Should we fly in the Lebesgue-designed airplane?—The correct defence of the Lebesgue integral

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Abstract

It is well-known that the Lebesgue integral generalises the Riemann integral. However, as is also well-known but less frequently well-explained, this generalisation alone is not the reason why the Lebesgue integral is important and needs to be a part of the arsenal of any mathematician, pure or applied. Those who understand the correct reasons for the importance of the Lebesgue integral realise there are at least two crucial differences between the Riemann and Lebesgue theories. One is the difference between the Dominated Convergence Theorem in the two theories, and another is the completeness of the normed vector spaces of integrable functions. Here topological interpretations are provided for the differences in the Dominated Convergence Theorems, and explicit counterexamples are given which illustrate the deficiencies of the Riemann integral. Also illustrated are the deleterious consequences of the defects in the Riemann integral on Fourier transform theory if one restricts to Riemann integrable functions.

Keywords. Lebesgue integral, Riemann integral.

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1. Introduction

The title of this paper is a reference to the well-known quote of the applied mathematician and engineer Richard W. Hamming (1915–1998):

> Does anyone believe that the difference between the Lebesgue and Riemann integrals can have physical significance, and that whether say, an airplane would or would not fly could depend on this difference? If such were claimed, I should not care to fly in that plane.

The statement by Hamming is open to many interpretations, but the interpretation of Hamming himself can be gleaned from [Hamming 1980] and, particularly, [Hamming 1998]; also see [Davis 1998] for a discussion of some of Hamming's views on mathematics and the "real world." Perhaps a fair summary of Hamming's views toward the Riemann and Lebesgue theories of integration is that the distinction between them is not apt to be seen in Nature. This seems about right to us. Unfortunately, however, this quote of Hamming's is often used in a confused manner that indicates the quoter's misunderstanding of the purpose and importance of the Lebesgue integral. Indeed, very often Hamming's quote is brought

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out as an excuse to disregard Lebesgue integration, the idea being that it is the product of some vapid pursuit of generality. Oxymoronically, this is often done simultaneously with the free use of results (like completeness of the L^p-spaces) which rely crucially on the distinctions between the Riemann and Lebesgue integrals.

That the value of the Lebesgue theory of integration (and all of the theories equivalent to or generalising it[1]) may not be appreciated fully by non-mathematicians should not be too surprising: the Lebesgue theory is subtle. Moreover, it is definitely not the case that the importance of the Lebesgue theory over the Riemann theory is explained clearly in all texts on integration theory; in fact, the important distinctions are rarely explicitly stated, though they are almost always implicitly present. What is most discomforting, however, is that mathematicians themselves sometimes offer an *incorrect* defence of the Lebesgue theory. For example, it is not uncommon to see defences made along the lines of, "The class of functions that can be integrated using the Lebesgue theory is larger than that using the Riemann theory" [Davis and Insall 2002]. Sometimes, playing into the existing suspicions towards unnecessary generality, it is asserted that the mere fact that the Lebesgue theory generalises the Riemann theory is sufficient to explain its superiority. These sorts of defences of the Lebesgue theory are certainly factual. But they are also emphatically *not* the sorts of reasons why the Lebesgue theory is important. The functions that can be integrated using the Lebesgue theory, but which cannot be integrated using the Riemann theory, are not important; just try showing a Lebesgue integrable but not Riemann integrable function to someone who is interested in applications of mathematics, and see if they think the function is important. This certainly must at least partially underlie Hamming's motivation for his quote.

The value of the Lebesgue theory over the Riemann theory is that it is superior, as a *theory of integration*. By this it is meant that there are theorems in the Lebesgue theory that are true and useful, but that are not true in the Riemann theory. Probably the most crucial such theorem is the powerful version of the Dominated Convergence Theorem that one has in the Lebesgue theory. This theorem is constantly used in the proof of many results that are important in applications. For example, the Dominated Convergence Theorem is used crucially in the proof of the completeness of L^p-spaces. In turn, the completeness of these spaces is an essential part of why these spaces are useful in, for example, the theory of signals and systems that is taught to all Electrical Engineering undergraduates. For instance, in many texts on signals and systems one can find the statement that the Fourier transform is an isomorphism of $\mathsf{L}^2(\mathbb{R};\mathbb{C})$. This statement is one that needs a sort of justification that is (understandably) not often present in such texts. But its presence can at least be justified by its being correct. With only the Riemann theory of integration at one's disposal, the statement is simply not correct. We illustrate this in Section 3.3.

In this paper we provide topological explanations for the differences between the Riemann and Lebesgue theories of integration. The intent is not to make these differences clear for non-mathematicians. Indeed, for non-mathematicians the contents of the preceding paragraphs, along with the statement that, "The Lebesgue theory of integration is to the Riemann theory of integration as the real numbers are to the rational numbers," (this is the content of our Example 3.7) seems about the best one can do. No, what we aim to do

[1] Hamming [1998] himself seems to find the Henstock integral, which generalises the Lebesgue integral, to be a more satisfactory construction than the Lebesgue integral.

in this paper is clarify *for mathematicians* the reasons for the superiority of the Lebesgue theory. We do this by providing two theorems, both topological in nature, that are valid for the Lebesgue theory and providing counterexamples illustrating that they are not true for the Riemann theory. We also illustrate the consequences of the topological deficiencies of the Riemann integral by explicitly depicting the limitations of the L^2-Fourier transform with the Riemann integral. One of the contributions in this paper is that we give the *correct* counterexamples. All too often one sees counterexamples that illustrate *some* point, but not always the one that one wishes to make. The core of what we say here exists in the literature in one form or another, although we have not seen in the literature counterexamples that illustrate what we illustrate in Examples 3.5 and 3.7. The principal objective here is to organise the results and examples in an explicit and compelling way.

In the event that the reader is consulting this paper in a panic just prior to boarding an airplane, let us answer the question posed in the title of the paper. The answer is, "The question is meaningless as the distinctions between the Riemann and Lebesgue integrals do not, and should not be thought to, contribute to such worldly matters as aircraft design." However, the salient point is that this is not a valid criticism of the Lebesgue integral. What follows is, we hope, a valid defence of the Lebesgue integral.

2. Spaces of functions

To keep things simple and to highlight the important parts of our presentation, in this section and in most of the rest of the paper we consider $\mathbb{R}$-valued functions defined on $I = [0,1] \subseteq \mathbb{R}$. Extensions to more general settings are performed mostly by simple changes of notation. The Lebesgue measure on $[0,1]$ is denoted by λ. In order to distinguish the Riemann and Lebesgue integrals we denote them by

$$\int_0^1 f(x)\,\mathrm{d}x, \qquad \int_I f\,\mathrm{d}\lambda,$$

respectively. In order to make our statements as strong as possible, by the Riemann integral we mean the improper Riemann integral to allow for unbounded functions [Marsden and Hoffman 1993, Section 8.5].

2.1. Normed vector spaces of integrable functions. We use slightly unconventional notation that is internally self-consistent and convenient for our purposes here.

Let us first provide the large vector spaces whose subspaces are of interest. By $\mathbb{R}^{[0,1]}$ we denote the set of all $\mathbb{R}$-valued functions on $[0,1]$. This is also the product of $\operatorname{card}([0,1])$ copies of $\mathbb{R}$, and we shall alternatively think of elements of $\mathbb{R}^{[0,1]}$ as being functions or elements of a product of sets, as is convenient for our purposes. We consider the standard $\mathbb{R}$-vector space structure on $\mathbb{R}^{[0,1]}$:

$$(f+g)(x) = f(x) + g(x), \quad (af)(x) = a(f(x)), \qquad f, g \in \mathbb{R}^{[0,1]},\ a \in \mathbb{R}.$$

In $\mathbb{R}^{[0,1]}$ consider the subspace

$$Z([0,1];\mathbb{R}) = \{f \in \mathbb{R}^{[0,1]} \mid \lambda(\{x \in [0,1] \mid f(x) \neq 0\}) = 0\}.$$

Then denote $\hat{\mathbb{R}}^{[0,1]} = \mathbb{R}^{[0,1]}/Z([0,1];\mathbb{R})$; this is then the set of equivalence classes of functions agreeing almost everywhere. We shall denote by $[f] = f + Z([0,1];\mathbb{R})$ the equivalence class of $f \in \mathbb{R}^{[0,1]}$.

Now let us consider subspaces of $\mathbb{R}^{[0,1]}$ and $\hat{\mathbb{R}}^{[0,1]}$ consisting of integrable functions. Let us denote by

$$\mathsf{R}^1([0,1];\mathbb{R}) = \{f\colon [0,1] \to \mathbb{R} \mid \ f \text{ is Riemann integrable}\}.$$

On $\mathsf{R}^1([0,1];\mathbb{R})$ define a seminorm $\|\cdot\|_1$ by

$$\|f\|_1 = \int_0^1 |f(x)| \, \mathrm{d}x,$$

and denote

$$\mathsf{R}_0([0,1];\mathbb{R}) = \{f \in \mathsf{R}^1([0,1];\mathbb{R}) \mid \ \|f\|_1 = 0\}.$$

Then we define

$$\hat{\mathsf{R}}^1([0,1];\mathbb{R}) = \mathsf{R}^1([0,1];\mathbb{R})/\mathsf{R}_0([0,1];\mathbb{R}),$$

and note that this $\mathbb{R}$-vector space is then equipped with the norm $\|[f]\|_1 = \|f\|_1$, accepting the slight abuse of notation of using $\|\cdot\|_1$ is different contexts.

