Semicontinuity of rank and nullity and some consequences

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Upper and lower semicontinuity. Let us first define what we mean by the two versions of semicontinuity.

1 Definition: (Upper and lower semicontinuity) Let X be a topological space. A function $f\colon X \to \mathbb{R}$ is

(i) ***upper semicontinuous*** if $f^{-1}(\,]-\infty, a[\,)$ is open for every $a \in \mathbb{R}$ and is

(ii) ***lower semicontinuous*** if $f^{-1}(\,]a, \infty[\,)$ is open for every $a \in \mathbb{R}$. •

One can easily verify that f is continuous if and only if it is both upper and lower semicontinuous. One might gain some insight into semicontinuity by showing that a set $A \subset X$ is open (resp. closed) if and only if the characteristic function χ_A is lower (resp. upper) semicontinuous. We shall refrain here from a detailed discussion of semicontinuity, although such a discussion is worth having. We refer the reader to [Willard 1970].

Semicontinuity of the rank and nullity of a matrix. Let $\boldsymbol{A} \in \mathrm{L}(\mathbb{R}^n; \mathbb{R}^m)$ be an $m \times n$ matrix. The ***rank*** of $\boldsymbol{A}$ is $\mathrm{rank}(\boldsymbol{A}) = \dim(\mathrm{image}(\boldsymbol{A}))$ and the ***nullity*** of $\boldsymbol{A}$ is $\mathrm{nullity}(\boldsymbol{A}) = \dim(\ker(\boldsymbol{A}))$.

2 Proposition: (Semicontinuity of rank and nullity) *Let X be a topological space and let $\boldsymbol{A}\colon X \to \mathrm{L}(\mathbb{R}^n; \mathbb{R}^m)$ be continuous. Then the functions $x \mapsto \mathrm{rank}(\boldsymbol{A}(x))$ and $x \mapsto \mathrm{nullity}(\boldsymbol{A}(x))$ are lower and upper semicontinuous, respectively.*

Proof: Let $a \in \mathbb{R}$ and let $x_0 \in \mathrm{rank}(\boldsymbol{A})^{-1}(\,]a, \infty[\,)$. Thus $k \triangleq \mathrm{rank}(\boldsymbol{A}(x_0)) > a$. There then exists $j_1, \dots, j_k \in \{1, \dots, n\}$ such that the columns $j_1, \dots, j_k$ are linearly independent. Therefore, there exists $i_1, \dots, i_k \in \{1, \dots, m\}$ such that the submatrix

$$\begin{bmatrix} \boldsymbol{A}_{i_1 j_1}(x_0) & \cdots & \boldsymbol{A}_{i_1 j_k}(x_0) \\ \vdots & \ddots & \vdots \\ \boldsymbol{A}_{i_k j_1}(x_0) & \cdots & \boldsymbol{A}_{i_k j_k}(x_0) \end{bmatrix}$$

has nonzero determinant. Since det is a continuous function on the set of $k \times k$ matrices, it follows that the function

$$x \mapsto \det \begin{bmatrix} \boldsymbol{A}_{i_1 j_1}(x) & \cdots & \boldsymbol{A}_{i_1 j_k}(x) \\ \vdots & \ddots & \vdots \\ \boldsymbol{A}_{i_k j_1}(x) & \cdots & \boldsymbol{A}_{i_k j_k}(x) \end{bmatrix}$$

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is continuous. Therefore, there exists a neighbourhood $\mathcal{U}$ of x_0 such that the matrix

$$\begin{bmatrix} \boldsymbol{A}_{i_1 j_1}(x) & \cdots & \boldsymbol{A}_{i_1 j_k}(x) \\ \vdots & \ddots & \vdots \\ \boldsymbol{A}_{i_k j_1}(x) & \cdots & \boldsymbol{A}_{i_k j_k}(x) \end{bmatrix}$$

has nonzero determinant for each $x \in \mathcal{U}$. Therefore, $\operatorname{rank}(\boldsymbol{A}(x)) \geq k > a$ for all $x \in \mathcal{U}$, and so $\mathcal{U} \subset \operatorname{rank}(\boldsymbol{A})^{-1}(\,]a, \infty[\,)$. Thus $\operatorname{rank}(\boldsymbol{A})^{-1}(\,]a, \infty[\,)$ is open, and so $\operatorname{rank}(\boldsymbol{A})$ is lower semicontinuous.

