Connections and spaces of connections

Andrew D. Lewis[*]

2007/06/26
Last updated: 2018/06/04

Abstract

Connections and the space of connections on a fibred manifold are described. Three different means of prescribing a connection are shown to be equivalent. The special case of linear connections is given particular attention. Nothing is new here.

1. Introduction

The two most commonly encountered characterisations of a connection on a bundle are (1) in terms of differentiation of sections of the bundle with respect to vector fields on the base space and (2) in terms of splitting the tangent space of the total space into a direct sum with one component being the vertical bundle. These characterisations are very often the most useful. However, they are not so convenient for understanding the structure of the set of all connections. There is an alternative characterisation that, while by no means unknown, is less often encountered, but which allows for a fairly easy explicit characterisation of the set of connections. In this note we give this alternative definition of a connection, establish explicitly its link with the more common definitions, and show how it can be used to provide the structure of the space of connections.

Everything we say can be found in scattered places. More or less everything is to be found, for example, in [Kolář, Michor, and Slovák 1993].

2. Connections on fibred manifolds

In this section we define what we mean by a connection and see how the definition we give is equivalent to two more familiar constructions associated with a connection: (1) a covariant derivative and (2) a splitting of the total space of the bundle on which the connection is being defined.

2.1. The definition of a connection. A ***fibred manifold*** is a triple $(\mathsf{Y}, \pi, \mathsf{X})$ where X and Y are differentiable manifolds and $\pi\colon \mathsf{Y} \to \mathsf{X}$ is a surjective submersion. By $\mathsf{J}^k(\pi)$ we denote the bundle of k-jets of sections of π. By $\pi_k\colon \mathsf{J}^k(\pi) \to \mathsf{X}$ and, for $k < l$, by $\pi^l_k\colon \mathsf{J}^l(\pi) \to \mathsf{J}^k(\pi)$ we denote the canonical projections. We denote by $\mathsf{V}(\pi) = \ker(T\pi)$ the vertical subbundle.

Our initial definition of a connection on $(\mathsf{Y}, \pi, \mathsf{X})$ is the following.

[*]Professor, Department of Mathematics and Statistics, Queen's University, Kingston, ON K7L 3N6, Canada
Email: andrew.lewis@queensu.ca, URL: http://www.mast.queensu.ca/~andrew/

2.1 Definition: A ***connection*** on a fibred manifold $(\mathsf{Y},\pi,\mathsf{X})$ is a section $S\colon \mathsf{Y}\to \mathsf{J}^1(\pi)$ of the bundle $\pi^1_0\colon \mathsf{J}^1(\pi)\to \mathsf{Y}$. •

In natural coordinates (x^j,y^a,y^b_k) for $\mathsf{J}^1(\pi)$ a connection S has the form $(x^j,y^a)\mapsto (x^j,y^a,S^b_k(x,y))$ which defines the ***connection coefficients*** S^a_j, $a\in\{1,\dots,m\}$, $j\in\{1,\dots,n\}$, for the connection.

There are two interesting constructions one can make associated with a connection on a fibred manifold, and we now describe these.

2.2. The horizontal subbundle associated with a connection. Let $(\mathsf{Y},\pi,\mathsf{X})$ be a fibred manifold. We suppose that X is n-dimensional and that Y is $(n+m)$-dimensional. One of the interpretations of a connection is that it provides a complement to $\mathsf{V}(\pi)$ in TY. Let us see how this interpretation is developed.

Let $p_1\in \mathsf{J}^1(\pi)$, let $p=\pi^1_0(p_1)$, and let $x=\pi(p)$. Let $\xi\colon \mathcal{U}\to \mathsf{Y}$ be a local section defined on a neighbourhood $\mathcal{U}$ containing x and satisfying $j_1\xi(x)=p_1$. Define $L_{p_1}\in \mathrm{Hom}_{\mathbb{R}}(\mathsf{T}_x\mathsf{X};\mathsf{T}_p\mathsf{Y})$ by $L_{p_1}(v_x)=T_x\xi(v_x)$.

2.2 Lemma: *Let $p_1\in \mathsf{J}^1(\pi)$, let $p=\pi^1_0(p_1)$, and let $x=\pi(p)$. The following statements hold:*

(i) L_{p_1} is a well-defined linear injection;

(ii) $T_p\pi\circ L_{p_1}=\mathrm{id}_{\mathsf{T}_x\mathsf{X}}$;

(iii) $\mathrm{image}(L_{p_1})$ is a complement to $\mathsf{V}_p(\pi)$ in $\mathsf{T}_p\mathsf{Y}$.

Moreover, if $p\in\mathsf{Y}$ and $x=\pi(p)$, and if $L\colon \mathsf{T}_x\mathsf{X}\to\mathsf{T}_p\mathsf{Y}$ is a linear map satisfying $T_p\pi\circ L=\mathrm{id}_{\mathsf{T}_x\mathsf{X}}$, then there exists a unique $p_1\in(\pi^1_0)^{-1}(p)$ such that $L=L_{p_1}$.