The preceding constructions can be carried out, replacing "R" with "L" and replacing the Riemann integral with the Lebesgue integral, to arrive at the seminormed vector space

$$\mathsf{L}^1([0,1];\mathbb{R}) = \{f\colon [0,1] \to \mathbb{R} \mid \ f \text{ is Lebesgue integrable}\}$$

with the seminorm

$$\|f\|_1 = \int_I |f| \, \mathrm{d}\lambda,$$

the subspace

$$\mathsf{L}_0([0,1];\mathbb{R}) = \{f \in \mathsf{L}^1([0,1];\mathbb{R}) \mid \ \|f\|_1 = 0\},$$

and the normed vector space

$$\hat{\mathsf{L}}^1([0,1];\mathbb{R}) = \mathsf{L}^1([0,1];\mathbb{R})/\mathsf{L}_0([0,1];\mathbb{R}).$$

We denote the norm on $\hat{\mathsf{L}}^1([0,1];\mathbb{R})$ by $\|\cdot\|_1$, this not being too serious an abuse of notation since $\hat{\mathsf{R}}^1([0,1];\mathbb{R})$ is a subspace of $\hat{\mathsf{L}}^1([0,1];\mathbb{R})$ with the restriction of the norm on $\hat{\mathsf{L}}^1([0,1];\mathbb{R})$ to $\hat{\mathsf{R}}^1([0,1];\mathbb{R})$ agreeing with the norm on $\hat{\mathsf{R}}^1([0,1];\mathbb{R})$. This is a consequence of the well-known fact that the Lebesgue integral generalises the Riemann integral [Cohn 2013, Theorem 2.5.4].

During the course of the development of the Lebesgue theory of integration one shows that

$$\mathsf{L}_0([0,1];\mathbb{R}) = Z([0,1];\mathbb{R})$$

[e.g., Cohn 2013, Corollary 2.3.11]. The corresponding assertion is not true for the Riemann theory.

2.1 Example: Let us denote by $F \in \mathbb{R}^{[0,1]}$ the characteristic function of $\mathbb{Q} \cap [0,1]$. This is perhaps the simplest and most commonly seen example of a function that is Lebesgue integrable but not Riemann integrable [Cohn 2013, Example 2.5.4]. Thus $F \notin \mathsf{R}_0([0,1];\mathbb{R})$. However, $F \in Z([0,1];\mathbb{R})$. •

While the preceding example is often used as an example of a function that is not Riemann integrable but is Lebesgue integrable, one needs to be careful about exaggerating the importance, even mathematically, of this example. In Examples 3.5 and 3.7 below we shall see that this example is not sufficient for demonstrating some of the more important deficiencies of the Riemann integral.

2.2. Pointwise convergence topologies. For $x \in [0,1]$ let us denote by $p_x \colon \mathbb{R}^{[0,1]} \to \mathbb{R}_{\geq 0}$ the seminorm defined by $p_x(f) = |f(x)|$. The family $(p_x)_{x\in[0,1]}$ of seminorms on $\mathbb{R}^{[0,1]}$ defines a locally convex topology. A basis of open sets for this topology is given by products of the form $\prod_{x\in[0,1]} U_x$ where $U_x \subseteq \mathbb{R}$ is open and where $U_x = \mathbb{R}$ for all but finitely many $x \in [0,1]$. A sequence $(f_j)_{j\in\mathbb{Z}_{>0}}$ in $\mathbb{R}^{[0,1]}$ converges to $f \in \mathbb{R}^{[0,1]}$ if and only if the sequence converges pointwise in the usual sense [Willard 1970, Theorem 42.2]. Let us, therefore, call this the ***topology of pointwise convergence*** and let us denote by $\mathsf{C}_p([0,1];\mathbb{R})$ the vector space $\mathbb{R}^{[0,1]}$ when equipped with this topology. For clarity, we shall prefix with "C_p" topological properties in the topology of pointwise convergence. Thus, for example, we shall say "C_p-open" to denote an open set in $\mathsf{C}_p([0,1];\mathbb{R})$.

We will be interested in bounded subsets of $\mathsf{C}_p([0,1];\mathbb{R})$. We shall use the characterisation of a bounded subset B of a topological $\mathbb{R}$-vector space V that a set is bounded if and only if, for every sequence $(v_j)_{j\in\mathbb{Z}_{>0}}$ in B and for every sequence $(a_j)_{j\in\mathbb{Z}_{>0}}$ in $\mathbb{R}$ converging to 0, it holds that the sequence $(a_j v_j)_{j\in\mathbb{Z}_{>0}}$ converges to zero in the topology of V [Rudin 1991, Theorem 1.30].

2.2 Proposition: *A subset $B \subseteq \mathsf{C}_p([0,1];\mathbb{R})$ is C_p-bounded if and only if there exists a nonnegative-valued $g \in \mathbb{R}^{[0,1]}$ such that*

$$B \subseteq \{f \in \mathsf{C}_p([0,1];\mathbb{R}) \mid |f(x)| \leq g(x) \text{ for every } x \in [0,1]\}.$$

Proof: Suppose that there exists a nonnegative-valued $g \in \mathbb{R}^{[0,1]}$ such that $|f(x)| \leq g(x)$ for every $x \in [0,1]$ if $f \in B$. Let $(f_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence in B and let $(a_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence in $\mathbb{R}$ converging to 0. If $x \in [0,1]$ then

$$\lim_{j\to\infty} |a_j f_j(x)| \leq \lim_{j\to\infty} |a_j| g(x) = 0,$$

which gives C_p-convergence of the sequence $(a_j f_j)_{j\in\mathbb{Z}_{>0}}$ to zero.

Next suppose that there exists no nonnegative-valued function $g \in \mathbb{R}^{[0,1]}$ such that $|f(x)| \leq g(x)$ for every $x \in [0,1]$ if $f \in B$. This means that there exists $x_0 \in [0,1]$ such that, for every $M \in \mathbb{R}_{>0}$, there exists $f \in B$ such that $|f(x_0)| > M$. Let $(a_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence in $\mathbb{R}$ converging to 0 and such that $a_j \neq 0$ for every $j \in \mathbb{Z}_{>0}$. Then let $(f_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence in B such that $|f_j(x_0)| > |a_j^{-1}|$ for every $j \in \mathbb{Z}_{>0}$. Then $|a_j f_j(x_0)| > 1$ for every $j \in \mathbb{Z}_{>0}$, implying that the sequence $(a_j f_j)_{j\in\mathbb{Z}_{>0}}$ cannot C_p-converge to zero. Thus B is not C_p-bounded. ■

Of course, in the theory of integration one is interested, not in pointwise convergence of arbitrary functions, but in pointwise convergence of measurable functions. Let us, therefore, denote

$$\mathsf{M}([0,1];\mathbb{R}) = \{f \in \mathbb{R}^{[0,1]} \mid f \text{ is Lebesgue measurable}\},$$

where we understand the topology on $\mathsf{M}([0,1];\mathbb{R})$ to be the subspace topology inherited from $\mathsf{C}_p([0,1];\mathbb{R})$. Standard theorems on measurable functions show that $\mathsf{M}([0,1];\mathbb{R})$ is a subspace [Cohn 2013, Proposition 2.1.6] and is C_p-sequentially closed [Cohn 2013, Proposition 2.1.5]. Moreover, the stronger assertion of closedness does not hold. The following result shows this, as well as giving topological properties of $Z([0,1];\mathbb{R})$.

2.3 Proposition: *The subspaces* $\mathsf{M}([0,1];\mathbb{R})$ *and* $Z([0,1];\mathbb{R})$ *of* $\mathsf{C}_p([0,1];\mathbb{R})$ *are not* C_p*-closed, but are* C_p*-sequentially closed.*

Proof: The C_p-sequential closedness of $\mathsf{M}([0,1];\mathbb{R})$ and $Z([0,1];\mathbb{R})$ follow from standard theorems, as pointed out above. We first show that $\mathsf{C}_p([0,1];\mathbb{R}) \setminus \mathsf{M}([0,1];\mathbb{R})$ is not C_p-open. Let $f \in \mathsf{C}_p([0,1];\mathbb{R}) \setminus \mathsf{M}([0,1];\mathbb{R})$ and let V be a C_p-open set containing f. Then V contains a basic neighbourhood. Thus there exists $\epsilon \in \mathbb{R}_{>0}$, $x_1, \dots, x_k \in [0,1]$, and a basic neighbourhood $U = \prod_{x\in[0,1]} U_x$ contained in V where

1. $U_{x_j} = (f(x_j) - \epsilon, f(x_j) + \epsilon)$ for $j \in \{1, \dots, k\}$ and
2. $U_x = \mathbb{R}$ for $x \in [0,1] \setminus \{x_1, \dots, x_k\}$.

Then the function $g \in \mathbb{R}^{[0,1]}$ defined by

$$g(x) = \begin{cases} f(x), & x \in \{x_1, \dots, x_k\}, \\ 0, & \text{otherwise} \end{cases}$$

is in $U \cap \mathsf{M}([0,1];\mathbb{R})$. Thus $g \in V$, so showing that every neighbourhood of f contains a member of $\mathsf{M}([0,1];\mathbb{R})$.

To show that $Z([0,1];\mathbb{R})$ is not C_p-closed we shall show that $\mathsf{C}_p([0,1];\mathbb{R}) \setminus Z([0,1];\mathbb{R})$ is not C_p-open. Let $f \in \mathsf{C}_p([0,1];\mathbb{R}) \setminus Z([0,1];\mathbb{R})$ be given by $f(x) = 1$ for all $x \in [0,1]$. Let V be a C_p-open subset containing f. Then V contains a basic neighbourhood from $\mathsf{C}_p([0,1];\mathbb{R})$, and in particular a basic neighbourhood of the form $U = \prod_{x\in[0,1]} U_x$ where the C_p-open sets $U_x \subseteq \mathbb{R}$, $x \in [0,1]$, have the following properties:

1. there exists $\epsilon \in (0,1)$ and a finite set $x_1, \dots, x_k \in [0,1]$ such that $U_{x_j} = (1-\epsilon, 1+\epsilon)$ for each $j \in \{1, \dots, k\}$;
2. for $x \in [0,1] \setminus \{x_1, \dots, x_k\}$ we have $U_x = \mathbb{R}$.

