An illustration of Wang's Theorem

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Abstract

Wang's Theorem characterises the set of invariant connections on a principal fibre bundle. Here the point of the theorem is illustrated via an example.

1. Setup

Let us define the ingredients for the example we use to illustrate Wang's Theorem.

1.1. The frame bundle of $\mathbb{R}^n$. We consider the principal bundle $L(\mathbb{R}^n)$, the frame bundle of $\mathbb{R}^n$. Thus a point in $L(\mathbb{R}^n)$ is a basis $\{\boldsymbol{X}_1, \dots, \boldsymbol{X}_n\}$ for the tangent space $\mathsf{T}_{\boldsymbol{x}}\mathbb{R}^n$ at some $\boldsymbol{x} \in \mathbb{R}^n$. We use the natural identification of $\mathsf{T}_{\boldsymbol{x}}\mathbb{R}^n$ with $\mathbb{R}^n$ so that we think of $\boldsymbol{X}_1, \dots, \boldsymbol{X}_n \in \mathbb{R}^n$. We use this same identification to allow us to write $L(\mathbb{R}^n)$ as a product: $L(\mathbb{R}^n) = \mathbb{R}^n \times \mathsf{GL}(n;\mathbb{R})$. Let us recall how this is done, explicitly. If $\{\boldsymbol{X}_1, \dots, \boldsymbol{X}_n\}$ is a basis for $\mathsf{T}_{\boldsymbol{x}}\mathbb{R}^n$ then we write

$$\boldsymbol{X}_j = \sum_{k=1}^n a_j^k \frac{\partial}{\partial x^k}, \qquad j \in \{1, \dots, n\},$$

defining some unique matrix $\boldsymbol{a} \in \mathsf{GL}(n;\mathbb{R}^n)$ by $\boldsymbol{a}(j,k) = a_k^j$, $j,k \in \{1, \dots, n\}$. We then make the identification

$$L(\mathbb{R}^n) \ni \{\boldsymbol{X}_1, \dots, \boldsymbol{X}_n\} \simeq (\boldsymbol{x}, \boldsymbol{a}) \in \mathbb{R}^n \times \mathsf{GL}(n;\mathbb{R}).$$

The principal $\mathsf{GL}(n;\mathbb{R})$-bundle structure of $L(\mathbb{R}^n)$ is then defined as follows. For $(\boldsymbol{x}, \boldsymbol{a}) \in L(\mathbb{R}^n)$ and for $\boldsymbol{b} \in \mathsf{GL}(n;\mathbb{R})$ define

$$(\boldsymbol{x}, \boldsymbol{a})\boldsymbol{b} = (\boldsymbol{x}, \boldsymbol{a}\boldsymbol{b}).$$

Summarising, in the language of Kobayashi and Nomizu [1963], we have a principal fibre bundle $\mathsf{P}(\mathsf{M}, \mathsf{G})$ with $\mathsf{P} = L(\mathbb{R}^n)$, $\mathsf{M} = \mathbb{R}^n$, and $\mathsf{G} = \mathsf{GL}(n;\mathbb{R})$.

1.2. The canonical flat connection on $L(\mathbb{R}^n)$. Let us define a $\mathfrak{gl}(n;\mathbb{R})$-valued one-form on $L(\mathbb{R}^n)$. First let us represent points in $\mathsf{T}(L(\mathbb{R}^n))$ in a convenient way, using the identifications above. We have

$$\begin{aligned} & L(\mathbb{R}^n) \simeq \mathbb{R}^n \times \mathsf{GL}(n;\mathbb{R}) \\ \implies \quad & \mathsf{T}(L(\mathbb{R}^n)) \simeq (\mathbb{R}^n \times \mathsf{GL}(n;\mathbb{R})) \times (\mathbb{R}^n \oplus \mathsf{L}(\mathbb{R}^n;\mathbb{R}^n)). \end{aligned}$$

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We therefore write a typical point in $\mathsf{T}(L(\mathbb{R}^n))$ as $((\boldsymbol{x},\boldsymbol{a}),(\boldsymbol{v},\boldsymbol{A}))$. We then take $\omega \in \mathsf{T}^*(L(\mathbb{R}^n)) \otimes \mathfrak{gl}(n;\mathbb{R})$ to be given by

$$\omega(\boldsymbol{x},\boldsymbol{a}) \cdot (\boldsymbol{v},\boldsymbol{A}) = \boldsymbol{a}^{-1}\boldsymbol{A}.$$

