A square Riemann integrable function whose Fourier transform is not square Riemann integrable

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2009/03/18
Last updated: 2009/09/15

Abstract

An explicit example of such a function is provided.

It is tacitly understood that the power of the Lebesgue integral (or one of its equivalents) is needed for many of the results that underpin modern system theory, particularly linear system theory. In particular, the completeness of the normed vector spaces of signals with integral norms, the L^p-spaces, is an essential component in the fundamental theorems of system theory. Yet, despite this well understood necessity of the Lebesgue integral machinery, many of us feel that we can safely restrict ourselves to thinking about nicer classes of functions than those that are Lebesgue integrable, possibly classes of functions that can be characterised by the "tamer" Riemann integral. But these feelings are invariably dashed by the poor behaviour of the Riemann integral under limit operations. In this paper we give an explicit illustration of this principle as it pertains to the Fourier transform.

Notation

Here is the notation we use in the paper.

If A is a subset of B we write $A \subseteq B$, and we denote proper inclusion by $A \subset B$. By χ_A we denote the characteristic function of a set A. By $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ we denote the integers, the rationals, the real numbers, and the complex numbers, respectively. By $\mathbb{Z}_{>0}$ and $\mathbb{R}_{>0}$ we denote the set of positive integers and positive real numbers, respectively.

The Lebesgue measure on $\mathbb{R}$ is denoted by λ.

The classical L^p-spaces are equivalence classes of signals under the equivalence relation of almost everywhere equality. We shall pay attention in this paper to the distinction between a function and the equivalence class of a function. For an interval $I \subseteq \mathbb{R}$ we denote by $\mathsf{L}^{(p)}(I;\mathbb{C})$ the set measurable of $\mathbb{C}$-valued functions on I for which the function $I \ni x \mapsto |f(x)|^p \in \mathbb{R}$ is Lebesgue integrable. If $f \in \mathsf{L}^{(p)}(I;\mathbb{C})$ then $[f]$ denotes the set of functions that agree almost everywhere with f. We denote by $\mathsf{L}^p(I;\mathbb{C})$ the set of equivalence classes. On $\mathsf{L}^{(p)}(I;\mathbb{C})$ we have the seminorm $\|\cdot\|_p$ defined by

$$\|f\|_p = \Big(\int_I |f(x)|^p \, \mathrm{d}x\Big)^{1/p},$$

and corresponding to this is the norm on $\mathsf{L}^p(I;\mathbb{C})$, also denoted by $\|\cdot\|_p$, defined by $\|[f]\|_p = \|f\|_p$.

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Research supported in part by a grant from the Natural Sciences and Engineering Research Council of Canada

We wish to be careful about what we mean by the Riemann integral so that we can properly compare the Riemann and Lebesgue integrals. The Riemann integral in which we are interested is the improper Riemann integral which is defined for possibly unbounded functions on possibly noncompact intervals; see [Marsden and Hoffman 1993, Section 8.5]. With this definition of the Riemann integral, the Lebesgue integral extends the Riemann integral in all cases. (In some texts on integration, there is an annoying tendency to give incomparable definitions of the Riemann and Lebesgue integrals with the result that neither of the sets of integrable functions is a subset of the other.) By $\mathsf{R}^{(p)}(I;\mathbb{C})$ we denote the subset of $\mathsf{L}^{(p)}(I;\mathbb{C})$ consisting of functions such that $x \mapsto |f(x)|^p$ is Riemann integrable. By $\mathsf{R}^p(I;\mathbb{C})$ we denote the set of equivalence classes of functions in $\mathsf{R}^{(p)}(I;\mathbb{C})$ under the equivalence relation $f \sim g$ if and only if $\|f-g\|_p = 0$. If one understands the relationship between the Riemann and Lebesgue integrals, one can see that $\mathsf{R}^p(I;\mathbb{C}) \subset \mathsf{L}^p(I;\mathbb{C})$.

The L^2-Fourier transform

Let us recall the definition of the Fourier transform for square integrable functions; we refer to Gasquet and Witomski [1999] for details concerning the Fourier transform. Let $[f] \in \mathsf{L}^2(\mathbb{R};\mathbb{C})$ and define

$$f_j(x) = \begin{cases} f(x), & x \in [-j,j], \\ 0, & \text{otherwise.} \end{cases}$$

Then $f_j \in \mathsf{L}^1(\mathbb{R};\mathbb{C}) \cap \mathsf{L}^2(\mathbb{R};\mathbb{C})$ and so one can define $\mathscr{F}(f_j)\colon \mathbb{R} \to \mathbb{C}$ by

$$\mathscr{F}(f_j)(y) = \int_{\mathbb{R}} f_j(x) \mathrm{e}^{-2\pi \mathrm{i} xy}\, \mathrm{d}x.$$

Then one can show that the sequence $([\mathscr{F}(f_j)])_{j\in\mathbb{Z}_{>0}}$ is a Cauchy sequence in $\mathsf{L}^2(\mathbb{R};\mathbb{C})$. By completeness of $\mathsf{L}^2(\mathbb{R};\mathbb{C})$ we define $\mathscr{F}([f])$ as the limit of this Cauchy sequence, calling this the ***L^2-Fourier transform*** of $[f]$. The same construction applies with "$\mathrm{e}^{-2\pi \mathrm{i} xy}$" in the integrand being replaced with "$\mathrm{e}^{2\pi \mathrm{i} xy}$." The result is then the inverse of the L^2-Fourier transform.