We claim that such a basic neighbourhood U contains a function from $Z([0,1];\mathbb{R})$. Indeed, the function

$$g(x) = \begin{cases} 1, & x \in \{x_1, \dots, x_k\}, \\ 0, & \text{otherwise} \end{cases}$$

is in $U \cap Z([0,1];\mathbb{R})$, and so is in $V \cap Z([0,1];\mathbb{R})$. This shows that $\mathsf{C}_p([0,1];\mathbb{R}) \setminus Z([0,1];\mathbb{R})$ is not C_p-open, as desired. ■

2.3. Almost everywhere pointwise convergence limit structures. For many applications, it is the space $\hat{\mathsf{L}}^1([0,1];\mathbb{R})$, not $\mathsf{L}([0,1];\mathbb{R})$, that is of interest, this by virtue of its possessing a norm and not a seminorm. (Of course, one might also be interested in $\hat{\mathsf{R}}^1([0,1];\mathbb{R})$, but the point of this paper is to clarify the ways in which this space is deficient.) Thus one is interested in subspaces of $\hat{\mathbb{R}}^{[0,1]}$. The largest such subspace in which we shall be interested is the image of the Lebesgue measurable functions in $\hat{\mathbb{R}}^{[0,1]}$ under the projection from $\mathbb{R}^{[0,1]}$:

$$\hat{\mathsf{M}}([0,1];\mathbb{R}) = \mathsf{M}([0,1];\mathbb{R})/Z([0,1];\mathbb{R}).$$

Note that the quotient is well-defined since completeness of the Lebesgue measure gives $Z([0,1];\mathbb{R}) \subseteq \mathsf{M}([0,1];\mathbb{R})$.

Now, one wishes to provide structure on $\hat{\mathsf{M}}([0,1];\mathbb{R})$ such that there is a notion of convergence which agrees with almost everywhere pointwise convergence. First let us be clear about what we mean by almost everywhere pointwise convergence relative to the various function spaces we are using.

2.4 Definition: (i) A sequence $(f_j)_{j\in\mathbb{Z}_{>0}}$ in $\mathsf{M}([0,1];\mathbb{R})$ is ***almost everywhere pointwise convergent*** to $f \in \mathsf{M}([0,1];\mathbb{R})$ if

$$\lambda(\{x \in [0,1] \mid (f_j(x)) \text{ does not converge to } f(x)\}) = 0.$$

(ii) A sequence $([f_j])_{j\in\mathbb{Z}_{>0}}$ in $\hat{\mathsf{M}}([0,1];\mathbb{R})$ is ***almost everywhere pointwise convergent*** to $[f] \in \hat{\mathsf{M}}([0,1];\mathbb{R})$ if

$$\lambda(\{x \in [0,1] \mid (f_j(x)) \text{ does not converge to } f(x)\}) = 0. \qquad \bullet$$

We should ensure that the definition of almost everywhere pointwise convergence in $\hat{\mathsf{M}}([0,1];\mathbb{R})$ is well-defined.

2.5 Lemma: *For a sequence $([f_j])_{j\in\mathbb{Z}_{>0}}$ in $\hat{\mathsf{M}}([0,1];\mathbb{R})$ and for $[f] \in \hat{\mathsf{M}}([0,1];\mathbb{R})$ the following statements are equivalent:*

(i) there exists a sequence $(g_j)_{j\in\mathbb{Z}_{>0}}$ in $\mathsf{M}([0,1];\mathbb{R})$ and $g \in \mathsf{M}([0,1];\mathbb{R})$ such that
  (a) $[g_j] = [f_j]$ for $j \in \mathbb{Z}_{>0}$,
  (b) $[g] = [f]$, and
  (c) $(g_j)_{j\in\mathbb{Z}_{>0}}$ converges pointwise almost everywhere to g.

(ii) for every sequence $(g_j)_{j\in\mathbb{Z}_{>0}}$ in $\mathsf{M}([0,1];\mathbb{R})$ and for every $g \in \mathsf{M}([0,1];\mathbb{R})$ satisfying
  (a) $[g_j] = [f_j]$ for $j \in \mathbb{Z}_{>0}$ and
  (b) $[g] = [f]$,

  it holds that $(g_j)_{j\in\mathbb{Z}_{>0}}$ converges pointwise almost everywhere to g.

Proof: It is clear that the second statement implies the first, so we only prove the converse. Thus we let $(g_j)_{j\in\mathbb{Z}_{>0}}$ in $\mathsf{M}([0,1];\mathbb{R})$ and $g \in \mathsf{M}([0,1];\mathbb{R})$ be such that

1. $[g_j] = [f_j]$ for $j \in \mathbb{Z}_{>0}$,
2. $[g] = [f]$, and
3. $(g_j)_{j\in\mathbb{Z}_{>0}}$ converges pointwise almost everywhere to g.

Let $(h_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence in $\mathsf{M}([0,1];\mathbb{R})$ and let $h \in \mathsf{M}([0,1];\mathbb{R})$ be such that

1. $[h_j] = [f_j]$ for $j \in \mathbb{Z}_{>0}$ and
2. $[h] = [f]$.

Define

$$A = \{x \in [0,1] \mid g(x) \neq f(x)\}, \quad B = \{x \in [0,1] \mid h(x) \neq f(x)\}$$

and, for $j \in \mathbb{Z}_{>0}$, define

$$A_j = \{x \in [0,1] \mid g_j(x) \neq f_j(x)\}, \quad B_j = \{x \in [0,1] \mid h_j(x) \neq f_j(x)\}$$

and note that

$$x \in [0,1] \setminus (A \cup B) = ([0,1] \setminus A) \cap ([0,1] \setminus B) \quad \Longrightarrow \quad h(x) = f(x) = g(x)$$

and

$$x \in [0,1] \setminus (A_j \cup B_j) = ([0,1] \setminus A_j) \cap ([0,1] \setminus B_j) \quad \Longrightarrow \quad h_j(x) = f_j(x) = g_j(x).$$

Thus,

$$x \in [0,1] \setminus \big((\cup_{j\in\mathbb{Z}_{>0}} A_j \cup B_j) \cup (A \cup B)\big) \quad \Longrightarrow \quad \lim_{j\to\infty} h_j(x) = \lim_{j\to\infty} g_j(x) = g(x) = h(x).$$

Since $(\cup_{j\in\mathbb{Z}_{>0}} A_j \cup B_j) \cup (A \cup B)$ is a countable union of sets of measure zero, it has zero measure, and so $(h_j)_{j\in\mathbb{Z}_{>0}}$ converges pointwise almost everywhere to h. ■

Now that we understand just what sort of convergence we seek in $\hat{\mathsf{M}}([0,1];\mathbb{R})$, we can think about how to achieve this. The obvious first guess is to use the quotient topology on $\hat{\mathsf{M}}([0,1];\mathbb{R})$ inherited from the C_p-topology on $\mathsf{M}([0,1];\mathbb{R})$. However, convergence in this topology fails to agree with almost everywhere pointwise convergence. Indeed, we have the following more sweeping statement.

2.6 Proposition: *Let $\mathscr{T}_{\text{a.e.}}$ be the set of topologies τ on $\hat{\mathsf{M}}([0,1];\mathbb{R})$ such that the convergent sequences in τ are precisely the almost everywhere pointwise convergent sequences. Then $\mathscr{T}_{\text{a.e.}} = \varnothing$.*

Proof: Suppose that $\mathscr{T}_{\text{a.e.}} \neq \varnothing$ and let $\tau \in \mathscr{T}_{\text{a.e.}}$. Let us denote by $z \in \mathsf{M}([0,1];\mathbb{R})$ the zero function. Let $(f_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence in $\mathsf{M}([0,1];\mathbb{R})$ converging in measure to z, i.e., for every $\epsilon \in \mathbb{R}_{>0}$,

$$\lim_{j\to\infty} \lambda(\{x \in [0,1] \mid |f_j(x)| \geq \epsilon\}) = 0,$$

but not converging pointwise almost everywhere to z [Cohn 2013, Example 3.1.1(b)]. Since almost everywhere pointwise convergence agrees with convergence in τ, there exists a neighbourhood U of $[z]$ in $\hat{\mathsf{M}}([0,1];\mathbb{R})$ such that the set

$$\{j \in \mathbb{Z}_{>0} \mid [f_j] \in U\}$$

is finite. As is well-known [Cohn 2013, Proposition 3.1.3], there exists a subsequence $(f_{j_k})_{k\in\mathbb{Z}_{>0}}$ of $(f_j)_{j\in\mathbb{Z}_{>0}}$ that converges pointwise almost everywhere to z. Thus the sequence $([f_{j_k}])_{k\in\mathbb{Z}_{>0}}$ converges pointwise almost everywhere to $[z]$, and so converges to $[z]$ in τ. Thus, in particular, the set

$$\{k \in \mathbb{Z}_{>0} \mid [f_{j_k}] \in U\}$$

is infinite, which is a contradiction. ■

It is moderately well-known that there can be no topology on $\hat{\mathsf{M}}([0,1];\mathbb{R})$ which gives rise to almost everywhere pointwise convergence. For instance, this is observed by [Fréchet 1921]. Our proof of Proposition 2.6 is adapted slightly from the observation of Ordman [1966]. The upshot of the result is that, if one is going to provide some structure with which to describe almost everywhere pointwise convergence, this structure must be something different than a topology. This was addressed by Arens [1950] who observed that the notion of convergence in measure *is* topological, but almost everywhere pointwise convergence is not. To structurally distinguish between the two sorts of convergence, Arens introduces the notion of a limit structure. This idea is discussed in some generality for Borel measurable functions by Höhle [2000] using multiple valued topologies. Here we will introduce the notion of a limit structure in as direct a manner as possible, commensurate with our objectives. Readers wishing to explore the subject in more detail are referred to [Beattie and Butzmann 2002].