Proof: (i) Suppose that $\xi_1,\xi_2\colon\mathcal{U}\to\mathsf{Y}$ satisfy $j_1\xi_1(x)=j_1\xi_2(x)=p_1$. This means that, for any smooth curve $\gamma\colon I\to\mathsf{X}$ satisfying $0\in\mathrm{int}(I)$ and $\gamma(0)=x$ it holds that $\frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0}(\xi_1\circ\gamma)=\frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0}(\xi_2\circ\gamma)$. This immediately gives $T_x\xi_1(v_x)=T_x\xi_2(v_x)$ for every $v_x\in\mathsf{T}_x\mathsf{X}$. Thus the definition of L_{p_1} is independent of the choice of local section ξ. Linearity of L_{p_1} is now obvious. To see that L_{p_1} is injective note that $\pi\circ\xi=\mathrm{id}_{\mathcal{U}}$ and so $T_{\xi(x)}\pi\circ T_x\xi=\mathrm{id}_{\mathsf{T}_x\mathsf{X}}$. Thus L_{p_1} possesses a left-inverse and so is injective.

(ii) This was proved as part of the proof of the previous assertion.

(iii) Suppose that $u_p\in\mathrm{image}(L_{p_1})\cap\mathsf{V}_p\mathsf{Y}$. Let $u_p\in\mathrm{image}(L_{p_1})$ write $u_p=L_{p_1}(v_x)$ for $v_x\in\mathsf{T}_x\mathsf{X}$. Then let $\gamma\colon I\to\mathsf{X}$ be a smooth curve satisfying $0\in\mathrm{int}(I)$ and $\gamma'(0)=v_x$. If $\xi\colon\mathcal{U}\to\mathsf{Y}$ is a local section satisfying $j_1\xi(x)=p_1$ then this means that $u_p=\frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0}(\xi\circ\gamma)$. Since u_p is vertical we have $T_p\pi(u_p)=0_x$. This in turn means that

$$0_x=\frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0}(\pi\circ\xi\circ\gamma)=\frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0}\gamma=v_x.$$

This means, therefore, that $u_p=0_p$. This gives $\mathrm{image}(L_{p_1})\cap\mathsf{V}_p\mathsf{Y}=\{0_p\}$. This part of the result then follows from a dimension count.

For the last assertion, let $(\mathcal{V},\psi)$ be an adapted chart for Y and let $(\mathcal{U},\phi)$ be the associated chart for X. Denote the coordinates for Y by (x^j,y^a). Suppose that $\phi(x)=\mathbf{0}_n$ and that $\psi(p)=\mathbf{0}_{n+m}$. An arbitrary linear map between $\mathsf{T}_x\mathsf{X}$ and $\mathsf{T}_p\mathsf{Y}$ will have the coordinate representation

$$L=A^j_k\mathrm{d}x^k\otimes\frac{\partial}{\partial x^j}+B^a_j\mathrm{d}x^j\otimes\frac{\partial}{\partial y^a}.$$

The condition that $T_p\pi\circ L=\mathrm{id}_{\mathsf{T}_x\mathsf{X}}$ is readily seen to imply that $A^j_k=\delta^j_k$. Now, if we define a local section ξ on $\mathcal{U}$ with local representative

$$\xi(\boldsymbol{x})=(\boldsymbol{x},\boldsymbol{B}\boldsymbol{x})$$

where $\boldsymbol{B}$ is the $m\times n$ matrix with components B^a_j, $a\in\{1,\dots,m\}$, $j\in\{1,\dots,n\}$, then it is immediate that if we take $p_1=j_1\xi(x)$ we have $L=L_{p_1}$. This gives the existence assertion. For uniqueness suppose that $L_{p_1}=L_{p_2}$ and let ξ_1 and ξ_2 be local sections such that $L_{p_1}=T_x\xi_1$ and $L_{p_2}=T_x\xi_2$. This means that $j_1\xi_1(x)=j_1\xi_2(x)$ and so $p_1=p_2$, as desired. ■

In natural coordinates (x^j,y^a,y^b_k) for $\mathsf{J}^1(\pi)$ we have

$$L_{p_1}=\delta^j_k\mathrm{d}x^k\otimes\frac{\partial}{\partial x^j}+y^a_j(p)\mathrm{d}x^j\otimes\frac{\partial}{\partial y^j}\in\mathsf{T}^*_x\mathsf{X}\otimes\mathsf{T}_p\mathsf{Y},$$

where (x^j,y^a,y^b_k) are the coordinate values for $p_1\in\mathsf{J}^1(\pi)$.

Now we define an endomorphism P^H_S of TY by

$$P^H_S(u_p)=L_{S(p)}\circ T_p\pi(u_p),\qquad u_p\in\mathsf{T}_p\mathsf{Y}.$$

The following assertions are more or less obvious.

2.3 Lemma: *The endomorphism $P^H_S\in\Gamma^\infty(\mathsf{T}^*\mathsf{Y}\otimes\mathsf{TY})$ has the following properties:*

(i) $\ker(P^H_S)=\mathsf{V}(\pi)$*;*

(ii) $\mathsf{TY}=\ker(P^H_S)\oplus\operatorname{image}(P^H_S)$*.*

Proof: (i) It is clear that $\mathsf{V}(\pi)\subset\ker(P^H_S)$. For the opposite inclusion, suppose that $P^H_S(u_p)=0_p$. Since $L_{S(p)}$ is injective this implies that $T_p\pi(u_p)=0_p$ whence u_p is vertical.

