Geometric analysis on real analytic manifolds

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Abstract

The continuity, in a suitable topology, of algebraic and geometric operations on real analytic manifolds and vector bundles is proved. This is carried out using recently arrived at seminorms for the real analytic topology. A new characterisation of the topology of the space of real analytic mappings between manifolds is also developed. To characterise these topologies, geometric decompositions of various jet bundles are given by use of connections. These decompositions are then used to characterise many of the standard operations from differential geometry: algebraic operations, tensor evaluation, various lifts of tensor fields, compositions of mappings, etc. Apart from the main results, numerous techniques are developed that will facilitate the performing of analysis on real analytic manifolds.

Keywords. geometric analysis, real analytic manifolds, locally convex topological vector spaces

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1. Introduction

Almost all operations/operators in differential geometry are formed by combining a few essential operations such as composition, prolongation, tensor evaluation, and/or some sort of lifting process. Typically, these operations are tacitly regarded as being continuous in some sense. In smooth differential geometry, "in some sense" usually means with respect to the smooth compact-open topology (the topology of uniform convergence of derivatives on compact sets) or, if one is interested in differential topology, a topology like the Whitney topology. An accounting of these sorts of topologies can be found in [Hirsch 1976, Michor 1980]. It is pretty easy to convince oneself of the continuity of standard operations in the smooth compact-open topology; one only needs to put suitable bounds on finitely many derivatives on compact sets. Thus there is some justification in not working this out carefully in the smooth case. However, if one is interested in continuity in the real analytic category, it is not very easy to convince oneself about the continuity of geometric operations. Indeed, the more one thinks about this, the harder the problem becomes.

A barrier right at the start is that the appropriate topology for real analytic functions (functions, for simplicity) is not so easily envisaged. While a suitable real analytic topology has been around since at least the work of Martineau [1966]—who provided two descriptions of such a topology, and showed that they agree—there has not been a "user-friendly" description of the real analytic topology, i.e., a description using seminorms, until quite recently. Some useful initial formulae are provided by Mujica [1984], and seminorms are provided in the lecture notes of [Domański 2012]. However, as far as we are aware, it is only in the technical note of Vogt [2013] that we see a proof of the suitability of these seminorms. These were adapted to the geometric setting for sections of a real analytic vector bundle by Jafarpour and Lewis [2014]. Part of this development was a decomposition of jet bundles using connections. The initial developments of that monograph are the starting point for our approach here.

Another complicating facet of the real analytic theory arises when one considers lifts from the base space to the total space of a real analytic vector bundle $\pi_{\mathsf{E}}\colon \mathsf{E} \to \mathsf{M}$, e.g., vertical lift of a section of E or horizontal lift of a vector field on M. The first of these operations requires no additional structure, but the second requires a connection. However, both require connections to study their real analytic continuity, because one needs to provide bounds for the jets on the codomain (i.e., on E) in terms of jets on the domain (i.e., on M). To provide seminorms, one also needs Riemannian and vector bundle metrics, and all of this data has to fit together nicely to provide the bounds required. For instance, one has a natural Riemannian metric on the manifold E arising from (1) a Riemannian metric on M, (2) a fibre metric on E, (3) an affine connection on M, and (4) a linear connection in E. This structure makes use of the resulting structure of $\pi_{\mathsf{E}}\colon \mathsf{E} \to \mathsf{M}$ being a Riemannian submersion, and using formulae of O'Neill [1968]. The determination of a systematic means to provide jet bundle estimates in this setting occupies us for a significant portion of the paper.

In order to illustrate the nature of the difficulties one encounters, let us consider a specific and illustrative instance of the sort of argument that one must piece together to prove continuity in the real analytic case. Suppose that we have a real analytic vector bundle $\pi_{\mathsf{E}}\colon \mathsf{E} \to \mathsf{M}$ with $\nabla^{\pi_{\mathsf{E}}}$ a real analytic linear connection in E. Let X be a real analytic vector field on M which we horizontally lift to a real analytic vector field X^{h} on E. To assess the continuity of the map $X \mapsto X^{\mathrm{h}}$ in the real analytic topology, one needs to compute jets of X^{h} and relate these to jets of X. Thus one needs to differentiate X^{h} arbitrarily many times. This differentiation must be done on E, as this is the base on which X^{h} is defined. Trying this directly in local coordinates is, in principle, possible, but it is pretty unlikely that one will be able to produce the refined estimates required in this way. Thus, in our approach, one needs an affine connection on E (thinking of E as just a manifold now). One can now see that there will be a complicated intermingling of the linear connection $\nabla^{\pi_{\mathsf{E}}}$, an affine connection ∇^{M} on M (to compute jets of X), and a fabricated affine connection ∇^{E} on E. This is only the beginning of the difficulties one faces. One also needs, not only formulae for the derivatives of X^{h}, but also recursive formulae relating how a derivative of X^{h} of order, say, k is related to the derivatives of X of orders $0, 1, \dots, k$. This recursive formulae is essential for being able to obtain growth estimates for the derivatives needed to relate the seminorms applied to X^{h} to those applied to X. Moreover, since the mapping $X \mapsto X^{\mathrm{h}}$ is injective, one might hope that the mapping is not just continuous, but is an homeomorphism onto its image. To prove this, one now needs to get estimates for the jets of X from formulae involving the jets of X^{h}. Thus one needs estimates that go "both ways." It is also worth mentioning that the estimates one needs from these recursive formulae are quite unforgiving, and so their form has to be very precisely managed. This requires extensive bookkeeping. This bookkeeping occupies us for a substantial portion of the paper. This is contrasted with the smooth case, where very coarse bounds suffice; we shall say a few words about this contrast at illustrative places in the paper.

Another difficulty is that the use of connections to compute derivatives for jets forces one to address the matter of whether the seminorms used for jets, and derived from the use of connections, are actually not dependent on the chosen connection. Thus one must compare iterated covariant derivatives with respect to different connections and show that these are related to one another in such a way that the resulting real analytic topology is well defined. This, in itself, is a substantial undertaking. It is done in an *ad hoc* way by Jafarpour and Lewis [2014, Lemma 2.5]; here we do this in a systematic and geometric way that offers many benefits towards the objectives of this paper, apart from rendering more attractive the computations of Jafarpour and Lewis.