For a set X let $\mathscr{F}(X)$ denote the set of filters on X and, for $x \in X$, denote by

$$\mathcal{F}_x = \{S \subseteq X \mid x \in S\}$$

the principal filter generated by $\{x\}$. If $(\Lambda, \preceq)$ is a directed set and if $\phi\colon \Lambda \to X$ is a Λ-net, we denote the tails of the net ϕ by

$$T_\phi(\lambda) = \{\phi(\lambda') \mid \lambda' \geq \lambda\}, \qquad \lambda \in \Lambda.$$

We then denote by

$$\mathcal{F}_\phi = \{S \subseteq X \mid T_\phi(\lambda) \subseteq S \text{ for some } \lambda \in \Lambda\}$$

the "tail filter" (also sometimes called the "Fréchet filter") associated to the Λ-net ϕ.

2.7 Definition: A ***limit structure*** on a set X is a subset $\mathscr{L} \subseteq \mathscr{F}(X) \times X$ with the following properties:

(i) if $x \in X$ then $(\mathcal{F}_x, x) \in \mathscr{L}$;

(ii) if $(\mathcal{F}, x) \in \mathscr{L}$ and if $\mathcal{F} \subseteq \mathcal{G} \in \mathscr{F}(X)$ then $(\mathcal{G}, x) \in \mathscr{L}$;

(iii) if $(\mathcal{F}, x), (\mathcal{G}, x) \in \mathscr{L}$ then $(\mathcal{F} \cap \mathcal{G}, x) \in \mathscr{L}$.

If $(\Lambda, \preceq)$ is a directed set, a Λ-net $\phi\colon \Lambda \to X$ is ***$\mathscr{L}$-convergent*** to $x \in X$ if $(\mathcal{F}_\phi, x) \in \mathscr{L}$. Let us denote by $\mathscr{S}(\mathscr{L})$ the set of $\mathscr{L}$-convergent $\mathbb{Z}_{>0}$-nets, i.e., the set of $\mathscr{L}$-convergent sequences. •

The intuition behind the notion of a limit structure is as follows. Condition (i) says that the trivial filter converging to x should be included in the limit structure, condition (ii) says that if a filter converges to x, then every coarser filter also converges to x, and condition (iii) says that "mixing" filters converging to x should give a filter converging to x. Starting

from the definition of a limit structure, one can reproduce many of the concepts from topology, e.g., openness, closedness, compactness, continuity. Since we are not going to make us of any of these general constructions, we merely refer the interested reader to [Beattie and Butzmann 2002]. The one notion we will use is the following: a subset A of a set X with a limit structure $\mathscr{L}$ is ***$\mathscr{L}$-sequentially closed*** if every $\mathscr{L}$-convergent sequence $(x_j)_{j\in\mathbb{Z}_{>0}}$ in A converges to $x \in A$.

We are interested in the special case of limit structures on a $\mathbb{R}$-vector space V; one will trivially see that there is nothing in the definitions that requires the field to be $\mathbb{R}$. For $\mathcal{F}, \mathcal{G} \in \mathscr{F}(\mathsf{V})$ and for $a \in \mathbb{R}$ we denote

$$\mathcal{F} + \mathcal{G} = \{A + B \mid A \in \mathcal{F},\ B \in \mathcal{G}\}, \quad a\mathcal{F} = \{aA \mid A \in \mathcal{F}\},$$

where, as usual,

$$A + B = \{u + v \mid u \in A,\ v \in B\}, \quad aA = \{au \mid u \in A\}.$$

We say that a limit structure $\mathscr{L}$ on a vector space V is ***linear*** if $(\mathcal{F}_1, v_1), (\mathcal{F}_2, v_2) \in \mathscr{L}$ implies that $(\mathcal{F}_1 + \mathcal{F}_2, v_1 + v_2) \in \mathscr{L}$ and if $a \in \mathbb{R}$ and $(\mathcal{F}, v) \in \mathscr{L}$ then $(a\mathcal{F}, av) \in \mathscr{L}$. Following the characterisation of bounded subsets of topological vector spaces, we say a subset $B \subseteq \mathsf{V}$ is ***$\mathscr{L}$-bounded*** if, for every sequence $(v_j)_{j\in\mathbb{Z}_{>0}}$ in B and for every sequence $(a_j)_{j\in\mathbb{Z}_{>0}}$ in $\mathbb{R}$ converging to 0, the sequence $(a_j v_j)_{j\in\mathbb{Z}_{>0}}$ is $\mathscr{L}$-convergent to zero.

For $[f] \in \hat{\mathsf{M}}([0,1];\mathbb{R})$ define

$$\mathscr{F}_{[f]} = \{\mathcal{F} \in \mathscr{F}(\hat{\mathsf{M}}([0,1];\mathbb{R})) \mid \mathcal{F}_\phi \subseteq \mathcal{F} \text{ for some } \mathbb{Z}_{>0}\text{-net } \phi \text{ such that } (\phi(j))_{j\in\mathbb{Z}_{>0}} \text{ is almost everywhere pointwise convergent to } [f]\}.$$

We may now define a limit structure on $\hat{\mathsf{M}}([0,1];\mathbb{R})$ as follows.

2.8 Theorem: *The subset of $\mathscr{F}(\hat{\mathsf{M}}([0,1];\mathbb{R})) \times \hat{\mathsf{M}}([0,1];\mathbb{R})$ defined by*

$$\mathscr{L}_\lambda = \{(\mathcal{F}, [f]) \mid \mathcal{F} \in \mathscr{F}_{[f]}\}$$

is a linear limit structure on $\hat{\mathsf{M}}([0,1];\mathbb{R})$. Moreover, a sequence $([f_j])_{j\in\mathbb{Z}_{>0}}$ is $\mathscr{L}_\lambda$-convergent to $[f]$ if and only if the sequence is almost everywhere pointwise convergent to $[f]$.

Proof: Let $[f] \in \hat{\mathsf{M}}([0,1];\mathbb{R})$. Consider the trivial $\mathbb{Z}_{>0}$-net $\phi_{[f]} \colon \mathbb{Z}_{>0} \to \hat{\mathsf{M}}([0,1];\mathbb{R})$ defined by $\phi_{[f]}(j) = [f]$. Since $\mathcal{F}_\phi = \mathcal{F}_{[f]}$ and since $(\mathcal{F}_\phi, [f]) \in \mathscr{L}_\lambda$, the condition (i) for a limit structure is satisfied.

Let $(\mathcal{F}, [f]) \in \mathscr{L}_\lambda$ and suppose that $\mathcal{F} \subseteq \mathcal{G}$. Then $\mathcal{F} \in \mathscr{F}_{[f]}$ and so $\mathcal{F} \supseteq \mathcal{F}_\phi$ for some $\mathbb{Z}_{>0}$-net ϕ that converges pointwise almost everywhere to $[f]$. Therefore, we immediately have $\mathcal{F}_\phi \subseteq \mathcal{G}$ and so $(\mathcal{G}, [f]) \in \mathscr{L}_\lambda$. This verifies condition (ii) in the definition of a limit structure.

Finally, let $(\mathcal{F}, [f]), (\mathcal{G}, [f]) \in \mathscr{L}_\lambda$ and let ϕ and ψ be $\mathbb{Z}_{>0}$-nets that converge pointwise almost everywhere to $[f]$ and satisfy $\mathcal{F}_\phi \subseteq \mathcal{F}$ and $\mathcal{F}_\psi \subseteq \mathcal{G}$. Define a $\mathbb{Z}_{>0}$-net $\phi \wedge \psi$ by

$$\phi \wedge \psi(j) = \begin{cases} \phi(\frac{1}{2}(j+1)), & j \text{ odd}, \\ \psi(\frac{1}{2}j), & j \text{ even}. \end{cases}$$

We first claim that $\phi \wedge \psi$ converges pointwise almost everywhere to $[f]$. Let

$$A = \big\{x \in [0,1] \;\big|\; \lim_{j\to\infty} \phi(j)(x) \neq f(x)\big\}, \quad B = \big\{x \in [0,1] \;\big|\; \lim_{j\to\infty} \psi(j)(x) \neq f(x)\big\}.$$

If $x \in [0,1] \setminus (A \cup B)$ then

$$\lim_{j\to\infty} \phi(j)(x) = \lim_{j\to\infty} \psi(j)(x) = f(x).$$

Thus, for $x \in [0,1] \setminus (A \cup B)$ and $\epsilon \in \mathbb{R}_{>0}$ there exists $N \in \mathbb{Z}_{>0}$ such that

$$|f(x) - \phi(j)(x)|, |f(x) - \psi(j)(x)| < \epsilon, \qquad j \geq N.$$

Therefore, for $j \geq 2N$ and for $x \in [0,1] \setminus (A \cup B)$ we have $|f(x) - \phi \wedge \psi(j)(x)| < \epsilon$ and so

$$\lim_{j\to\infty} \phi \wedge \psi(j)(x) = f(x), \qquad x \in [0,1] \setminus (A \cup B).$$

Since $\lambda(A \cup B) = 0$ it indeed follows that $\phi \wedge \psi$ converges pointwise almost everywhere to $[f]$.