(ii) Since $T_p\pi$ is surjective it follows that $\operatorname{image}(P^H_S(p))=\operatorname{image}(L_{S(p)})$ which is complementary to $\mathsf{V}_p(\pi)$ by Lemma 2.2. ■

The endomorphism P^H_S is called the ***horizontal projection*** associated with the connection S and the endomorphism $P^V_S=\mathrm{id}_{\mathsf{TY}}-P^H_S$ is called the ***vertical projection***. It is easy to see that P^V_S is a projection onto $\mathsf{V}(\pi)$ and P^H_S is a projection onto a subbundle, denoted by $\mathsf{H}(\pi)$, that is complimentary to $\mathsf{V}(\pi)$. In adapted coordinates (x^j,y^a) we have

$$\begin{aligned}P^H_S&=\delta^j_k\mathrm{d}x^k\otimes\frac{\partial}{\partial x^j}+S^a_j\mathrm{d}x^j\otimes\frac{\partial}{\partial y^a},\\ P^V_S&=\delta^a_b\mathrm{d}y^b\otimes\frac{\partial}{\partial y^a}-S^a_j\mathrm{d}x^j\otimes\frac{\partial}{\partial y^a}.\end{aligned}$$

Thus in this way we derive a splitting of TY associated to a connection.

One can also recover a connection from a splitting. Thus suppose that $\mathsf{TY}=\mathsf{V}(\pi)\oplus\mathsf{H}(\pi)$ for some subbundle $\mathsf{H}(\pi)$ and denote by P^H the projection onto $\mathsf{H}(\pi)$. For each $p\in\mathsf{Y}$ the restriction of $T_p\pi$ to $\mathsf{H}_p(\pi)$ is an isomorphism onto $\mathsf{T}_x\mathsf{X}$ where $x=\pi(p)$. Thus the inverse of this isomorphism defines a map L from $\mathsf{T}_x\mathsf{X}$ to $\mathsf{T}_p\mathsf{Y}$ satisfying $T_p\pi\circ L=\mathrm{id}_{\mathsf{T}_x\mathsf{X}}$. By Lemma 2.2 it follows that there exists a unique $p_1\in(\pi^1_0)^{-1}(p)$ such that $L=L_{p_1}$. We define a connection by $S(p)=p_1$.

Let us summarise this construction.

2.4 Proposition: *For a fibred manifold* $(\mathsf{Y},\pi,\mathsf{X})$ *let* $\mathsf{H}(\pi)$ *be a subbundle of* TY *complementary to* $\mathsf{V}(\pi)$*. If* $P^H\in\Gamma^\infty(\mathsf{T}^*\mathsf{Y}\otimes\mathsf{TY})$ *is the projection onto* $\mathsf{H}(\pi)$ *along* $\mathsf{V}(\pi)$ *then there exists a unique connection* S *on* π *for which* $P^H=P^H_S$*.*

Quite obviously, if P^H is given in adapted coordinates by

$$P^H=\delta^j_k\mathrm{d}x^k\otimes\frac{\partial}{\partial x^j}+P^a_j\mathrm{d}x^j\otimes\frac{\partial}{\partial y^a},$$

then the connection coefficients of S are exactly P^a_j, $a\in\{1,\dots,m\}$, $j\in\{1,\dots,n\}$.

It is evident that the constructions in this section establish a 1–1 correspondence between connections on π and complements to $\mathsf{V}(\pi)$ in TY. This immediately gives the following result.

2.5 Proposition: *If* $(\mathsf{Y},\pi,\mathsf{X})$ *is a fibred manifold with* Y *paracompact then there exists a connection on* π*.*

Proof: Since Y is paracompact it possesses a Riemannian metric. A subbundle to $\mathsf{V}(\pi)$ can then be taken to be the orthogonal complement. This subbundle then prescribes a unique connection according to Proposition 2.4. ■

2.3. The covariant derivative associated with a connection. In this section we introduce the relationship between a connection and a covariant derivative operator on sections of π. For the following definition we recall the fact that $\pi^1_0\colon\mathsf{J}^1(\pi)\to\mathsf{Y}$ is an affine bundle modelled on $\pi^*\mathsf{T}^*\mathsf{X}\otimes_{\mathsf{Y}}\mathsf{V}(\pi)$, where $\pi^*\mathsf{T}^*\mathsf{X}$ denotes the pull-back of $\mathsf{T}^*\mathsf{X}$ to Y by π.

2.6 Definition: If $S\colon\mathsf{Y}\to\mathsf{J}^1(\pi)$ is a connection on a fibred manifold $(\mathsf{Y},\pi,\mathsf{X})$ and if $\xi\colon\mathcal{U}\to\mathsf{Y}$ is a local section of π, the ***S-covariant differential*** of ξ is the local section $\overset{S}{\nabla}\xi$ of $\mathsf{T}^*\mathsf{X}\otimes\xi^*\mathsf{V}(\pi)$ defined by

$$\overset{S}{\nabla}\xi(x)=j_1\xi(x)-S(\xi(x)).$$ •

If Z is a vector field on X then $\overset{S}{\nabla}_Z\xi=\overset{S}{\nabla}\xi(Z)$ is the ***S-covariant derivative*** of ξ with respect to Z. Note that $\overset{S}{\nabla}_Z\xi$ is a section of $\xi^*\mathsf{V}(\pi)$. The coordinate expression for $\overset{S}{\nabla}\xi$ is

$$\overset{S}{\nabla}\xi=\Big(\frac{\partial\xi^a}{\partial x^j}-S^a_j\Big)\mathrm{d}x^j\otimes\frac{\partial}{\partial y^a}.$$

The following result records the basic properties of the S-covariant derivative. We let $\nu_{\mathsf{Y}}\colon\mathsf{V}(\pi)\to\mathsf{Y}$ denote the canonical projection. We also denote by $\pi_{\mathsf{TX}}\colon\mathsf{TX}\to\mathsf{X}$ the tangent bundle projection.

