we have

$$\mathsf{G}(\mathsf{R}_{\mathrm{el}},(t,\boldsymbol{x},\boldsymbol{v},\boldsymbol{a})) = \{\boldsymbol{u}\in\mathbb{R}^n \mid\ \boldsymbol{D}_2^2L(\boldsymbol{x},\boldsymbol{v})\boldsymbol{u}=\boldsymbol{0}\}.$$

   We identify $S_2(\mathbb{R})\otimes\mathbb{R}^n$ with $\mathbb{R}^n$. •

5.2. The reduced symbol for linear partial differential equations. Of course, Proposition 5.2 applies to any partial differential equation, including linear equations. However, in the linear case we have additional structure that we would like to keep track of. As in the general case, we first have to make some constructions.

We let $\pi\colon\mathsf{E}\to\mathsf{X}$ be a vector bundle and let $\mathsf{R}_k\subset\mathsf{J}_k\pi$ be a kth-order inhomogeneous linear partial differential equation. Thus R_k is an affine subbundle of $\pi_k\colon\mathsf{J}_k\pi\to\mathsf{X}$. In the usual way, we denote by $\hat{\pi}_k\colon\mathsf{R}_k\to\mathsf{X}$ and $\hat{\pi}_l^k\colon\mathsf{R}_k\to\mathsf{J}_l\pi$, $l<k$, the canonical projections. Let us denote by $L(\mathsf{R}_k)\subset\mathsf{J}_k\pi$ the vector subbundle associated with the affine subbundle R_k and denote by $L(\pi_k)\colon L(\mathsf{R}_k)\to\mathsf{X}$ the vector bundle projection. Note that the maps $\hat{\pi}_l^k$, $l<k$, are morphisms of affine bundles. As such they have linear parts, denoted by $L(\hat{\pi}_l^k)\colon L(\mathsf{R}_k)\to\mathsf{J}_l\pi$, which are morphisms of vector bundles.

The inclusion map $\iota_k\colon\mathsf{R}_k\to\mathsf{J}_k\pi$ is a morphism of affine bundles. It therefore has associated to it its linear part, denoted by $L(\iota_k)\colon L(\mathsf{R}_k)\to\mathsf{J}_k\pi$, which is a morphism of vector bundles. Thus we have the short exact sequence

$$0_{\mathsf{X}}\longrightarrow L(\mathsf{R}_k)\xrightarrow{L(\iota_k)}\mathsf{J}_k\pi\longrightarrow\mathrm{coker}(L(\iota_k))\longrightarrow 0_{\mathsf{X}}$$

of vector bundles over X. We denote by $\theta_{k,x}^0$ the restriction to $\ker(\pi_{k-1}^k)_x$ of the canonical projection from $\mathsf{J}_k\pi_x$ to $\mathrm{coker}(L(\iota_k))_x$.

In the homogeneous case, the situation simplifies. For the record, let us state these obvious simplifications.

5.6 Proposition: (Simplifications for homogeneous linear partial differential equations) *Let $\pi\colon \mathsf{E} \to \mathsf{X}$ be a vector bundle and let $\mathsf{R}_k \subset \mathsf{J}_k\pi$ be a homogeneous linear partial differential equation. Then*

(i) $L(\mathsf{R}_k) = \mathsf{R}_k$,

(ii) $L(\pi_k) = \pi_k$,

(iii) $L(\pi^k_l) = \pi^k_l$, $l < k$, *and*

(iv) $L(\iota_k) = \iota_k$.

Note that $\pi^k_{k-1}\colon \mathsf{J}_k\pi \to \mathsf{J}_{k-1}\pi$ is a morphism of vector bundles over X. Therefore, $\ker(\pi^k_{k-1})$ is to be thought of as a vector bundle over X. Indeed, there is a natural isomorphism $\phi^0_k\colon S_k(\mathsf{T}^*\mathsf{X}) \otimes \mathsf{E} \to \ker(\pi^k_{k-1})$ of vector bundles. We denote by $\phi^0_{k,x}$ the restriction of ϕ^0_k to the fibre over $x \in \mathsf{X}$. We also denote by $\kappa_k\colon S_k(\mathsf{T}^*\mathsf{X}) \otimes \mathsf{E} \to \mathsf{X}$ the vector bundle projection.

In the linear case, the vector bundle map ϵ_k has a direct representation with vector bundles, and not just their vertical bundles. Indeed, we have a vector bundle morphism $\epsilon^0_k\colon S_k(\mathsf{T}^*\mathsf{X}) \otimes \mathsf{E} \to \mathsf{J}_k\pi$ such that the sequence

$$0_{\mathsf{X}} \longrightarrow S_k(\mathsf{T}^*\mathsf{X}) \otimes \mathsf{E} \xrightarrow{\epsilon^0_k \circ (\phi^0_k)^{-1}} \mathsf{J}_k\pi \xrightarrow{\pi^k_{k-1}} \mathsf{J}_{k-1}\pi \longrightarrow 0_{\mathsf{X}}$$

is an exact sequence of vector bundles over X. With this we can define the symbol of a morphism of vector bundles. The terminology of "reduced symbol" we give here is not standard in the literature, where the name symbol is used for both the general object and for the special object in the linear case.

5.7 Definition: (Reduced symbol of a vector bundle morphism) Let $\pi\colon \mathsf{E} \to \mathsf{X}$ and $\tau\colon \mathsf{F} \to \mathsf{X}$ be vector bundles and let $\Phi\colon \mathsf{J}_k\pi \to \mathsf{F}$ be a vector bundle morphism over id_{X}. The morphism $\sigma^0(\Phi) = \Phi \circ \epsilon^0_k$ of the vector bundles $S_k(\mathsf{T}^*\mathsf{X}) \otimes \mathsf{E}$ and F is the ***reduced symbol*** of Φ. •

We denote by $\sigma^0(\Phi)_x$ the restriction of $\sigma^0(\Phi)$ to the fibre over $x \in \mathsf{X}$.

The following result gives the more or less obvious relationship between the symbol and the reduced symbol. It relies on the fact that the vertical bundle of a vector bundle is isomorphic to the pull-back of the vector bundle itself by the vector bundle projection.

5.8 Proposition: (Relationship between symbol and reduced symbol) *Let $\pi\colon \mathsf{E} \to \mathsf{X}$ and $\tau\colon \mathsf{F} \to \mathsf{X}$ be vector bundles and let $\Phi\colon \mathsf{J}_k\pi \to \mathsf{F}$ be a vector bundle morphism over id_{X}. Then the following diagram commutes:*

$$\begin{tikzcd}
\pi_k^*(S_k(\mathsf{T}^*\mathsf{X}) \otimes \mathsf{E}) \arrow[r, "\sigma(\Phi)"] \arrow[d, "\pi_k^*\kappa_k"'] & \tau^*\mathsf{F} \arrow[d, "\tau^*\tau"] \\
S_k(\mathsf{T}^*\mathsf{X}) \otimes \mathsf{E} \arrow[r, "\sigma^0(\Phi)"'] & \mathsf{F}
\end{tikzcd}$$

The idea is that the symbol is simply the reduced symbol, taking into account that the bundles forming the domain and codomain for the symbol are isomorphic to pull-back bundles.

We can now state the analogue of Proposition 5.2 for linear partial differential equations.

5.9 Proposition: (Equivalent characterisations of the reduced symbol for linear partial differential equations) *Let* $\pi\colon \mathsf{E} \to \mathsf{X}$ *be a vector bundle and let* $\mathsf{R}_k \subset \mathsf{J}_k\pi$ *be a kth-order inhomogeneous linear partial differential equation, supposing as in Proposition 3.10 that* $\mathsf{R}_k = \ker_\eta(\Phi)$ *for a vector bundle* $\tau\colon \mathsf{F} \to \mathsf{X}$*, a section* η *of this vector bundle, and a vector bundle morphism* $\Phi\colon \mathsf{J}_k\pi \to \mathsf{F}$ *over* id_{X}*. Let* $x \in \mathsf{X}$*. For a subspace* $\mathsf{G}^0(\mathsf{R}_k, x)$ *of* $S_k(\mathsf{T}^*_x\mathsf{X}) \otimes \mathsf{E}_x$*, the following statements are equivalent:*

(i) $\phi^0_{k,x}(\mathsf{G}^0(\mathsf{R}_k, x)) = L(\mathsf{R}_k)_x \cap \ker(\pi^k_{k-1})_x$*;*

(ii) the sequence

$$0 \longrightarrow \phi^0_{k,x}(\mathsf{G}^0(\mathsf{R}_k, x)) \longrightarrow L(\mathsf{R}_k)_x \xrightarrow{L(\hat{\pi}^k_{k-1})} \mathsf{J}_{k-1}\pi_x$$

is exact;

(iii) the sequence

$$0 \longrightarrow \phi^0_{k,x}(\mathsf{G}^0(\mathsf{R}_k, x)) \longrightarrow \ker(\pi^k_{k-1})_x \xrightarrow{\theta^0_{k,x}} \mathrm{coker}(L(\iota_k))_x$$

is exact;

(iv) the sequence

$$0 \longrightarrow \mathsf{G}^0(\mathsf{R}_k, x) \longrightarrow S_k(\mathsf{T}^*_x\mathsf{X}) \otimes \mathsf{E}_x \xrightarrow{\sigma^0(\Phi)_x} \mathsf{F}_x$$

is exact.

Moreover, $\mathsf{G}^0(\mathsf{R}_k, x) = \mathsf{G}(\mathsf{R}_k, p_k)$ *for every* $p_k \in \hat{\pi}_k^{-1}(x)$*.*

Proof: The proof follows, in much the same way as does the proof of Proposition 5.2, from Lemma 1 in that proposition. Thus all we shall do here is give the relevant commutative diagrams, leaving to the reader to draw the necessary conclusions from the lemma.

(i) $\iff$ (ii) We consider the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & L(\mathsf{R}_k)_x \cap \ker(\pi^k_{k-1})_x \arrow[r] \arrow[d] & L(\mathsf{R}_k)_x \arrow[r, "\pi^k_{k-1}|L(\mathsf{R}_k)_x"] \arrow[d] & \mathsf{J}_{k-1}\pi_x \arrow[d, equal] & \\
0 \arrow[r] & \ker(\pi^k_{k-1})_x \arrow[r] & \mathsf{J}_k\pi_x \arrow[r, "\pi^k_{k-1}"'] & \mathsf{J}_{k-1}\pi_x \arrow[r] & 0
\end{tikzcd}$$

(i) $\iff$ (iii) Here we consider the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & L(\mathsf{R}_k)_x \cap \ker(\pi^k_{k-1})_x \arrow[r] \arrow[d] & \ker(\pi^k_{k-1})_x \arrow[r, "\theta^0_{k,x}"] \arrow[d] & \mathsf{J}_k\pi_x/L(\mathsf{R}_k)_x \arrow[d, equal] & \\
0 \arrow[r] & L(\mathsf{R}_k)_x \arrow[r] & \mathsf{J}_k\pi_x \arrow[r] & \mathsf{J}_k\pi_x/L(\mathsf{R}_k)_x \arrow[r] & 0
\end{tikzcd}$$

(i) $\iff$ (iv) The commutative diagram that is used here is

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & & \\ 0 \arrow[r] & L(\mathsf{R}_k)_x \cap \ker(\pi^k_{k-1})_x \arrow[r] \arrow[d] & \ker(\pi^k_{k-1})_x \arrow[r, "\Phi|\ker(\pi^k_{k-1})"] \arrow[d] & \mathsf{F}_x \arrow[d, equal] & \\ 0 \arrow[r] & L(\mathsf{R}_k)_x \arrow[r] & \mathsf{J}_k\pi_x \arrow[r, "\Phi"'] & \mathsf{F}_x \arrow[r] & 0 \end{tikzcd}$$

For the final assertion of the proposition, we need only apply Proposition 5.8 and part (iv) above, understanding that $\mathsf{V}\hat{\pi}_k \simeq \pi_k^* L(\mathsf{R}_k)$, that $\mathsf{V}\pi^k_{k-1} \simeq \pi_k^*\kappa_k$, and that $\mathsf{V}\tau \simeq \tau^*\mathsf{F}$. ■

For linear partial differential equations we then have the following particular notion of symbol.

5.10 DEFINITION: (Symbol of a linear partial differential equation) Let $\pi\colon \mathsf{E} \to \mathsf{X}$ be a vector bundle and let $\mathsf{R}_k \subset \mathsf{J}_k\pi$ be an inhomogeneous linear partial differential equation. For $x \in \mathsf{X}$, the subspace $\mathsf{G}^0(\mathsf{R}_k)_x$ satisfying any one of the equivalent characterisations of Proposition 5.9 is the ***reduced symbol*** of R_k at x. The ***reduced symbol bundle*** of R_k is then the family $\mathsf{G}^0(\mathsf{R}_k)$ of subspaces of the vector bundle $S_k(\mathsf{T}^*\mathsf{X}) \otimes \mathsf{E}$ defined by $\mathsf{G}^0(\mathsf{R}_k)_x = \mathsf{G}^0(\mathsf{R}_k, x)$. •

5.3. The prolongation of the symbol. The manner in which symbols and prolongation interact is a little obtuse in certain versions of the standard treatment. Part of the reason for this is that it is possible to extract the symbol from the partial differential equation (after all, it is simply a subspace of a tensor product of the form $S_k(\mathsf{V}^*)\otimes\mathsf{W}$), i.e., a kth-order tableau, and treat it prolongation as algebra. There are simplifying advantages to this because in this process one isolates certain of the key concepts associated with the symbol (e.g., Spencer cohomology and involutivity). However, one can lose the relationship of the symbol and its prolongations with the partial differential equation. In this section we provide the usual purely algebraic characterisation of the prolongation of a symbol, giving the relationship with prolongation of partial differential equations in the next section. We shall see that the "solution" versus the "equation" point of view for partial differential equations comes up here, and we give both characterisations, also explaining the relationship between them.

It is useful to keep in mind in this section that all constructions are done at a fixed point $p_k \in \mathsf{R}_k$, and so are entirely algebraic. The partial differential equation itself does not play a rôle here, except to hand us the symbol to get started.

In this section we consider a fibred manifold $\pi\colon \mathsf{Y} \to \mathsf{X}$ and a kth-order partial differential equation $\mathsf{R}_k \subset \mathsf{J}_k\pi$. We fix $p_k \in \mathsf{R}_k$ and let $y = \pi^k_0(p_k)$ and $x = \pi(y)$. The symbol of R_k at p_k is $\mathsf{G}(\mathsf{R}_k, p_k) \subset S_k(\mathsf{T}^*_x\mathsf{X})\otimes\mathsf{V}_y\pi$. In order to give the appropriate algebraic characterisation of prolongation, we first need to provide some constructions concerning symmetric tensors.