We next claim that $\mathcal{F}_{\phi\wedge\psi} \subseteq \mathcal{F} \cap \mathcal{G}$. Indeed, let $S \in \mathcal{F}_{\phi\wedge\psi}$. Then there exists $N \in \mathbb{Z}_{>0}$ such that $T_{\phi\wedge\psi}(N) \subseteq S$. Therefore, there exists $N_\phi, N_\psi \in \mathbb{Z}_{>0}$ such that $T_\phi(N_\phi) \subseteq S$ and $T_\psi(N_\psi) \subseteq S$. That is, $S \in \mathcal{F}_\phi \cap \mathcal{F}_\psi \subseteq \mathcal{F} \cap \mathcal{G}$. This shows that $(\mathcal{F} \cap \mathcal{G}, [f]) \in \mathscr{L}_\lambda$ and so shows that condition (iii) in the definition of a limit structure holds.

Thus we have shown that $\mathscr{L}_\lambda$ is a limit structure. Let us show that it is a linear limit structure. Let $(\mathcal{F}_1, [f_1]), (\mathcal{F}_2, v_2) \in \mathscr{L}_\lambda$. Thus there exists $\mathbb{Z}$-nets ϕ_1 and ϕ_2 in $\hat{\mathsf{M}}([0,1];\mathbb{R})$ converging pointwise almost everywhere to $[f_1]$ and $[f_2]$, respectively, and such that $\mathcal{F}_{\phi_1} \subseteq \mathcal{F}_1$ and $\mathcal{F}_{\phi_2} \subseteq \mathcal{F}_2$. Let us denote by $(f_{1,j})_{j\in\mathbb{Z}_{>0}}$ and $(f_{2,j})_{j\in\mathbb{Z}_{>0}}$ sequences in $\mathsf{M}([0,1];\mathbb{R})$ such that $[f_{1,j}] = \phi_1(j)$ and $[f_{2,j}] = \phi_2(j)$ for $j \in \mathbb{Z}_{>0}$. Then, as in the proof of Lemma 2.5, there exists a subset $A \subseteq [0,1]$ of zero measure such that

$$\lim_{j\to\infty} f_{j,1}(x) = f_1(x), \quad \lim_{j\to\infty} f_{2,j}(x) = f_2(x), \qquad x \in [0,1] \setminus A.$$

Thus, for $x \in [0,1] \setminus A$,

$$\lim_{j\to\infty} (f_{1,j} + f_{2,j})(x) = (f_1 + f_2)(x).$$

This shows that the $\mathbb{Z}_{>0}$-net $\phi_1 + \phi_2$ converges pointwise almost everywhere to $[f_1 + f_2]$. Since $\mathcal{F}_{\phi_1+\phi_2} \subseteq \mathcal{F}_1 + \mathcal{F}_2$, it follows that $(\mathcal{F}_1 + \mathcal{F}_2, [f_1 + f_2]) \in \mathscr{L}_\lambda$. An entirely similarly styled argument gives $(a\mathcal{F}, av) \in \mathscr{L}_\lambda$ for $(\mathcal{F}, v) \in \mathscr{L}_\lambda$.

We now need to show that $\mathscr{S}(\mathscr{L}_\lambda)$ consists exactly of the almost everywhere pointwise convergent sequences. The very definition of $\mathscr{L}_\lambda$ ensures that if a $\mathbb{Z}_{>0}$-net ϕ is almost everywhere pointwise convergent then $\phi \in \mathscr{S}(\mathscr{L}_\lambda)$. We prove the converse, and so let ϕ be $\mathscr{L}_\lambda$-convergent to $[f]$. Therefore, by definition of $\mathscr{L}_\lambda$, there exists a $\mathbb{Z}_{>0}$-net ψ converging pointwise almost everywhere to $[f]$ such that $\mathcal{F}_\psi \subseteq \mathcal{F}_\phi$.

1 Lemma: *There exists of a subsequence ψ' of ψ such that $\mathcal{F}_{\psi'} = \mathcal{F}_\phi$.*

Proof: Let $n \in \mathbb{Z}_{>0}$ and note that $T_\psi(n) \in \mathcal{F}_\psi \subseteq \mathcal{F}_\phi$. Thus there exists $k \in \mathbb{Z}_{>0}$ such that $T_\phi(k) \subseteq T_\psi(n)$. Then define

$$k_n = \min\{k \in \mathbb{Z}_{>0} \mid T_\phi(k) \subseteq T_\psi(n)\},$$

the minimum being well-defined since

$$k > k' \quad \Longrightarrow \quad T_\phi(k) \subseteq T_\phi(k').$$

This uniquely defines, therefore, a sequence $(k_n)_{n\in\mathbb{Z}_{>0}}$. Moreover, if $n_1 > n_2$ then $T_\psi(n_2) \subseteq T_\psi(n_1)$ which implies that $T_\phi(k_{n_2}) \subseteq T_\psi(n_1)$. Therefore, $k_{n_2} \geq k_{n_1}$, showing that the sequence $(k_n)_{n\in\mathbb{Z}_{>0}}$ is nondecreasing.

Now define $\theta\colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ as follows. If $j < k_n$ for every $n \in \mathbb{Z}_{>0}$ then define $\theta(j)$ in an arbitrary manner. If $j \geq k_1$ then note that $\phi(j) \in T_\phi(k_1) \subseteq T_\psi(1)$. Thus there exists (possibly many) $m \in \mathbb{Z}_{>0}$ such that $\phi(j) = \psi(m)$. If $j \geq k_n$ for $n \in \mathbb{Z}_{>0}$ then there exists (possibly many) $m \geq n$ such that $\phi(j) = \psi(m)$. Thus for any $j \in \mathbb{Z}_{>0}$ we can define $\theta(j) \in \mathbb{Z}_{>0}$ such that $\phi(j) = \psi(\theta(j))$ if $j \geq k_1$ and such that $\theta(j) \geq n$ if $j \geq k_n$.

Note that any function $\theta\colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ as constructed above is unbounded. Therefore, there exists a strictly increasing function $\rho\colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that $\operatorname{image}(\rho) = \operatorname{image}(\theta)$. We claim that $\mathcal{F}_\rho = \mathcal{F}_\theta$. First let $n \in \mathbb{Z}_{>0}$ and let $j \geq k_{\rho(n)}$. Then $\theta(j) \geq \rho(n)$. Since $\operatorname{image}(\rho) = \operatorname{image}(\theta)$ there exists $m \in \mathbb{Z}_{>0}$ such that $\rho(m) = \theta(j) \geq \rho(n)$. Since ρ is strictly increasing, $m \geq n$. Thus $\theta(j) \in T_\rho(n)$ and so $T_\theta(k_{\rho(n)}) \subseteq T_\rho(n)$. This implies that $\mathcal{F}_\rho \subseteq \mathcal{F}_\theta$.

Conversely, let $n \in \mathbb{Z}_{>0}$ and let $r_n \in \mathbb{Z}_{>0}$ be such that

$$\rho(r_n) > \max\{\theta(1), \dots, \theta(n)\};$$

this is possible since ρ is unbounded. If $j \geq r_n$ then

$$\rho(j) \geq \rho(r_n) > \max\{\theta(1), \dots, \theta(n)\}.$$

Since $\operatorname{image}(\rho) = \operatorname{image}(\theta)$ we have $\rho(j) = \theta(m)$ for some $m \in \mathbb{Z}_{>0}$. We must have $m > n$ and so $\rho(j) \in T_\theta(n)$. Thus $T_\rho(r_n) \subseteq T_\theta(n)$ and so $\mathcal{F}_\theta \subseteq \mathcal{F}_\rho$.

To arrive at the conclusions of the lemma we first note that, by definition of θ, $\mathcal{F}_\phi = \mathcal{F}_{\psi\circ\theta}$. We now define $\psi' = \psi \circ \rho$ and note that

$$\mathcal{F}_\phi = \mathcal{F}_{\psi\circ\theta} = \psi(\mathcal{F}_\theta) = \psi(\mathcal{F}_\rho) = \mathcal{F}_{\psi\circ\rho},$$

as desired. ▼

Since a subsequence of an almost everywhere pointwise convergent sequence is almost everywhere pointwise convergent to the same limit, it follows that ψ', and so ϕ, converges almost everywhere pointwise to $[f]$. ■

The preceding theorem seems to be well-known; see [Beattie and Butzmann 2002] where, in particular, the essential lemma in the proof is given. Nonetheless, we have never seen the ingredients of the proof laid out clearly in one place, so the result is worth recording.

Let us record a characterisation of $\mathscr{L}_\lambda$-bounded subsets of $\hat{\mathsf{M}}([0,1];\mathbb{R})$.

2.9 Proposition: *A subset $B \subseteq \hat{\mathsf{M}}([0,1];\mathbb{R})$ is $\mathscr{L}_\lambda$-bounded if and only if there exists a nonnegative-valued $g \in \mathsf{M}([0,1];\mathbb{R})$ such that*

$$B \subseteq \{[f] \in \hat{\mathsf{M}}([0,1];\mathbb{R}) \mid |f(x)| \leq g(x) \text{ for almost every } x \in [0,1]\}.$$

Proof: We first observe that the condition that $|f(x)| \leq g(x)$ for almost every $x \in [0,1]$ is independent of the choice of representative f from the equivalence class $[f]$.

Suppose that there exists a nonnegative-valued $g \in \mathsf{M}([0,1];\mathbb{R})$ such that, if $[f] \in B$, then $|f(x)| \leq g(x)$ for almost every $x \in [0,1]$. Let $([f_j])_{j\in\mathbb{Z}_{>0}}$ be a sequence in B and let $(a_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence in $\mathbb{R}$ converging to zero. For $j \in \mathbb{Z}_{>0}$ define

$$A_j = \{x \in [0,1] \mid |f_j(x)| \leq g(x)\}.$$

Note that if $x \in [0,1] \setminus (\cup_{j\in\mathbb{Z}_{>0}} A_j)$ then

$$\lim_{j\to\infty} |a_j f_j(x)| \leq \lim_{j\to\infty} |a_j| g(x) = 0.$$

Since $\lambda(\cup_{j\in\mathbb{Z}_{>0}} A_j) = 0$ this implies that the sequence $(a_j[f_j])_{j\in\mathbb{Z}_{>0}}$ is $\mathscr{L}_\lambda$-convergent to zero. One may show that this argument is independent of the choice of representatives f_j from the equivalence classes $[f_j]$, $j \in \mathbb{Z}_{>0}$.