Note that this has the properties of a connection. Indeed, note that the action of $\mathsf{GL}(n;\mathbb{R}^n)$ on $L(\mathbb{R}^n)$ given by

$$(\boldsymbol{x},\boldsymbol{a})\boldsymbol{b} = (\boldsymbol{x},\boldsymbol{a}\boldsymbol{b})$$

induces by differentiation the action on $\mathsf{T}(L(\mathbb{R}^n))$ given by

$$(\boldsymbol{x},\boldsymbol{a},\boldsymbol{v},\boldsymbol{A})\boldsymbol{b} = (\boldsymbol{x},\boldsymbol{a}\boldsymbol{b},\boldsymbol{v},\boldsymbol{A}\boldsymbol{b}).$$

Thus

$$\omega(\boldsymbol{x},\boldsymbol{a}\boldsymbol{b}) \cdot (\boldsymbol{v},\boldsymbol{A}\boldsymbol{b}) = \boldsymbol{b}^{-1}\boldsymbol{a}^{-1}\boldsymbol{A}\boldsymbol{b} = \mathrm{Ad}_{\boldsymbol{b}^{-1}}(\omega(\boldsymbol{x},\boldsymbol{a}) \cdot (\boldsymbol{v},\boldsymbol{A})),$$

giving the desired equivariance. Also,

$$\omega(\boldsymbol{x},\boldsymbol{a}) \cdot (\boldsymbol{0},\boldsymbol{A}) = \boldsymbol{a}^{-1}\boldsymbol{A},$$

and notice that $(\boldsymbol{0},\boldsymbol{A})$ is the infinitesimal generator at $(\boldsymbol{x},\boldsymbol{a})$ associated with the Lie algebra element $\boldsymbol{a}^{-1}\boldsymbol{A}$.

1.3. A group of transformations of $\mathbb{R}^n$. We let K be the group of affine transformations of $\mathbb{R}^n$. Thus, an element of $\phi \in \mathsf{K}$ assumes the following form:

$$\phi(\boldsymbol{x}) = \boldsymbol{m}\boldsymbol{x} + \boldsymbol{r},$$

where $\boldsymbol{m} \in \mathsf{GL}(n;\mathbb{R})$ and $\boldsymbol{r} \in \mathbb{R}^n$. We wish to use this group of transformations to induce a group of automorphisms of the frame bundle $L(\mathbb{R}^n)$. We do this by observing that the transformation $\phi \in \mathsf{K}$ induces a transformation of $\mathsf{T}\mathbb{R}^n$ is the obvious way: $T\phi\colon \mathsf{T}\mathbb{R}^n \to \mathsf{T}\mathbb{R}^n$. Explicitly,

$$\phi(\boldsymbol{x}) = \boldsymbol{m}\boldsymbol{x} + \boldsymbol{r} \qquad \Longrightarrow \qquad T\phi(\boldsymbol{x},\boldsymbol{v}) = (\boldsymbol{m}\boldsymbol{x} + \boldsymbol{r}, \boldsymbol{m}\boldsymbol{v}).$$

In terms of frames, $\phi \in \mathsf{K}$ gives rise to the automorphism $L\phi$ of $L(\mathbb{R}^n)$ defined by

$$L\phi(\{\boldsymbol{X}_1,\dots,\boldsymbol{X}_n\}) = \{T\phi(\boldsymbol{X}_1),\dots,T\phi(\boldsymbol{X}_n)\}.$$

Let us compute this transformation in terms of the identification of $L(\mathbb{R}^n)$ with $\mathbb{R}^n \times \mathsf{GL}(n;\mathbb{R})$.

1.1 Lemma: *If* $\phi(\boldsymbol{x}) = \boldsymbol{m}\boldsymbol{x} + \boldsymbol{r}$ *then* $L\phi(\boldsymbol{x},\boldsymbol{a}) = (\boldsymbol{m}\boldsymbol{x} + \boldsymbol{r}, \boldsymbol{m}\boldsymbol{a})$.