Let V be a finite-dimensional $\mathbb{R}$-vector space. For $l, m \in \mathbb{Z}_{\geq 0}$ define $\Delta_{l,m}\colon S_{l+m}(\mathsf{V}^*) \to S_m(\mathsf{V}^*) \otimes S_l(\mathsf{V}^*)$ by

$$\Delta_{l,m}(A)((v_1, \dots, v_m), (v_{m+1}, \dots, v_{l+m})) = A(v_1, \dots, v_{l+m}).$$

In terms of bases for the various objects induced by a basis for $\{e_1,\dots,e_n\}$ for V, we have

$$\Delta_{l,m}(A_{j_1\cdots j_{l+m}}e^{j_1}\otimes\cdots\otimes e^{j_{l+m}}) \\ = A_{j_1\cdots j_m j_{m+1}\cdots j_{l+m}}(e^{j_1}\otimes\cdots\otimes e^{j_m})\otimes(e^{j_{m+1}}\otimes\cdots\otimes e^{j_{l+m}}).$$

Note that $\Delta_{l,m}$ is simply the natural inclusion of $S_{l+m}(\mathsf{V}^*)$ in $S_m(\mathsf{V}^*)\otimes S_l(\mathsf{V}^*)$.

Now let us return to the symbol $\mathsf{G}(\mathsf{R}_k,p_k)\subset S_k(\mathsf{T}^*_x\mathsf{X})\otimes\mathsf{V}_y\pi$. Since $\mathsf{G}(\mathsf{R}_k,p_k)$ is a subspace of $S_k(\mathsf{T}^*_x\mathsf{X})\otimes\mathsf{V}_y\pi$ we can regard it as the kernel of a linear map taking values in some vector space W_{p_k}. Indeed, one can do this is many ways, three of which are given by Proposition 5.2:

$$\mathsf{G}(\mathsf{R}_k,p_k)\simeq\ker(V_{p_k}\hat{\pi}^k_{k-1}),\quad \mathsf{G}(\mathsf{R}_k,p_k)\simeq\ker(\theta_{k,p_k}),\quad \mathsf{G}(\mathsf{R}_k,p_k)=\ker(\sigma(\Phi)_{p_k}),$$

where $(\mathcal{U},\mathsf{Z},\tau,\Phi,\eta)$ is a local defining equation for R_k. Of course, the advantage of the first two representations is that they do not depend on choosing a local defining equation, so they are often the ones we will deal with. However, it is illustrative to simply suppose that $\mathsf{G}(\mathsf{R}_k,p_k)=\ker(L_{p_k})$ for some linear map $L_{p_k}\in\operatorname{Hom}_{\mathbb{R}}(S_k(\mathsf{T}^*_x\mathsf{X})\otimes\mathsf{V}_y\pi;\mathsf{W}_{p_k})$.

For $l\in\mathbb{Z}_{\geq 0}$ the map $\Delta_{k,l}\colon S_{k+l}(\mathsf{T}^*_x\mathsf{X})\to S_l(\mathsf{T}^*_x\mathsf{X})\otimes S_k(\mathsf{T}^*_x\mathsf{X})$ can be naturally extended to

$$\Delta_{k,l}\otimes\mathrm{id}_{\mathsf{V}_y\pi}\colon S_{k+l}(\mathsf{T}^*_x\mathsf{X})\otimes\mathsf{V}_y\pi\to S_l(\mathsf{T}^*_x\mathsf{X})\otimes S_k(\mathsf{T}^*_x\mathsf{X})\otimes\mathsf{V}_y\pi.$$

With the above as motivation, we make a definition concerning the prolongation of the map L_{p_k}. The astute reader will at this point realise that we do not need to use the particular vector space $\mathsf{T}^*_x\mathsf{X}$ and that a significant simplification of notation would take place if we were to simply replace this with a general finite-dimensional vector space V. However, let us leave this for the future, and instead for now maintain the connection to the partial differential equation R_k.

5.11 Definition: (Prolongation of a linear map) Let $\mathsf{G}(\mathsf{R}_k,p_k)\subset S_k(\mathsf{T}^*\mathsf{X})\otimes\mathsf{V}_y\pi$ and suppose that $\mathsf{G}(\mathsf{R}_k,p_k)=\ker(L_{p_k})$ for some $L_{p_k}\in\operatorname{Hom}_{\mathbb{R}}(S_k(\mathsf{T}^*_x\mathsf{X})\otimes\mathsf{V}_y\pi;\mathsf{W}_{p_k})$. Then the map

$$\rho_l(L_{p_k})\colon S_{k+l}(\mathsf{T}^*_x\mathsf{X})\otimes\mathsf{V}_y\pi\to S_l(\mathsf{T}^*_x\mathsf{X})\otimes\mathsf{W}_{p_k}$$

defined by $(\mathrm{id}_{S_l(\mathsf{T}^*_x\mathsf{X})}\otimes L_{p_k})\circ(\Delta_{k,l}\otimes\mathrm{id}_{\mathsf{V}_y\pi})$ is the ***lth prolongation*** of L_{p_k}. •

The reader may wish to make an analogy in our development between $\rho_l(L_{p_k})$ for the prolongation of the symbol and $\rho_l(\Phi)$ for the prolongation of a morphism giving a local defining equation. The intuition is that the equations giving the kernel of L_{p_k} give the symbol, and the equations giving the kernel of $\rho_l(L_{p_k})$ are obtained by differentiating the equations for the kernel of L_{p_k} with respect to the independent variable in $\mathsf{T}^*_x\mathsf{X}$. We shall subsequently explain this point of view more fully when we distill away the superfluous notation coming from the partial differential equation.

Let us now develop another representation of what we will call the prolongation of the symbol. Note that, for $l\in\mathbb{Z}_{\geq 0}$, both $S_l(\mathsf{T}^*_x\mathsf{X})\otimes\mathsf{G}(\mathsf{R}_k,p_k)$ and $S_{k+l}(\mathsf{T}^*_x\mathsf{X})\otimes\mathsf{V}_y\pi$ can be thought of as subspaces of $(\otimes_{j=1}^{k+l}\mathsf{T}^*_x\mathsf{X})\otimes\mathsf{V}_y\pi$. Therefore, their intersection makes sense as a subspace of $(\otimes_{j=1}^{k+l}\mathsf{T}^*_x\mathsf{X})\otimes\mathsf{V}_y\pi$. This is what we will mean when we write this intersection in the sequel.

We may now state, using the above two constructions, two equivalent characterisations of the prolongation of the symbol.

5.12 Proposition: (Characterisations of the prolongation of the symbol) *Let* $\mathsf{G}(\mathsf{R}_k, p_k) \subset S_k(\mathsf{T}^*_x\mathsf{X}) \otimes \mathsf{V}_y\pi$ *be given by* $\mathsf{G}(\mathsf{R}_k, p_k) = \ker(L_{p_k})$ *for* $L_{p_k} \in \mathrm{Hom}_{\mathbb{R}}(S_k(\mathsf{T}^*_x\mathsf{X}) \otimes \mathsf{V}_y\pi; \mathsf{W}_{p_k})$. *Then the following statements for a subspace* $\rho_l(\mathsf{G}(\mathsf{R}_k, p_k)) \subset S_{k+l}(\mathsf{T}^*_x\mathsf{X} \otimes \mathsf{V}_y\pi)$ *are equivalent:*

(i) $\rho_l(\mathsf{G}(\mathsf{R}_k, p_k)) = (S_l(\mathsf{T}^*_x\mathsf{X}) \otimes \mathsf{G}(\mathsf{R}_k, p_k)) \cap (S_{k+l}(\mathsf{T}^*_x\mathsf{X}) \otimes \mathsf{V}_y\pi)$*;*

(ii) the sequence

$$0 \longrightarrow \rho_l(\mathsf{G}(\mathsf{R}_k, p_k)) \longrightarrow S_{k+l}(\mathsf{T}^*_x\mathsf{X}) \otimes \mathsf{V}_y\pi \xrightarrow{\rho_l(L_{p_k})} S_l(\mathsf{T}^*_x\mathsf{X}) \otimes \mathsf{W}_{p_k}$$

is exact.

Proof: To facilitate typesetting, let us denote

$$\mathsf{U} = (S_l(\mathsf{T}^*_x\mathsf{X}) \otimes \mathsf{G}(\mathsf{R}_k, p_k)) \cap (S_{k+l}(\mathsf{T}^*_x\mathsf{X}) \otimes \mathsf{V}_y\pi).$$

We then have the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathsf{U} \arrow[r] \arrow[d] & S_{k+l}(\mathsf{T}^*_x\mathsf{X}) \otimes \mathsf{V}_y\pi \arrow[r] \arrow[d] & S_l(\mathsf{T}^*_x\mathsf{X}) \otimes \mathsf{W}_{p_k} \arrow[d, equal] \\
0 \arrow[r] & S_l(\mathsf{T}^*_x\mathsf{X}) \otimes \mathsf{G}(\mathsf{R}_k, p_k) \arrow[r] & S_l(\mathsf{T}^*_x\mathsf{X}) \otimes S_k(\mathsf{T}^*_x\mathsf{X}) \otimes \mathsf{V}_y\pi \arrow[r] & S_l(\mathsf{T}^*_x\mathsf{X}) \otimes \mathsf{W}_{p_k}
\end{tikzcd}$$

where the bottom right horizontal arrow is $\mathrm{id}_{S_l(\mathsf{T}^*_x\mathsf{X})} \otimes L_{p_k}$. By definition of L_{p_k} the bottom row is exact. Now, by Lemma 1 of Proposition 5.2 it follows that the diagram is exact if the top right horizontal arrow is the restriction of $\mathrm{id}_{S_l(\mathsf{T}^*_x\mathsf{X})} \otimes L_{p_k}$ to $S_{k+l}(\mathsf{T}^*_x\mathsf{X}) \otimes \mathsf{V}_y\pi$. The proposition will now follow if we can show that this restriction is equal to $\rho_l(L_{p_k})$. However, this follows since the restriction of $\mathrm{id}_{S_l(\mathsf{T}^*_x\mathsf{X})} \otimes L_{p_k}$ to $S_{k+l}(\mathsf{T}^*_x\mathsf{X}) \otimes \mathsf{V}_y\pi$ is given by composing $\mathrm{id}_{S_l(\mathsf{T}^*_x\mathsf{X})} \otimes L_{p_k}$ with the inclusion of $S_{k+l}(\mathsf{T}^*_x\mathsf{X}) \otimes \mathsf{V}_y\pi$ in $S_l(\mathsf{T}^*_x\mathsf{X}) \otimes S_k(\mathsf{T}^*_x\mathsf{X}) \otimes \mathsf{V}_y\pi$. However, since this inclusion is exactly $\Delta_{k,l} \otimes \mathrm{id}_{\mathsf{V}_y\pi}$, the result follows from the definition of $\rho_l(L_{p_k})$. ■

In the first characterisation of $\rho_l(\mathsf{G}(\mathsf{R}_k, p_k))$, the intuition is that "$S_l(\mathsf{T}^*_x\mathsf{X}) \otimes \mathsf{G}(\mathsf{R}_k, p_k)$" refers to the notion that the lth prolongation corresponds to taking l derivatives, and the "$S_{k+l}(\mathsf{T}^*_x\mathsf{X}) \otimes \mathsf{V}_y\pi$" refers to the notion that derivatives yield symmetric tensors.

We may now define the prolongation of the symbol.

5.13 Definition: (Prolongation of the symbol) Let $\pi\colon \mathsf{Y} \to \mathsf{X}$ be a fibred manifold, let $\mathsf{R}_k \subset \mathsf{J}_k\pi$ be a kth-order partial differential equation, let $p_k \in \mathsf{R}_k$, $y = \pi^k_0(p_k)$, and $x = \pi(y)$, and let $\mathsf{G}(\mathsf{R}_k, p_k)$ be the symbol of R_k at p_k. For $l \in \mathbb{Z}_{\geq 0}$ the subspace $\rho_l(\mathsf{G}(\mathsf{R}_k, p_k))$ is the ***lth prolongation*** of the symbol at p_k. The family $\rho_l(\mathsf{G}(\mathsf{R}_k))$ of subspaces of the vector bundle $\pi^*_k S_{k+l}(\mathsf{T}^*\mathsf{X}) \otimes (\pi^k_0)^*\mathsf{V}\pi$ defined by $\rho_l(\mathsf{G}(\mathsf{R}_k))_{p_k} = \rho_l(\mathsf{G}(\mathsf{R}_k, p_k))$ is the ***lth prolongation*** of the symbol. •

5.14 Remark: (Base space for the symbol) Note that the lth prolongation of the symbol is defined as a family of vector spaces over R_k, not over $\rho_l(\mathsf{R}_k)$. This will be explained in more detail in the next section. •

Let us next prove that, just as for prolongation of partial differential equations, the order in which one prolongs a symbol is not of consequence. This should not be surprising, given that the symbol is itself a partial differential equation as we shall see in Section 5.6.

5.15 Proposition: (The mth prolongation of the lth prolongation is the $(l+m)$th prolongation) *Let $\pi\colon \mathsf{Y} \to \mathsf{X}$ be a fibred manifold and let $\mathsf{R}_k \subset \mathsf{J}_k\pi$ be a kth-order partial differential equation. For $p_k \in \mathsf{R}_k$ and $l, m \in \mathbb{Z}_{>0}$, $\rho_m(\rho_l(\mathsf{G}(\mathsf{R}_k, p_k))) = \rho_{l+m}(\mathsf{G}(\mathsf{R}_k, p_k))$.*

Proof: Suppose that $\mathsf{G}(\mathsf{R}_k, p_k) = \ker(L_{p_k})$ for $L_{p_k} \in \mathrm{Hom}_{\mathbb{R}}(S_k(\mathsf{T}^*_x\mathsf{X}) \otimes \mathsf{V}_y\pi; \mathsf{W}_{p_k})$. We have the following commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & \rho_{l+m}(\mathsf{G}(\mathsf{R}_k, p_k)) \arrow[r] \arrow[d, dashed, equal] & S_{k+l+m}(\mathsf{T}^*_x\mathsf{X}) \otimes \mathsf{V}_y\pi \arrow[r, "\rho_{l+m}(L_{p_k})"] \arrow[d, equal] & S_{l+m}(\mathsf{T}^*_x\mathsf{X}) \otimes \mathsf{W}_{p_k} \arrow[d] \\
0 \arrow[r] & \rho_m(\rho_l(\mathsf{G}(\mathsf{R}_k, p_k))) \arrow[r] & S_{k+l+m}(\mathsf{T}^*_x\mathsf{X}) \otimes \mathsf{V}_y\pi \arrow[r, "\rho_m(\rho_l(L_{p_k}))"'] & S_m(\mathsf{T}^*_x\mathsf{X}) \otimes S_l(\mathsf{T}^*_x\mathsf{X}) \otimes \mathsf{W}_{p_k}
\end{tikzcd}$$

which is moreover exact by Proposition 5.12. Now let $v \in \rho_{l+m}(\mathsf{G}(\mathsf{R}_k, p_k))$. Thinking of v as an element of $S_{k+l+m}(\mathsf{T}^*_x\mathsf{X}) \otimes \mathsf{V}_y\pi$ we then have $\rho_m(\rho_l(L_{p_k})) = 0$ by exactness of the above diagram. Thus $v \in \rho_m(\rho_l(\mathsf{R}_k, p_k))$, again by exactness of the above diagram. Now let $v \in \rho_m(\rho_l(\mathsf{G}(\mathsf{R}_k, p_k)))$ so that, thinking of v as an element of $S_{k+l+m}(\mathsf{T}^*_x\mathsf{X}) \otimes \mathsf{V}_y\pi$, we have $\rho_m(\rho_l(L_{p_k}))(v) = 0$. By commutativity of the above diagram,

$$(\Delta_{l,m} \otimes \mathrm{id}_{\mathsf{W}_{p_k}}) \circ \rho_{l+m}(L_{p+k})(v) = 0,$$

recalling that $\Delta_{l,m} \otimes \mathrm{id}_{\mathsf{W}_{p_k}}$ is the canonical inclusion of $S_{l+m}(\mathsf{T}^*_x\mathsf{X}) \otimes \mathsf{W}_{p_k}$ in $S_m(\mathsf{T}^*_x\mathsf{X}) \otimes S_l(\mathsf{T}^*_x\mathsf{X}) \otimes \mathsf{W}_{p_k}$. Since $\Delta_{l,m} \otimes \mathrm{id}_{\mathsf{W}_{p_k}}$ is injective this means that $\rho_{l+m}(L_{p+k})(v) = 0$ so $v \in \rho_{l+m}(\mathsf{G}(\mathsf{R}_k, p_k))$. Thus the dashed equals sign is an actual equals sign in the diagram above. ■

5.4. The symbol of the prolongation. As we have already discussed, the preceding development of the prolongation of the symbol has, at first glance, not much to do with the partial differential equation; it appears to be an isolated algebraic construction. However, this is not the case and, in fact, the symbol is saying very important things about the partial differential equation. In this section we reconnect the prolongation of the symbol to the partial differential equation by explaining its relationship to the symbol of the prolongation, a notion that makes sense when certain regularity conditions are imposed.