A limit of Riemann integrable functions that is not Riemann integrable

We now turn our attention to the essential construction of the paper, that of a sequence in $\mathsf{R}^1((0,1);\mathbb{C})$ that converges in $\mathsf{L}^1((0,1);\mathbb{C})$ to a function that is not Riemann integrable. The standard example of such a sequence has as its limit the characteristic function of the rational numbers in $(0,1)$. This example is not adequate for us since the characteristic function of the rational numbers in $(0,1)$ is equivalent to the zero function which *is* Riemann integrable. Since we are interested, not in functions, but equivalence classes of functions, this construction is, therefore, inadequate for our purposes.

In order to illustrate the character of the construction, we provide a general construction in which our specific example is embedded. We begin with a lemma.

1 Lemma: *There exists an open set $U \subseteq (0,1)$ such that $\lambda(\mathrm{bd}(U)) > 0$.*

Proof: Let $(q_j)_{j\in\mathbb{Z}_{>0}}$ be an enumeration of the rational numbers in $(0,1)$. Let $\ell \in (0,1)$ and for $j \in \mathbb{Z}_{>0}$ define

$$I_j = (0,1) \cap (q_j - \tfrac{\ell}{2^{j+1}}, q_j + \tfrac{\ell}{2^{j+1}})$$

to be the interval of length $\frac{\ell}{2^j}$ centred at q_j. Then define $U_k = \cup_{j=1}^k I_j$, $k \in \mathbb{Z}_{>0}$, and $U = \cup_{k\in\mathbb{Z}_{>0}} U_k$. Since $\mathbb{Q} \cap (0,1) \subseteq U$ we have $\mathrm{cl}(U) = [0,1]$. Thus $[0,1] = U \cup \mathrm{bd}(U)$ and so

$$\lambda([0,1]) \leq \lambda(U) + \lambda(\mathrm{bd}(U)).$$

Since

$$\lambda(U) \leq \sum_{j=1}^{\infty} \lambda(I_j) \leq \sum_{j=1}^{\infty} \frac{\ell}{2^j} = \ell,$$

it follows that $\lambda(\mathrm{bd}(U)) \geq 1 - \ell > 0$. ■

The set U we construct in the lemma has the benefit of being more or less easily understandable. But it might be criticised as being somewhat degenerate on the grounds that its closure is quite large compared to the set itself. For example, the closure has measure 1 whereas the set has measure $\ell \in (0,1)$. Also, $U \subset \mathrm{int}(\mathrm{cl}(U))$. However, with a little more work one can devise nicer open sets having the properties asserted in the lemma. For example, Börger [1999] gives an example of an open set $U \subseteq (0,1)$ such that $\mathrm{int}(\mathrm{cl}(U)) = U$ and $\lambda(\mathrm{bd}(U)) > 0$.

The next lemma gives a property of the characteristic function of any set U such as in Lemma 1.

2 Lemma: *If U is as in Lemma 1 then χ_U is not Riemann integrable and is the pointwise limit of Riemann integrable functions.*

Proof: Recall from [e.g., Marsden and Hoffman 1993] that a function is Riemann integrable if and only if its set of discontinuities has measure zero. The characteristic function of a subset A is, therefore, Riemann integrable if and only if $\lambda(\mathrm{bd}(A)) = 0$. From this we immediately have that χ_U is not Riemann integrable.

As U is an open subset of $\mathbb{R}$ it is the countable union of open disjoint open intervals, say $U = \cup_{j\in\mathbb{Z}_{>0}} I_j$. Denote $U_k = \cup_{j=1}^k I_j$, $k \in \mathbb{Z}_{>0}$. Each of the characteristic functions χ_{I_j} is Riemann integrable as the characteristic function of an interval is clearly Riemann integrable. Therefore, χ_{U_k} is Riemann integrable since it is a finite sum of Riemann integrable functions. Since $U = \cup_{k\in\mathbb{Z}_{>0}} U_k$, χ_U is obviously the pointwise limit of the sequence $(\chi_{U_k})_{k\in\mathbb{Z}_{>0}}$ of Riemann integrable functions. ■

Note that χ_U is Lebesgue integrable by the Dominated Convergence Theorem.

In the proof of the preceding lemma we defined a sequence $(U_k)_{k\in\mathbb{Z}_{>0}}$ of open sets such that $U_k \subseteq U_{k+1}$, $k \in \mathbb{Z}_{>0}$, and $U = \cup_{k\in\mathbb{Z}_{>0}} U_k$. The sequence $(\chi_{U_k})_{k\in\mathbb{Z}_{>0}}$ converges pointwise to χ_U. The following lemma gives convergence in $\mathsf{L}^1((0,1);\mathbb{C})$.