Conversely, suppose that there exists no nonnegative-valued function $g \in \mathsf{M}([0,1];\mathbb{R})$ such that, for every $[f] \in B$, $|f(x)| \leq g(x)$ for almost every $x \in [0,1]$. This means that there exists a set $E \subseteq [0,1]$ of positive measure such that, for any $M \in \mathbb{R}_{>0}$, there exists $[f] \in B$ such that $|f(x)| > M$ for almost every $x \in E$. Let $(a_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence in $\mathbb{R}$ converging to 0 and such that $a_j \neq 0$ for every $j \in \mathbb{Z}_{>0}$. Then let $([f_j])_{j\in\mathbb{Z}_{>0}}$ be a sequence in B such that $|f_j(x)| > |a_j^{-1}|$ for almost every $x \in E$ and for every $j \in \mathbb{Z}_{>0}$. Define

$$A_j = \{x \in E \mid |f_j(x)| > |a_j^{-1}|\}.$$

If $x \in E \setminus (\cup_{j\in\mathbb{Z}_{>0}} A_j)$ then $|a_j f_j(x)| > 1$ for every $j \in \mathbb{Z}_{>0}$. Since $\lambda(E \setminus (\cup_{j\in\mathbb{Z}_{>0}} A_j)) > 0$ it follows that $(a_j[f_j])_{j\in\mathbb{Z}_{>0}}$ cannot $\mathscr{L}_\lambda$-converge to zero, and so B is not $\mathscr{L}_\lambda$-bounded. ■

3. Two topological distinctions between the Riemann and Lebesgue theories of integration

In this section we give topological characterisations of the differences between the Riemann and Lebesgue theories. In Section 3.3 we also explicitly see how these distinctions lead to a deficiency in the Fourier transform theory using the Riemann integral.

3.1. The Dominated Convergence Theorems. Both the Lebesgue and Riemann theories of integration possess a Dominated Convergence Theorem. This gives us two versions of the Dominated Convergence Theorem that we can compare and contrast. Moreover, there are also "pointwise convergent" and "almost everywhere pointwise convergent" versions of both theorems. Typically, the "pointwise convergent" version is stated for the Riemann integral[2] and the "almost everywhere pointwise convergent" version is stated for the Lebesgue

[2]Since the "almost everywhere pointwise convergent" version actually requires the Lebesgue theory of integration.

integral. However, both versions are valid for both integrals, so this gives, in actuality, four theorems to compare and contrast. What we do here is state both versions of the Dominated Convergence Theorem for the Lebesgue integral using topological and limit structures, and we give counterexamples illustrating why these statements are not valid for the Riemann integral.

Let us first state the various Dominated Convergence Theorems in their usual form. The Dominated Convergence Theorem—including the pointwise convergent and almost everywhere pointwise convergent statements—for the Riemann integral is the following.

3.1 Theorem: *Let $(f_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence of $\mathbb{R}$-valued functions on $[0,1]$ satisfying the following conditions:*

(i) $f_j \in \mathsf{R}^1([0,1];\mathbb{R})$ *for each* $j \in \mathbb{Z}_{>0}$*;*

(ii) there exists a nonnegative-valued $g \in \mathsf{R}^1([0,1];\mathbb{R})$ *such that* $|f_j(x)| \le g(x)$ *for every (resp. almost every)* $x \in [0,1]$ *and for every* $j \in \mathbb{Z}_{>0}$*;*

(iii) the limit $\lim_{j\to\infty} f_j(x)$ *exists for every (resp. almost every)* $x \in [0,1]$*;*

(iv) the function $f\colon [0,1] \to \mathbb{R}$ *defined by* $f(x) = \lim_{j\to\infty} f_j(x)$ *is in* $\mathsf{R}^1([0,1];\mathbb{R})$ *(resp. there exists* $f \in \mathsf{R}^1([0,1];\mathbb{R})$ *such that* $\lim_{j\to\infty} f_j(x) = f(x)$ *for almost every* $x \in [0,1]$*).*

Then

$$\lim_{j\to\infty} \int_0^1 f_j(x)\,\mathrm{d}x = \int_0^1 f(x)\,\mathrm{d}x.$$

For the Lebesgue integral we have the following Dominated Convergence Theorem(s).

3.2 Theorem: *Let $(f_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence of $\mathbb{R}$-valued functions on $[0,1]$ satisfying the following conditions:*

(i) f_j *is measurable for each* $j \in \mathbb{Z}_{>0}$*;*

(ii) there exists a nonnegative-valued $g \in \mathsf{L}^1([0,1];\mathbb{R})$ *such that* $|f_j(x)| \le g(x)$ *for every (resp. almost every)* $x \in [0,1]$ *and for every* $j \in \mathbb{Z}_{>0}$*;*

(iii) the limit $\lim_{j\to\infty} f_j(x)$ *exists for every (resp. almost every)* $x \in [0,1]$*.*

Then the function $f\colon [0,1] \to \mathbb{R}$ *defined by*

$$f(x) = \begin{cases} \lim_{j\to\infty} f_j(x), & \text{the limit exists,} \\ 0, & \text{otherwise} \end{cases}$$

and each of the functions f_j*,* $j \in \mathbb{Z}_{>0}$*, are in* $\mathsf{L}^1([0,1];\mathbb{R})$ *and*

$$\lim_{j\to\infty} \int_I f_j\,\mathrm{d}\lambda = \int_I f\,\mathrm{d}\lambda.$$

Our statements make it clear that there is one real difference between the Riemann and Lebesgue theories: the condition of integrability of the limit function f is an *hypothesis* in the Riemann theory but a *conclusion* in the Lebesgue theory. This distinction is crucial and explains why the Lebesgue theory is more powerful than the Riemann theory. Moreover, the structure we introduced in Section 2 allows for an elegant expression of this distinction. The result which follows is simply a rephrasing of the Dominated Convergence Theorem(s) for the Lebesgue integral, and follows from that theorem, along with Theorem 2.8 and Proposition 2.9.

3.3 Theorem: *The following statements hold:*

(i) C_p*-bounded subsets of* $\mathsf{L}^1([0,1];\mathbb{R})$ *are* C_p*-sequentially closed;*

(ii) $\mathscr{L}_\lambda$*-bounded subsets of* $\hat{\mathsf{L}}^1([0,1];\mathbb{R})$ *are* $\mathscr{L}_\lambda$*-sequentially closed.*

The necessity of the weaker conclusions for the Dominated Convergence Theorem(s) for the Riemann integral is illustrated by the following examples. First we show why part (i) of Theorem 3.3 does not hold for the Riemann integral.

3.4 Example: By means of an example, we show that there are C_p-bounded subsets of the seminormed vector space $\mathsf{R}^1([0,1];\mathbb{R})$ that are not sequentially closed in the topology of $\mathsf{C}_p([0,1];\mathbb{R})$. Let us denote

$$B = \{f \in \mathsf{R}^1([0,1];\mathbb{R}) \mid |f(x)| \leq 1\},$$

noting by Proposition 2.2 that B is C_p-bounded. Let $(q_j)_{j\in\mathbb{Z}_{>0}}$ be an enumeration of the rational numbers in $[0,1]$ and define a sequence $(F_k)_{k\in\mathbb{Z}_{>0}}$ in $\mathsf{R}^1([0,1];\mathbb{R})$ by

$$F_k(x) = \begin{cases} 1, & x \in \{q_1, \dots, q_k\}, \\ 0, & \text{otherwise.} \end{cases}$$

The sequence converges in $\mathsf{C}_p([0,1];\mathbb{R})$ to the characteristic function of $\mathbb{Q} \cap [0,1]$; let us denote this function by F. This limit function is not Riemann integrable and so not in $\mathsf{R}^1([0,1];\mathbb{R})$. Thus B is not C_p-sequentially closed. •

Next we show why part (ii) of Theorem 3.3 does not hold for the Riemann integral.

3.5 Example: We give an example that shows that $\mathscr{L}_\lambda$-bounded subsets of the normed vector space $\hat{\mathsf{R}}^1([0,1];\mathbb{R})$ are not $\mathscr{L}_\lambda$-sequentially closed. We first remark that the construction of Example 3.4, projected to $\hat{\mathsf{R}}^1([0,1];\mathbb{R})$, does not suffice because $[F]$ is equal to the equivalence class of the zero function which *is* Riemann integrable, even though F is not, cf. the statements following Example 3.117 in [Kurtz 2004]. The fact that $[F]$ contains functions that are Riemann integrable and functions that are not Riemann integrable is a reflection of the fact that $\mathsf{R}_0([0,1];\mathbb{R})$ is not sequentially closed. This is a phenomenon of interest, but it is not what is of interest here.