As usual, we need some notation. We let $\pi\colon \mathsf{Y} \to \mathsf{X}$ be a fibred manifold, let $\mathsf{R}_k \subset \mathsf{J}_k\pi$ be a kth-order partial differential equation, and let $l \in \mathbb{Z}_{>0}$. As usual, we denote by $\rho_l(\mathsf{R}_k)$ the lth prolongation of R_k.

The vector bundle $\mathsf{V}\pi^{k+1}_{k+l-1}$ is isomorphic to the vector bundle $\pi^*_{k+l}S_{k+l}(\mathsf{T}^*\mathsf{X}) \otimes (\pi^{k+l}_0)^*\mathsf{V}\pi$. Thus, given $p_{k+l} \in \mathsf{J}_{k+l}\pi$, there exists an isomorphism $\phi_{k+1,p_{k+l}}$ from $S_{k+l}(\mathsf{T}^*_x\mathsf{X}) \otimes \mathsf{V}_y\pi$ where $y = \pi^{k+l}_0(p_{k+l})$ and $x = \pi(y)$.

We denote by $\iota_{k+l}\colon \rho_l(\mathsf{R}_k) \to \mathsf{J}_{k+l}\pi$ the inclusion. If $\rho_l(\mathsf{R}_k)$ is a fibred submanifold of $\mathsf{J}_{k+l}\pi$ then we have the exact sequence

$$0_{\mathsf{X}} \longrightarrow \mathsf{V}\hat{\pi}_{k+l}\mathsf{V}\pi_{k+l} \longrightarrow \mathrm{coker}(V\iota_{k+l}) \qquad 0_{\mathsf{X}}$$

of vector bundles over X. In this case we denote by $\theta_{k+l,p_{k+l}}$ the restriction to $\mathsf{V}_{p_{k+l}}\pi^{k+l}_{k+l-1}$ of the canonical projection from $\mathsf{V}_{p_{k+l}}\pi_{k+l}$ to $\operatorname{coker}(V\iota_{k+l})$.

To define the prolongation of the symbol of a morphism, we use the following result.

5.16 Proposition: (Symbol of prolongation of morphism) *Let* $\pi\colon \mathsf{Y}\to\mathsf{X}$ *and* $\tau\colon\mathsf{Z}\to\mathsf{X}$ *be a fibred manifolds, let* $\Phi\colon \mathsf{J}_k\pi\to\mathsf{Z}$ *be a morphism over* id_{X}*, and let* $l\in\mathbb{Z}_{>0}$*. Then there exists a unique morphism* $\sigma_l(\Phi)\colon \pi^*_{k+l}S_{k+l}(\mathsf{T}^*\mathsf{X})\otimes(\pi^{k+l}_0)^*\mathsf{V}\pi\to\tau^*_lS_l(\mathsf{T}^*\mathsf{X})\otimes(\tau^l_0)^*\mathsf{V}\tau$ *over* Φ *such that the diagram*

$$\begin{tikzcd}
0_{\mathsf{X}} \arrow[d] & 0_{\mathsf{X}} \arrow[d] \\
\pi^*_{k+l}S_{k+l}(\mathsf{T}^*\mathsf{X})\otimes(\pi^{k+l}_0)^*\mathsf{V}\pi \arrow[r, "\sigma_l(\Phi)"] \arrow[d, "\epsilon_{k+l}"'] & \tau^*_lS_l(\mathsf{T}^*\mathsf{X})\otimes(\tau^l_0)^*\mathsf{V}\tau \arrow[d, "\epsilon_l"] \\
\mathsf{V}\pi_{k+l} \arrow[r, "V(\rho_l(\Phi))"] \arrow[d] & \mathsf{V}\tau_l \arrow[d] \\
\mathsf{V}\pi_{k+l-1} \arrow[r, "V(\rho_{l-1}(\Phi))"] \arrow[d] & \mathsf{V}\tau_{l-1} \arrow[d] \\
0_{\mathsf{X}} & 0_{\mathsf{X}}
\end{tikzcd}$$

commutes and has exact columns.

Proof: The exactness of the columns is a consequence of the definition of the vector bundle morphisms ϵ_r. The bottom square commutes as a consequence of Proposition 4.14. Moreover, since the bottom square commutes, it follows that $\rho_l(\Phi)$ maps $\mathsf{V}_{p_{k+l}}\pi^{k+l}_{k+l-1}$ to $\mathsf{V}_{\Phi(p_{k+l})}\tau^l_{l-1}$. Thus the existence and uniqueness of $\sigma_l(\Phi)$ follows since the top horizontal arrow in the diagram is prescribed by it being the restriction of $\rho_l(\Phi)$ to $\pi^*_{k+l}S_{k+l}(\mathsf{T}^*\mathsf{X})\otimes(\pi^{k+l}_0)^*\mathsf{V}\pi\simeq\mathsf{V}\pi^{k+l}_{k+l-1}$.

It remains to show that $\sigma_l(\Phi)$ is fibred over Φ. Thus we must show that the diagram

$$\begin{tikzcd}
\pi^*_{k+l}S_{k+l}(\mathsf{T}^*\mathsf{X})\otimes(\pi^{k+l}_0)^*\mathsf{V}\pi \arrow[r, "\sigma_l(\Phi)"] \arrow[d] & \tau^*_lS_l(\mathsf{T}^*\mathsf{X})\otimes(\tau^l_0)^*\mathsf{V}\tau \arrow[d] \\
\mathsf{J}_k\pi \arrow[r, "\Phi"'] & \mathsf{Z}
\end{tikzcd}$$

commutes. This, however, follows from the fact that the diagram

$$\begin{tikzcd}
\pi_{k+l}^* S_{k+l}(\mathsf{T}^*\mathsf{X}) \otimes (\pi_0^{k+l})^*\mathsf{V}\pi \arrow[r, "\sigma_l(\Phi)"] \arrow[d] & \tau_l^* S_l(\mathsf{T}^*\mathsf{X}) \otimes (\tau_0^l)^*\mathsf{V}\tau \arrow[d] \\
\mathsf{V}\pi_{k+l} \arrow[r, "V(\rho_l(\Phi))"] \arrow[d, dotted, no head] & \mathsf{V}\tau_l \arrow[d, dotted, no head] \\
\mathsf{V}\pi_{k+1} \arrow[r, "V(\rho_1(\Phi))"] \arrow[d] & \mathsf{V}\tau_1 \arrow[d] \\
\mathsf{V}\pi_k \arrow[r, "V\Phi"] \arrow[d] & \mathsf{V}\tau \arrow[d] \\
\mathsf{J}_k\pi \arrow[r, "\Phi"'] & \mathsf{Z}
\end{tikzcd}$$

commutes by successive applications of Proposition 4.14. ■

We now make a definition based on the preceding result.

5.17 Definition: (Prolongation of symbol of morphism) Let $\pi\colon \mathsf{Y} \to \mathsf{X}$ and $\tau\colon \mathsf{Z} \to \mathsf{X}$ be fibred manifolds, let $\Phi\colon \mathsf{J}_k\pi \to \mathsf{Z}$ be a morphism fibred over id_{X}, and let $l \in \mathbb{Z}_{>0}$. The morphism $\sigma_l(\Phi)$ of Proposition 5.16 is the ***lth prolongation*** of the symbol $\sigma(\Phi)$ of Φ. •

We can now state our main result concerning the relationship between prolongations of symbols and symbols of prolongations.

5.18 Proposition: (The symbol of the prolongation is the prolongation of the symbol) *Let $\pi\colon \mathsf{Y} \to \mathsf{X}$ be a fibred manifold, let $\mathsf{R}_k \subset \mathsf{J}_k\pi$ be a partial differential equation, and let $l \in \mathbb{Z}_{>0}$. Suppose that $\rho_j(\mathsf{R}_k)$ is a fibred submanifold of $\pi_{k+j}\colon \mathsf{J}_{k+j}\pi \to \mathsf{X}$ for $j \in \{l-1, l\}$. Let $p_{k+l} \in \rho_l(\mathsf{R}_k)$ with $p_{k+l-1} = \pi_{k+l-1}^{k+l}(p_{k+l})$, $p_k = \pi_k^{k+l}(p_{k+l})$, $y = \pi_0^k(p_k)$ and $x = \pi(y)$. Then, for a subspace $\mathsf{U}_{p_k} \subset S_{l+k}(\mathsf{T}_x^*\mathsf{X}) \otimes \mathsf{V}_y\pi$, the following statements are equivalent:*

(i) $\mathsf{U}_{p_k} = \rho_l(\mathsf{G}(\mathsf{R}_k, p_k))$;

(ii) $\phi_{k+l,p_{k+l}}(\mathsf{U}_{p_k}) = \mathsf{V}_{p_{k+l}}\hat{\pi}_{k+l} \cap \mathsf{V}_{p_{k+l}}\pi_{k+l-1}^{k+l}$;

(iii) the sequence

$$0 \longrightarrow \phi_{k+l,p_{k+l}}(\mathsf{U}_{p_k}) \longrightarrow \mathsf{V}_{p_{k+l}}\hat{\pi}_{k+l} \xrightarrow{V\hat{\pi}_{k+l-1}^{k+l}} \mathsf{V}_{p_{k+l-1}}\hat{\pi}_{k+l-1}$$

is exact;

(iv) the sequence

$$0 \longrightarrow \phi_{k+l,p_{k+l}}(\mathsf{U}_{p_k}) \longrightarrow \mathsf{V}_{p_{k+l}}\pi_{k+l-1}^{k+l} \xrightarrow{\theta_{k+l,p_{k+l}}} \operatorname{coker}(V_{p_{k+l}}\iota_{k+l})$$

is exact.

If $(\mathcal{U}, \mathsf{Z}, \tau, \Phi, \eta)$ is a local defining equation for R_k about $x \in \mathsf{X}$, then the preceding characterisations are equivalent to:

(v) the sequence

$$0 \longrightarrow \mathsf{U}_{p_k} \longrightarrow S_{k+l}(\mathsf{T}_x^*\mathsf{X}) \otimes \mathsf{V}_y\pi \xrightarrow{\sigma_l(\Phi)} S_l(\mathsf{T}_x^*\mathsf{X}) \otimes \mathsf{V}_{\Phi(p_k)}\tau$$

is exact.

Proof: The equivalence of statements (ii)–(v) follows in the same manner as the equivalence of the four statements of Proposition 5.2. All one need do is appropriately fill in the vertices in the commutative diagrams from the proof of Proposition 5.2. We leave this straightforward exercise to the reader.

To complete the proof we shall show the equivalence of parts (i) and (iv). Let us denote by $\nu\colon \operatorname{coker}(\iota_k) \to \mathsf{R}_k$ the vector bundle associated with the inclusion $\iota_k\colon \mathsf{R}_k \to \mathsf{J}_k\pi$. The exact sequence

$$0 \longrightarrow \mathsf{G}(\mathsf{R}_k)_{p_k} \longrightarrow \mathsf{V}_{p_k}\pi^k_{k-1} \longrightarrow \operatorname{coker}(V_{p_k}\iota_k)$$

from Proposition 5.2 induces an exact sequence

$$0 \longrightarrow \mathsf{G}(\mathsf{R}_k, p_k) \longrightarrow S_k(\mathsf{T}_x^*\mathsf{X}) \otimes \mathsf{V}_y\pi \longrightarrow \operatorname{coker}(V_{p_k}\iota_k)$$

under the canonical isomorphism of $\mathsf{V}_{p_k}\pi^k_{k-1}$ with $S_k(\mathsf{T}_x^*\mathsf{X}) \otimes \mathsf{V}_y\pi$. Associated to this is the exact sequence defining the prolongation as in Proposition 5.12:

$$0 \longrightarrow \rho_l(\mathsf{G}(\mathsf{R}_k, p_k)) \longrightarrow S_{k+l}(\mathsf{T}_x^*\mathsf{X}) \otimes \mathsf{V}_y\pi \longrightarrow S_l(\mathsf{T}_x^*\mathsf{X}) \otimes \operatorname{coker}(V_{p_k}\iota_k) \tag{5.1}$$

The sequence

$$0_{\mathsf{X}} \longrightarrow \mathsf{V}\hat{\pi}_k \longrightarrow \mathsf{V}\pi_k \longrightarrow \operatorname{coker}(V\iota_k) \longrightarrow 0_{\mathsf{X}}$$

is an exact sequence of vector bundles over R_k which gives rise to the exact sequence

$$0_{\mathsf{X}} \longrightarrow \mathsf{J}_l(\mathsf{V}\hat{\pi}_k) \longrightarrow \mathsf{J}_l(\mathsf{V}\pi_k) \longrightarrow \mathsf{J}_l\nu \longrightarrow 0_{\mathsf{X}}$$

of vector bundles over $\rho_l(\mathsf{R}_k)$. The canonical isomorphism of the jet bundle of a vertical bundle and the vertical bundle of a jet bundle, along with the definition of prolongation, gives the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathsf{V}\hat{\pi}_{k+l} \arrow[r] \arrow[d] & \mathsf{V}(\mathsf{J}_{k+l}\pi)|\rho_l(\mathsf{R}_k) \arrow[r] \arrow[d] & \mathsf{J}_l\nu|\rho_l(\mathsf{R}_k) \arrow[d, equal] \\
0 \arrow[r] & \mathsf{V}((\hat{\pi}_k)_l)|\rho_l(\mathsf{R}_k) \arrow[r] & \mathsf{V}((\pi_k)_l)|\rho_l(\mathsf{R}_k) \arrow[r] & \mathsf{J}_l\nu|\rho_l(\mathsf{R}_k)
\end{tikzcd}$$

whose bottom row is exact. An application of Lemma 1 from the proof of Proposition 5.2 gives the exactness of the sequence

$$0 \longrightarrow \mathsf{V}_{p_{k+l}}\hat{\pi}_{k+l} \longrightarrow \mathsf{V}_{p_{k+l}}\pi_{k+l} \longrightarrow \mathsf{J}_l\nu_{p_{k+l}} \tag{5.2}$$

Now we consider the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & \phi_{k+l,p_{k+l}}(\rho_l(\mathsf{G}(\mathsf{R}_k,p_k))) \arrow[r, "f_{11}"] \arrow[d, "f_{21}"'] & \mathsf{V}_{p_{k+l}}\pi^{k+l}_{k+l-1} \arrow[r, "f_{12}"] \arrow[d, "f_{22}"'] & \mathsf{V}_{p_{k+l}}\nu^l_{l-1} \arrow[d, "f_{23}"'] \\
0 \arrow[r] & \mathsf{V}_{p_{k+l}}\hat{\pi}_{k+l} \arrow[r, "f_{31}"] \arrow[d, "f_{41}"'] & \mathsf{V}_{p_{k+l}}\pi_{k+l} \arrow[r, "f_{32}"] \arrow[d, "f_{42}"'] & \mathsf{J}_l\nu_{p_{k+l}} \arrow[d, "f_{43}"'] \\
0 \arrow[r] & \mathsf{V}_{p_{k+l-1}}\hat{\pi}_{k+l-1} \arrow[r, "f_{51}"] & \mathsf{V}_{p_{k+l-1}}\pi_{k+l-1} \arrow[r, "f_{52}"] \arrow[d] & \mathsf{J}_{l-1}\nu_{p_{k+l-1}} \arrow[d] \\
 & & 0 & 0
\end{tikzcd}$$

where we abbreviate the names of the maps for clarity. For each arrow the actual definition of the map should be clear given the domain and codomain. The exactness of the top row follows from (5.1). The exactness of the second and third rows follows from the exact sequence (5.2). The exactness of the second and third columns is immediate.