We mention that the idea of obtaining recursive formulae for derivatives is given in a local setting by Thilliez [1997] during the course of the proof of his Proposition 2.5, and can be applied to the mapping $\mathrm{C}^{\omega}(\mathsf{N}) \ni f \mapsto \Phi^* f \in \mathrm{C}^{\omega}(\mathsf{M})$ of pull-back by a real analytic mapping $\Phi \in \mathrm{C}^{\omega}(\mathsf{M}; \mathsf{N})$. We are able to extend the ideas in Thilliez' computations to general classes of geometric operations. For example, as we mention above, a local working out of the estimates for the horizontal lift operation seems like it will be very difficult. However, once one *does* get these things to work out, it is relatively straightforward to prove the main results of the paper, which are the continuity of the fundamental geometric operations mentioned in the first paragraph.

One of the features of the paper is that almost all constructions are done intrinsically. While this may seem to unnecessarily complicate things, this is not, in fact, so. Even were one to work locally, there would still arise two difficult problems that we overcome in our approach, but that still must be overcome in a local approach: (1) the difficulty of lifts as described in detail above; (2) the verification that the topologies do not depend on various choices made (charts in the local calculations, and metrics and connections in the intrinsic calculations). Thus, while the intrinsic calculations are sometimes complicated, they are only a little more complicated than the necessarily already complicated local calculations. And we believe that the intrinsic approach is ultimately easier to use, once one understands how to use it. An objective of this paper is to do a lot of the tedious hard work required to produce methods and results that are themselves more or less straightforward.

As a side-benefit to our approach, we also are able to easily provide proofs in the finitely differentiable and smooth cases. We point out the relevant places where modifications can be made to the real analytic proofs to give the results in the finitely differentiable and smooth cases.

1.1. Organisation of paper. In Section 2 we review the definition of the real analytic topology and the geometric seminorms for this topology as constructed in [Jafarpour and Lewis 2014].

Section 3 is the first of three sections, forming the bulk of the paper in terms of words used, where we provide a host of geometric constructions whose bearing on the main goal of the paper will be difficult to glean on a first reading. Some sketchy motivation for the constructions of Sections 3, 4, and 5 is outlined above in our discussion of the difficulties one will encounter trying to prove continuity of the horizontal lift mapping $X \mapsto X^{\mathrm{h}}$. In Section 3 we perform constructions with functions, vector fields, and tensors on the total space of a vector bundle. These form the basis for derivative computations done in Section 4. Particularly, in Section 4.1 we give $\pi_{\mathsf{E}}\colon \mathsf{E} \to \mathsf{M}$ the structure of a Riemannian submersion, following O'Neill [1968]. This allows us to relate, in a natural way, constructions on E with those on M. In Section 5 we provide the crucial recursive formulae that relate derivatives on E with those on M. We do this for a few of the standard geometric lifts one has for a vector bundle with a linear connection. Some of these we do because they are intrinsically interesting. Some we do because they are required for our general approach, even if one is not interested in them *per se*.

In Section 6 we give fibre norms for various jet bundles that are used to define seminorms corresponding to the geometric constructions of interest. In Section 7 we put all of our work from Sections 3–6 to use to prove Lemma 7.8, the technical lemma which makes everything work. The lemma gives a very precise estimate for the fibre norms of derivatives of coefficients that arise in the recursive constructions of Section 5. There is no wiggle room in the form of the required estimate, and this is one of the reasons why the computations of Sections 3–5 are so laboriously carried out; these computations need to be understood at a high resolution. Once we have these estimates, however, in Section 8 we show that the fibre-norms for jet bundles obtained in Section 6 behave in the proper way as to make the topologies we construct independent of our choices of connections and metrics. This is stated as Lemma 8.7. The actual proving of the independence of the topologies is carried out by proving in Theorem 8.10 that the topologies are each the same as a topology described using local forms of the seminorms. This device of using a local description carries two benefits.

1. It provide the local description of the seminorms. While our approach is intrinsic as much as this is possible, sometimes in practice one must work locally, and having the explicit local formulae for the fibre norms is beneficial.
2. While we have tried to make our treatment intrinsic, there is a crucial point where a *local* estimate for the growth of derivatives becomes unavoidable, resting as it does on the Cauchy estimates for holomorphic functions. In our proof of Theorem 8.10 is where this seemingly unavoidable local estimate is not avoided.

In Section 9 we prove continuity of some representative and some important geometric constructions. There is a long list of these constructions and we only give representatives; we hope that the tools we develop in the paper, and put to use in Section 9, will make it easy for researchers down the road to prove some important results in the real analytic setting where continuity is crucial.

1.2. Notation and background. We shall quickly review the notation we use.

Basic terminology and notation. When A is a subset of a set X, we write $A \subseteq X$. If we wish to exclude the possibility that $A = X$, we write $A \subset X$. For a family of sets $(X_i)_{i\in I}$, we denote by $\prod_{i\in I} X_i$ the product of these sets. By $\mathrm{pr}_j\colon \prod_{i\in I} X_i \to X_j$ we denote the projection onto the jth factor. The identity map on a set X is denoted by id_X.

By $\mathbb{Z}$ we denote the set of integers. We use the notation $\mathbb{Z}_{>0}$ and $\mathbb{Z}_{\geq 0}$ to denote the subsets of positive and nonnegative integers. By $\mathbb{R}$ we denote the sets of real numbers. By $\mathbb{R}_{>0}$ we denote the subset of positive real numbers.

Algebra and linear algebra. By $\mathfrak{S}_k$ we denote the permutation group of $\{1,\dots,k\}$. For $k,l \in \mathbb{Z}_{\geq 0}$, we denote by $\mathfrak{S}_{k,l}$ the subset of $\mathfrak{S}_{k+l}$ consisting of permutations σ satisfying

$$\sigma(1) < \cdots < \sigma(k), \quad \sigma(k+1) < \cdots < \sigma(k+l).$$

We also denote by $\mathfrak{S}_{k|l}$ the subgroup of $\mathfrak{S}_{k+l}$ having the form

$$\begin{pmatrix} 1 & \cdots & k & k+1 & \cdots & k+l \\ \sigma_1(1) & \cdots & \sigma_1(k) & k+\sigma_2(1) & \cdots & k+\sigma_2(l) \end{pmatrix}$$

for $\sigma_1 \in \mathfrak{S}_k$ and $\sigma_2 \in \mathfrak{S}_l$. We note that $\mathfrak{S}_{k|l}\backslash\mathfrak{S}_{k+l} \simeq \mathfrak{S}_{k,l}$, so that (1) if $\sigma \in \mathfrak{S}_{k+l}$, then $\sigma = \sigma_1 \circ \sigma_2$ for $\sigma_1 \in \mathfrak{S}_{k|l}$ and $\sigma_2 \in \mathfrak{S}_{k,l}$ and (2) $\mathrm{card}(\mathfrak{S}_{k,l}) = \frac{(k+l)!}{k!l!}$.