2.7 Lemma: *For a local section* ξ *of* π*, for vector fields* $Z,Z_1,Z_2\in\Gamma^\infty(\pi_{\mathsf{TX}})$*, and for functions* $f_1,f_2\in C^\infty(\mathsf{X})$*, the following statements hold:*

(i) $\nu_{\mathsf{Y}}\circ\overset{S}{\nabla}_Z\xi(x)=\xi(x)$*;*

(ii) $\overset{S}{\nabla}_{f_1Z_1+f_2Z_2}\xi=f_1\overset{S}{\nabla}_{Z_1}\xi+f_2\overset{S}{\nabla}_{Z_2}\xi$*.*

Proof: (i) Since $j_1\xi(x),S(\xi(x))\in(\pi^1_0)^{-1}(\xi(x))$ it follows that $j_1\xi(x)-S(\xi(x))\in\mathsf{V}_{\xi(x)}(\pi)$.

(ii) This is clear from the definition. ■

One can also recover a connection from a covariant derivative operator. We do this as follows. Given a connection S let us define $\Phi_S\colon \mathsf{J}^1(\pi) \to \mathsf{V}(\pi)$ by

$$\Phi_S(p_1) = p_1 - S(\pi_0^1(p_1)),$$

and note that Φ_S is a fibred morphism over id_{X}. Moreover, if $\xi\colon \mathcal{U} \to \mathsf{Y}$ is a local section of π then $\Phi_S(j_1\xi(x)) = \overset{S}{\nabla}\xi(x)$. Through this bundle map we can easily establish the relationship between a covariant derivative and a connection. To do so let us first establish the essential properties of Φ_S.

2.8 Lemma: *The bundle map Φ_S has the following properties:*

(i) $\Phi_S(p_1) \in \mathsf{V}_{\pi_0^1(p_1)}(\pi)$ for every $p_1 \in \mathsf{J}^1(\pi)$;

(ii) $\Phi_S(p_1) - \Phi_S(p_1') = p_1 - p_1'$ if $\pi_0^1(p_1) = \pi_0^1(p_1')$, where we use the fact that $(\pi_0^1)^{-1}(p)$ is an affine space modelled on $\mathsf{T}^*_{\pi(p)}\mathsf{X} \otimes \mathsf{V}_p(\pi)$ for each $p \in \mathsf{Y}$.

Proof: (i) This is clear since $S(p) \in (\pi_0^1)^{-1}(p)$ for every $p \in \mathsf{Y}$.

(ii) We have

$$\begin{aligned}\Phi_S(p_1) - \Phi_S(p_1') &= (p_1 - S(\pi_0^1(p_1))) - (p_1' - S(\pi_0^1(p_1'))) \\ &= (p_1 - p_1') - (S(\pi_0^1(p_1)) - S(\pi_0^1(p_1'))) = p_1 - p_1',\end{aligned}$$

as desired. ■

2.9 Remark: Condition (ii) amounts to saying that the symbol of Φ_S is the identity map. •

2.10 Proposition: *If $\Phi\colon \mathsf{J}^1(\pi) \to \mathsf{V}(\pi)$ is a bundle map over id_{X} satisfying*

(i) $\Phi(p_1) \in \mathsf{V}_{\pi_0^1(p_1)}(\pi)$ for every $p_1 \in \mathsf{J}^1(\pi)$ and

(ii) $\Phi(p_1) - \Phi(p_1') = p_1 - p_1'$ if $\pi_0^1(p_1) = \pi_0^1(p_1')$,

then there exists a unique connection S for which $\Phi = \Phi_S$.

Proof: Let $p \in \mathsf{Y}$, let $p_1 \in (\pi_0^1)^{-1}(p)$, and define $S(p) = \Phi_S(p_1) - p_1$. We claim that $S(p)$ is independent of the choice of p_1. Indeed, let $p_1, p_1' \in (\pi_0^1)^{-1}(p)$ and note that

$$(\Phi_S(p_1') - p_1') - (\Phi_S(p_1) - p_1) = (\Phi_S(p_1' - p_1)) - (p_1' - p_1) = (p_1' - p_1) - (p_1' - p_1) = 0_p,$$

giving $\Phi_S(p_1') - p_1' = \Phi_S(p_1) - p_1$, as desired. Thus S is a connection, and it is, moreover, clear that $\Phi_S = \Phi$. Uniqueness is clear. ■

The constructions in this section provide, therefore, a 1–1 correspondence between connections on π and covariant differentiation of sections of π.

3. Linear connections on vector bundles

In the case when the bundle is question has the structure of a vector bundle, the general definition of a connection still applies. However, it is interesting to also consider the vector bundle structure. For a vector bundle $\pi\colon \mathsf{E} \to \mathsf{X}$ recall that the jet bundles $\mathsf{J}^k(\pi)$ are also vector bundles over X.

3.1. The definition of a linear connection. It is straightforward to adapt the usual definition of a connection to incorporate the vector bundle structure.