The proposition will follow if we can show that the first column is exact. We should first show that the first column makes sense in that $\phi_{k+l,p_{k+l}}(\rho_l(\mathsf{G}(\mathsf{R}_k,p_k)))$ is a subspace of $\mathsf{V}_{p_{k+l}}\hat{\pi}_{k+l}$. This follows, however, since the map f_{12} is the restriction of the projection f_{22} to $\mathsf{V}_{p_{k+l}}\pi^{k+l}_{k+l-1}$ and since $\phi_{k+l,p_{k+l}}(\rho_l(\mathsf{G}(\mathsf{R}_k,p_k)))$ is a subspace of $\mathsf{V}_{p_{k+l}}\pi^{k+l}_{k+l-1}$.

Let $v \in \mathrm{image}(f_{21})$ so that $v = f_{21}(v_1)$ for some $v_1 \in \phi_{k+l,p_{k+l}}(\rho_l(\mathsf{G}(\mathsf{R}_k,p_k)))$. Then $f_{31}(v) = f_{22}\circ f_{11}(v_1)$ by commutativity of the diagram. Thus $f_{31}(v) \in \mathrm{image}(f_{22}) = \ker(f_{42})$ so that

$$f_{42}\circ f_{31}(v) = f_{51}\circ f_{41}(v) = 0$$

by commutativity. Thus $f_{41}(v) = 0$ since f_{51} is injective. This gives $\mathrm{image}(f_{21}) \subset \ker(f_{41})$.

Now let $v \in \ker(f_{41})$. Then $f_{51}\circ f_{41}(v) = 0$ or $f_{42}\circ f_{31}(v) = 0$ by commutativity of the diagram. Thus $f_{31}(v) \in \ker(f_{42})$ and so, by exactness of the second column, there exists $v_1 \in \mathsf{V}_{p_{k+l}}\pi^{k+l}_{k+l-1}$ such that $f_{22}(v_1) = f_{31}(v)$. By exactness of the second row, $f_{32}\circ f_{31}(v) = 0$, or $f_{32}\circ f_{22}(v_1) = 0$. Commutativity gives $f_{23}\circ f_{12}(v_1) = 0$. Exactness of the third column gives $f_{12}(v_1) = 0$ and so exactness of the first row gives the existence of $v_2 \in \phi_{k+l,p_{k+l}}(\rho_l(\mathsf{G}(\mathsf{R}_k,p_k)))$ such that $f_{11}(v_2) = v_1$. Then commutativity gives

$$f_{31}\circ f_{21}(v_2) = f_{22}(v_1) = f_{31}(v).$$

Injectivity of f_{31} then gives $v = f_{21}(v_2)$. Thus $\ker(f_{41}) \subset \mathrm{image}(f_{21})$. ■

With this result we can state a definition of what we mean by the symbol of the prolongation and affix some notation to this.

5.19 Definition: (Symbol of the prolongation) Let $\pi\colon \mathsf{Y} \to \mathsf{X}$ be a fibred manifold, let R_k be a kth-order partial differential equation, let $l \in \mathbb{Z}_{>0}$, and suppose that $\rho_l(\mathsf{R}_k)$ and $\rho_{l-1}(\mathsf{R}_k)$ are fibred submanifolds of $\mathsf{J}_{k+l}\pi$ and $\mathsf{J}_{k+l-1}\pi$, respectively. Let $p_{k+l} \in \rho_l(\mathsf{R}_k)$ and denote $p_k = \pi^{k+l}_k(p_{k+l})$, $y = \pi^k_0(p_k)$, and $x = \pi(y)$. The ***symbol*** of $\rho_l(\mathsf{R}_k)$ at p_{k+l} is the subspace $\rho_l(\mathsf{G}(\mathsf{R}_k, p_k))$ of $S_{k+l}(\mathsf{T}^*_x\mathsf{X}) \otimes \mathsf{V}_y\pi$. The ***symbol bundle*** of $\rho_l(\mathsf{R}_k)$ is the family $\rho_l(\mathsf{G}(\mathsf{R}_k))$ of subspaces of the vector bundle $\pi_k^* S_{k+l}(\mathsf{T}^*\mathsf{X}) \otimes (\pi_0^k)^*\mathsf{V}\pi$ over R_k defined by $\rho_l(\mathsf{G}(\mathsf{R}_k))_{p_k} = \rho_l(\mathsf{G}(\mathsf{R}_k, p_k))$. •

We have the same sort of semicontinuity condition on the dimension of the symbol of the prolongation as we have for the symbol itself.

5.20 Proposition: (Semicontinuity of the dimension of the prolongation of the symbol) *Let $\pi\colon \mathsf{Y} \to \mathsf{X}$ be a fibred manifold, let R_k be a kth-order partial differential equation, and let $l \in \mathbb{Z}_{>0}$. Then the function $\mathsf{R}_k \ni p_k \mapsto \dim(\rho_l(\mathsf{G}(\mathsf{R}_k))_{p_k})$ is upper semicontinuous.*

As was the case in our notation for the symbol, we have a slight, but definitely existent, notational distinction between the symbol of the prolongation as a subspace of $S_{k+l}(\mathsf{T}^*_x\mathsf{X}) \otimes \mathsf{V}_y\pi$ and as a subspace in the vector bundle $\pi_k^* S_{k+l}(\mathsf{T}^*\mathsf{X}) \otimes (\pi_0^k)^*\mathsf{V}\pi$. Note that Proposition 5.18 ensures that there is no notational incompatibility between Definitions 5.13 and 5.19. Note, however, that Definition 5.13 is more general since no regularity assumptions need to be made on the prolongations of R_k. Indeed, the words "symbol of the prolongation" tacitly assume some regularity of the prolongations. However, the "prolongation of the symbol" makes no such assumptions.

5.21 Remark: (The base space for the symbol of the prolongation) The base space for the symbol bundle $\rho_l(\mathsf{G}(\mathsf{R}_k))$ of the lth prolongation is R_k. However, as per characterisations (ii)–(v) of Proposition 5.18, it is more geometrically natural to think of the base space to be $\rho_l(\mathsf{R}_k)$. The difficulty with this is that $\rho_l(\mathsf{R}_k)$ may not be a manifold, and so is not always suitable as the base space for a vector bundle. As we shall see, we will want to be able to use the words "$\rho_l(\mathsf{G}(\mathsf{R}_k))$ is a vector bundle" without knowing that $\rho_l(\mathsf{R}_k)$ is a manifold. Nevertheless, in understanding the geometry of the symbol of the prolongation, it is recommended to keep in mind that the natural base space is $\rho_l(\mathsf{R}_k)$. •

5.5. The symbol of the prolongation for linear partial differential equations. The constructions in Section 5.3 concerning the prolongation of the symbol translate verbatim to linear partial differential equations. This is a consequence of the fact that the constructions are, in some sense, independent of the differential equation. Somewhat more precisely, the constructions do not care whether the original symbol came from a linear partial differential equation or not. However, the constructions of Section 5.4 can be specialised in the case of linear partial differential equations in exactly the same way that the constructions of Section 5.2 specialise the constructions of Section 5.1. In this section we carry out this specialisation. We merely state the results, as the proofs follow from those in Section 5.4 in the same manner as the proofs in Section 5.2 follow from those in Section 5.1.

As usual with things related to the symbol, we first need some notation. We let $\pi\colon \mathsf{E} \to \mathsf{X}$ be a vector bundle with R_k an inhomogeneous linear partial differential equation. As we say in Section 5.2, $\ker(\pi_{k+l-1}^{k+l})$ is a vector bundle over X which is isomorphic to $S_{k+l}(\mathsf{T}^*\mathsf{X}) \otimes \mathsf{E}$. We denote the isomorphism by $\phi^0_{k+l}\colon S_{k+l}(\mathsf{T}^*\mathsf{X}) \otimes \mathsf{E} \to \ker(\pi_{k+l-1}^{k+l})$, and denote by $\phi^0_{k+l,x}$ the restriction to the fibre over $x \in \mathsf{X}$.

If $\rho_l(\mathsf{R}_k)$ is an affine subbundle, the inclusion $\iota_{k+l}\colon \rho_l(\mathsf{R}_k) \to \mathsf{J}_{k+1}\pi$ is a morphism of affine bundles and we denote its linear part by $L(\iota_{k+lk})\colon L(\rho_l(\mathsf{R}_k)) \to \mathsf{J}_{k+l}\pi$. We have the short exact sequence

$$\mathsf{0}_{\mathsf{X}} \longrightarrow L(\rho_l(\mathsf{R}_k)) \xrightarrow{L(\iota_{k+l})} \mathsf{J}_{k+l}\pi \longrightarrow \operatorname{coker}(L(\iota_{k+l})) \longrightarrow \mathsf{0}_{\mathsf{X}}$$

of vector bundles over X. We denote by $\theta^0_{k+l,x}$ the restriction to $\ker(\pi^{k+l}_{k+l-1})$ of the canonical projection from $\mathsf{J}_{k+l}\pi_x$ to $\operatorname{coker}(L(\iota_{k+l}))_x$.

We can also simply the notion of the prolongation of a morphism when the fibred manifolds are vector bundles. To do so, we use the following result which is exactly analogous to Proposition 5.16.

5.22 Proposition: (Symbol of prolongation of vector bundle map) *Let $\pi\colon \mathsf{E} \to \mathsf{X}$ and $\tau\colon \mathsf{F} \to \mathsf{X}$ be a vector bundles, let $\Phi\colon \mathsf{J}_k\pi \to \mathsf{F}$ be a vector bundle map over* id_{X}*, and let $l \in \mathbb{Z}_{>0}$. Then there exists a unique morphism $\sigma^0_l(\Phi)\colon S_{k+l}(\mathsf{T}^*\mathsf{X}) \otimes \mathsf{E} \to S_l(\mathsf{T}^*X) \otimes \mathsf{F}$ over Φ such that the diagram*

$$\begin{tikzcd}
\mathsf{0}_{\mathsf{X}} \arrow[d] & \mathsf{0}_{\mathsf{X}} \arrow[d] \\
S_{k+l}(\mathsf{T}^*\mathsf{X}) \otimes \mathsf{E} \arrow[r, "\sigma^0_l(\Phi)"] \arrow[d] & S_l(\mathsf{T}^*X) \otimes \mathsf{F} \arrow[d] \\
\mathsf{J}_{k+l}\pi \arrow[r, "\rho_l(\Phi)"] \arrow[d] & \mathsf{J}_l\tau \arrow[d] \\
\mathsf{J}_{k+l-1}\pi \arrow[r, "\rho_{l-1}(\Phi)"] \arrow[d] & \mathsf{J}_{l-1}\tau \arrow[d] \\
\mathsf{0}_{\mathsf{X}} & \mathsf{0}_{\mathsf{X}}
\end{tikzcd}$$

commutes and has exact columns.

We then make the following definition.

5.23 Definition: (Prolongation of symbol of morphism of vector bundles) Let $\pi\colon \mathsf{E} \to \mathsf{X}$ and $\tau\colon \mathsf{F} \to \mathsf{X}$ be vector bundles, let $\Phi\colon \mathsf{J}_k\pi \to \mathsf{F}$ be a vector bundle map over id_{X}, and let $l \in \mathbb{Z}_{>0}$. The morphism $\sigma^0_l(\Phi)$ is the ***lth prolongation*** of the reduced symbol $\sigma^0(\Phi)$ of Φ. •

The following result relates the prolongation of the reduced symbol and the prolongation of the symbol.

5.24 Proposition: (Relationship between prolongations of symbol and reduced symbol) *Let $\pi\colon \mathsf{E} \to \mathsf{X}$ and $\tau\colon \mathsf{F} \to \mathsf{X}$ be vector bundles, let $\Phi\colon \mathsf{J}_k\pi \to \mathsf{F}$ be a vector bundle map over* id_{X}*, and let $l \in \mathbb{Z}_{>0}$. Then the following diagram commutes:*

$$\begin{tikzcd}
\pi^*_{k+l}(S_{k+l}(\mathsf{T}^*\mathsf{X}) \otimes \mathsf{E}) \arrow[r, "\sigma_l(\Phi)"] \arrow[d] & \tau^*_l(S_l(\mathsf{T}^*\mathsf{X}) \otimes \mathsf{F}) \arrow[d] \\
S_{k+l}(\mathsf{T}^*\mathsf{X}) \otimes \mathsf{E} \arrow[r, "\sigma^0_l(\Phi)"'] & S_l(\mathsf{T}^*\mathsf{X}) \otimes \mathsf{F}
\end{tikzcd}$$

We can now state the basic result concerning the prolongation of the symbol for linear partial differential equations.