We denote by $\mathbb{R}^n$ the n-fold Cartesian product of $\mathbb{R}$. A point in $\mathbb{R}^n$ will typically be denoted in a bold font, e.g., $\boldsymbol{x} = (x_1,\dots,x_n)$. We denote the standard basis for $\mathbb{R}^n$ by $(\boldsymbol{e}_1,\dots,\boldsymbol{e}_n)$.

For $\mathbb{R}$-vector spaces U and V, we denote by $\mathrm{Hom}_{\mathbb{R}}(\mathsf{U};\mathsf{V})$ the set of $\mathbb{R}$-linear mappings from U to V. We denote $\mathrm{End}_{\mathbb{R}}(\mathsf{V}) = \mathrm{Hom}_{\mathbb{R}}(\mathsf{V};\mathsf{V})$. We denote by $\mathsf{V}^* = \mathrm{Hom}_{\mathbb{R}}(\mathsf{V};\mathbb{R})$ the algebraic dual. If $v \in \mathsf{V}$ and $\alpha \in \mathsf{V}^*$, we will denote the evaluation of α on v at various points by $\alpha(v)$, $\alpha \cdot v$, or $\langle \alpha; v\rangle$, whichever seems most pleasing to us at the moment. If $A \in \mathrm{Hom}_{\mathbb{R}}(\mathsf{U};\mathsf{V})$, we denote by $A^* \in \mathrm{Hom}_{\mathbb{R}}(\mathsf{V}^*;\mathsf{U}^*)$ the dual of A. If $S \subseteq \mathsf{V}$, then we denote by

$$\mathrm{ann}(S) = \{\alpha \in \mathsf{V}^* \mid \alpha(v) = 0,\ v \in S\}$$

the annihilator subspace.

For a $\mathbb{R}$-vector space V, $\mathrm{T}^k(\mathsf{V})$ is the k-fold tensor product of V with itself. For $r,s \in \mathbb{Z}_{>0}$, we denote

$$\mathrm{T}^r_s(\mathsf{V}) = \underbrace{\mathsf{V} \otimes \cdots \otimes \mathsf{V}}_{r \text{ times}} \otimes \underbrace{\mathsf{V}^* \otimes \cdots \otimes \mathsf{V}^*}_{s \text{ times}}.$$

By $\mathrm{S}^k(\mathsf{V})$ we denote the k-fold symmetric tensor product of V with itself, and we think of this as a subset of $\mathrm{T}^k(\mathsf{V})$. For $A \in \mathrm{S}^k(\mathsf{V})$ and $B \in \mathrm{S}^l(\mathsf{V})$, we define the symmetric tensor product of A and B to be

$$A \odot B = \sum_{\sigma \in \mathfrak{S}_{k,l}} \sigma(A \otimes B).$$

We define $\mathrm{Sym}_k\colon \mathrm{T}^k(\mathsf{V}) \to \mathrm{S}^k(\mathsf{V})$ by

$$\mathrm{Sym}_k(v_1 \otimes \cdots \otimes v_k) = \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}.$$

We note that we have the alternative formula

$$A \odot B = \frac{(k+l)!}{k!l!} \mathrm{Sym}_{k+l}(A \otimes B) \tag{1.1}$$

for the product of $A \in \mathrm{S}^k(\mathsf{V})$ and $B \in \mathrm{S}^l(\mathsf{V})$. We recall that

$$\dim_{\mathbb{R}}(\mathrm{S}^k(\mathsf{V})) = \binom{\dim_{\mathbb{R}}(\mathsf{V}) + k - 1}{k}, \tag{1.2}$$

when V is finite-dimensional.

For a $\mathbb{R}$-vector space V, let us denote

$$\mathrm{T}^{\le m}(\mathsf{V}) = \bigoplus_{j=0}^{m} \mathrm{T}^j(\mathsf{V}), \quad \mathrm{S}^{\le m}(\mathsf{V}) = \bigoplus_{j=0}^{m} \mathrm{S}^j(\mathsf{V}),$$

and define

$$\begin{aligned} \mathrm{Sym}_{\le m}\colon \mathrm{T}^{\le m}(\mathsf{V}) &\to \mathrm{S}^{\le m}(\mathsf{V}) \\ (A_0, A_1, \dots, A_m) &\mapsto (A_0, \mathrm{Sym}_1(A_1), \dots, \mathrm{Sym}_m(A_m)). \end{aligned}$$

For $\mathbb{R}$-inner product spaces $(\mathsf{U}, \mathbb{G}_{\mathsf{U}})$ and $(\mathsf{V}, \mathbb{G}_{\mathsf{V}})$, we denote the transpose of $L \in \mathrm{Hom}_{\mathbb{R}}(\mathsf{U}; \mathsf{V})$ as the linear map $L^T \in \mathrm{Hom}_{\mathbb{R}}(\mathsf{V}; \mathsf{U})$ defined by

$$\mathbb{G}_{\mathsf{V}}(L(u), v) = \mathbb{G}_{\mathsf{U}}(u, L^T(v)), \qquad u \in \mathsf{U},\ v \in \mathsf{V}.$$

Topology. We shall not use any particular notation for the Euclidean norm for $\mathbb{R}^n$, and so will just denote this norm by

$$\|\boldsymbol{x}\| = \left(\sum_{j=1}^{n} |x_j|^2\right)^{1/2}.$$

It is sometimes convenient to use other norms for $\mathbb{R}^n$, particularly the 1- and ∞-norms defined, as usual, by

$$\|\boldsymbol{x}\|_1 = \sum_{j=1}^{n} |x_j|, \quad \|\boldsymbol{x}\|_\infty = \sup\{|x_j| \mid j \in \{1, \dots, n\}\}.$$

The following relationships between these norms are useful:

$$\begin{gathered} \|x\| \le \|\boldsymbol{x}\|_1 \le \sqrt{n}\|\boldsymbol{x}\|, \quad \|\boldsymbol{x}\|_\infty \le \|\boldsymbol{x}\| \le \sqrt{n}\|\boldsymbol{x}\|_\infty, \\ \|\boldsymbol{x}\|_\infty \le \|\boldsymbol{x}\|_1 \le n\|\boldsymbol{x}\|_\infty. \end{gathered} \tag{1.3}$$

If we are using a norm whose definition is evident from context, we will simply denote it by $\|\cdot\|$, accepting that context will ensure that there is no confusion.