Now let $a \in \mathbb{R}$ and let $x_0 \in \operatorname{nullity}(\boldsymbol{A})^{-1}(\,]-\infty, a[\,)$. Thus $k \triangleq \operatorname{nullity}(\boldsymbol{A}(x_0)) < a$. Recall that the kernel of a matrix is equal to the orthogonal complement of the image of its transpose. This means that there exists $i_1, \dots, i_{m-k} \in \{1, \dots, m\}$ such that the rows $i_1, \dots, i_{m-k}$ form a basis for the orthogonal complement to $\ker(\boldsymbol{A})$. Therefore, there exists $j_1, \dots, j_{m-k} \in \{1, \dots, n\}$ such that the submatrix

$$\begin{bmatrix} \boldsymbol{A}_{i_1 j_1}(x_0) & \cdots & \boldsymbol{A}_{i_1 j_{m-k}}(x_0) \\ \vdots & \ddots & \vdots \\ \boldsymbol{A}_{i_{m-k} j_1}(x_0) & \cdots & \boldsymbol{A}_{i_{m-k} j_{m-k}}(x_0) \end{bmatrix}$$

has nonzero determinant. As above, there exists a neighbourhood $\mathcal{U}$ of x_0 such that this same submatrix has a nonzero determinant. This shows that $\operatorname{rank}(\boldsymbol{A}^T(x)) \geq k$ in some neighbourhood $\mathcal{U}$ of x_0. Therefore, $\operatorname{nullity}(\boldsymbol{A}(x)) \leq k$ in the same neighbourhood $\mathcal{U}$. This shows that $\mathcal{U} \subset \operatorname{nullity}(\boldsymbol{A})^{-1}(\,]-\infty, a[\,)$, and so $\operatorname{nullity}(\boldsymbol{A})^{-1}(\,]-\infty, a[\,)$ is open, giving upper semicontinuity of $\operatorname{nullity}(\boldsymbol{A})$. ■

One sees that the upshot of the lower semicontinuity of $\operatorname{rank}(\boldsymbol{A})$ is that the rank of $\boldsymbol{A}$ does not decrease in a sufficiently small neighbourhood of any point x. Similarly, the upshot of the upper semicontinuity of $\operatorname{nullity}(\boldsymbol{A})$ is that the nullity of $\boldsymbol{A}$ does not increase in a sufficiently small neighbourhood of any point x.

It is often of interest to suppose that a point x is of "constant rank" or "constant nullity."[1]

3 Definition: (Regular and singular points) Let X be a topological space and let $\boldsymbol{A}\colon X \to \mathrm{L}(\mathbb{R}^n; \mathbb{R}^m)$ be continuous. A point $x_0 \in X$ is

(i) a ***rank regular point*** for $\boldsymbol{A}$ if there exists a neighbourhood $\mathcal{U}$ of x_0 such that $\operatorname{rank}(\boldsymbol{A}(x)) = \operatorname{rank}(\boldsymbol{A}(x_0))$ for each $x \in \mathcal{U}$, is

(ii) a ***nullity regular point*** for $\boldsymbol{A}$ if there exists a neighbourhood $\mathcal{U}$ of x_0 such that $\operatorname{nullity}(\boldsymbol{A}(x)) = \operatorname{nullity}(\boldsymbol{A}(x_0))$ for each $x \in \mathcal{U}$, is

(iii) a ***rank singular point*** if it is not a rank regular point, and is

(iv) a ***nullity singular point*** if it is not a nullity regular point. •

The sets of rank and nullity regular points have a topological property that is sometimes useful.

[1] We shall pretend that we do not know about the rank–nullity formula here since it is sometimes convenient to talk separately about rank and nullity without reference to the other.

4 Proposition: (Regular points are open and dense) *Let X be a topological space and let $\boldsymbol{A}\colon X \to \mathrm{L}(\mathbb{R}^n;\mathbb{R}^m)$ be continuous. Then the sets of rank regular points and nullity regular points are open and dense.*

Proof: Let us denote by R_{rank} the set of rank regular points and let $x_0 \in R_{\mathrm{rank}}$. Then, by definition of R_{rank}, there exists a neighbourhood $\mathcal{U}$ of x_0 such that $\mathcal{U} \subset R_{\mathrm{rank}}$. Thus R_{rank} is open. Now let $x_0 \in X$ and let $\mathcal{U}$ be a neighbourhood of x_0. Since the function rank is bounded, there exists a least integer M such that $\mathrm{rank}(\boldsymbol{A}(x)) \le M$ for each $x \in \mathcal{U}$. Moreover, since rank is integer-valued, there exists $x' \in \mathcal{U}$ such that $\mathrm{rank}(\boldsymbol{A}(x')) = M$. Now, by lower semicontinuity of rank, there exists a neighbourhood $\mathcal{U}'$ of x' such that $\mathrm{rank}(\boldsymbol{A}(x)) \ge M$ for all $x \in \mathcal{U}'$. By definition of M we also have $\mathrm{rank}(\boldsymbol{A}(x)) \le M$ for each $x \in \mathcal{U}'$. Thus $x' \in R_{\mathrm{rank}}$, and so $x_0 \in \mathrm{cl}(R_{\mathrm{rank}})$. Therefore R_{rank} is dense.