5.25 PROPOSITION: (The symbol of the prolongation is the prolongation of the symbol for linear partial differential equations) *Let $\pi\colon \mathsf{E} \to \mathsf{X}$ be a vector bundle and let $\mathsf{R}_k \subset \mathsf{J}_k\pi$ be a kth-order inhomogeneous linear partial differential equation, supposing as in Proposition 3.10 that $\mathsf{R}_k = \ker_\eta(\Phi)$ for a vector bundle $\tau\colon \mathsf{F} \to \mathsf{X}$, a section η of this vector bundle, and a vector bundle morphism $\Phi\colon \mathsf{J}_k\pi \to \mathsf{F}$ over id_{X}. Let $l \in \mathbb{Z}_{>0}$ and suppose that $\rho_j(\mathsf{R}_k)$ are affine subbundles for $j \in \{l-1, l\}$. Let $x \in \mathsf{X}$. Then, for a subspace $\mathsf{U}_x \subset S_{l+k}(\mathsf{T}_x^*\mathsf{X}) \otimes \mathsf{E}_x$, the following statements are equivalent:*

(i) $\mathsf{U}_x = \rho_l(\mathsf{G}^0(\mathsf{R}_k, x))$;

(ii) $\phi^0_{k+l,x}(\mathsf{U}_x) = L(\rho_l(\mathsf{R}_k))_x \cap \ker(\pi^{k+l}_{k+l-1})_x$;

(iii) the sequence

$$0 \longrightarrow \phi^0_{k+l,x}(\mathsf{U}_x) \longrightarrow L(\rho_l(\mathsf{R}_k))_x \xrightarrow{L(\hat{\pi}^{k+l}_{k+l-1})} L(\rho_{l-1}(\mathsf{R}_k))_x$$

is exact;

(iv) the sequence

$$0 \longrightarrow \phi^0_{k+l,x}(\mathsf{U}_x) \longrightarrow \ker(\pi^{k+l}_{k+l-1})_x \xrightarrow{\theta^0_{k+l,x}} \operatorname{coker}(L(\iota_{k+l})_x)$$

is exact;

(v) the sequence

$$0 \longrightarrow \mathsf{U}_x \longrightarrow S_k(\mathsf{T}_x^*\mathsf{X}) \otimes \mathsf{E}_x \xrightarrow{\sigma^0_l(\Phi)} S_l(\mathsf{T}_x^*\mathsf{X}) \otimes \mathsf{F}_x$$

is exact.

Moreover, $\rho_l(\mathsf{G}^0(\mathsf{R}_k, x)) = \rho_l(\mathsf{G}(\mathsf{R}_k, p_k))$ for every $p_k \in \hat{\pi}_k^{-1}(x)$.

Based on the preceding result we make the following definition.

5.26 DEFINITION: (Symbol of the prolongation for linear partial differential equations) Let $\pi\colon \mathsf{E} \to \mathsf{X}$ be a vector bundle, let $\mathsf{R}_k \subset \mathsf{J}_k\pi$ be a inhomogeneous linear partial differential equation, and let $l \in \mathbb{Z}_{>0}$. Suppose that $\rho_j(\mathsf{R}_k)$ are affine subbundles for $j \in \{l-1, l\}$. For $x \in \mathsf{X}$, the subspace $\rho_l(\mathsf{G}^0(\mathsf{R}_k, x))$ of $S_{k+l}(\mathsf{T}_x^*\mathsf{X}) \otimes \mathsf{E}_x$ satisfying any one of the equivalent conditions of Proposition 5.25 is the ***reduced symbol*** of $\rho_l(\mathsf{R}_k)$ at x. The ***reduced symbol bundle*** of $\rho_l(\mathsf{R}_k)$ is the family $\rho_l(\mathsf{G}^0(\mathsf{R}_k))$ of subspaces of the vector bundle $S_{k+l}(\mathsf{T}^*\mathsf{X}) \otimes \mathsf{E}$ defined by $\rho_l(\mathsf{G}^0(\mathsf{R}_k))_x = \rho_l(\mathsf{G}^0(\mathsf{R}_k, x))$. •

5.6. The symbol as a partial differential equation. In this section we shall show that the symbol is itself a partial differential equation in a few different ways. In order to state these characterisations, we recall the following.

1. Let $\nu\colon \mathsf{V}\pi \to \mathsf{X}$ be the natural projection. Then $\mathsf{V}\pi_k$ and $\mathsf{J}_k\nu$ are isomorphic as vector bundles over $\mathsf{J}_k\pi$.

2. (We refer ahead to Section 8.1 for the notation we use here.) If $(\xi, \mathcal{U})$ is a local section of $\pi\colon \mathsf{Y} \to \mathsf{X}$ then the vector bundles $\mathsf{J}_k(\xi^*\nu_{\mathcal{U}})$ and $j_k\xi^*(\mathsf{J}_k\nu_{\mathcal{U}})$ over $\mathcal{U}$ are isomorphic. In particular, the pull-back bundle $j_k\xi^*(\mathsf{J}_k\nu_{\mathcal{U}})$ is the kth jet bundle of a vector bundle over X.

With these observations, and with the definition of the symbol, we have the following result.

5.27 PROPOSITION: (Characterisations of the symbol as a partial differential equation) *Let $\pi\colon \mathsf{Y} \to \mathsf{X}$ be a fibred manifold, let $\mathsf{R}_k \subset \mathsf{J}_k\pi$ be a kth-order partial differential equation, and let $p_k \in \mathsf{R}_k$ with $y = \pi_0^k(p_k)$ and $x = \pi(y)$. Then the following statements hold:*

(i) $\mathsf{G}(\mathsf{R}_k, p_k) \subset S_k(\mathsf{T}_x^*\mathsf{X}) \otimes \mathsf{V}_y\pi$ *is a kth-order tableau;*

(ii) if $\mathsf{G}(\mathsf{R}_k)$ is a vector bundle, then $\phi_k(\mathsf{G}(\mathsf{R}_k))$ is a partial differential equation of homogeneous degree k associated with the fibred manifold $\nu\colon \mathsf{V}\pi \to \mathsf{X}$;

(iii) if $\mathsf{Y} = \mathsf{E}$ is a vector bundle, if R_k is an inhomogeneous linear partial differential equation, and if $\mathsf{G}^0(\mathsf{R}_k)$ is a vector bundle, then $\phi_k(\mathsf{G}^0(\mathsf{R}_k))$ is a homogeneous linear partial differential equation of homogeneous degree k associated with the vector bundle $\pi\colon \mathsf{E} \to \mathsf{X}$;

(iv) if $(\xi, \mathcal{U})$ is a local section of $\pi\colon \mathsf{Y} \to \mathsf{X}$ and if $\mathsf{G}(\mathsf{R}_k)|\pi_k^{-1}(\mathcal{U})$ is a vector bundle, then $j_k^*(\phi_k(\mathsf{G}(\mathsf{R}_k))|\pi_k^{-1}(\mathcal{U}))$ is a homogeneous linear partial differential equation of homogeneous degree k associated with the vector bundle $\xi^*\nu_{\mathcal{U}}\colon \xi^*\mathsf{V}\pi_{\mathcal{U}} \to \mathcal{U}$.

The first (and simplest) interpretation of the symbol as a kth-order tableau is the most useful, and will be explored in detail subsequently.

5.7. Coordinate formulae. Here we shall merely give the form for the symbol and its prolongations in the case when the partial differential equation is determined by a local defining equation. Thus we let $\pi\colon \mathsf{Y} \to \mathsf{X}$ be a fibred manifold, let $\mathsf{R}_k \subset \mathsf{J}_k\pi$ be a kth-order partial differential equation, and let $l \in \mathbb{Z}_{>0}$. We suppose that $(\mathcal{U}, \mathsf{Z}, \tau, \Phi, \eta)$ is a local defining equation and that $\mathcal{U}$ is the domain of a coordinate chart for X. Thus the local representative of Φ is given by

$$(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{p}_1, \dots, \boldsymbol{p}_k) \mapsto (\boldsymbol{x}, \boldsymbol{\Phi}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{p}_1, \dots, \boldsymbol{p}_k)),$$

where we adopt the notation first introduced in Section 3.5. The symbol for Φ is the highest-order term in the derivative. That is to say, the local representative of $\sigma(\Phi)$ at a point $p_k \in \mathsf{J}_k\pi$ is the map $\sigma(\boldsymbol{\Phi})$ from $L^k_{\mathrm{sym}}(\mathbb{R}^n; \mathbb{R}^m)$ to $\mathbb{R}^r$ given by

$$\sigma(\boldsymbol{\Phi})(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{p}_1, \dots, \boldsymbol{p}_k) \cdot \boldsymbol{A} = \boldsymbol{D}_{k+2}\boldsymbol{\Phi}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{p}_1, \dots, \boldsymbol{p}_k) \cdot \boldsymbol{A}, \qquad \boldsymbol{A} \in L^k_{\mathrm{sym}}(\mathbb{R}^n; \mathbb{R}^m).$$

The lth prolongation, $\sigma_l(\Phi)$, of $\sigma(\Phi)$ has as its local representative the map $\sigma_l(\boldsymbol{\Phi})$ from $L^{k+l}_{\mathrm{sym}}(\mathbb{R}^n; \mathbb{R}^m)$ to $L^l_{\mathrm{sym}}(\mathbb{R}^n; \mathbb{R}^r)$ defined by

$$\begin{aligned}(\sigma_l(\boldsymbol{\Phi})(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{p}_1, \dots, \boldsymbol{p}_k) \cdot \boldsymbol{A}) \cdot (\boldsymbol{v}_1, \dots, \boldsymbol{v}_l) \\ = \boldsymbol{D}_{k+2}\boldsymbol{\Phi}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{p}_1, \dots, \boldsymbol{p}_k) \cdot ((\boldsymbol{v}_1, \dots, \boldsymbol{v}_l) \lrcorner \boldsymbol{A}), \\ \boldsymbol{A} \in L^{k+l}_{\mathrm{sym}}(\mathbb{R}^n; \mathbb{R}^m),\ \boldsymbol{v}_1, \dots, \boldsymbol{v}_l \in \mathbb{R}^n.\end{aligned}$$

6. Projection

Prolongation is the process of producing equations of higher-order by successive differentiations. One can think of the symbol as telling us how prolongation affects the highest-order derivatives. By projection we investigate how prolongation affects the derivatives of lower-order than the top order. In the case where we prolong by one derivative and then project, the two operations are precisely connected by the symbol, at least provided that certain regularity conditions are satisfied.

6.1. Definitions. The projection is simple to define.

6.1 Definition: (Projection) Let $\pi\colon \mathsf{Y} \to \mathsf{X}$ be a fibred manifold, let $\mathsf{R}_k \subset \mathsf{J}_k\pi$ be a kth-order partial differential equation, and let $l, m \in \mathbb{Z}_{\geq 0}$. The ***mth projection*** to the lth prolongation is the subset of $\rho_l(\mathsf{R}_k)$ defined by $\rho_l^{(m)}(\mathsf{R}_k) = \pi_{k+l}^{k+l+m}(\rho_{l+m}(\mathsf{R}_k))$. •

The first thing to note is that $\rho_l^{(m)}(\mathsf{R}_k)$ is actually a subset of $\rho_l(\mathsf{R}_k)$ by virtue of Proposition 4.3. Generally $\rho_l^{(m)}(\mathsf{R}_k)$ will not be a submanifold. The case where it is is of interest.

6.2 Definition: (Sufficiently regular partial differential equation) Let $\pi\colon \mathsf{Y} \to \mathsf{X}$ be a fibred manifold and let R_k be a regular kth-order partial differential equation. We say that R_k is ***sufficiently regular*** if $\hat{\pi}_{k+l}^{k+l+m}\colon \rho_{l+m}(\mathsf{R}_k) \to \rho_l(\mathsf{R}_k)$ has locally constant rank for each $l, m \in \mathbb{Z}_{\geq 0}$. •

If R_k is sufficiently regular it follows that $\rho_l^{(m)}(\mathsf{R}_k)$ is a fibred submanifold of $\rho_l(\mathsf{R}_k)$ for each $l, m \in \mathbb{Z}_{\geq 0}$.

Let us consider some examples. Generally the process of projection is sometimes not that straightforward to perform as it will involve solving implicit algebraic equations.

6.3 Examples: 1. Let us consider the Laplacian partial differential equation R_{lap}. From our previous work we have

$$\mathsf{R}_{\text{lap}} = \{\, (x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) \in \mathsf{J}_2\pi \mid \ u_{xx} + u_{yy} = 0\}$$

and

$$\rho_1(\mathsf{R}_{\text{lap}}) = \{(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, u_{xxx}, u_{xxy}, u_{xyx}, u_{xyy}, u_{yyy}) | \\ u_{xx} + u_{yy} = 0,\ u_{xxx} + u_{yyx} = 0,\ u_{xxy} + u_{yyy} = 0\}.$$

We see that there are no constraints put on derivatives of order two or lower by the first prolongation. Therefore, $\rho_0^{(1)}(\mathsf{R}_{\text{lap}}) = \mathsf{R}_{\text{lap}}$.

2. For the partial differential equation R_{der} we have

$$\mathsf{R}_{\text{der}} = \{\, (x, y, f, f_x, f_y) \in \mathsf{J}_1\pi \mid \ f_x = \alpha(x, y),\ f_y = \beta(x, y)\}$$

and

$$\rho_1(\mathsf{R}_{\text{der}}) = \{(x, y, f_x, f_y, f_{xx}, f_{xy}, f_{yy}) | \ f_x = \alpha(x, y),\ f_y = \beta(x, y), \\ f_{xx} = \tfrac{\partial\alpha}{\partial x}(x, y),\ f_{xy} = \tfrac{\partial\alpha}{\partial y}(x, y),\ f_{xy} = \tfrac{\partial\beta}{\partial x}(x, y),\ f_{yy} = \tfrac{\partial\beta}{\partial y}(x, y)\}.$$

There are no constraints placed on the first-order or zeroth-order derivatives by the first prolongation that do not already exist in the partial differential equation itself. However, there are constraints placed on the points in the space of independent variables (x, y). Indeed, we must have $\frac{\partial\alpha}{\partial y}(x, y) = \frac{\partial\beta}{\partial x}(x, y)$. Therefore,

$$\rho_0^{(1)}(\mathsf{R}_{\mathrm{der}}) = \{(x, y, f, f_x, f_y) \mid\ f_x = \alpha(x, y),\ f_y = \beta(x, y),\ \tfrac{\partial\alpha}{\partial y}(x, y) = \tfrac{\partial\beta}{\partial x}(x, y)\}.$$

In particular, if $\frac{\partial\alpha}{\partial y}(x, y) = \frac{\partial\beta}{\partial x}(x, y)$ for all $(x, y) \in \mathbb{R}^2$, then $\rho_0^{(1)}(\mathsf{R}_{\mathrm{der}}) = \mathsf{R}_{\mathrm{der}}$.