For $\boldsymbol{x} \in \mathbb{R}^n$ and $r \in \mathbb{R}_{>0}$, we denote by

$$\mathsf{B}(r, \boldsymbol{x}) = \{\boldsymbol{y} \in \mathbb{R}^n \mid \|\boldsymbol{y} - \boldsymbol{x}\| < r\}$$

and

$$\overline{\mathsf{B}}(r, \boldsymbol{x}) = \{\boldsymbol{y} \in \mathbb{R}^n \mid \|\boldsymbol{y} - \boldsymbol{x}\| \leq r\}$$

the ***open*** and ***closed balls*** of radius r centred at $\boldsymbol{x}$. As with the notation for norms, we shall often use the preceding notation for balls in settings different from $\mathbb{R}^n$, and accept the abuse of notation.

Differential calculus. If $\mathcal{U} \subseteq \mathbb{R}^n$ is open and if $\boldsymbol{\Phi}\colon \mathcal{U} \to \mathbb{R}^m$ is differentiable at $\boldsymbol{x} \in \mathcal{U}$, we denote its derivative by $\boldsymbol{D\Phi}(\boldsymbol{x})$. Higher-order derivatives, when they exist, are denoted by $\boldsymbol{D}^k\boldsymbol{\Phi}(\boldsymbol{x})$, k being the order of differentiation. We recall that, if $\boldsymbol{\Phi}\colon \mathcal{U} \to \mathbb{R}^m$ is of class C^k, $k \in \mathbb{Z}_{>0}$, then $\boldsymbol{D}^k\boldsymbol{\Phi}(\boldsymbol{x})$ is symmetric. We shall sometimes find it convenient to use multi-index notation for derivatives. A ***multi-index*** with length n is an element of $\mathbb{Z}^n_{\geq 0}$, i.e., an n-tuple $I = (i_1, \dots, i_n)$ of nonnegative integers. If $\boldsymbol{\Phi}\colon \mathcal{U} \to \mathbb{R}^m$ is a smooth function, then we denote

$$\boldsymbol{D}^I\boldsymbol{\Phi}(\boldsymbol{x}) = \boldsymbol{D}_1^{i_1} \cdots \boldsymbol{D}_n^{i_n}\boldsymbol{\Phi}(\boldsymbol{x}).$$

We will use the symbol $|I| = i_1 + \cdots + i_n$ to denote the order of the derivative. Another piece of multi-index notation we shall use is

$$\boldsymbol{a}^I = a_1^{i_1} \cdots a_n^{i_n},$$

for $\boldsymbol{a} \in \mathbb{R}^n$ and $I \in \mathbb{Z}^n_{\geq 0}$. Also, we denote $I! = i_1! \cdots i_n!$.

Differential geometry. We shall adopt the notation and conventions of smooth differential geometry of [Abraham, Marsden, and Ratiu 1988]. We shall also make use of real analytic differential geometry. There are no useful textbook references dedicated to real analytic differential geometry, but the book of [Cieliebak and Eliashberg 2012] contains much of what we shall need. Throughout the paper, unless otherwise stated, manifolds are connected, second countable, Hausdorff manifolds. The assumption of connectedness can be dispensed with but is convenient as it allows one to not have to worry about manifolds with components of different dimensions and vector bundles with fibres of different dimensions.

We shall work with regularity classes $r \in \{\infty, \omega\}$, "∞" meaning smooth, "ω" meaning real analytic. Sometimes we do not require infinite differentiability, but will hypothesise it anyway. Other times we will precisely specify the regularity needed; but we will be a little sloppy with this as (1) it is not crucial to the purposes of this paper and (2) it is typically easy to know when infinite differentiability is hypothesised but not required.

The tangent bundle of a manifold M is denoted by $\pi_{\mathsf{TM}}\colon \mathsf{TM} \to \mathsf{M}$ and the cotangent bundle by $\pi_{\mathsf{T}^*\mathsf{M}}\colon \mathsf{T}^*\mathsf{M} \to \mathsf{M}$.

We denote by $\mathrm{C}^r(\mathsf{M}; \mathsf{N})$ the set of mappings from a manifold M to a manifold N of class C^r. When $\mathsf{N} = \mathbb{R}$, we denote by $\mathrm{C}^r(\mathsf{M}) = \mathrm{C}^r(\mathsf{M}; \mathbb{R})$ the set of scalar-valued functions of class C^r. For $\Phi \in \mathrm{C}^1(\mathsf{M}; \mathsf{N})$, $T\Phi\colon \mathsf{TM} \to \mathsf{TN}$ denotes the derivative of Φ, and $T_x\Phi = T\Phi|\mathsf{T}_x\mathsf{M}$. For $f \in \mathrm{C}^r(\mathsf{M})$, we denote by $\mathrm{d}f \in \Gamma^{r-1}(\mathsf{T}^*\mathsf{M})$ the ***differential*** of f, defined by

$$T_x f(v_x) = (f(x), \langle \mathrm{d}f(x); v_x\rangle, \qquad v_x \in \mathsf{T}_x\mathsf{M}.$$

We denote by $T_x^*\Phi$ the dual of $T_x\Phi$. For a vector field X and a differentiable function f, $\mathscr{L}_X f$ denotes the Lie derivative of f with respect to X. We might also write $Xf = \mathscr{L}_X f$. For differentiable vector fields X and Y, we denote by $[X, Y]$ the Lie bracket of these vector fields. For $X \in \Gamma^r(\mathsf{TM})$, the flow of X is denoted by Φ_t^X, meaning that, for $x \in \mathsf{M}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_t^X(x) = X \circ \Phi_t^X(x), \quad \Phi_0^X(x) = x.$$

The Lie derivative for vector fields extends to a derivation of the tensor algebra for a manifold. Specifically, for $X \in \Gamma^\infty(\mathsf{TM})$, we denote

$$\mathscr{L}_X f = \langle \mathrm{d}f; X\rangle, \quad \mathscr{L}_X Y = [X, Y], \qquad f \in \mathrm{C}^\infty(\mathsf{M}),\ X \in \Gamma^\infty(\mathsf{TM}).$$

For $\alpha \in \Gamma^\infty(\mathsf{T}^*\mathsf{M})$, we can then define its Lie derivative with respect to X by

$$\langle \mathscr{L}_X\alpha; Y\rangle = \mathscr{L}_X\langle\alpha; Y\rangle - \langle\alpha; \mathscr{L}_X Y\rangle, \qquad Y \in \Gamma^\infty(\mathsf{TM}).$$

The Lie derivative of a tensor field $A \in \Gamma^\infty(\mathrm{T}^r_s(\mathsf{TM}))$ is then defined by

$$\begin{aligned}&\mathscr{L}_X A(\alpha^1, \dots, \alpha^r, X_1, \dots, X_s) = \mathscr{L}_X(A(\alpha^1, \dots, \alpha^r, X_1, \dots, X_s))\\ &- \sum_{j=1}^r A(\alpha^1, \dots, \mathscr{L}_X\alpha^j, \dots, \alpha^r, X_1, \dots, X_s) - \sum_{j=1}^s A(\alpha^1, \dots, \alpha^r, X_1, \dots, \mathscr{L}_X X_j, \dots, X_s).\end{aligned} \tag{1.4}$$

Of course, these constructions make sense for tensor fields and vector fields that are less regular than smooth.