The argument for the set of nullity regular points proceeds along identical lines except that M is chosen to be the maximal integer such that $\mathrm{nullity}(\boldsymbol{A}(x)) \ge M$ for each $x \in \mathcal{U}$, and one instead uses upper semicontinuity of nullity. ■

5 Remark: (Singular points of measure zero?) One often sees the confounding of "open and dense" and "complement has measure zero" in discussions of rank (or nullity, but let's focus on rank for concreteness) regular points. Note that one can have open and dense subsets of $\mathbb{R}^n$ for which the complement has arbitrarily large measure. Therefore, there is no correspondence between "open and dense" and "complement has measure zero." If we replace the topological space X with a real analytic manifold and require that the matrix function be real analytic, then the set of rank singular points *is* of measure zero. However, much more is true in this case: the set of rank singular points is a proper analytic subset (meaning it is locally the intersection of the set of zeros of a finite number of analytic functions). Measure zero is a worthlessly coarse description of a proper analytic subset. Therefore, "measure zero" should be left out of this discussion, and one should use "open and dense" in the continuous case and "complement being a proper analytic subset" in the analytic case. •

Consequences. The constructions in the preceding section involve matrices. Let us see how these may be used to infer similar properties of rank and nullity for more geometric constructions.

First we let $r \in \mathbb{Z}_{\ge 0} \cup \{\infty\} \cup \{\omega\}$ and let M and N be C^r-manifolds. We let $\pi\colon \mathsf{E} \to \mathsf{M}$ and $\tau\colon \mathsf{F} \to \mathsf{N}$ be C^r vector bundles with finite-dimensional fibres and we let $f\colon \mathsf{E} \to \mathsf{F}$ be a C^r-vector bundle map over $f_0\colon \mathsf{M} \to \mathsf{N}$. We have functions $\mathrm{rank}(f), \mathrm{nullity}(f)\colon \mathsf{M} \to \mathbb{Z}_{\ge 0}$ defined by asking that $\mathrm{rank}(f)(x)$ be the rank of $f|\mathsf{E}_x$ and $\mathrm{nullity}(f)(x)$ be the nullity of $f|\mathsf{E}_x$. We define the set of ***rank regular points*** (resp. ***nullity regular points***) to be the points $x_0 \in \mathsf{M}$ for which there exists a neighbourhood $\mathcal{U}$ of x_0 such that $\mathrm{rank}(f)|\mathcal{U}$ (resp. $\mathrm{nullity}(f)|\mathcal{U}$) is constant. The set of ***rank singular points*** (resp. ***nullity singular points***) is the complement in M of the set of rank regular points (resp. nullity regular points).

We now have the following result concerning rank, nullity, and regular points.

6 Proposition: (Rank, nullity, and regular points for vector bundle maps) *Let r, M, N, E, F, f, and f_0 be as above. Then the functions* $\mathrm{rank}(f)$ *and* $\mathrm{nullity}(f)$ *are lower*

semicontinuous and upper semicontinuous, respectively. Moreover, the sets of rank regular points and nullity regular points are open and dense.

Proof: Let $a \in \mathbb{R}$ and let $x_0 \in \operatorname{rank}(f)^{-1}(]a,\infty[)$. Thus $k \triangleq \operatorname{rank}(f)(x_0) > a$. Choose vector bundle charts $(\mathcal{U},\phi)$ and $(\mathcal{V},\psi)$ for E and F, respectively, supposing that $f(\mathcal{U}) \subset \mathcal{V}$, this being possible by continuity of f. Denote by $(\mathcal{U}_0,\phi_0)$ and $(\mathcal{V}_0,\psi_0)$ the corresponding charts for M and N, respectively. Write the local representative of f in these charts by

$$\mathcal{U}' \times \mathbb{R}^n \ni (\boldsymbol{x},\boldsymbol{v}) \mapsto (\boldsymbol{f}_0(\boldsymbol{x}), \boldsymbol{A}(\boldsymbol{x}) \cdot \boldsymbol{v}) \in \mathcal{V}' \times \mathbb{R}^m,$$