3. For $\mathsf{R}_{\mathrm{ode}}$ we have

$$\mathsf{R}_{\mathrm{ode}} = \{\,(t, x, y, x_t, y_t) \mid\ x_t = f(x, y, t),\ \alpha y_t = g(t, x, y)\}$$

and

$$\begin{aligned}\rho_1(\mathsf{R}_{\mathrm{ode}}) = \{(t, x, y, x_t, y_t, x_{tt}, y_{tt})|\ & x_t = f(t, x, y),\ \alpha y_t = g(t, x, y),\\ & x_{tt} = \tfrac{\partial f}{\partial t}(t, x, y) + \tfrac{\partial f}{\partial x}(t, x, y)x_t + \tfrac{\partial f}{\partial y}(t, x, y)y_t,\\ & \alpha y_{tt} = \tfrac{\partial g}{\partial t}(t, x, y) + \tfrac{\partial g}{\partial x}(t, x, y)x_t + \tfrac{\partial g}{\partial y}(t, x, y)y_t\}.\end{aligned}$$

Note that the equation

$$x_{tt} = \tfrac{\partial f}{\partial t}(t, x, y) + \tfrac{\partial f}{\partial x}(t, x, y)x_t + \tfrac{\partial f}{\partial y}(t, x, y)y_t$$

places no restrictions on the derivatives of order one or lower since x_{tt} is arbitrary. However, the equation

$$\alpha y_{tt} = \tfrac{\partial g}{\partial t}(t, x, y) + \tfrac{\partial g}{\partial x}(t, x, y)x_t + \tfrac{\partial g}{\partial y}(t, x, y)y_t$$

does place restrictions on the first-order derivatives in the case that $\alpha = 0$. Thus we have $\rho_0^{(1)}(\mathsf{R}_{\mathrm{ode}}) = \mathsf{R}_{\mathrm{ode}}$ in the case when $\alpha \neq 0$, and

$$\begin{aligned}\rho_0^{(1)}(\mathsf{R}_{\mathrm{ode}}) = \{(t, x, y, x_t, y_t)|\ & x_t = f(x, y, t),\ g(t, x, y) = 0,\\ & \tfrac{\partial g}{\partial t}(t, x, y) + \tfrac{\partial g}{\partial x}(t, x, y)x_t + \tfrac{\partial g}{\partial y}(t, x, y)y_t = 0\}\end{aligned}$$

in the case when $\alpha = 0$. Thus, in the case when $\alpha = 0$, there are possibly constraints placed on membership in $\rho_0^{(1)}(\mathsf{R}_{\mathrm{ode}})$ that are not already present in the constraints placed on membership in R_k. •

6.2. Prolongation, projection, and the symbol. Next we relate the three constructions of prolongation, projection, and symbol. This relationship helps to explain why the symbol is important. Indeed, as we shall see in Section 7, one way of constructing solutions to partial differential equations is “order-by-order.” In this technique, the symbol provides a measure of one’s ability to do this, and sometimes a means of understanding the character of the solutions that one gets in this manner. But this is getting slightly ahead of ourselves.

The main result is the following.

6.4 Proposition: (Prolongation, projection, and symbol) *Let $\pi\colon \mathsf{Y} \to \mathsf{X}$ be a fibred manifold, let $\mathsf{R}_k \subset \mathsf{J}_k\pi$ be a kth-order partial differential equation, and let $l \in \mathbb{Z}_{\geq 0}$. Consider the following statements:*

1. *$\rho_l(\mathsf{R}_k)$ is a fibred submanifold of $\pi_{k+l}\colon \mathsf{J}_{k+l}\pi \to \mathsf{X}$;*
2. *$\rho_{l+1}(\mathsf{R}_k)$ is a fibred submanifold of $\pi_{k+l+1}\colon \mathsf{J}_{k+l+1}\pi \to \mathsf{X}$;*
3. *$\hat{\pi}^{k+l+1}_{k+l}$ has locally constant rank;*
4. *$\hat{\pi}^{k+l+1}_{k+l}$ is surjective;*
5. *$\hat{\pi}^{k+l+1}_{k+l}$ is an epimorphism;*
6. *$\rho_{l+1}(\mathsf{G}(\mathsf{R}_k))$ is a vector bundle;*
7. *$\rho^{(1)}_l(\mathsf{R}_k)$ is a fibred submanifold of $\pi_{k+l}\colon \mathsf{J}_{k+l} \to \mathsf{X}$;*
8. *$\rho_{l+1}(\mathsf{R}_k)$ is an affine bundle over $\rho^{(1)}_l(\mathsf{R}_k)$ modelled on $\rho_{l+1}(\mathsf{G}(\mathsf{R}_k))$.*

Then the following implications hold:

(i) $3 \Longleftrightarrow 6$;

(ii) $2 \cap 3 \Longrightarrow 7$;

(iii) $1 \cap 6 \Longrightarrow 3 \cap 7 \cap 8$;

(iv) $1 \cap 4 \cap 6 \Longrightarrow 2 \cap 5$;

(v) $2 \cap 5 \Longrightarrow 1 \cap 6$.

Proof: (i) This follows immediately from part (iii) of Proposition 5.18.

(ii) This follows from general properties of fibred morphisms.

(iii) Since $\rho_{l+1}(\mathsf{R}_k) = \rho_1(\rho_l(\mathsf{R}_k))$ we may without loss of generality suppose that $l = 0$. Let us suppose that $(\mathcal{U}, \mathsf{Z}, \tau, \Phi, \eta)$ is a local defining equation for R_k and that, for simplicity and without loss of generality, $\mathcal{U} = \mathsf{X}$. That $\hat{\pi}^{k+1}_k$ has locally constant rank follows immediately from part (i). Thus 3 holds. Next we show that $\rho^{(1)}_0(\mathsf{R}_k)$ is a fibred submanifold and that $\rho_1(\mathsf{R}_k)$ is an affine bundle over $\rho^{(1)}_0(\mathsf{R}_k)$ modelled on $\rho_1(\mathsf{G}(\mathsf{R}_k))$. By definition of the prolongation of the symbol of Φ we have the commutative and exact diagram:

$$
\begin{tikzcd}
0_{\mathsf{X}} \arrow[d] & 0_{\mathsf{X}} \arrow[d] \\
\pi^*_{k+1} S_{k+1}(\mathsf{T}^*\mathsf{X}) \otimes (\pi^{k+1}_0)^* \mathsf{V}\pi \arrow[r, "\sigma_1(\Phi)"] \arrow[d, "\epsilon_{k+1}"'] & \tau_1^* \mathsf{T}^*\mathsf{X} \otimes (\tau^1_0)^* \mathsf{V}\tau \arrow[d, "\epsilon_1"] \\
\mathsf{V}\pi_{k+1} \arrow[r, "V\rho_1(\Phi)"] \arrow[d] & \mathsf{V}\tau_1 \arrow[d] \\
\mathsf{V}\pi_k \arrow[r, "V\Phi"] \arrow[d] & \mathsf{V}\tau \arrow[d] \\
0_{\mathsf{X}} & 0_{\mathsf{X}}
\end{tikzcd}
$$

It follows that the rank of the vector bundle map $V\rho_1(\Phi)$ is equal to the sum of the ranks of $\sigma_1(\Phi)$ and $V\Phi$. Since these latter are constant, it follows that $\rho_1(\Phi)$ has constant rank. Thus $\rho_1(\mathsf{R}_k)$ is a fibred submanifold of $\mathsf{J}_{k+1}\pi$. By part (ii) it follows that $\rho^{(1)}_0(\mathsf{R}_k)$ is a fibred submanifold of $\mathsf{J}_k\pi$; that is, 7 holds. Moreover, we now have the following exact and commutative diagram:

$$\begin{tikzcd}
0_{\mathsf{X}} \arrow[r] & \rho_1(\mathsf{G}(\mathsf{R}_k)) \arrow[r] \arrow[d, dashed] & \pi_{k+1}^* S_{k+1}(\mathsf{T}^*\mathsf{X}) \otimes (\pi_0^{k+1})^*\mathsf{V}\pi \arrow[r, "\sigma_1(\Phi)"] \arrow[d, dashed] & \pi^*\mathsf{T}^*\mathsf{X} \otimes \mathsf{V}\pi \arrow[d, dashed] \\
0_{\mathsf{X}} \arrow[r] & \rho_1(\mathsf{R}_k) \arrow[r] \arrow[d] & \mathsf{J}_{k+1}\pi \arrow[r, "\rho_1(\Phi)"] \arrow[d] & \mathsf{J}_1\tau \arrow[d] \\
0_{\mathsf{X}} \arrow[r] & \mathsf{R}_k \arrow[r] & \mathsf{J}_k\pi \arrow[r, "\Phi"] & \mathsf{Z}
\end{tikzcd}$$

This gives 8.

(iv) From part (iii) we know that $\rho_1(\mathsf{R}_k)$ is an affine bundle over $\rho_0^{(1)}(\mathsf{R}_k)$. Since $\hat{\pi}_k^{k+1}$ is surjective, $\rho_0^{(1)}(\mathsf{R}_k) = \mathsf{R}_k$ and so $\hat{\pi}_k^{k+1}$ is an epimorphism.

(v) Since $\hat{\pi}_{k+l}^{k+l+1}$ is an epimorphism, it has locally constant rank and so, by part (ii) it follows that $\rho_l(\mathsf{R}_k)$ is a fibred submanifold. It also follows from (i) that $\rho_{l+1}(\mathsf{G}(\mathsf{R}_k))$ is a vector bundle. ■

7. Formal matters

As we have seen in our examples, a partial differential equation may not have solutions. However, it is not normally clear when this is the case, and it is helpful to be able to talk about another sort of "solution" that always exists: a "formal solution" of finite order. Note that formal solutions, particularly formal solutions of finite order, are *not* interesting *per se*. However, it is useful to be able to talk about them along the way to developing a theory for the existence of actual solutions.

7.1. Formal solutions. We begin with the definition.

7.1 Definition: (Formal solution, local formal solution) Let $\pi\colon \mathsf{Y} \to \mathsf{X}$ be a fibred manifold and let $\mathsf{R}_k \subset \mathsf{J}_k\pi$ be a kth-oder partial differential equation.

(i) For $l \in \mathbb{Z}_{\geq 0}$, a ***formal solution*** at $x \in \mathsf{X}$ of order $k+l$ to R_k is an element of $\hat{\pi}_{k+l}^{-1}(x)$.

(ii) A ***local formal solution*** of order k to R_k is a pair $(\xi_k, \mathcal{U})$ where $\mathcal{U}$ is an open subset of X and $\xi_k\colon \mathcal{U} \to \hat{\pi}_k^{-1}(\mathcal{U})$ is a section.

(iii) If R_k is regular and if $l \in \mathbb{Z}_{>0}$, then a ***local formal solution*** of order $k+l$ to R_k is a pair $(\xi_{k+l}, \mathcal{U})$ where $\mathcal{U}$ is an open subset of X and $\xi_{k+l}\colon \mathcal{U} \to \hat{\pi}_{k+l}^{-1}(\mathcal{U})$ is a section. •

Let us see what a local formal solution is and is not. Let us restrict the discussion to formal solutions of order k, since the general case offers no further burdens in interpretation. First of all, a local formal solution $(\xi_k, \mathcal{U})$ does not necessarily have the property that $\xi_k = j_k\xi$ for a local section $(\xi, \mathcal{U})$. Thus, to a local formal solution there does not necessarily correspond a local solution. What one *can* say is this. Suppose that we have an adapted chart $(\mathcal{V}, \psi)$ for Y that induces a chart $(\mathcal{U}, \phi)$ for X. Let us denote coordinates in the adapted chart by $(\boldsymbol{x}, \boldsymbol{y}) \in \phi(\mathcal{U}) \times \mathcal{V}' \subset \mathbb{R}^n \times \mathbb{R}^m$. Let $x_0 \in \mathcal{U}$ be a point in the chart domain and denote $\boldsymbol{x}_0 = \phi(x_0)$. Then write

$$(\boldsymbol{x}_0, \boldsymbol{\xi}(\boldsymbol{x}_0), \boldsymbol{D\xi}(\boldsymbol{x}_0), \dots, \boldsymbol{D}^k\boldsymbol{\xi}(\boldsymbol{x}_0))$$

as the coordinate representation for $\xi_k(x_0)$, where $\boldsymbol{x} \mapsto (\boldsymbol{x}, \boldsymbol{\xi}(\boldsymbol{x}))$ is the coordinate representation of a local section ξ for which $j_k\xi(x_0) = \xi_k(x_0)$. We may then define a local section $(\eta, \mathcal{U})$ of $\pi\colon \mathsf{Y} \to \mathsf{X}$ by

$$\eta(x) = \psi^{-1}\Big(\phi(x), \sum_{j=0}^{k} \frac{1}{j!}\boldsymbol{D}^j\boldsymbol{\xi}(\boldsymbol{x}_0)\cdot(\boldsymbol{x}-\boldsymbol{x}_0,\dots,\boldsymbol{x}-\boldsymbol{x}_0)\Big).$$

Note that $j_k\eta(x_0) = \xi_k(x_0)$, but that we do not necessarily have $j_k\eta(x) = \xi_k(x_0)$ for $x \neq x_0$. Thus a local formal solution merely represents a section that satisfies the partial differential equation at a single point.

The obvious relationship between solutions and formal solutions is the following.

7.2 PROPOSITION: (Solutions give rise to formal solutions) *Let $\pi\colon \mathsf{Y} \to \mathsf{X}$ be a fibred manifold and let R_k be a kth-order partial differential equation. Then the following statements hold:*

(i) if $(\xi, \mathcal{U})$ is a local solution of R_k then $(j_k\xi, \mathcal{U})$ is a local formal solution of order k;

(ii) if R_k is regular, if $l \in \mathbb{Z}_{>0}$, and if $(\xi, \mathcal{U})$ is a local solution of R_k then $(j_{k+l}\xi, \mathcal{U})$ is a local formal solution of order $k+l$.

The extent to which the converse of the previous result does not hold is exactly what the subject of integrability of partial differential equations deals with.

7.2. Formal integrability. We now study the idea of constructing solutions to a partial differential equation by constructing the Taylor's series of a solution, coefficient by coefficient. Let us first illustrate the concept with an example.

7.3 EXAMPLE: (Taylor series solution for partial differential equation) Let us take $\mathsf{X} = \mathbb{R}^2$, $\mathsf{Y} = \mathbb{R}^2 \times \mathbb{R}$, $\pi((x, y), f) = (x, y)$, and consider the first-order partial differential equation $\mathsf{R}_{\mathrm{der}}$ of Example 3.2–2. We consider two special cases.

1. First we take $\alpha(x, y) = y$ and $\beta(x, y) = -x$. Thus we wish to solve the partial differential equation

$$\frac{\partial f}{\partial x}(x, y) = y, \quad \frac{\partial f}{\partial y}(x, y) = -x.$$

We seek a Taylor series solution about $(x_0, y_0) = (0, 0)$ with initial condition $f(0, 0) = 0$. From the differential equation we immediately read off $\frac{\partial f}{\partial x}(0, 0) = 0$ and $\frac{\partial f}{\partial y}(0, 0) = 0$. Differentiating the partial differential equation once (i.e., prolonging it once) we get the equations

$$\frac{\partial^2 f}{\partial x^2}(0, 0) = 0, \quad \frac{\partial^2 f}{\partial x \partial y}(0, 0) = 1$$

from the first equation and

$$\frac{\partial^2 f}{\partial x \partial y}(0, 0) = -1, \quad \frac{\partial^2 f}{\partial y^2}(0, 0) = 0$$

from the second equation. We immediately see that this process of constructing a Taylor series solution is doomed, since we have arrived at incompatible conditions on $\frac{\partial^2 f}{\partial x \partial y}(0, 0)$. This is, of course, exactly a reflection of what we have seen already in Example 3.7–2.