3.1 Definition: A ***linear connection*** on a vector bundle $(\mathsf{E},\pi,\mathsf{X})$ is a connection $S\colon \mathsf{E}\to \mathsf{J}^1(\pi)$ that is also a vector bundle morphism. •

In coordinates (x^j,u^a,u^b_k) for $\mathsf{J}^1(\pi)$ adapted to vector bundle coordinates (x^j,u^a) for E, a linear connection has the form $(x^j,u^a)\mapsto(x^j,u^a,S^b_{jc}u^c)$ which defines the ***connection coefficients*** S^a_{jb}, $a,b\in\{1,\dots,m\}$, $j\in\{1,\dots,n\}$.

Let us now describe the constructions associated with a linear connection.

3.2. The horizontal subbundle associated with a linear connection. As with a connection in the more general sense, given a linear connection S on a vector bundle $(\mathsf{E},\pi,\mathsf{X})$ we have associated horizontal and vertical projections $P^H_S,P^V_S\colon \mathsf{TE}\to\mathsf{TE}$. To understand how the linearity of the connection S manifests itself in the splitting of TE we shall provide some structure for the vertical projection. To do this we shall first understand the structure of the vertical bundle $\mathsf{V}(\pi)$.

We let $\nu_{\mathsf{E}}\colon \mathsf{V}(\pi)\to\mathsf{E}$ be the restriction of the tangent bundle projection to $\mathsf{V}(\pi)$. We denote by $\pi^*\pi\colon\pi^*\mathsf{E}\to\mathsf{E}$ the pull-back bundle. The map $\mathrm{vlft}\colon\pi^*\mathsf{E}\to\mathsf{V}(\pi)$ defined by

$$\mathrm{vlft}(e_x,w_x)=\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0}(e_x+sw_x)$$

is a vector bundle isomorphism such that the diagram

$$\begin{array}{ccc}\pi^*\mathsf{E} & \xrightarrow{\mathrm{vlft}} & \mathsf{V}(\pi)\\ \Big\downarrow{\scriptstyle \mathrm{pr}_1} & & \Big\downarrow{\scriptstyle \nu_{\mathsf{E}}}\\ \mathsf{E} & = & \mathsf{E}\end{array}\tag{3.1}$$

commutes. Let us also denote by $\nu_{\mathsf{X}}\colon\mathsf{V}(\pi)\to\mathsf{X}$ the composition $\nu_{\mathsf{X}}=\pi\circ\nu_{\mathsf{E}}$. We claim that the bundle $\nu_{\mathsf{X}}\colon\mathsf{V}(\pi)\to\mathsf{X}$ has a natural vector bundle structure.

3.2 Lemma: *Let $Z\colon\mathsf{X}\to\mathsf{TX}$ be the zero section. The map*

$$\mathsf{V}(\pi)\ni u_e\mapsto(u_e,\pi(e))\in\mathsf{TE}\times\mathsf{X}$$

is a bijection onto $Z^*\mathsf{TE}$. Moreover, this bijection commutes with the following classes of charts for $\mathsf{V}(\pi)$ and $Z^*\mathsf{TE}$, respectively:

(i) the restriction to $\mathsf{V}(\pi)$ of the natural charts for TE induced by vector bundle charts for E;

(ii) the vector bundle charts for $Z^*\mathsf{TE}$ induced by natural charts for TX and vector bundle charts for E.

Proof: Note that

$$Z^*\mathsf{TE}=\{(u,x)\in\mathsf{TE}\times\mathsf{X}\mid\ T\pi_{\mathsf{TE}}(u)=Z(x)\}=\{(u,x)\in\mathsf{TE}\times\mathsf{M}\mid\ T\pi_{\mathsf{TE}}(u)=0_{\pi(e)}\}.$$

This shows that the points $(u,x)\in Z^*\mathsf{TE}$ are in 1–1 correspondence with the point u which must lie in $\mathsf{V}(\pi)$.

For the second assertion of the lemma, let us consider coordinates (x^j) and $(\tilde{x}^j)$ for X, natural coordinates (x^j,v^k) and $(\tilde{x}^j,\tilde{v}^k)$ for TX, and vector bundle coordinates (x^j,u^a)

and $(\tilde{x}^j, \tilde{u}^a)$ for E. We then have transition maps for the vector bundle structure of E that define the relation $\tilde{u}^a = A^a_b(x)u^b$. In natural and vector bundle coordinates an element of $\mathsf{V}(\pi)$ is represented as $(x^j, u^a, 0, w^b)$ and $(\tilde{x}^j, \tilde{u}^a, 0, \tilde{w}^b)$. The transition functions for the vector bundle structure give $\tilde{w}^a = A^a_b(x)w^b$. Now, an element of $Z^*\mathsf{TE}$ in natural and vector bundle coordinates has the form $((x^j, u^a, 0, w^b), x^j)$ and $((\tilde{x}^j, \tilde{u}^a, 0, \tilde{w}^b), \tilde{x}^j)$. The transition functions for the vector bundle structure give $\tilde{w}^a = A^a_b(x)w^b$. Since the bijection from $\mathsf{V}(\pi)$ to $Z^*\mathsf{TE}$ has the form $(x^j, u^a, 0, w^b) \mapsto ((x^j, u^a, 0, w^b), x^j)$ it follows that the bijection commutes with the various natural charts. ■