Let $\pi_{\mathsf{E}}\colon \mathsf{E} \to \mathsf{M}$ be a vector bundle of class C^r. We shall sometimes denote the fibre over $x \in \mathsf{M}$ by E_x, noting that this has the structure of a $\mathbb{R}$-vector space. If $A \subseteq \mathsf{M}$, we denote $\mathsf{E}|A = \pi_{\mathsf{E}}^{-1}(A)$. By $\Gamma^r(\mathsf{E})$ we denote the set of sections of E of class C^r. This space has the structure of a $\mathbb{R}$-vector space with the vector space operations

$$(\xi + \eta)(x) = \xi(x) + \eta(x), \quad (a\xi)(x) = a(\xi(x)), \qquad x \in \mathsf{M},$$

and of a $\mathrm{C}^r(\mathsf{M})$-module with the additional operation of multiplication

$$(f\xi)(x) = f(x)\xi(x), \qquad x \in \mathsf{M},$$

for $f \in \mathrm{C}^r(\mathsf{M})$, $\xi, \eta \in \Gamma^r(\mathsf{E})$, and $a \in \mathbb{R}$. By $\mathscr{G}^r_{\mathsf{E}}$ we denote the sheaf of C^r-sections of E. Thus

$$\mathscr{G}^r_{\mathsf{E}}(\mathcal{U}) = \Gamma^r(\mathsf{E}|\mathcal{U})$$

when $\mathcal{U} \subseteq \mathsf{M}$ is open. By $\mathbb{R}^k_{\mathsf{M}}$ we denote the trivial bundle $\mathbb{R}^k_{\mathsf{M}} = \mathsf{M} \times \mathbb{R}^k$ with vector bundle projection being projection onto the first factor. The dual bundle E^* of a vector bundle E is the set of vector bundle mappings from E to $\mathbb{R}_{\mathsf{M}}$ over id_{M}. We note that there is a natural identification of $\Gamma^r(\mathbb{R}_{\mathsf{M}})$ with $\mathrm{C}^r(\mathsf{M})$. Given a C^r-vector bundle $\pi_{\mathsf{E}}\colon \mathsf{E} \to \mathsf{M}$ and a mapping $\Phi \in \mathrm{C}^r(\mathsf{N}; \mathsf{M})$, we denote by $\Phi^*\pi_{\mathsf{E}}\colon \Phi^*\mathsf{E} \to \mathsf{N}$ the pull-back bundle. For C^r-vector bundles $\pi_{\mathsf{E}}\colon \mathsf{E} \to \mathsf{M}$ and $\pi_{\mathsf{F}}\colon \mathsf{F} \to \mathsf{M}$ over the same base, we denote by $\mathrm{VB}^r(\mathsf{E}; \mathsf{F})$ the set of C^r-vector bundle mappings from E to F over id_{M}.

Riemannian geometry and connections. We shall make use of basic constructions from Riemannian geometry. We also work a great deal with connections, both affine connections and linear connections in vector bundles. We refer to [Kobayashi and Nomizu 1963] as a standard reference, and [Kolář, Michor, and Slovák 1993] is also a useful reference.

First suppose that $r \in \{\infty, \omega\}$. A ***C^r-fibre metric*** on a C^r-vector bundle $\pi_{\mathsf{E}}\colon \mathsf{E} \to \mathsf{M}$ is $\mathbb{G}_{\pi_{\mathsf{E}}} \in \Gamma^r(\mathrm{S}^2(\mathsf{E}^*))$ such that $\mathbb{G}_{\pi_{\mathsf{E}}}(x)$ is an inner product on E_x for each $x \in \mathsf{M}$. The associated norm on fibres we denote by $\|\cdot\|_{\mathbb{G}}$. In case E is the tangent bundle of M, then a fibre metric is a Riemannian metric, and we will use the notation $\mathbb{G}_{\mathsf{M}}$ in this case.

A linear connection in a vector bundle $\pi_{\mathsf{E}}\colon \mathsf{E} \to \mathsf{M}$ will be denoted by $\nabla^{\pi_{\mathsf{E}}}$. In case E is the tangent bundle of M, then a linear connection is called an affine connection, and we will denote it by ∇^{M}. A linear connection in a vector bundle $\pi_{\mathsf{E}}\colon \mathsf{E} \to \mathsf{M}$ induces a splitting of the short exact sequence

$$0 \longrightarrow \ker(T\pi_{\mathsf{E}}) \longrightarrow \mathsf{TE} \xrightarrow{T\pi_{\mathsf{E}}} \mathsf{TM} \longrightarrow 0$$

For $e \in \mathsf{E}$, we thus have a splitting of the tangent space $\mathsf{T}_e\mathsf{E} \simeq \mathsf{T}_{\pi_{\mathsf{E}}(e)}\mathsf{M} \oplus \mathsf{E}_{\pi_{\mathsf{E}}(e)}$. The first component in this splitting we call ***horizontal*** and denote by $\mathsf{H}_e\mathsf{E}$, and the second we call ***vertical*** and denote by $\mathsf{V}_e\mathsf{E}$. By hor and ver we denote the projections onto the horizontal and vertical subspaces, respectively.