where $\mathcal{U}'$ and $\mathcal{V}'$ are some open subsets of appropriately dimensioned Euclidean spaces, and $\boldsymbol{f}_0 \colon \mathcal{U}' \to \mathcal{V}'$ and $\boldsymbol{A} \colon \mathcal{U}' \to \mathrm{L}(\mathbb{R}^n;\mathbb{R}^m)$ are maps of class C^r. By upper semicontinuity of the map $\boldsymbol{x} \mapsto \boldsymbol{A}(\boldsymbol{x})$ there exists a neighbourhood $\mathcal{U}_0'$ of $\phi_0(x_0)$ such that $\mathcal{U}_0 \subset \operatorname{rank}(\boldsymbol{A})^{-1}(]a,\infty[)$. The neighbourhood $\phi_0^{-1}(\mathcal{U}_0')$ of x_0 then is a subset of $\operatorname{rank}(f)^{-1}(]a,\infty[)$, giving lower semicontinuity of $\operatorname{rank}(f)$.

An entirely similar proof shows that $\operatorname{nullity}(f)$ is upper semicontinuous.

The proof that the set of rank regular points is open and dense goes exactly as in the proof of Proposition 4 for matrices. Similarly, the openness and denseness of the set of nullity regular points can be proved. ■

Of course, Remark 5 can be inserted into the discussion here.

Now let us turn to another geometric construction, a generalised subbundle.

7 Definition: (Generalised subbundle) Let $r \in \mathbb{Z}_{\geq 0} \cup \{\infty\} \cup \{\omega\}$ and let $\pi \colon \mathsf{E} \to \mathsf{M}$ be a C^r-vector bundle with finite-dimensional fibres. A subset $\mathcal{F} \subset \mathsf{E}$ is a ***C^r-generalised subbundle*** if $\mathcal{F}_x \triangleq \mathsf{E}_x \cap \mathcal{F}$ is a subspace of E_x for each $x \in \mathsf{M}$ and if, for each $x_0 \in \mathsf{M}$, there exists a neighbourhood $\mathcal{U}$ of x_0 and a collection $\{\xi_a\}_{a\in A}$ of C^r-sections of $\mathsf{E}|\mathcal{U}$ such that $\mathcal{F}_x = \operatorname{span}_{\mathbb{R}}(\xi_a(x)|\ a \in A)$ for each $x \in \mathcal{U}$. The local sections $\{\xi_a\}_{a\in A}$ are called ***local generators*** for $\mathcal{F}$. •

8 Remark: (Cardinality of local generators) If $r = \omega$ in the preceding definition, the Noetherian property of the ring of germs of analytic functions ensures that the index set A can be chosen to be finite. However, if $r \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, one cannot generally place bounds on the number of local generators. •

The ***rank*** of a generalised subbundle $\mathcal{F}$ is the function $\operatorname{rank}(\mathcal{F}) \colon \mathsf{M} \to \mathbb{Z}_{\geq 0}$ defined by $\operatorname{rank}(\mathcal{F}) = \dim(\mathcal{F}_x)$. A point $x \in \mathsf{M}$ is a ***regular point*** for $\mathcal{F}$ if there exists a neighbourhood $\mathcal{U}$ of x such that $\operatorname{rank}(\mathcal{F})|\mathcal{U}$ is constant. A point in M that is not a regular point is a ***singular point***.

We then have the (by now expected) following result describing rank and the set of regular points.

9 Proposition: (Rank and regular points for generalised subbundles) *Let r, M, E, and $\mathcal{F}$ be as above. Then the function* $\operatorname{rank}(\mathcal{F})$ *is lower semicontinuous and the set of regular points of $\mathcal{F}$ is open and dense.*

Proof: This does not quite just follow from our constructions with matrices, although it could be made to by extending our notion of matrix to allow for arbitrary numbers of rows and columns (i.e., the number of rows and columns are allowed to be any cardinal number). But rather than do this, let us proceed directly.