To better understand the vector bundle structure of $\nu_{\mathsf{X}} \colon \mathsf{V}(\pi) \to \mathsf{X}$ we note that, as manifolds, $\pi^*\mathsf{E} = \mathsf{E} \oplus \mathsf{E}$ where $\mathsf{E} \oplus \mathsf{E}$ denotes the Whitney sum of vector bundles over X. Therefore, we have the diffeomorphism $\mathrm{vlft} \colon \mathsf{E} \oplus \mathsf{E} \to \mathsf{V}(\pi)$. Using vector bundle coordinates and Lemma 3.2 one can easily see that vlft is, in fact, an isomorphism of vector bundles over X. That is to say, the diagram

$$\begin{CD} \mathsf{E} \oplus \mathsf{E} @>{\mathrm{vlft}}>> \mathsf{V}(\pi) \\ @V{\pi \oplus \pi}VV @VV{\nu_{\mathsf{X}}}V \\ \mathsf{X} @= \mathsf{X} \end{CD} \tag{3.2}$$

commutes, where $\pi \oplus \pi \colon \mathsf{E} \oplus \mathsf{E} \to \mathsf{X}$ is the vector bundle projection. If we further consider the rôle of Lemma 3.2 in the preceding diagram we have the diagram

$$\begin{CD} \mathsf{E} \oplus \mathsf{E} @>{\mathrm{vlft}}>> \mathsf{V}(\pi) \\ @V{\pi \oplus \pi}VV @VV{T\pi|\mathsf{V}(\pi)}V \\ \mathsf{X} @>>{Z}> \mathsf{TX} \end{CD}$$

which is a vector bundle morphism.

Now that we better understand the structure of $\mathsf{V}(\pi)$ we can use this structure to see how a linear connection gives rise to a more structured vertical projection. We first make the following observation about the vertical projection P^V_S.

3.3 Lemma: *If $S \colon \mathsf{E} \to \mathsf{J}^1(\pi)$ is a (not necessarily linear) connection on the vector bundle $(\mathsf{E}, \pi, \mathsf{X})$ then the vertical projection $P^V_S \colon \mathsf{TE} \to \mathsf{TE}$ is a vector bundle morphism with respect to the diagram*

$$\begin{CD} \mathsf{TE} @>{P^V_S}>> \mathsf{V}(\pi) \\ @V{\pi_{\mathsf{TE}}}VV @VV{\nu_{\mathsf{E}}}V \\ \mathsf{E} @= \mathsf{E} \end{CD}$$

If additionally S is linear then P^V_S is also a vector bundle morphism with respect to the diagram

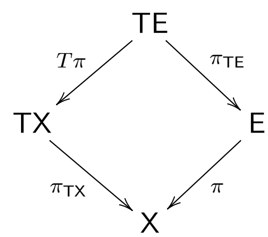
$$\begin{CD} \mathsf{TE} @>{P^V_S}>> \mathsf{V}(\pi) \\ @V{T\pi}VV @VV{\nu_{\mathsf{X}}}V \\ \mathsf{TX} @>>{\pi_{\mathsf{TX}}}> \mathsf{X} \end{CD}$$

Proof: As P_S^V is a $(1,1)$-tensor field on E the first diagram commutes and the top arrow is a morphism of vector bundles. So it is only the bottom diagram that is of concern. For commutativity of the bottom diagram we let $u \in \mathsf{TE}$ be such that $u \in \mathsf{T}_e\mathsf{E}$ and $T\pi(u) = v \in \mathsf{T}_x\mathsf{X}$. Then $\nu_{\mathsf{X}} \circ P_S^V(u) = x$. We also have

$$\pi_{\mathsf{TX}} \circ T\pi(u) = \pi_{\mathsf{TX}}(v_x) = x,$$

which gives commutativity of the bottom diagram. Thus it remains to show that the top arrow in the bottom diagram is a vector bundle morphism.

Let $v_x \in \mathsf{T}_x\mathsf{X}$ for $x \in \mathsf{X}$. We must show that $P_S^V|T\pi^{-1}(v_x)$ is a linear map from $T\pi^{-1}(v_x)$ to $\nu_{\mathsf{X}}^{-1}(x)$. Let $u_1, u_2 \in T\pi^{-1}(v_x)$ and let $e_1, e_2 \in \mathsf{E}$ be such that $u_1 \in \mathsf{T}_{e_1}\mathsf{E}$ and $u_2 \in \mathsf{T}_{e_2}\mathsf{E}$. Let us agree that $u_1 + u_2$ refers to addition in the vector space $T\pi^{-1}(v_x)$. Since the diagram



commutes we have $\pi(e_1) = \pi(e_2)$ and so we may add e_1 and e_2 using the vector bundle structure of E. Using linearity of S we then compute

$$\begin{aligned} P_S^V(u_1 + u_2) &= u_1 + u_2 - L_{S(e_1+e_2)} \circ T\pi(u_1 + u_2) \\ &= u_1 + u_2 - L_{S(e_1)}(v_x) - L_{S(e_2)}(v_x) \\ &= P_S^V(u_1) + P_S^V(u_2), \end{aligned}$$

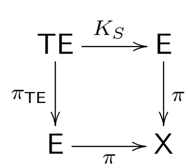
as desired. ■

Using this property of the vertical projection for a connection we make the following definition.

3.4 Definition: For a vector bundle $(\mathsf{E}, \pi, \mathsf{X})$ and a (not necessarily linear) connection $S\colon \mathsf{E} \to \mathsf{J}^1(\pi)$, the ***connector*** for S is the map $K_S\colon \mathsf{TE} \to \mathsf{E}$ defined by $K_S = \mathrm{pr}_2 \circ \mathrm{vlft}^{-1} \circ P_S^V$. •

Note that one can define the connector for any connection on a vector bundle E. But the following lemma gives the structure of the connector that arises from the linearity of S.