We note that covariant differentiation with respect to a vector field X of sections of E, along with Lie differentiation of functions, gives rise to covariant differentiation of tensors, just as we saw above for $\mathscr{L}_X$. A little more generally, if we have vector bundles $\pi_{\mathsf{E}}\colon \mathsf{E} \to \mathsf{M}$ and $\pi_{\mathsf{F}}\colon \mathsf{F} \to \mathsf{E}$, and linear connections $\nabla^{\pi_{\mathsf{E}}}$ and $\nabla^{\pi_{\mathsf{F}}}$, then we have a connection in $\mathsf{E} \otimes \mathsf{F}$ denoted by $\nabla^{\pi_{\mathsf{E}} \otimes \pi_{\mathsf{F}}}$ and defined by

$$\nabla^{\pi_{\mathsf{E}} \otimes \pi_{\mathsf{F}}}(\xi \otimes \eta) = (\nabla^{\pi_{\mathsf{E}}}\xi) \otimes \eta + \xi \otimes (\nabla^{\pi_{\mathsf{F}}}\eta).$$

Jet bundles. We shall make extensive use of jet bundles of various sorts. We can recommend [Saunders 1989] and [Kolář, Michor, and Slovák 1993, §12] as useful references.

Let M be a C^r-manifold and let $m \in \mathbb{Z}_{\geq 0}$. For $x \in \mathsf{M}$ and $a \in \mathbb{R}$, by $\mathsf{J}^m_{(x,a)}(\mathsf{M};\mathbb{R})$ we denote the m-jets of functions at x taking value a at x. For a C^r-function f defined in a neighbourhood of x, we denote by $j_m f(x) \in \mathsf{J}^m_{(x,f(x))}\mathsf{M}$ the m-ket of f. Of particular interest is the set $\mathsf{T}^{*m}_x\mathsf{M} = \mathsf{J}^m_{(x,0)}(\mathsf{M};\mathbb{R})$ of jets of functions taking the value 0 at x. This has the structure of a $\mathbb{R}$-algebra with the algebra structure defined by the three operations

$$\begin{aligned} j_m f(x) + j_m g(x) &= j_m(f+g)(x), \\ (j_m f(x))(j_m g(x)) &= j_m(fg)(x), \\ a(j_m f(x)) &= j_m(af)(x), \end{aligned}$$

for functions f and g and for $a \in \mathbb{R}$. We denote

$$\mathsf{T}^{*m}\mathsf{M} = \overset{\circ}{\bigcup_{x \in \mathsf{M}}} \mathsf{T}^{*m}_x\mathsf{M}.$$

For $m, l \in \mathbb{Z}_{\geq 0}$ with $m \geq l$, we have projections $\rho^m_l\colon \mathsf{T}^{*m}\mathsf{M} \to \mathsf{T}^{*l}\mathsf{M}$. Note that $\mathsf{T}^{*0}\mathsf{M} \simeq \mathsf{M}$ and that $\mathsf{T}^{*1}\mathsf{M} \simeq \mathsf{T}^*\mathsf{M}$. We abbreviate $\rho_m \triangleq \rho^m_0\colon \mathsf{T}^{*m}\mathsf{M} \to \mathsf{M}$ which has the structure of a vector bundle.

Let $\pi_{\mathsf{E}}\colon \mathsf{E} \to \mathsf{M}$ be a C^r-vector bundle. For $x \in \mathsf{M}$ and $m \in \mathbb{Z}_{\geq 0}$, $\mathsf{J}^m_x\mathsf{E}$ denotes the set of m-jets of sections of E at x. For a C^r-section ξ defined in some neighbourhood of x, $j_m\xi(x) \in \mathsf{J}^m_x\mathsf{E}$ denotes the m-jet of ξ. We denote by $\mathsf{J}^m\mathsf{E} = \mathring{\cup}_{x\in\mathsf{M}} \mathsf{J}^m_x\mathsf{E}$ the bundle of m-jets. For $m, l \in \mathbb{Z}_{\geq 0}$ with $m \geq l$, we denote by $\pi^m_l\colon \mathsf{J}^m\mathsf{E} \to \mathsf{J}^l\mathsf{E}$ the projection. Note that $\mathsf{J}^0\mathsf{E} \simeq \mathsf{E}$. We abbreviate $\pi_m \triangleq \pi_{\mathsf{E}} \circ \pi^m_0\colon \mathsf{J}^m\mathsf{E} \to \mathsf{M}$, and note that $\mathsf{J}^m\mathsf{E}$ has the structure of a vector bundle over M, with addition and scalar multiplication defined by

$$j_m\xi(x) + j_m\eta(x) = j_m(\xi + \eta)(x), \quad a(j_m\xi(x)) = j_m(a\xi)(x)$$

for sections ξ and η and for $a \in \mathbb{R}$. One can show that

$$\mathsf{J}^m\mathsf{E} \simeq (\mathbb{R}_{\mathsf{M}} \oplus \mathsf{T}^{*m}\mathsf{M}) \otimes \mathsf{E}. \tag{1.5}$$

2. The topology for sections of a real analytic vector bundle

In this section we shall provide a quick overview of the usual topology for real analytic sections of a real analytic vector bundle, and will give three descriptions of this topology, two due to Martineau [1966] and one via seminorms given by Jafarpour and Lewis [2014], based on the note of Vogt [2013].

2.1. Martineau's descriptions of the real analytic topology. We shall give a brief characterisation of two topologies for the space $\Gamma^\omega(\mathsf{E})$ of real analytic sections of a real analytic vector bundle $\pi_{\mathsf{E}}\colon \mathsf{E} \to \mathsf{M}$. The original work of Martineau [1966] describes these topologies for the space of real analytic functions, but it is evident that the same considerations apply to sections of a general vector bundle. Each description offers benefits in terms of providing immediately some useful properties of the topology, although showing that they agree is something of an undertaking, and we shall make some comments in this direction.

Both characterisations rely on the fact that a real analytic vector bundle $\pi_{\mathsf{E}}\colon \mathsf{E} \to \mathsf{M}$ can be complexified to an holomorphic vector bundle $\pi_{\overline{\mathsf{E}}}\colon \overline{\mathsf{E}} \to \overline{\mathsf{M}}$, following Whitney and Bruhat [1959]. We denote by $\Gamma^{\mathrm{hol}}(\overline{\mathsf{E}})$ the space of holomorphic sections of this vector bundle, which we equip with its usual compact-open topology, i.e., the topology of uniform convergence on compact sets. This renders $\Gamma^{\mathrm{hol}}(\overline{\mathsf{E}})$ a Fréchet space. For a subset $A \subseteq \overline{\mathsf{M}}$, we denote by $\mathscr{G}^{\mathrm{hol}}_{A,\overline{\mathsf{E}}}$ the space of germs of holomorphic sections of $\overline{\mathsf{E}}$ about A. The space $\mathscr{G}^{\mathrm{hol}}_{A,\overline{\mathsf{E}}}$ has the direct limit topology over the directed set of neighbourhoods of A.