3 Lemma: *The sequence $([\chi_{U_k}])_{k\in\mathbb{Z}_{>0}}$ is a Cauchy sequence in $\mathsf{L}^1((0,1);\mathbb{C})$.*

Proof: We let $(I_j)_{j\in\mathbb{Z}_{>0}}$ be the sequence of disjoint open intervals defined in the proof of Lemma 2. Since $\cup_{j\in\mathbb{Z}_{>0}} I_j = U \subset (0,1)$,

$$\sum_{j=1}^{\infty} \lambda(I_j) < 1.$$

Thus, for $\epsilon > 0$ there exists $N \in \mathbb{Z}_{>0}$ such that

$$\sum_{j=N+1}^{\infty} \lambda(I_j) < \epsilon.$$

Then, for $k \geq N$ we have

$$\int_0^1 |\chi_U(x) - \chi_{U_k}(x)| \, \mathrm{d}x = \sum_{j=N+1}^{\infty} \lambda(I_j) < \epsilon,$$

which gives the result. ■

Since we are interested in equivalence classes of functions, the following result will be important to us.

4 Lemma: *If U is as in Lemma 1 and if $f\colon (0,1) \to \mathbb{R}$ is any function agreeing almost everywhere with χ_U, then f is not Riemann integrable.*

Proof: It suffices to show that f is discontinuous on a set of positive measure. We shall show that f is discontinuous on the set $f^{-1}(0) \cap \mathrm{bd}(U)$. Indeed, let $x \in f^{-1}(0) \cap \mathrm{bd}(U)$. Then, for any $\epsilon > 0$ we have $(x-\epsilon, x+\epsilon) \cap U \neq \varnothing$ since $x \in \mathrm{bd}(U)$. Since $(x-\epsilon, x+\epsilon) \cap U$ is a nonempty open set, it has positive measure. Therefore, since χ_U and f agree almost everywhere, there exists $y \in (x-\epsilon, x+\epsilon) \cap U$ such that $f(y) = 1$. Since this holds for every $\epsilon \in \mathbb{R}_{>0}$ and since $f(x) = 0$, it follows that f is discontinuous at x. Finally, it suffices to show that $f^{-1}(0) \cap \mathrm{bd}(U)$ has positive measure. But this follows since $\mathrm{bd}(U) \subseteq \chi_U^{-1}(0)$ has positive measure and since χ_U and f agree almost everywhere. ■

To summarise:

There exists a sequence $(F_j)_{j \in \mathbb{Z}_{>0}}$ of functions $F_j\colon (0,1) \to \mathbb{C}$ such that

(i) $F_j(x) \in \{0,1\}$ for every $x \in (0,1)$,

(ii) F_j is Riemann integrable for each $j \in \mathbb{Z}_{>0}$,

(iii) the sequence $(F_j)_{j \in \mathbb{Z}_{>0}}$ converges pointwise to a function F such that $[F] \in \mathsf{L}^1((0,1);\mathbb{C}) \setminus \mathsf{R}^1((0,1);\mathbb{C})$.

A square Riemann integrable function whose Fourier transform is not square Riemann integrable

We denote by F the function defined above, but now extended to be defined on $\mathbb{R}$ by taking it to be zero off $(0,1)$. We have $F \in \mathsf{L}^1(\mathbb{R};\mathbb{C}) \cap \mathsf{L}^2(\mathbb{R};\mathbb{C})$ since F is bounded and measurable with compact support. Now define $f\colon \mathbb{R} \to \mathbb{C}$ by

$$f(x) = \int_{\mathbb{R}} F(y) \mathrm{e}^{2\pi \mathrm{i} xy} \, \mathrm{d}y;$$

thus f is the inverse Fourier transform of F. Since $F \in \mathsf{L}^1(\mathbb{R};\mathbb{C})$ it follows that f is uniformly continuous and decays to zero at infinity. Therefore, $f|[-R,R]$ is continuous and bounded, and hence Riemann integrable, for every $R > 0$. Since $F \in \mathsf{L}^2(\mathbb{R};\mathbb{C})$ we have $f \in \mathsf{L}^2(\mathbb{R};\mathbb{C})$ which implies that

$$\int_{-R}^{R} |f(x)|^2 \, \mathrm{d}x \leq \int_{\mathbb{R}} |f|(x)^2 \, \mathrm{d}x, \qquad R > 0.$$

Thus the limit

$$\lim_{R\to\infty} \int_{-R}^{R} |f(x)|^2 \, \mathrm{d}x$$

exists. This is exactly the condition for square Riemann integrability of f as a function on an unbounded domain [Marsden and Hoffman 1993, Section 8.5]. Now, since $[f] = \mathscr{F}^{-1}([F])$ by definition, we have $\mathscr{F}([f]) = [F]$. We showed above that $[F] \notin \mathsf{R}^1(\mathbb{R};\mathbb{C})$ and, since $|F|^2 = F$, $[F] \notin \mathsf{R}^2(\mathbb{R};\mathbb{C})$. Thus f is the desired square Riemann integrable function whose Fourier transform is not square Riemann integrable.

References

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