3.5 Proposition: *For a (not necessarily linear) connection S the connector K_S is a vector bundle morphism with respect to the diagram*



If S is additionally linear then K_S is a vector bundle morphism with respect to the diagram

$$\begin{array}{ccc} \mathsf{TE} & \xrightarrow{K_S} & \mathsf{E} \\ {\scriptstyle T\pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathsf{TX} & \xrightarrow[\pi_{\mathsf{TX}}]{} & \mathsf{X} \end{array}$$

Proof: This follows from combining Lemmata 3.2 and 3.3 along with the joint properties of vlft described in diagrams (3.1) and (3.2). ■

3.3. The covariant derivative associated with a linear connection. Now let us see how the linearity of a connection alters the picture of covariant differentiation. For a general connection $S\colon \mathsf{E} \to \mathsf{J}^1(\pi)$ on a vector bundle $(\mathsf{E}, \pi, \mathsf{X})$ and a section ξ of π we have $\overset{S}{\nabla}\xi$ as a section of $\mathsf{T}^*\mathsf{X} \otimes \xi^*\mathsf{V}(\pi)$. This induces, even when the connection is not linear, an interpretation of $\overset{S}{\nabla}\xi$ as a section of $\mathsf{T}^*\mathsf{X} \otimes \mathsf{E}$ using the fact that $\mathsf{V}_{\xi(x)}(\pi) \simeq \mathsf{E}_x$. However, when the connection is additionally linear there are further properties of the covariant differential.

3.6 Proposition: *If S is a linear connection on a vector bundle $(\mathsf{E}, \pi, \mathsf{X})$ then the covariant derivative has the following properties for vector fields $Z, Z_1, Z_2 \in \Gamma^\infty(\pi_{\mathsf{TX}})$, functions $f, f_1, f_2 \in C^\infty(\mathsf{X})$, and sections $\xi, \xi_1, \xi_2 \in \Gamma^\infty(\pi)$:*

(i) $\overset{S}{\nabla}_Z(\xi_1 + \xi_2) = \overset{S}{\nabla}_Z\xi_1 + \overset{S}{\nabla}_Z\xi_2$;

(ii) $\overset{S}{\nabla}_Z(f\xi) = f\overset{S}{\nabla}_Z\xi + (\mathscr{L}_Z f)\xi$;

(iii) $\overset{S}{\nabla}_{f_1Z_1+f_2Z_2}\xi = f_1\overset{S}{\nabla}_{Z_1}\xi + f_2\overset{S}{\nabla}_{Z_2}\xi$.

Proof: (i) We have

$$\begin{aligned}\overset{S}{\nabla}(\xi_1 + \xi_2)(x) &= j_1(\xi_1 + \xi_2)(x) - S(\xi_1 + \xi_2)(x) \\ &= j_1\xi_1(x) - S(\xi_1(x)) + j_1\xi_2(x) - S(\xi_2(x)) \\ &= \overset{S}{\nabla}\xi_1(x) + \overset{S}{\nabla}\xi_2(x).\end{aligned}$$

(ii) We have

$$\begin{aligned}\overset{S}{\nabla}(f\xi)(x) &= j_1(f\xi)(x) - S(f(x)\xi(x)) \\ &= f(x)j_1\xi(x) + \boldsymbol{d}f(x) \otimes \xi(x) - f(x)S(\xi(x)) \\ &= f(x)\overset{S}{\nabla}\xi(s) + \boldsymbol{d}f(x) \otimes \xi(x),\end{aligned}$$

and the result follows directly from this.

(iii) This was already shown in Lemma 2.7. ■

4. Spaces of connections

Now that we understand what a connection is we can talk about the set of all such things. It turns out that the jet bundle characterisation of a connection is extremely convenient for doing this. As we shall see the set of connections has the structure of the set of sections of an affine bundle. First we consider the case of the set of general connections.

4.1. The space of connections on a fibred manifold. The structure of the set of connections on a general fibred manifold follows directly from the definition of a connection as a section of π_0^1. Indeed, we immediately have the following result.

4.1 Proposition: *If* $(\mathsf{Y}, \pi, \mathsf{X})$ *is a fibred manifold then the set of connections on* π *is the set of sections of the affine bundle* $\pi_0^1 \colon \mathsf{Y} \to \mathsf{J}^1(\pi)$ *which is modelled on the vector bundle* $\pi^*\mathsf{T}^*\mathsf{X} \otimes \mathsf{V}(\pi)$*.*

Not much else to say here, really.

4.2. The space of linear connections on a vector bundle. For a vector bundle $(\mathsf{E}, \pi, \mathsf{X})$ a linear connection is, by our definition, an element of $\Gamma^\infty(\pi_0^1) \cap \Gamma^\infty(\mathsf{E}^* \otimes \mathsf{J}^1(\pi))$. Thus a linear connection has "access" to the affine structure of $\pi_0^1 \colon \mathsf{J}^1(\pi) \to \mathsf{E}$ and the vector bundle structure of $\mathsf{E}^* \otimes \mathsf{J}^1(\pi)$. After distillation this structure is most simply represented as follows.