In the first description of the topology of $\Gamma^\omega(\mathsf{E})$, we note that, if $\xi \in \Gamma^\omega(\mathsf{E})$, then there is some neighbourhood $\overline{\mathcal{U}}$ of M in $\overline{\mathsf{M}}$ to which ξ admits a unique holomorphic extension $\overline{\xi} \in \Gamma^{\mathrm{hol}}(\overline{\mathsf{E}}|\overline{\mathcal{U}})$. Thus we have a mapping

$$\Gamma^\omega(\mathsf{E}) \ni \xi \mapsto \overline{\xi} \in \mathscr{G}^{\mathrm{hol}}_{\mathsf{M},\overline{\mathsf{E}}}.$$

This map is easily seen to be an isomorphism of vector spaces, and so equips $\Gamma^\omega(\mathsf{E})$ with the direct limit topology for the space of germs of sections of $\overline{\mathsf{E}}$ about $\mathsf{M} \subseteq \overline{\mathsf{M}}$. This immediately shows that $\Gamma^\omega(\mathsf{E})$ is ultrabornological [Jarchow 1981, Corollaries 13.1.4 and 13.1.5].

The other description of a locally convex topology first fixes a compact subset $\mathcal{K} \subseteq \mathsf{M}$. We note, then, that $\mathcal{K}$ possesses a countable collection of neighbourhoods in $\overline{\mathsf{M}}$ that are cofinal in the directed set of all neighbourhoods. Thus the direct limit topology of $\mathscr{G}^{\mathrm{hol}}_{\mathcal{K},\overline{\mathsf{E}}}$ is that of a countable direct limit of Fréchet spaces. Indeed, by working instead with bounded sections, one ensures that one has a countable direct limit of Banach spaces. One can additionally and importantly show that the linking maps for the direct limit are compact, indeed nuclear. Thus the topology of $\mathscr{G}^{\mathrm{hol}}_{\mathcal{K},\overline{\mathsf{E}}}$ inherits many nice properties: it is a webbed, nuclear, Suslin space by [Jarchow 1981, Corollary 5.3.3], [Kriegl and Michor 1997, Theorem 8.4] and [Schwartz 1974, Example II.2(E)], respectively. We next note that we have a natural mapping $\xi \mapsto [\overline{\xi}]_{\mathcal{K}}$ from $\Gamma^\omega(\mathsf{E})$ to $\mathscr{G}^{\mathrm{hol}}_{\mathcal{K},\overline{\mathsf{E}}}$ by taking the germ about $\mathcal{K}$ of an holomorphic extension. Now, since M is second countable, it possesses a countable compact exhaustion $(\mathcal{K}_j)_{j\in\mathbb{Z}_{>0}}$, and one can then reasonably easily see that $\Gamma^\omega(\mathsf{E})$ is the inverse limit (as a vector space) of the inverse system $\mathscr{G}^{\mathrm{hol}}_{\mathcal{K}_j,\overline{\mathsf{E}}}$, $j \in \mathbb{Z}_{>0}$, (with linking maps given by restriction). We can then give $\Gamma^\omega(\mathsf{E})$ the inverse limit topology. The resulting topology is webbed, nuclear, and Suslin by [Jarchow 1981, Corollary 5.3.3], [Jarchow 1981, Corollary 21.2.3], and [Bogachev 2007, Lemma 6.6.5(ii) and (iii)]. It is not, however, metrisable as follows from [Vogt 2010, Theorem 10].

One of the contributions of Martineau [1966] is to show that the two preceding topologies for $\Gamma^\omega(\mathsf{E})$ agree. Martineau's original proof was by showing that $\cup_{j\in\mathbb{Z}_{>0}}(\mathscr{G}^{\mathrm{hol}}_{\mathcal{K}_j,\overline{\mathsf{E}}})^*$ is a dense subspace of the dual of $\Gamma^\omega(\mathsf{E})$ equipped with the direct limit topology, using earlier results in [Martineau 1963] on analytic functionals. A modern approach, using homological methods, equates an inverse limit being ultrabornological with the vanishing of Proj^1, where Proj is a functor on inverse systems devised by Palamodov [1968]. In all cases, showing equality of the two topologies is not straightforward.

The inverse limit description of the topology for $\Gamma^\omega(\mathsf{E})$ is the one that is most closely connected with our approach here, since the seminorms we give are essentially for $\mathscr{G}^{\mathrm{hol}}_{\mathcal{K},\overline{\mathsf{E}}}$ for a compact subset $\mathcal{K} \subseteq \mathsf{M} \subseteq \overline{\mathsf{M}}$. It is to the description of these seminorms that we now turn.

2.2. Decompositions for jet bundles. A prominent rôle in our characterisation of the topology for real analytic sections is played by jets and a decomposition of jets using connections. The reason for this is that the seminorms we define are given in terms of infinite jets of real analytic sections.

Let $\pi_{\mathsf{E}}\colon \mathsf{E} \to \mathsf{M}$ be a smooth vector bundle. We suppose that we have a linear connection $\nabla^{\pi_{\mathsf{E}}}$ on the vector bundle E and an affine connection ∇^{M} on M. We then have induced connections, that we also denote by $\nabla^{\pi_{\mathsf{E}}}$ and ∇^{M}, in various tensor bundles of E and TM, respectively. The connections $\nabla^{\pi_{\mathsf{E}}}$ and ∇^{M} extend naturally to connections in various tensor products of TM and E, all of these being denoted by $\nabla^{\mathsf{M},\pi_{\mathsf{E}}}$. Note that

$$\nabla^{\mathsf{M},\pi_{\mathsf{E}},m}\xi \triangleq \underbrace{\nabla^{\mathsf{M},\pi_{\mathsf{E}}}\cdots(\nabla^{\mathsf{M},\pi_{\mathsf{E}}}}_{m-1\text{ times}}(\nabla^{\pi_{\mathsf{E}}}\xi)) \in \Gamma^\infty(\mathrm{T}^m(\mathsf{T}^*\mathsf{M})\otimes\mathsf{E}). \tag{2.1}$$

Now, given $\xi \in \Gamma^\infty(\mathsf{E})$ and $m \in \mathbb{Z}_{\geq 0}$, we define

$$D^m_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}}(\xi) = \mathrm{Sym}_m \otimes \mathrm{id}_{\mathsf{E}}(\nabla^{\mathsf{M},\pi_{\mathsf{E}},m}\xi) \in \Gamma^\infty(\mathrm{S}^m(\mathsf{T}^*\mathsf{M})\otimes\mathsf{E}),$$

We take the convention that $D^0_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}}(\xi) = \xi$.