The construction we use is the following. Let $(q_j)_{j\in\mathbb{Z}_{>0}}$ be an enumeration of the rational numbers in $[0,1]$. Let $\ell \in (0,1)$ and for $j \in \mathbb{Z}_{>0}$ define

$$I_j = [0,1] \cap (q_j - \tfrac{\ell}{2^{j+1}}, q_j + \tfrac{\ell}{2^{j+1}})$$

to be the interval of length $\frac{\ell}{2^j}$ centred at q_j. Then define $A_k = \cup_{j=1}^k I_j$, $k \in \mathbb{Z}_{>0}$, and $A = \cup_{j\in\mathbb{Z}_{>0}} A_j$. Also define $G_k = \chi_{A_k}$, $k \in \mathbb{Z}_{>0}$, and $G = \chi_A$ be the characteristic functions of A_k and A, respectively. Note that A_k is a union of a finite number of intervals and so G_k is Riemann integrable for each $k \in \mathbb{Z}_{>0}$. However, we claim that G is not Riemann integrable. Indeed, the characteristic function of a set is Riemann integrable if and only the boundary of the set has measure zero; this is a direct consequence of Lebesgue's theorem stating that a function is Riemann integrable if and only if its set of discontinuities has measure zero [Cohn 2013, Theorem 2.5.4]. Note that since $\mathrm{cl}(\mathbb{Q} \cap [0,1]) = [0,1]$ we have

$$[0,1] = \mathrm{cl}(A) = A \cup \mathrm{bd}(A).$$

Thus

$$\lambda([0,1]) \leq \lambda(A) + \lambda(\mathrm{bd}(A)).$$

Since

$$\lambda(A) \leq \sum_{j=1}^{\infty} \lambda(I_j) \leq \ell,$$

it follows that $\lambda(\mathrm{bd}(A)) \geq 1 - \ell \in \mathbb{R}_{>0}$. Thus G is not Riemann integrable, as claimed.

It is clear that $(G_k)_{k\in\mathbb{Z}_{>0}}$ is C_p-convergent to G. Therefore, by Theorem 2.8 it follows that $([G_k])_{k\in\mathbb{Z}_{>0}}$ is $\mathscr{L}_\lambda$-convergent to $[G]$. We claim that $[G] \notin \hat{\mathsf{R}}^1([0,1];\mathbb{R})$. This requires that we show that if $g \in \mathsf{C}_p([0,1];\mathbb{R})$ satisfies $[g] = [G]$, then g is not Riemann integrable. To show this, it suffices to show that g is discontinuous on a set of positive measure. We shall show that g is discontinuous on the set $g^{-1}(0) \cap \mathrm{bd}(A)$. Indeed, let $x \in g^{-1}(0) \cap \mathrm{bd}(A)$. Then, for any $\epsilon \in \mathbb{R}_{>0}$ we have $(x-\epsilon, x+\epsilon) \cap A \neq \varnothing$ since $x \in \mathrm{bd}(A)$. Since $(x-\epsilon, x+\epsilon) \cap A$ is a nonempty open set, it has positive measure. Therefore, since G and g agree almost everywhere, there exists $y \in (x-\epsilon, x+\epsilon) \cap A$ such that $g(y) = 1$. Since this holds for every $\epsilon \in \mathbb{R}_{>0}$ and since $g(x) = 0$, it follows that g is discontinuous at x. Finally, it suffices to show that $g^{-1}(0) \cap \mathrm{bd}(A)$ has positive measure. But this follows since $\mathrm{bd}(A) = G^{-1}(0)$ has positive measure and since G and g agree almost everywhere.

To complete the example, we note that the sequence $([G_k])_{j\in\mathbb{Z}_{>0}}$ is in the set

$$B = \{[f] \in \hat{\mathsf{R}}^1([0,1];\mathbb{R}) \mid |f(x)| \leq 1 \text{ for almost every } x \in [0,1]\},$$

which is $\mathscr{L}_\lambda$-bounded by Proposition 2.9. The example shows that this $\mathscr{L}_\lambda$-bounded subset of $\hat{\mathsf{R}}^1([0,1];\mathbb{R})$ is not $\mathscr{L}_\lambda$-sequentially closed. •

3.2. Completeness of spaces of integrable functions. In Section 2.1 we constructed the two normed vector spaces $\hat{\mathsf{R}}^1([0,1];\mathbb{R})$ and $\hat{\mathsf{L}}^1([0,1];\mathbb{R})$. Using the Dominated Convergence Theorem, one proves the following well-known and important result [Cohn 2013, Theorem 3.4.1].

3.6 Theorem: $(\hat{\mathsf{L}}^1([0,1];\mathbb{R}), \|\cdot\|_1)$ *is a Banach space.*

It is generally understood that $(\hat{\mathsf{R}}^1([0,1];\mathbb{R}), \|\cdot\|_1)$ is *not* a Banach space. However, we have not seen this demonstrated in a sufficiently compelling manner, so the following example will hopefully be interesting in this respect.

3.7 Example: Let us consider the sequence of functions $(G_k)_{k\in\mathbb{Z}_{>0}}$ in $\mathsf{R}^1([0,1];\mathbb{R})$ constructed in Example 3.5. We also use the pointwise limit function G defined in that same example. We shall freely borrow the notation introduced in this example.

We claim that the sequence $([G_k])_{k\in\mathbb{Z}_{>0}}$ is Cauchy in $\hat{\mathsf{R}}^1([0,1];\mathbb{R})$. Let $\epsilon \in \mathbb{R}_{>0}$. Note that $\sum_{j=1}^{\infty} \lambda(I_j) \leq \ell$. This implies that there exists $N \in \mathbb{Z}_{>0}$ such that $\sum_{j=k+1}^{m} \lambda(I_j) < \epsilon$ for all $k, m \geq N$. Now note that for $k, m \in \mathbb{Z}_{>0}$ with $m > k$, the functions G_k and G_m agree except on a subset of $I_{k+1} \cup \cdots \cup I_m$. On this subset, G_m has value 1 and G_k has value 0. Thus

$$\int_0^1 |G_m(x) - G_k(x)| \, \mathrm{d}x \leq \lambda(I_{k+1} \cup \cdots \cup I_m) \leq \sum_{j=k+1}^{m} \lambda(I_j).$$

Thus we can choose $N \in \mathbb{Z}_{>0}$ sufficiently large that $\|G_m - G_k\|_1 < \epsilon$ for $k, m \geq N$. Thus the sequence $([G_k])_{k\in\mathbb{Z}_{>0}}$ is Cauchy, as claimed.

We next show that the sequence $([G_k])_{k\in\mathbb{Z}_{>0}}$ converges to $[G]$ in $\hat{\mathsf{L}}^1([0,1];\mathbb{R})$. In Example 3.5 we showed that $([G_k])_{k\in\mathbb{Z}_{>0}}$ is $\mathscr{L}_\lambda$-convergent to $[G]$). Since the sequence $([G - G_k])_{k\in\mathbb{Z}_{>0}}$ is in the subset

$$\{[f] \in \hat{\mathsf{L}}^1([0,1];\mathbb{R}) \mid |f(x)| \leq 1 \text{ for almost every } x \in [0,1]\},$$

and since this subset is $\mathscr{L}_\lambda$-bounded by Proposition 2.9, it follows from Theorem 3.3(ii) that

$$\lim_{k\to\infty} \|G - G_k\|_1 = \int_I \lim_{k\to\infty} |G - G_k| \, \mathrm{d}\lambda = 0.$$

This gives us the desired convergence of $([G_k])_{k\in\mathbb{Z}_{>0}}$ to $[G]$ in $\hat{\mathsf{L}}^1([0,1];\mathbb{R})$. However, in Example 3.5 we showed that $[G] \notin \hat{\mathsf{R}}^1([0,1];\mathbb{R})$. Thus the Cauchy sequence $([G_k])_{k\in\mathbb{Z}_{>0}}$ in $\hat{\mathsf{R}}^1([0,1];\mathbb{R})$ is not convergent in $\hat{\mathsf{R}}([0,1];\mathbb{R})$, giving the desired incompleteness of $(\hat{\mathsf{R}}([0,1];\mathbb{R}), \|\cdot\|_1)$. •

In [Kurtz 2004, Example 3.117] the sequence $(f_j)_{j\in\mathbb{Z}_{>0}}$ defined by

$$f_j(x) = \begin{cases} 0, & x \in [0, \frac{1}{j}], \\ x^{-1/2}, & x \in (\frac{1}{j}, 1] \end{cases}$$

is shown to be Cauchy in $\hat{\mathsf{R}}^1([0,1];\mathbb{R})$, but not convergent in $\hat{\mathsf{R}}^1([0,1];\mathbb{R})$. This sequence, however, is not as interesting as that in our preceding example since the limit function $f \in \hat{\mathsf{L}}^1([0,1];\mathbb{R})$ in Kurtz's example *is* Riemann integrable using the usual rule for defining the improper Riemann integral for unbounded functions. In the construction used in Example 3.7, the limit function in $\mathsf{L}^1([0,1];\mathbb{R})$ is not Riemann integrable in the sense of bounded functions defined on compact intervals, i.e., in the sense of the usual construction involving approximation above and below by step functions.

In [Dickmeis, Mevissen, Nessel, and van Wickeren 1988] a convergence structure is introduced on the set of Riemann integrable functions that is sequentially complete. The idea in this work is to additionally constrain convergence in $\hat{\mathsf{L}}^1([0,1];\mathbb{R})$ in such a way that Riemann integrability is preserved by limits.

3.3. The L^2-Fourier transform for the Riemann integral. In order to illustrate why it is important that spaces of integrable functions be complete, we consider the theory of the L^2-Fourier transform restricted to square Riemann integrable functions. Let us first recall the essential elements of the theory.