Let $a \in \mathbb{R}$ and let $x_0 \in \operatorname{rank}(\mathcal{F})^{-1}(\,]a,\infty[\,)$. Thus $k \triangleq \dim(\mathcal{F}_{x_0}) > a$. This means that there are k sections $\xi_{a_1},\dots,\xi_{a_k}$ defined in a neighbourhood $\mathcal{U}$ of x_0 such that $\mathcal{F}_{x_0} = \operatorname{span}_{\mathbb{R}}(\xi_{a_1}(x_0),\dots,\xi_{a_k}(x_0))$. Now choose a vector bundle chart $(\mathcal{U},\phi)$ about x_0 so that the local sections $\xi_{a_1},\dots,\xi_{a_k}$ have local representatives

$$\boldsymbol{x} \mapsto (\boldsymbol{x}, \boldsymbol{\xi}_{a_j}(\boldsymbol{x})), \qquad j \in \{1,\dots,k\}.$$

Let $(\mathcal{U}_0,\phi_0)$ be the induced chart for M and let $\boldsymbol{x}_0 = \phi_0(x_0)$. The vectors $\{\boldsymbol{\xi}_{a_1}(\boldsymbol{x}_0),\dots,\boldsymbol{\xi}_{a_k}(\boldsymbol{x}_0)\}$ are then linearly independent. Therefore there exists $j_1,\dots,j_k \in \{1,\dots,n\}$ (supposing that n is the dimension of the fibres on $\mathcal{U}_0$) such that the matrix

$$\begin{bmatrix} \boldsymbol{\xi}_{a_1 j_1}(\boldsymbol{x}_0) & \cdots & \boldsymbol{\xi}_{a_k j_1}(\boldsymbol{x}_0) \\ \vdots & \ddots & \vdots \\ \boldsymbol{\xi}_{a_1 j_k}(\boldsymbol{x}_0) & \cdots & \boldsymbol{\xi}_{a_k j_k}(\boldsymbol{x}_0) \end{bmatrix}$$

has nonzero determinant, where $\boldsymbol{\xi}_{a_i j_l}(\boldsymbol{x}_0)$ is the j_lth component of $\boldsymbol{\xi}_{a_i}$, $i,l \in \{1,\dots,k\}$. By continuity of the determinant there exists a neighbourhood $\mathcal{U}'$ of $\boldsymbol{x}_0$ such that the matrix

$$\begin{bmatrix} \boldsymbol{\xi}_{a_1 j_1}(\boldsymbol{x}) & \cdots & \boldsymbol{\xi}_{a_k j_1}(\boldsymbol{x}) \\ \vdots & \ddots & \vdots \\ \boldsymbol{\xi}_{a_1 j_k}(\boldsymbol{x}) & \cdots & \boldsymbol{\xi}_{a_k j_k}(\boldsymbol{x}) \end{bmatrix}$$

has nonzero determinant for every $\boldsymbol{x} \in \mathcal{U}'$. Thus the vectors $\{\boldsymbol{\xi}_{a_1}(\boldsymbol{x}),\dots,\boldsymbol{\xi}_{a_k}(\boldsymbol{x})\}$ are linearly independent for every $\boldsymbol{x} \in \mathcal{U}'$. Therefore, the local sections $\xi_{a_1},\dots,\xi_{a_k}$ are linearly independent on $\phi_0^{-1}(\mathcal{U}')$, and so $\phi_0^{-1}(\mathcal{U}') \subset \operatorname{rank}(\mathcal{F})^{-1}(\,]a,\infty[\,)$ which gives lower semicontinuity of $\operatorname{rank}(\mathcal{F})$.

The openness and denseness of the set of regular points for $\mathcal{F}$ can be proved in a manner analogous to the proof of Proposition 4. ■

Clearly at this point one can insert Remark 5.

Where does the analogue of nullity arise in the discussion of generalised subbundles? Let r, M, E, and $\mathcal{F}$ be as above, and define $\operatorname{ann}(\mathcal{F})$ to be the subset of the dual bundle E^* defined by

$$\operatorname{ann}(\mathcal{F})_x \triangleq \operatorname{ann}(\mathcal{F}) \cap \mathsf{E}^*_x = \operatorname{ann}(\mathcal{F}_x).$$

This is the annihilator of $\mathcal{F}$. The ***rank*** of $\operatorname{ann}(\mathcal{F})$ is the function $\operatorname{rank}(\operatorname{ann}(\mathcal{F}))(x) = \dim(\operatorname{ann}(\mathcal{F}_x))$. A ***regular point*** for $\operatorname{ann}(\mathcal{F})$ is a point $x \in \mathsf{M}$ such that there exists a neighbourhood $\mathcal{U}$ of x for which $\operatorname{rank}(\operatorname{ann}(\mathcal{F}))|\mathcal{U}$ is constant. A ***singular point*** for $\operatorname{ann}(\mathcal{F})$ is a point that is not a regular point. One can then show that the rank of $\operatorname{ann}(\mathcal{F})$ is upper semicontinuous and that the set of regular points is open and dense. We leave this as an elementary exercise that can be performed following the proof of upper semicontinuity of nullity in Proposition 2.

References

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