Proof: Let $\{\boldsymbol{X}_1,\dots,\boldsymbol{X}_n\}$ be the frame associated with $(\boldsymbol{x},\boldsymbol{a})$ so that

$$\boldsymbol{X}_j = \sum_{k=1}^n \boldsymbol{a}(k,j)\frac{\partial}{\partial x^k}, \qquad j \in \{1,\dots,n\}.$$

Then, thinking of $\boldsymbol{X}_j$ as being a vector in $\mathbb{R}^n$ with components $(\boldsymbol{a}(1,j),\dots,\boldsymbol{a}(n,j))$ we have

$$T\phi(\boldsymbol{x},\boldsymbol{X}_j) = (\boldsymbol{m}\boldsymbol{x} + \boldsymbol{r}, \boldsymbol{m}\boldsymbol{X}_j).$$

Note that the kth component of $\boldsymbol{m}\boldsymbol{X}_j$ is $\sum_{l=1}^n \boldsymbol{m}(k,l)\boldsymbol{a}(l,j) = (\boldsymbol{m}\boldsymbol{a})(k,j)$. That is,

$$\boldsymbol{m}\boldsymbol{X}_j = \sum_{k=1}^n (\boldsymbol{m}\boldsymbol{a})(k,j)\frac{\partial}{\partial x^k},$$

giving the result. ■

In summary, we have K acting an a group of automorphisms of $L(\mathbb{R}^n)$ with

$$\phi(\boldsymbol{x}) = \boldsymbol{m}\boldsymbol{x} + \boldsymbol{r} \quad\implies\quad \phi(\boldsymbol{x},\boldsymbol{a}) = (\boldsymbol{m}\boldsymbol{x} + \boldsymbol{r}, \boldsymbol{m}\boldsymbol{a}),$$

where we introduce our abuse of notation by now writing ϕ in place of $L\phi$.

1.4. The infinitesimal group action. We shall need the infinitesimal generators for the group of affine transformations. In order to do this, I suppose we need a convenient representation of the group K so we can describe its Lie algebra appropriately. If $\phi \in \mathsf{K}$ is given by $\phi(\boldsymbol{x}) = \boldsymbol{m}\boldsymbol{x} + \boldsymbol{r}$ let us define an $(n+1)\times(n+1)$ matrix

$$\boldsymbol{A}(\phi) = \begin{bmatrix} \boldsymbol{m} & \boldsymbol{r} \\ \boldsymbol{0} & 1 \end{bmatrix}.$$

This matrix is evidently invertible. Note that

$$\boldsymbol{A}(\phi)\begin{bmatrix} \boldsymbol{x} \\ 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{m}\boldsymbol{x} + \boldsymbol{r} \\ 1 \end{bmatrix},$$

so the matrix $\boldsymbol{A}(\phi)$ captures the action of K on $\mathbb{R}^n$ in some way. Moreover, this realises K as a subgroup of $\mathsf{GL}(n+1;\mathbb{R})$, and so allows us to use matrices to represent the Lie algebra. An element in the Lie algebra of K is represented in the form

$$\begin{bmatrix} \boldsymbol{A} & \boldsymbol{v} \\ \boldsymbol{0} & 0 \end{bmatrix} \tag{1.1}$$

for $\boldsymbol{A} \in \mathfrak{gl}(n;\mathbb{R})$ and $\boldsymbol{v} \in \mathbb{R}_{<0}$. We may verify that

$$\exp\left(\begin{bmatrix} \boldsymbol{A} & \boldsymbol{v} \\ \boldsymbol{0} & 0 \end{bmatrix}\right) = \begin{bmatrix} \exp(\boldsymbol{A}) & \boldsymbol{B}_{\boldsymbol{A}}\boldsymbol{v} \\ \boldsymbol{0} & 1 \end{bmatrix},$$

where

$$\boldsymbol{B}_{\boldsymbol{A}} = \boldsymbol{I}_n + \frac{\boldsymbol{A}}{2!} + \frac{\boldsymbol{A}^2}{3!} + \cdots.$$

This allows us to compute the infinitesimal generator on $L(\mathbb{R}^n)$ associated to the Lie algebra element (1.1):

$$\begin{aligned} \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp\left(\begin{bmatrix} \boldsymbol{A}t & \boldsymbol{v}t \\ \boldsymbol{0} & 0 \end{bmatrix}\right)(\boldsymbol{x},\boldsymbol{a}) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\exp(\boldsymbol{A}t)\boldsymbol{x} + t\boldsymbol{B}_{\boldsymbol{A}t}\boldsymbol{v}, \exp(\boldsymbol{A}t)\boldsymbol{a}) \\ &= (\boldsymbol{A}\boldsymbol{x} + \boldsymbol{v}, \boldsymbol{A}\boldsymbol{a}). \end{aligned}$$