For $p \in [1, \infty)$ we denote by $\mathsf{L}^p(\mathbb{R};\mathbb{C})$ the set of $\mathbb{C}$-valued functions f defined on $\mathbb{R}$ which satisfy

$$\int_{\mathbb{R}} |f|^p \, \mathrm{d}\lambda < \infty,$$

where we now denote by λ the Lebesgue measure on $\mathbb{R}$. We let

$$\mathsf{L}_0(\mathbb{R};\mathbb{C}) = \Big\{ f \in \mathsf{L}^p(\mathbb{R};\mathbb{C}) \;\Big|\; \int_{\mathbb{R}} |f|^p \, \mathrm{d}\lambda = 0 \Big\}$$

and denote

$$\hat{\mathsf{L}}^p(\mathbb{R};\mathbb{C}) = \mathsf{L}^p(\mathbb{R};\mathbb{C})/\mathsf{L}_0(\mathbb{R};\mathbb{C}).$$

As we have done previously, we denote $[f] = f + \mathsf{L}_0(\mathbb{R};\mathbb{C})$. If we define

$$\|[f]\|_p = \Big(\int_{\mathbb{R}} |f|^p \,\mathrm{d}\lambda\Big)^{1/p}$$

then one shows that $(\hat{\mathsf{L}}^p(\mathbb{R};\mathbb{C}), \|\cdot\|_p)$ is a Banach space [Cohn 2013, Theorem 3.4.1].

For $a \in \mathbb{C}$ let us denote $\exp_a \colon \mathbb{R} \to \mathbb{C}$ by $\exp_a(x) = \mathrm{e}^{ax}$. For $[f] \in \hat{\mathsf{L}}^1(\mathbb{R};\mathbb{C})$ one defines $\mathscr{F}_1([f]) \colon \mathbb{R} \to \mathbb{C}$ by

$$\mathscr{F}_1([f])(\omega) = \int_{\mathbb{R}} f \exp_{-2\pi\mathrm{i}\omega} \,\mathrm{d}\lambda$$

the ***L^1-Fourier transform*** of $[f] \in \hat{\mathsf{L}}^1(\mathbb{R};\mathbb{C})$. If we define $\mathsf{C}^0_{0,\mathrm{uc}}(\mathbb{R};\mathbb{C})$ to be the set of uniformly continuous $\mathbb{C}$-valued functions f on $\mathbb{R}$ that satisfy $\lim_{|x|\to\infty}|f(x)| = 0$, then $(\mathsf{C}^0_{0,\mathrm{uc}}(\mathbb{R};\mathbb{C}), \|\cdot\|_\infty)$ is a Banach space with

$$\|f\|_\infty = \sup\{|f(x)| \mid x \in \mathbb{R}\}.$$

Moreover, $\mathscr{F}_1([f]) \in \mathsf{C}^0_{0,\mathrm{uc}}(\mathbb{R};\mathbb{C})$ and the linear map $\mathscr{F}_1 \colon \hat{\mathsf{L}}^1(\mathbb{R};\mathbb{C}) \to \mathsf{C}^0_{0,\mathrm{uc}}(\mathbb{R};\mathbb{C})$ is continuous [Gasquet and Witomski 1999, Theorem 17.1.3].

For $[f] \in \hat{\mathsf{L}}^2(\mathbb{R};\mathbb{C})$ the preceding construction cannot be applied verbatim since $\hat{\mathsf{L}}^2(\mathbb{R};\mathbb{C})$ is not a subspace of $\hat{\mathsf{L}}^1(\mathbb{R};\mathbb{C})$. However, one can make an adaptation as follows [Gasquet and Witomski 1999, Lesson 22]. One shows that $\hat{\mathsf{L}}^1(\mathbb{R};\mathbb{C}) \cap \hat{\mathsf{L}}^2(\mathbb{R};\mathbb{C})$ is dense in $\hat{\mathsf{L}}^2(\mathbb{R};\mathbb{C})$. One can do this explicitly by defining, for $[f] \in \hat{\mathsf{L}}^2(\mathbb{R};\mathbb{C})$, a sequence $([f_j])_{j\in\mathbb{Z}_{>0}}$ in $\hat{\mathsf{L}}^1(\mathbb{R};\mathbb{C}) \cap \hat{\mathsf{L}}^2(\mathbb{R};\mathbb{C})$ by

$$f_j(x) = \begin{cases} f(x), & x \in [-j, j], \\ 0, & \text{otherwise}, \end{cases}$$

and showing, using the Cauchy–Bunyakovsky–Schwarz inequality, that this sequence converges in $\hat{\mathsf{L}}^2(\mathbb{R};\mathbb{C})$ to $[f]$. Moreover, one can show that the sequence $(\mathscr{F}_1([f_j]))_{j\in\mathbb{Z}_{>0}}$ is a Cauchy sequence $\hat{\mathsf{L}}^2(\mathbb{R};\mathbb{C})$ and so converges to some element of $\hat{\mathsf{L}}^2(\mathbb{R};\mathbb{C})$ that we denote by $\mathscr{F}_2([f])$, the ***$\hat{\mathsf{L}}^2$-Fourier transform*** of $[f] \in \hat{\mathsf{L}}^2(\mathbb{R};\mathbb{C})$. The map $\mathscr{F}_2 \colon \hat{\mathsf{L}}^2(\mathbb{R};\mathbb{C}) \to \hat{\mathsf{L}}^2(\mathbb{R};\mathbb{C})$ so defined is, moreover, a Hilbert space isomorphism. The inverse has the property that

$$\mathscr{F}_2^{-1}([f])(x) = \int_{\mathbb{R}} f \exp_{2\pi\mathrm{i}x} \,\mathrm{d}\lambda$$

for almost every $x \in \mathbb{R}$, where the same constructions leading to the definition of $\mathscr{F}_2$ for functions in $\hat{\mathsf{L}}^2(\mathbb{R};\mathbb{C})$ are applied.

Let us see that $\mathscr{F}_2$ restricted to the subspace of square Riemann integrable functions is problematic. We denote by $\mathsf{R}^p(\mathbb{R};\mathbb{C})$ the collection of functions $f \colon \mathbb{R} \to \mathbb{C}$ which satisfy

$$\int_{-\infty}^{\infty} |f(x)|^p \,\mathrm{d}x < \infty,$$

where we use, as above, the Riemann integral for possibly unbounded functions defined on unbounded domains [Marsden and Hoffman 1993, Section 8.5]. We also define

$$\mathsf{R}_0(\mathbb{R};\mathbb{C}) = \Big\{ f \in \mathsf{R}^p(\mathbb{R};\mathbb{C}) \;\Big|\; \int_{-\infty}^{\infty} |f(x)|^p \,\mathrm{d}x = 0 \Big\}$$

and denote

$$\hat{\mathsf{R}}^p(\mathbb{R};\mathbb{C}) = \mathsf{R}^p(\mathbb{R};\mathbb{C})/\mathsf{R}_0(\mathbb{R};\mathbb{C}).$$

As we have done previously, we denote $[f] = f + \mathsf{R}_0(\mathbb{R};\mathbb{C})$. If we define

$$\|[f]\|_p = \Big(\int_{-\infty}^{\infty} |f(x)|^p \,\mathrm{d}x\Big)^{1/p}$$

then $(\hat{\mathsf{R}}^p(\mathbb{R};\mathbb{C}), \|\cdot\|_p)$ is a normed vector space. It is not a Banach space since the example of Example 3.7 can be extended to $\hat{\mathsf{R}}^p(\mathbb{R};\mathbb{C})$ by taking all functions to be zero outside the interval $[0,1]$.

Let us show that $\mathscr{F}_2|\hat{\mathsf{R}}^2(\mathbb{R};\mathbb{C})$ does not take values in $\hat{\mathsf{R}}^2(\mathbb{R};\mathbb{C})$, and thus show that the "$\hat{\mathsf{R}}^2$-Fourier transform" is not well-defined. We denote by G the function defined in Example 3.5, but now extended to be defined on $\mathbb{R}$ by taking it to be zero off $[0,1]$. We have $G \in \mathsf{L}^1(\mathbb{R};\mathbb{C}) \cap \mathsf{L}^2(\mathbb{R};\mathbb{C})$ since G is bounded and measurable with compact support. Now define $F\colon \mathbb{R} \to \mathbb{C}$ by

$$F(x) = \int_{\mathbb{R}} G \exp_{2\pi \mathrm{i} x} \,\mathrm{d}\lambda;$$

thus F is the inverse Fourier transform of G. Since $G \in \mathsf{L}^1(\mathbb{R};\mathbb{C})$ it follows that $F \in \mathsf{C}^0_{0,\mathrm{uc}}(\mathbb{R};\mathbb{C})$. Therefore, $F|[-R,R]$ is continuous and bounded, and hence Riemann integrable for every $R \in \mathbb{R}_{>0}$. Since $G \in \mathsf{L}^2(\mathbb{R};\mathbb{C})$ we have $F \in \mathsf{L}^2(\mathbb{R};\mathbb{C})$ which implies that

$$\int_{-R}^{R} |F(x)|^2 \,\mathrm{d}x = \int_{[-R,R]} |F|^2 \,\mathrm{d}\lambda \le \int_{\mathbb{R}} |F|^2 \,\mathrm{d}\lambda, \qquad R \in \mathbb{R}_{>0}.$$

Thus the limit

$$\lim_{R\to\infty} \int_{-R}^{R} |F(x)|^2 \,\mathrm{d}x$$

exists. This is exactly the condition for Riemann integrability of F as a function on an unbounded domain [Marsden and Hoffman 1993, Section 8.5]. Now, since $[F] = \mathscr{F}_2^{-1}([G])$ by definition, we have $\mathscr{F}_2([F]) = [G]$. In Example 3.5 we showed that $[G]|[0,1] \notin \hat{\mathsf{R}}^1([0,1];\mathbb{C})$. From this we conclude that $[G] \notin \hat{\mathsf{R}}^1(\mathbb{R};\mathbb{C})$ and, since $|G|^2 = G$, $[G] \notin \hat{\mathsf{R}}^2(\mathbb{R};\mathbb{C})$. Thus $\mathscr{F}_2(\hat{\mathsf{R}}^2(\mathbb{R};\mathbb{C})) \not\subseteq \hat{\mathsf{R}}^2(\mathbb{R};\mathbb{C})$, as it was desired to show.

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