4.2 Proposition: *If* $(\mathsf{E}, \pi, \mathsf{X})$ *is a vector bundle then the set of linear connections on* π *is the set of sections of an affine subbundle of the vector bundle* $\mathsf{E}^* \otimes \mathsf{J}^1(\pi)$ *(over* X*) modelled on the vector bundle* $\mathsf{E}^* \otimes \mathsf{T}^*\mathsf{X} \otimes \mathsf{E}$*.*

Proof: Note that a section S of $\mathsf{E}_x^* \otimes \mathsf{J}^1(\pi)_x$ is a linear connection on π if and only if

$$\mathrm{id}_{\mathsf{E}_x^*} \otimes \pi_0^1(S(x)) = \mathrm{id}_{\mathsf{E}_x} \in \mathsf{E}_x^* \otimes \mathsf{E}_x.$$

This immediately gives the set of linear connections as sections of an affine subbundle as it is the set of solutions to a fibrewise linear equation. Thus we need only show that the model vector bundle is $\mathsf{E}^* \otimes \mathsf{T}^*\mathsf{X} \otimes \mathsf{E}$. Let $S_1, S_2 \in \Gamma^\infty(\mathsf{E}^* \otimes \mathsf{J}^1(\pi))$ be linear connections and note that

$$\mathrm{id}_{\mathsf{E}_x^*} \otimes \pi_0^1(S_1(x) - S_2(x)) = 0_{\mathsf{E}_x^* \otimes \mathsf{E}_x}.$$

Thus $S_1 - S_2$ is a section of $\ker(\mathrm{id}_{\mathsf{E}^*} \otimes \pi_0^1)$; in other words a section of $\mathsf{E}^* \otimes \mathsf{T}^*\mathsf{X} \otimes \mathsf{E}$. This shows that the model vector bundle for the set of linear connections is contained in $\mathsf{E}^* \otimes \mathsf{T}^*\mathsf{X} \otimes \mathsf{E}$. To give the converse inclusion, let $S \in \Gamma^\infty(\mathsf{E}^* \otimes \mathsf{J}^1(\pi))$ be a linear connection and let $A \in \Gamma^\infty(\mathsf{E}^* \otimes \mathsf{T}^*\mathsf{X} \otimes \mathsf{E})$. Then

$$\mathrm{id}_{\mathsf{E}_n^*} \otimes \pi_0^1(S(x) + A(x)) = \mathrm{id}_{\mathsf{E}_n^*} \otimes \pi_0^1(S(x)) = \mathrm{id}_{\mathsf{E}_x^* \otimes \mathsf{E}_x},$$

and so $S + A$ is a linear connection. This shows that the model vector bundle for the set of linear connections contains $\mathsf{E}^* \otimes \mathsf{T}^*\mathsf{X} \otimes E^*$. ■

4.3. The space of affine connections on a manifold. In the particular case of a linear connection on the vector bundle $(\mathsf{TX}, \pi_{\mathsf{TX}}, \mathsf{X})$ (i.e., what is often called an affine connection on X) we have the following result which follows immediately from Proposition 4.2.

4.3 Proposition: *The set of affine connections on a manifold* X *is the set of sections of an affine subbundle of the vector bundle* $\mathsf{T}^*\mathsf{X} \otimes \mathsf{J}^1(\pi_{\mathsf{TX}})$ *(over* X*) modelled on the vector bundle* $\mathsf{T}^*\mathsf{X} \otimes \mathsf{T}^*\mathsf{X} \otimes \mathsf{TX}$*.*

For colour we also have the following result.

4.4 Proposition: *The set of torsion-free affine connections on a manifold* X *is the set of sections of an affine subbundle of the vector bundle* $\mathsf{T}^*\mathsf{X} \otimes \mathsf{J}^1(\pi_{\mathsf{TX}})$ *(over* X*) modelled on the vector bundle* $\mathrm{S}^2(\mathsf{T}^*\mathsf{X}) \otimes \mathsf{TX}$.

Proof: Let S_1 and S_2 be two torsion-free affine connections and let

$$A = S_1 - S_2 \in \Gamma^\infty(\mathsf{T}^*\mathsf{X} \otimes \mathsf{T}^*\mathsf{X} \otimes \mathsf{TX}).$$

Since S_1 and S_2 are torsion free we have

$$\overset{S_1}{\nabla}_Z W - \overset{S_1}{\nabla}_W Z = [Z, W], \quad \overset{S_2}{\nabla}_Z W - \overset{S_2}{\nabla}_W Z = [Z, W]$$

for vector fields Z and W on X. Thus

$$\overset{S_1}{\nabla}_Z W - \overset{S_1}{\nabla}_W Z = \overset{S_2}{\nabla}_Z W - \overset{S_2}{\nabla}_W Z$$

for all vector fields Z and W on X. Using the definition of the relationship between the covariant derivative and the connection this implies that

$$S_1(Z, W) - S_1(Z, W) = S_2(Z, W) - S_2(W, Z) \quad \Longrightarrow \quad (S_1 - S_2)(Z, W) - (S_1 - S_2)(W, Z) = 0$$

and so A is a section of $\mathrm{S}^2(\mathsf{T}^*\mathsf{X}) \otimes \mathsf{TX}$. Reversing the computation shows that if S is a torsion-free affine connection and if A is a section of $\mathrm{S}^2(\mathsf{T}^*\mathsf{X}) \otimes \mathsf{TX}$ then $S + A$ is a torsion-free affine connection. ■

References

Kolář, I., Michor, P. W., and Slovák, J. [1993] *Natural Operations in Differential Geometry*, Springer-Verlag: New York/Heidelberg/Berlin, ISBN: 978-3-540-56235-1.