2. Wang's Theorem in general and in particular

Here we recall Wang's Theorem (see [Wang 1958]) which describes the set of K-invariant connections on a principal bundle $\mathsf{P}(\mathsf{M},\mathsf{G})$ is the case when K acts transitively on M. We do this in general first, and then apply the constructions to the frame bundle example in the preceding section.

2.1. Wang's Theorem in general. Let $\mathsf{P}(\mathsf{M},\mathsf{G})$ be a principal fibre bundle and let $u_0 \in \mathsf{P}$ with $x_0 = \pi(u_0) \in \mathsf{M}$. Let K be a Lie group of automorphisms of P. Denote by

$$\mathsf{J} = \{\phi \in \mathsf{K} \mid \ \phi(\pi^{-1}(x_0)) = \pi^{-1}(x_0)\}$$

the isotropy group of x_0. (We follow Kobayashi and Nomizu by not hanging a u_0 on J as would be appropriate.) Define a homomorphism $\lambda\colon \mathsf{J} \to \mathsf{G}$ by asking that $\phi(u_0) = u_0\lambda(\phi)$.

The theorem of interest is the following.

2.1 Theorem: (Wang's Theorem) *Let $\mathsf{P}(\mathsf{M},\mathsf{G})$ be a principal fibre bundle and let K be a Lie group of automorphisms of P having the property that, if $x_1, x_2 \in \mathsf{M}$, then there exists $\phi \in \mathsf{K}$ such that $\phi(\pi^{-1}(x_1)) = \pi^{-1}(x_2)$. Denote by J the isotropy group of x_0 and let $\lambda\colon \mathsf{J} \to \mathsf{G}$ be the homomorphism defined above. Then there is a 1–1 correspondence between the following sets of objects:*

(i) the connections on P that are invariant under ϕ for every $\phi \in \mathsf{K}$;

(ii) the linear maps $\Lambda\colon \mathfrak{k} \to \mathfrak{g}$ for which

(a) $\Lambda(\mathrm{Ad}_j(\xi)) = \mathrm{Ad}_{\lambda(j)}(\Lambda(\xi))$ for every $j \in \mathsf{J}$ and $\xi \in \mathfrak{k}$ and

(b) $\Lambda|\mathfrak{j} = T_e\lambda$.

Explicitly, if ω is the connection form for an object from (i) then the corresponding object from (ii) is given by $\Lambda(\xi) = \omega(\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\exp(t\xi)(u_0))$. Conversely, if Λ is an object from (ii) then the connection form for the corresponding object from (i) is given by

$$\omega(X_{u_0}) = \Lambda(\xi) - A$$

where $\xi \in \mathfrak{k}$ and $A \in \mathfrak{g}$ are chosen so that

$$\begin{aligned} T_{u_0}\pi\big(\tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\exp(t\xi)(u_0)\big) &= T_{u_0}\pi(X_{u_0}), \\ \tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\,u_0\exp(At) &= \tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\exp(t\xi)(u_0) - X_{u_0}. \end{aligned}$$

Wang's Theorem is not so easy to interpret in the beginning. So let us offer some comments that we hope will be helpful.

1. Note that the action of K on P induces a linear map from $\mathfrak{k}$ to $\mathsf{T}_{u_0}\mathsf{P}$ sending a Lie algebra element to the value of its infinitesimal generator at u_0. The image of $\mathfrak{k}$ under this action is the tangent space to the K-orbit through u_0. One might expect this tangent space to have a vertical component saying what K does in the fibre direction and a horizontal component saying what K does in the horizontal direction. Of course, one needs a connection to be unambiguous about this. To wit…
2. A connection in P gives a splitting $\mathsf{T}_{u_0}\mathsf{P} = \mathsf{H}_{u_0}\mathsf{P} \oplus \mathsf{V}_{u_0}\mathsf{P} \simeq \mathsf{T}_{x_0}\mathsf{M} \oplus \mathfrak{g}$. This decomposition then combines with the map from $\mathfrak{k}$ to $\mathsf{T}_{u_0}\mathsf{P}$ to give a map from $\mathfrak{k}$ to $\mathfrak{g}$. This is the map Λ. This can thus be thought of as the vertical projection of the infinitesimal action of $\mathfrak{k}$ on P.