2. Next we suppose that $\frac{\partial\alpha}{\partial y}(x,y) = \frac{\partial\beta}{\partial x}(x,y)$ for all $(x,y) \in \mathsf{X}$. We then wish to solve the partial differential equation

$$\frac{\partial f}{\partial x}(x,y) = \alpha(x,y), \quad \frac{\partial f}{\partial y}(x,y) = \beta(x,y).$$

We seek a Taylor series solution about $(x_0,y_0) = (0,0)$ taking initial condition $f(0,0) = 1$. The partial differential equation gives $\frac{\partial f}{\partial x}(0,0) = 0$ and $\frac{\partial f}{\partial y}(0,0)$. Taking the first prolongation of the partial differential equation gives

$$\frac{\partial^2 f}{\partial x^2}(0,0) = \frac{\partial\alpha}{\partial x}(0,0), \quad \frac{\partial^2 f}{\partial x\partial y}(0,0) = \frac{\partial\alpha}{\partial y}(0,0)$$

from the first equation and

$$\frac{\partial^2 f}{\partial x\partial y}(0,0) = \frac{\partial\beta}{\partial x}(0,0), \quad \frac{\partial^2 f}{\partial y^2}(0,0) = \frac{\partial\beta}{\partial y}(0,0)$$

from the second equation. Since we are assuming that $\frac{\partial\alpha}{\partial y} = \frac{\partial\beta}{\partial x}$, these equations yield the second partial derivatives of f. Now let us consider the second prolongation, which gives

$$\begin{aligned}\frac{\partial^3 f}{\partial x^3}(0,0) = \frac{\partial^2\alpha}{\partial x^2}(0,0), &\quad \frac{\partial^3 f}{\partial x^2\partial y}(0,0) = \frac{\partial^2\alpha}{\partial x\partial y}(0,0),\\ \frac{\partial^3 f}{\partial x^2\partial y}(0,0) = \frac{\partial^2\alpha}{\partial x\partial y}(0,0), &\quad \frac{\partial^3 f}{\partial x\partial y^2}(0,0) = \frac{\partial^2\alpha}{\partial y^2}(0,0)\end{aligned}$$

from the first equation and

$$\begin{aligned}\frac{\partial^3 f}{\partial x^2\partial y}(0,0) = \frac{\partial^2\beta}{\partial x^2}(0,0), &\quad \frac{\partial^3 f}{\partial x\partial y^2}(0,0) = \frac{\partial^2\beta}{\partial x\partial y}(0,0),\\ \frac{\partial^3 f}{\partial x\partial y^2}(0,0) = \frac{\partial^2\beta}{\partial x\partial y}(0,0), &\quad \frac{\partial^3 f}{\partial y^3}(0,0) = \frac{\partial^2\beta}{\partial y^2}(0,0)\end{aligned}$$

from the second equation. One can see that the condition that $\frac{\partial\alpha}{\partial y} = \frac{\partial\beta}{\partial x}$ implies that these conditions on the third partial derivatives of f are compatible. What is not obvious, but nonetheless true, is that this process may be continued term-by-term for the coefficients of the Taylor series. In this particular case, this is a consequence of the formal Poincaré lemma. •

With this example in mind, let us turn to generalities. The objective of the formal theory of partial differential equations is that one wishes to construct solutions of the partial differential equation, order-by-order in the Taylor series. Let us attach some language to this order-by-order process in order to make sure we understand it.

7.4 Definition: (Extension of formal solution) Let $\pi\colon \mathsf{Y} \to \mathsf{X}$ be a fibred manifold and let R_k be a kth-order partial differential equation. Let $l,m \in \mathbb{Z}_{\geq 0}$. If $p_{k+l} \in \rho_l(\mathsf{R}_k)_x$ is a formal solution at x of order $k+l$, an ***extension*** of p_{k+l} to order $k+l+m$ is an element $p_{k+l+m} \in \rho_{l+m}(\mathsf{R}_k)_x$ such that $\hat{\pi}^{k+l+m}_{k+l}(p_{k+l+m}) = p_{k+l}$. •

Generally, given $p_{k+l} \in \rho_l(\mathsf{R}_k)_x$ there may not exist an extension to some higher order. This is easily demonstrated with an example.

7.5 Example: (Nonexistence of extensions) We consider Example 7.3–1 where the partial differential equation R_{der} is given by

$$\mathsf{R}_{\text{der}} = \{\, (x, y, f, f_x, f_y) \mid \;\; f_x = y,\;\; f_y = -x\} \,.$$

An example of a formal solution of order 1 at $(x, y) = (0, 0)$ is the element $(0, 0, 0, 0, 0)$ of R_{der}. Following Example 4.6–2 we ascertain that the first prolongation of R_{der} is

$$\begin{aligned}\rho_1(\mathsf{R}_{\text{der}}) = \{(x, y, f_x, f_y, f_{,x}, f_{,y}, f_{x,x}, f_{x,y}, f_{y,x}, f_{y,y})|\;\; & f_x = y,\;\; f_y = -x,\\ f_{,x} = f_x,\;\; f_{,y} = f_y,\;\; f_{x,x} = 0,\;\; f_{x,y} = 1,\;\; & f_{y,x} = -1,\;\; f_{y,y} = 0,\;\; f_{x,y} = f_{y,x}\}.\end{aligned}$$

Thus $\rho_1(\mathsf{R}_{\text{der}}) = \emptyset$, and clearly it follows that the formal solution above of order 1 cannot be extended to a formal solution of order 2. •

The following result presents the most common means of ensuring that formal solutions of any order can be constructed.

7.6 Proposition: (Condition for existence of infinite order formal solutions) *Let $\pi\colon \mathsf{Y} \to \mathsf{X}$ be a fibred manifold and let R_k be a kth-order partial differential equation. If there exists $m \in \mathbb{Z}_{\geq 0}$ such that the maps $\hat{\pi}^{k+m+l+1}_{k+m+l}\colon \rho_{m+l+1}(\mathsf{R}_k) \to \rho_{m+l}(\mathsf{R}_k)$, $l \in \mathbb{Z}_{\geq 0}$, are surjective, then, given $p_{k+m} \in \rho_m(\mathsf{R}_k)$, there exists $p_\infty \in \rho_\infty(\mathsf{R}_k)$ such that $\hat{\pi}^{\infty}_{k+m}(p_\infty) = p_{k+m}$.*

Proof: We claim that for each $l \in \mathbb{Z}_{\geq 0}$ there exists $p_{k+m+l} \in \rho_{m+l}(\mathsf{R}_k)$ such that $\hat{\pi}^{k+m+l}_{k+m}(p_{k+m+l}) = p_{k+m}$. We prove this by induction on l. This is trivial for $l = 0$, so suppose it true for $l = j$. Thus there exists $p_{k+m+j} \in \rho_{m+j}(\mathsf{R}_k)$ such that $\hat{\pi}^{k+m+j}_{k+m}(p_{k+m+j}) = p_{k+m}$. Since $\hat{\pi}^{k+m+j+1}_{k+m+j}$ is surjective there exists $p_{k+m+j+1} \in \rho_{m+j+1}(\mathsf{R}_k)$ such that $\hat{\pi}^{k+m+j+1}_{k+m+j}(p_{k+m+j+1}) = p_{k+m+j}$. Then $\hat{\pi}^{k+m+j+1}_{k+m}(p_{k+m+j+1}) = p_{k+m}$, and so the claim holds for $l = j + 1$. Now define $p_\infty \in \rho_\infty(\mathsf{R}_k)$ by $p_\infty(l) = p_{k+m+l}$ to give the proposition. ■

The proposition motivates the following definition.

7.7 Definition: (Formal integrability) Let $\pi\colon \mathsf{Y} \to \mathsf{X}$ be a fibred manifold, let $\mathsf{R}_k \subset \mathsf{J}_k\pi$ be a kth-order partial differential equation, and let $m \in \mathbb{Z}_{\geq 0}$. Then R_k is ***formally integrable at order m*** if

(i) $\rho_{l+m}(\mathsf{R}_k)$ is a fibred submanifold of $\pi_{k+m+l}\colon \mathsf{J}_{k+m+l}\pi \to \mathsf{X}$ for each $l \in \mathbb{Z}_{\geq 0}$ and

(ii) the maps $\hat{\pi}^{k+m+l+1}_{k+m+l}\colon \rho_{m+l+1}(\mathsf{R}_k) \to \rho_{m+l}(\mathsf{R}_k)$ are epimorphisms of fibred manifolds for each $l \in \mathbb{Z}_{\geq 0}$. •

The matter of checking formal integrability using the definition is infeasible. Therefore, one wishes to derive simple conditions which ensure formal integrability via "finite" computations. This is what the subject for formal integrability of partial differential equations is all about. We shall exclusively be concerned with formal integrability at order 0.

The following result gives a useful characterisation of formal integrability, and helps to illuminate the rôle of the symbol in the "order-by-order" manner of determining formal solutions of infinite order.

7.8 Proposition: (Alternative characterisation of formal integrability) *Let* $\pi\colon \mathsf{Y} \to \mathsf{X}$ *be a fibred manifold and let* R_k *be a* k*th-order partial differential equation. For* $m \in \mathbb{Z}_{\geq 0}$ *the following statements are equivalent:*

(i) R_k *is formally integrable at order* m*;*

(ii) the following statements hold:

- *(a)* $\rho_m(\mathsf{R}_k)$ *is a fibred submanifold of* $\pi_{k+m}\colon \mathsf{J}_{k+m}\pi \to \mathsf{X}$*;*
- *(b)* $\rho_{m+l+1}(\mathsf{G}(\mathsf{R}_k))$ *is a vector bundle for all* $l \in \mathbb{Z}_{\geq 0}$*;*
- *(c)* $\hat{\pi}^{k+m+l+1}_{k+m+l}$ *is surjective for each* $l \in \mathbb{Z}_{\geq 0}$*.*

Proof: Suppose that R_k is formally integrable at order m. Clearly then, $\hat{\pi}^{k+m+l+1}_{k+m+l}$ is surjective for $l \in \mathbb{Z}_{\geq 0}$. By Proposition 6.4(v) it follows that $\rho_{m+l+1}(\mathsf{G}(\mathsf{R}_k))$ is a vector bundle. It is also immediate that $\rho_m(\mathsf{R}_k)$ is a fibred submanifold.

If the second conditions of the proposition holds, we prove by induction on l that $\rho_{m+l+1}(\mathsf{R}_k)$ is a fibred submanifold and that $\hat{\pi}^{k+m+l+1}_{k+m+l}$ is an epimorphism. For $l = 0$ this follows from Proposition 6.4(iv). Now suppose that $\rho_{m+l+1}(\mathsf{R}_k)$ is a fibred submanifold and that $\hat{\pi}^{k+m+l+1}_{k+m+l}$ is an epimorphism for $l \in \{1, \dots, r-1\}$. Then, again by Proposition 6.4(iv), it follows that $\rho_{m+r+1}(\mathsf{R}_k)$ is a fibred submanifold and that $\hat{\pi}^{k+m+r+1}_{k+m+r}$ is an epimorphism. ■

An important question is what formal integrability actually buys us in terms of existence of local solutions to the partial differential equation. This is answered by the following theorem of Malgrange [1972a, 1972b].

7.9 Theorem: (Formal integrability gives existence in the analytic case) *Let* $\pi\colon \mathsf{Y} \to \mathsf{X}$ *be an analytic fibred manifold with* R_k *an analytic partial differential equation of order* k*. If* R_k *is formally integrable at order* $m \in \mathbb{Z}_{\geq 0}$ *and if* $p_{k+m} \in \rho_m(\mathsf{R}_k)_x$*, then there exists an analytic local solution* $(\xi, \mathcal{U})$ *with* $x \in \mathcal{U}$ *and* $j_{k+m}\xi(x) = p_{k+m}$*.*

Note that there will generally be many solutions, so this is an existence theorem. To get uniqueness one needs further conditions on the solution, typically coming in the form of boundary conditions.

8. Linearisation

Although linearisation is not typically part of the standard treatment of partial differential equations, it is nonetheless insightful to study this notion in a systematic way. Before we give the definition of the linearisation, we first need to perform some constructions with various bundles and jet bundles. The idea will be that we arrive at two notions of linearisation. One is linearisation about a local solution of a partial differential equation, and the other is linearisation about a local *formal* solution of a partial differential equation. In each case we arrive at a linear partial differential equation for a vector bundle over X. We denote by $\nu\colon \mathsf{V}\pi \to \mathsf{X}$ the canonical projection and recall that there is a canonical isomorphism from $\mathsf{V}\pi_k$ to $\mathsf{J}_k\nu$ as vector bundles over $\mathsf{J}_k\pi$.

8.1. Some jet bundle constructions. In order to arrive at the linearisation as a linear partial differential equation, we choose a local section $(\xi, \mathcal{U})$ of $\pi\colon \mathsf{Y} \to \mathsf{X}$. Corresponding to this is the local section $(j_k\xi, \mathcal{U})$ of $\pi_k\colon \mathsf{J}_k\pi \to \mathsf{X}$. Let $\mathsf{Y}_{\mathcal{U}} = \pi^{-1}(\mathcal{U})$, $\pi_{\mathcal{U}} = \pi|\mathsf{Y}_{\mathcal{U}}$,

$\mathsf{V}\pi_{\mathcal{U}} = \mathsf{V}\pi|\mathsf{Y}_{\mathcal{U}}$, and $\nu_{\mathcal{U}} = \nu|\mathsf{V}\pi_{\mathcal{U}}$. Then the pull-back bundle $\xi^*\nu_{\mathcal{U}} \colon \xi^*\mathsf{V}\pi_{\mathcal{U}} \to \mathcal{U}$ is a vector bundle whose fibre over $x \in \mathcal{U}$ is naturally isomorphic to $\mathsf{V}_{\xi(x)}\pi$. We wish to describe the jet bundles of this pull-back bundle. To do so, note that if $\eta \colon \mathcal{U} \to \xi^*\mathsf{V}\pi_{\mathcal{U}}$ is a section, then there corresponds a section $\zeta_\eta \colon \mathcal{U} \to \mathsf{V}\pi_{\mathcal{U}}$ defined by $\zeta(x) = \eta(x)$, where we think of $\eta(x)$ as being an element of $\mathsf{V}_{\xi(x)}\pi_{\mathcal{U}} \subset \nu_{\mathcal{U}}^{-1}(x)$. With this notation we state the following result.

8.1 Proposition: (The jet bundle of a pull-back is the pull-back of the jet bundle) *With the above notation, define a map $\psi_k \colon \mathsf{J}_k(\xi^*\nu_{\mathcal{U}}) \to j_k\xi^*(\mathsf{J}_k\nu_{\mathcal{U}})$ by*

$$\psi_k(j_k\eta(x)) = j_k\zeta_\eta(x).$$

Then ψ_k is an isomorphism of vector bundles over $\mathcal{U}$.

Proof: This is easily verified in coordinates. ■

Let us understand the proposition in the case when $k = 1$. Let us suppose that $\mathsf{X} = \mathcal{U} \subset \mathbb{R}^n$ and that $\mathsf{Y} = \mathcal{U} \times \mathcal{V}$ with $\mathcal{V} \subset \mathbb{R}^m$. Let us denote by $\boldsymbol{\xi} \colon \mathcal{U} \to \mathcal{V}$ the map defined by $\xi(\boldsymbol{x}) = (\boldsymbol{x}, \boldsymbol{\xi}(\boldsymbol{x}))$. Let us give coordinates to all the various objects:

1. X: $\boldsymbol{x} \in \mathcal{U}$;
2. Y: $(\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{U} \times \mathcal{V}$;
3. $\mathsf{V}\pi$: $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{v}) \in \mathcal{U} \times \mathcal{V} \times \mathbb{R}^m$;
4. $\mathsf{J}_1\pi$: $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{p}) \in \mathcal{U} \times \mathcal{V} \times L(\mathbb{R}^n; \mathbb{R}^m)$;
5. $\mathsf{J}_1\nu$: $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{p}, \boldsymbol{v}, \boldsymbol{p}') \in \mathcal{U} \times \mathcal{V} \times L(\mathbb{R}^n; \mathbb{R}^m) \times \mathbb{R}^m \times L(\mathbb{R}^n; \mathbb{R}^m)$ (here the last two coordinates are for the fibres for the vector bundle over $\mathsf{J}_1\pi$).