3. Let us examine the conditions satisfied by Λ.
   (a) The equivariance condition for $\mathsf{J} \subset \mathsf{K}$ is exactly that induced by the G-invariance resulting from the principal bundle structure, and via the homomorphism $\lambda\colon \mathsf{J} \to \mathsf{G}$. This is rather analogous to the equivariance condition a connection must satisfy.
   (b) The condition that Λ agrees with $T_e\lambda$ on j reflects the fact that the vertical component of the infinitesimal group action is prescribed by the principal bundle structure via the homomorphism λ. This is rather analogous to the condition on a connection that, when evaluated on a vertical vector, it must return the Lie algebra element whose infinitesimal generator gives the vertical vector.

2.2. Wang's Theorem in particular. We consider the constructions of the preceding section with $\mathsf{P} = L(\mathbb{R}^n)$, $\mathsf{M} = \mathbb{R}^n$, $\mathsf{G} = \mathsf{GL}(n;\mathbb{R})$, and K the affine transformation group of $\mathbb{R}^n$ acting on $L(\mathbb{R}^n)$. Let $\boldsymbol{x}_0 \in \mathbb{R}^n$ and suppose that $\phi \in \mathsf{K}$ is given by $\phi(\boldsymbol{x}) = \boldsymbol{m}\boldsymbol{x} + \boldsymbol{r}$. Then $\phi \in \mathsf{J}$ (using reference point $(\boldsymbol{x}_0, \boldsymbol{a}_0)$) if and only if

$$\boldsymbol{m}\boldsymbol{x}_0 + \boldsymbol{r} = \boldsymbol{x}_0 \quad \Longleftrightarrow \quad \boldsymbol{r} = (\boldsymbol{I}_n - \boldsymbol{m})\boldsymbol{x}_0.$$

Thus, if $\phi \in \mathsf{J}$ is given by $\phi(\boldsymbol{x}) = \boldsymbol{m}\boldsymbol{x} + \boldsymbol{r}$ then

$$\phi(\boldsymbol{x}_0, \boldsymbol{a}_0) = (\boldsymbol{x}_0, \boldsymbol{m}\boldsymbol{a}_0) = (\boldsymbol{x}_0, \boldsymbol{a}_0\boldsymbol{a}_0^{-1}\boldsymbol{m}\boldsymbol{a}_0) \quad \Longrightarrow \quad \lambda(\phi) = \boldsymbol{a}_0^{-1}\boldsymbol{m}\boldsymbol{a}_0.$$

Let us keep things simple and consider $\boldsymbol{x}_0 = \boldsymbol{0}$ and $\boldsymbol{a}_0 = \boldsymbol{I}_n$ whence

$$\mathsf{J} = \{\phi \in \mathsf{K} \mid \ \phi(\boldsymbol{x}) = \boldsymbol{m}\boldsymbol{x},\ \boldsymbol{m} \in \mathsf{GL}(n;\mathbb{R})\}$$

and $\lambda(\phi) = \boldsymbol{m}$ if $\phi(\boldsymbol{x}) = \boldsymbol{m}\boldsymbol{x}$.

One may verify that the canonical flat connection on $L(\mathbb{R}^n)$ is K-invariant.

2.2 Lemma: (K-invariance of the canonical flat connection) *The connection of Section 1.2 is invariant under the automorphism group of automorphisms of $L(\mathbb{R}^n)$ defined in Section 1.3.*

Proof: We need only show that the connection form is K-invariant. Let $\phi \in \mathsf{K}$ and write

$$\phi(\boldsymbol{x}, \boldsymbol{a}) = (\boldsymbol{m}\boldsymbol{x} + \boldsymbol{r}, \boldsymbol{m}\boldsymbol{a}).$$

Thus

$$T_{(\boldsymbol{x},\boldsymbol{a})}\phi(\boldsymbol{v}, \boldsymbol{A}) = (\boldsymbol{m}\boldsymbol{v}, \boldsymbol{m}\boldsymbol{A}).$$