Now let us, in these coordinates, identify the various pull-back bundles that will be needed:

6. $j_1\xi^*(\mathsf{J}_1\nu)$: $\{((\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{p}, \boldsymbol{v}, \boldsymbol{p}'), \boldsymbol{x}) \mid \ \boldsymbol{y} = \boldsymbol{\xi}(\boldsymbol{x}),\ \boldsymbol{p} = \boldsymbol{D\xi}(\boldsymbol{x})\}$;
7. $\xi^*\nu$: $\{((\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{v}), \boldsymbol{x}) \mid \ \boldsymbol{y} = \boldsymbol{\xi}(\boldsymbol{x})\}$.

This then gives the following coordinates for the following objects:

8. $j_1\xi^*(\mathsf{J}_1\nu)$: $(\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{p}') \in \mathcal{U} \times \mathbb{R}^m \times L(\mathbb{R}^n; \mathbb{R}^m)$ (the last two coordinates are for the fibres of the vector bundle over X);
9. $\xi^*\nu$: $(\boldsymbol{x}, \boldsymbol{v}) \in \mathcal{U} \times \mathbb{R}^m$ (the last coordinate is for the fibres of the vector bundle over X);
10. $\mathsf{J}_1(\xi^*\nu)$: $(\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{p}) \in \mathcal{U} \times \mathbb{R}^m \times L(\mathbb{R}^n; \mathbb{R}^m)$ (the last two coordinates are for the fibres of the vector bundle over X).

A section η of $\xi^*\mathsf{V}\pi$ is given by $\boldsymbol{x} \mapsto (\boldsymbol{x}, \boldsymbol{\eta}(\boldsymbol{x}))$ and the corresponding section ζ_η of $\mathsf{V}\pi$ is given by $\boldsymbol{x} \mapsto (\boldsymbol{x}, \boldsymbol{\xi}(\boldsymbol{x}), \boldsymbol{\eta}(\boldsymbol{x}))$. Then $j_1\eta$ is given in coordinates by $\boldsymbol{x} \mapsto (\boldsymbol{x}, \boldsymbol{\eta}(\boldsymbol{x}), \boldsymbol{D\eta}(\boldsymbol{x}))$ and $j_1\zeta_\eta$ is given by

$$\boldsymbol{x} \mapsto (\boldsymbol{x}, \boldsymbol{\xi}(\boldsymbol{x}), \boldsymbol{D\xi}(\boldsymbol{x}), \boldsymbol{\eta}(\boldsymbol{x}), \boldsymbol{D\eta}(\boldsymbol{x})).$$

The isomorphism ψ_1 from $\mathsf{J}_1(\xi^*\nu)$ to $\xi_1^*(\mathsf{J}_1\nu)$ is then given in coordinates by

$$(\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{p}) \mapsto (\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{p}).$$

8.2. Linearisation of partial differential equations. Now let us insert a kth-order partial differential equation $\mathsf{R}_k \subset \mathsf{J}_k\pi$ into the picture. The restriction of π_k to R_k we denote by $\hat{\pi}_k \colon \mathsf{R}_k \to \mathsf{X}$.

8.2 Definition: (Linearisation of a partial differential equation) Let $\pi \colon \mathsf{Y} \to \mathsf{X}$ be a fibred manifold and let $\mathsf{R}_k \subset \mathsf{J}_k\pi$ be a kth-order partial differential equation.

(i) If $(\xi, \mathcal{U})$ is a local solution of R_k then the ***linearisation*** of R_k about ξ is the kth-order linear partial differential equation $\mathscr{L}(\mathsf{R}_k, \xi) = j_k\xi^*(\ker(T\hat{\pi}_k))$ which we think of as a subbundle of the bundle $\mathsf{J}_k(\xi^*\nu_{\mathcal{U}})$ of a k-jets of the vector bundle $\xi^*\nu_{\mathcal{U}} \colon \xi^*\mathsf{V}\pi_{\mathcal{U}} \to \mathcal{U}$.

(ii) If $(\xi_k, \mathcal{U})$ is a local formal solution of R_k then the ***formal linearisation*** of R_k about ξ_k is the zeroth-order linear partial differential equation $\mathscr{L}_k(\mathsf{R}_k, \xi_k) = \ker(T\hat{\pi}_k) \subset \mathsf{V}\pi_k$ which we think of as a vector bundle over $\mathsf{R}_k \cap \pi_k^{-1}(\mathcal{U})$. •

The idea of linearisation, in the broadest sense, is that it is what arises when one varies a solution. Let us now see that this interpretation agrees with our definition. To do this we introduce the notion of a variation of a solution and of a formal solution.

8.3 Definition: (Variation of solution and formal solution) Let $\pi \colon \mathsf{Y} \to \mathsf{X}$ be a fibred manifold and let R_k be a kth-order partial differential equation. For a local solution $(\xi, \mathcal{U})$ and a local formal solution $(\xi_k, \mathcal{U})$,

(i) a ***variation*** of $(\xi, \mathcal{U})$ is a C^∞-map $\sigma_\xi \colon \mathcal{U} \times I \to \mathsf{Y}$ such that

  (a) $I \subset \mathbb{R}$ is an interval for which $0 \in \operatorname{int}(I)$,

  (b) $\sigma_\xi(x, 0) = \xi(x)$ for all $x \in \mathcal{U}$, and

  (c) $\pi \circ \sigma_\xi(x, s) = x$ for all $x \in \mathcal{U}$ and $s \in I$,

and

(ii) a ***variation*** of $(\xi_k, \mathcal{U})$ is a C^∞-map $\sigma_{\xi_k} \colon \mathcal{U} \times I \to \mathsf{R}_k$ such that

  (a) $I \subset \mathbb{R}$ is an interval for which $0 \in \operatorname{int}(I)$,

  (b) $\sigma_{\xi_k}(x, 0) = \xi_k(x)$ for all $x \in \mathcal{U}$, and

  (c) $\hat{\pi}_k \circ \sigma_{\xi_k}(x, s) = x$ for all $x \in \mathcal{U}$ and $s \in I$. •

Given a variation σ_ξ of a local solution $(\xi, \mathcal{U})$ we denote by $j_k\sigma_\xi \colon \mathcal{U} \times I \to \mathsf{J}_k\pi$ the map defined by asking that, for each $s \in I$, $x \mapsto j_k\sigma_\xi(x, s)$ be the k-jet of the local section $x \mapsto \sigma_\xi(x, s)$.

We now characterise the formal linearisation in terms of variations.

8.4 Proposition: (Characterisation of formal linearisation) *Let $\pi \colon \mathsf{Y} \to \mathsf{X}$ be a fibred manifold and let R_k be a kth-order partial differential equation. Suppose that $(\xi_k, \mathcal{U})$ is a local formal solution of order k and let $\sigma_{\xi_k} \colon \mathcal{U} \times I \to \mathsf{R}_k$ be a variation of $(\xi_k, \mathcal{U})$. Then*

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} \sigma_{\xi_k}(x, s) \in \mathscr{L}_k(\mathsf{R}_k, \xi_k)_{\xi_k(x)}.$$

Conversely, if $X \in \mathscr{L}_k(\mathsf{R}_k, \xi_k)_{p_k}$, then there exists a local formal solution $(\xi_k, \mathcal{U})$ of order k with $x_0 = \pi_k(p_k) \in \mathcal{U}$ and a variation $\sigma_{\xi_k} \colon \mathcal{U} \times I \to \mathsf{Y}$ such that $X = \frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0} \sigma_{\xi_k}(x_0, s)$.

Proof: Since $\hat{\pi}_k \circ \sigma_{\xi_k}(x_0, s) = x$ for all $s \in I$, the first part of the result follows by differentiating this expression with respect to s.

For the second assertion, since R_k is a fibred submanifold of $\mathsf{J}_k\pi$ we choose an adapted chart $(\mathcal{V}, \psi)$ for $\mathsf{J}_k\pi$ such that

1. $p_k \in \mathcal{V}$,
2. $(\mathcal{U}, \phi)$ denotes the induced chart for X,
3. ψ takes values in $\mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}^s$ where $n = \dim(\mathsf{X})$ and where $r, s \in \mathbb{Z}_{>0}$, and
4. there exists an open subset $\mathcal{W} \subset \mathbb{R}^r$ such that $\psi|(\pi_k^{-1}(\mathcal{U}) \cap \mathsf{R}_k)$ is a diffeomorphism onto $\phi(\mathcal{U}) \times \mathcal{W} \times \{\mathbf{0}\}$.

Suppose that the local representative of p_k in the chart $(\mathcal{V}, \psi)$ is $(\boldsymbol{x}_0, \boldsymbol{z}_0, \mathbf{0})$, and define a local formal solution $(\xi_k, \mathcal{U})$ by asking that its local representative be $\boldsymbol{x} \mapsto (\boldsymbol{x}, \boldsymbol{z}_0, \mathbf{0})$. Let $\boldsymbol{X} \oplus \mathbf{0} \in \mathbb{R}^r \oplus \mathbb{R}^s$ be the principal part of the local representative for X and define σ_{ξ_k} by

$$\sigma_{\xi_k}(x, s) = \psi^{-1}(\phi(x), \boldsymbol{z}_0 + s\boldsymbol{X}, \mathbf{0}),$$

noting that this makes sense for s sufficiently small since $\mathcal{W}$ is open. It is clear that $\frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0}\sigma_{\xi_k}(x_0, s) = X$. ■

Thus there is essentially a 1–1 correspondence between the formal linearisation about $(\xi_k, \mathcal{U})$ and tangent vectors to variations of $(\xi_k, \mathcal{U})$. For the linearisation, this is no longer true; it is only the case that tangent vectors to variations give rise to elements in the linearisation, but not necessarily conversely. This is not surprising since the same statement is true at the level of solutions and formal solutions as well.

8.5 Proposition: (Characterisation of linearisation) *Let $\pi\colon \mathsf{Y} \to \mathsf{X}$ be a fibred manifold and let R_k be a kth-order partial differential equation. Suppose that $(\xi, \mathcal{U})$ is a local solution and let $\sigma_\xi\colon \mathcal{U} \times I \to \mathsf{R}_k$ be a variation of $(\xi, \mathcal{U})$. Then*

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} j_k\sigma_\xi(x, s) \in \mathscr{L}_k(\mathsf{R}_k, \xi)_x.$$

Proof: This follows from Proposition 8.4 since $(j_k\xi, \mathcal{U})$ is a local formal solution and since $j_k\sigma_\xi$ is a variation of $(j_k\xi, \mathcal{U})$. ■

8.3. Linearisation and prolongation. We have seen that the linearisation can be roughly thought of as measuring the richness of the set of solutions at first order. We have also seen that in the construction, order-by-order, of solutions to partial differential equations, one is interested in the surjectivity of the maps $\hat{\pi}^{k+l+1}_{k+l}\colon \rho_{l+1}(\mathsf{R}_k) \to \rho_l(\mathsf{R}_k)$. This surjectivity allows us to extend a formal solution of order $k+l$ to a formal solution of order $k+l+1$. The point now is to study the process of extension via first-order approximation, i.e., via linearisation.

We first define an appropriate variation of a local formal solution.

8.6 Definition: (Top-order variation) Let $\pi\colon \mathsf{Y} \to \mathsf{X}$ be a fibred manifold, let R_k be a kth-order partial differential equation, and let $l \in \mathbb{Z}_{\geq 0}$, supposing that $\rho_{l+1}(\mathsf{R}_k)$ is a fibred submanifold of $\pi_{k+l+1}\colon \mathsf{J}_{k+l+1}\pi \to \mathsf{X}$. For a local formal solution $(\xi_{k+l+1}, \mathcal{U})$ of order $k+l+1$, a ***top-order variation*** is a C^∞-map $\sigma_{\xi_{k+l+1}}\colon \mathcal{U} \times I \to \rho_{l+1}(\mathsf{R}_k)$ such that

(i) $I \subset \mathbb{R}$ is an interval for which $0 \in \operatorname{int}(I)$,

(ii) $\sigma_{\xi_{k+l+1}}(x, 0) = \xi_{k+l+1}(x)$ for all $x \in \mathcal{U}$, and

(iii) $\hat{\pi}^{k+l+1}_{k+l} \circ \sigma_{\xi_{k+l+1}}(x,s) = \hat{\pi}^{k+l+1}_{k+l} \circ \xi_{k+l+1}(x)$ for all $s \in I$. •

The following result relates top-order variations to the symbol.

8.7 Proposition: (Top-order variations and the symbol) *Let $\pi \colon \mathsf{Y} \to \mathsf{X}$ be a fibred manifold, let R_k be a kth-order partial differential equation, and let $l \in \mathbb{Z}_{\geq 0}$, supposing that $\rho_l(\mathsf{R}_k)$ is a fibred submanifold of $\pi_{k+l} \colon \mathsf{J}_{k+l}\pi \to \mathsf{X}$ and that $\rho_{l+1}(\mathsf{G}(\mathsf{R}_k))$ is a vector bundle. If $\sigma_{\xi_{k+l+1}}$ is a top-order variation of a local formal solution $(\xi_{k+l+1}, \mathcal{U})$ of order $k+l+1$, then*

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} \sigma_{\xi_{k+l+1}}(x,s) \in \phi_{k+l+1}(\rho_{l+1}(\mathsf{G}(\mathsf{R}_k, \xi_{k+l+1}(x)))).$$

Conversely, if $X \in \phi_{k+l+1}(\rho_{l+1}(\mathsf{G}(\mathsf{R}_k, p_{k+l+1})))$, then there exists a local formal solution $(\xi_{k+l+1}, \mathcal{U})$ of order $k+l+1$ with $x_0 = \pi_{k+l+1}(p_{k+l+1}) \in \mathcal{U}$ and a top-order variation $\sigma_{\xi_{k+l+1}}$ such that $\frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0} \sigma_{\xi_{k+l+1}}(x_0,s) = X$.

Proof: By Proposition 6.4(iii) we know that $\rho_{l+1}(\mathsf{R}_k)$ is an affine bundle over $\rho^{(1)}_l(\mathsf{R}_k)$ modelled on $\rho_{l+1}(\mathsf{G}(\mathsf{R}_k))$. The first part of the result follows by differentiating the equality

$$\hat{\pi}^{k+l+1}_{k+l} \circ \sigma_{\xi_{k+l+1}}(x,s) = \hat{\pi}^{k+l+1}_{k+l} \circ \xi_{k+l+1}(x)$$

with respect to s. The second part of the result can be easily proved along the lines of Proposition 8.4, using the fact that $\phi_{k+l}(\rho_{l+1}(\mathsf{G}(\mathsf{R}_k))) = \ker(\hat{\pi}^{k+l+1}_{k+l})$ from Proposition 5.18. ■

Thus we see one interpretation of the symbol. It measures ways a formal solution of order $k+l+1$ can be perturbed to arrive at other formal solutions of order $k+l+1$ that are extensions of the same formal solution of order $k+l$.

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