Therefore,

$$\begin{aligned}
\phi^*\omega(\boldsymbol{x}, \boldsymbol{a}) \cdot (\boldsymbol{v}, \boldsymbol{A}) &= \omega(T_{(\boldsymbol{x},\boldsymbol{a})}\phi(\boldsymbol{v}, \boldsymbol{A})) \\
&= \omega(\boldsymbol{m}\boldsymbol{x} + \boldsymbol{r}, \boldsymbol{m}\boldsymbol{a}) \cdot (\boldsymbol{m}\boldsymbol{v}, \boldsymbol{m}\boldsymbol{A}) \\
&= (\boldsymbol{m}\boldsymbol{a})^{-1}\boldsymbol{m}\boldsymbol{A} = \boldsymbol{a}^{-1}\boldsymbol{A} = \omega(\boldsymbol{x}, \boldsymbol{a}) \cdot (\boldsymbol{v}, \boldsymbol{A}),
\end{aligned}$$

as desired. ■

Thus, according to Wang's Theorem, the connection ω defines a linear map $\Lambda\colon \mathfrak{k} \to \mathfrak{gl}(n;\mathbb{R})$ having a couple of properties. Wang tells us how to determine Λ. Indeed, our computations of Section 1.4 give

$$\begin{aligned}\Lambda\left(\begin{bmatrix}\boldsymbol{A} & \boldsymbol{v}\\ \boldsymbol{0} & 0\end{bmatrix}\right) &= \omega\left(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\exp\left(\begin{bmatrix}\boldsymbol{A}t & \boldsymbol{v}t\\ \boldsymbol{0} & 0\end{bmatrix}\right)(\boldsymbol{0},\boldsymbol{I}_n)\right)\\ &= \omega(\boldsymbol{0},\boldsymbol{I}_n)\cdot(\boldsymbol{v},\boldsymbol{A}) = \boldsymbol{A}.\end{aligned}$$

Thus Λ returns the "obvious" vertical part of $\mathfrak{k}$. Had we chosen a more interesting K-invariant connection, we would get a more interesting map Λ.

For fun, let us verify that Λ has the two properties asserted in Wang's Theorem. First the equivariance property. Let $j \in \mathsf{J}$ so that $j(\boldsymbol{x}) = \boldsymbol{m}\boldsymbol{x}$ for some $\boldsymbol{m}$ and so that $\lambda(j) = \boldsymbol{m}$. Then

$$\begin{aligned}\Lambda\left(\mathrm{Ad}_j\begin{bmatrix}\boldsymbol{A} & \boldsymbol{v}\\ \boldsymbol{0} & 0\end{bmatrix}\right) &= \Lambda\left(\begin{bmatrix}\boldsymbol{m} & \boldsymbol{0}\\ \boldsymbol{0} & 1\end{bmatrix}\begin{bmatrix}\boldsymbol{A} & \boldsymbol{v}\\ \boldsymbol{0} & 0\end{bmatrix}\begin{bmatrix}\boldsymbol{m}^{-1} & \boldsymbol{0}\\ \boldsymbol{0} & 1\end{bmatrix}\right) = \Lambda\left(\begin{bmatrix}\boldsymbol{m}\boldsymbol{A}\boldsymbol{m}^{-1} & \boldsymbol{m}\boldsymbol{v}\\ \boldsymbol{0} & 0\end{bmatrix}\right)\\ &= \boldsymbol{m}\boldsymbol{A}\boldsymbol{m}^{-1} = \mathrm{Ad}_{\lambda(j)}\Lambda\left(\begin{bmatrix}\boldsymbol{A} & \boldsymbol{v}\\ \boldsymbol{0} & 0\end{bmatrix}\right),\end{aligned}$$

as desired. Now we check the restriction condition. The elements of $\mathfrak{j}$ have the form

$$\begin{bmatrix}\boldsymbol{A} & \boldsymbol{0}\\ \boldsymbol{0} & 0\end{bmatrix},$$

and so it is immediate that

$$\Lambda\left(\begin{bmatrix}\boldsymbol{A} & \boldsymbol{0}\\ \boldsymbol{0} & 0\end{bmatrix}\right) = \boldsymbol{A} = T_{\boldsymbol{I}_n}\lambda(\boldsymbol{A})$$

since λ is simply the identity map.

References

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