The following lemma is then key for our presentation, and is proved in [Jafarpour and Lewis 2014, Lemma 2.1] by means of induction and a diagram chase.

2.1 Lemma: (Decomposition of jet bundles) *The map*

$$\begin{aligned} S^m_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}}\colon \mathsf{J}^m\mathsf{E} &\to \bigoplus_{j=0}^m (\mathrm{S}^j(\mathsf{T}^*\mathsf{M})\otimes\mathsf{E}) \\ j_m\xi(x) &\mapsto (\xi(x), D^1_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}}(\xi)(x),\dots,D^m_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}}(\xi)(x)) \end{aligned}$$

is an isomorphism of vector bundles, and, for each $m\in\mathbb{Z}_{>0}$, the diagram

$$\begin{array}{ccc} \mathsf{J}^{m+1}\mathsf{E} & \xrightarrow{S^{m+1}_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}}} & \oplus_{j=0}^{m+1}(\mathrm{S}^j(\mathsf{T}^*\mathsf{M})\otimes\mathsf{E}) \\ {\scriptstyle \pi^{m+1}_m}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}^{m+1}_m} \\ \mathsf{J}^m\mathsf{E} & \xrightarrow[S^m_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}}]{} & \oplus_{j=0}^{m}(\mathrm{S}^j(\mathsf{T}^*\mathsf{M})\otimes\mathsf{E}) \end{array}$$

commutes, where pr^{m+1}_m is the obvious projection, stripping off the last component of the direct sum.

There are a couple of special cases of interest.

1. Jets of functions fit into the framework of the lemma by using the trivial line bundle $\mathbb{R}_{\mathsf{M}}=\mathsf{M}\times\mathbb{R}$. The identification of a function with a section of this bundle is specified by $f\mapsto\xi_f$, with $\xi_f(x)=(x,f(x))$. In this case, the bundle has a canonical flat connection defined by $\nabla^{\pi_{\mathsf{E}}}f=\mathrm{d}f$. Therefore, the decomposition of Lemma 2.1 is determined by an affine connection ∇^{M} on M, and so we have a mapping
$$\begin{aligned} S^m_{\nabla^{\mathsf{M}}}\colon \mathsf{J}^m(\mathsf{M};\mathbb{R}) &\to \bigoplus_{j=0}^m \mathrm{S}^j(\mathsf{T}^*\mathsf{M}) \\ f(x) &\mapsto (f(x),\mathrm{d}f(x),\dots,\mathrm{Sym}_m\circ\nabla^{\mathsf{M},m-1}\mathrm{d}f(x)). \end{aligned} \tag{2.2}$$
This can be restricted to $\mathsf{T}^{*m}\mathsf{M}$ to give the mapping
$$\begin{aligned} S^m_{\nabla^{\mathsf{M}}}\colon \mathsf{T}^{*m}\mathsf{M} &\to \bigoplus_{j=1}^m \mathrm{S}^j(\mathsf{T}^*\mathsf{M}) \\ f(x) &\mapsto (\mathrm{d}f(x),\dots,\mathrm{Sym}_m\circ\nabla^{\mathsf{M},m-1}\mathrm{d}f(x)), \end{aligned} \tag{2.3}$$
adopting a mild abuse of notation. We recall that $\mathsf{T}^{*m}_x\mathsf{M}$ is an $\mathbb{R}$-algebra, and the induced $\mathbb{R}$-algebra structure on $\oplus_{j=1}^m\mathrm{S}^j(\mathsf{T}^*_x\mathsf{M})$ is that of polynomial functions that vanish at 0 and with degree at most m.

2. Another special case is that of jets of vector fields. In this case, the vector bundle is $\pi_{\mathsf{TM}}\colon\mathsf{TM}\to\mathsf{M}$. We can make use of an affine connection ∇^{M} on M to provide everything we need to define the mapping
$$\begin{aligned} S^m_{\nabla^{\mathsf{M}}}\colon \mathsf{J}^m\mathsf{TM} &\to \bigoplus_{j=0}^m (\mathrm{S}^j(\mathsf{T}^*\mathsf{M})\otimes\mathsf{TM}) \\ X(x) &\mapsto (X(x),\nabla^{\mathsf{M}}X(x),\dots,\mathrm{Sym}_m\circ\nabla^{\mathsf{M},m}X(x)). \end{aligned} \tag{2.4}$$
Of course, this applies equally well to jets of one-forms on M, or any other sections of tensor bundles associated with the tangent bundle.
This case of vector fields is the setting of Jafarpour and Lewis [2014] in their study of flows of time-varying vector fields.

2.3. Fibre norms for jet bundles of vector bundles. Our discussion begins with general constructions for the fibres of jet bundles. Thus we let $r \in \{\infty, \omega\}$ and let $\pi_{\mathsf{E}} \colon \mathsf{E} \to \mathsf{M}$ be a C^r-vector bundle. We shall suppose that we have a C^r-affine connection ∇^{M} on M and a C^r-vector bundle connection $\nabla^{\pi_{\mathsf{E}}}$ in E, as in Section 2.2. This allows us to give the decomposition of $\mathsf{J}^m\mathsf{E}$ as in Lemma 2.1. By additionally supposing that we have a C^r-Riemannian metric $\mathbb{G}_{\mathsf{M}}$ on M and a C^r-fibre metric $\mathbb{G}_{\pi_{\mathsf{E}}}$ on E, we shall give a C^r-fibre norm on $\mathsf{J}^m\mathsf{E}$. Note that the existence of the metrics and connections is ensured by [Jafarpour and Lewis 2014, Lemma 2.4].

The first step in making the construction is the following result concerning inner products on tensor products.

2.2 Lemma: (Inner products on tensor products) *Let U and V be finite-dimensional $\mathbb{R}$-vector spaces and let $\mathbb{G}$ and $\mathbb{H}$ be inner products on U and V, respectively. Then the element $\mathbb{G} \otimes \mathbb{H}$ of $\mathrm{T}^2(\mathsf{U}^* \otimes \mathsf{V}^*)$ defined by*

$$\mathbb{G} \otimes \mathbb{H}(u_1 \otimes v_1, u_2 \otimes v_2) = \mathbb{G}(u_1, u_2)\mathbb{H}(v_1, v_2)$$

is an inner product on $\mathsf{U} \otimes \mathsf{V}$.

Proof: Let $(e_1, \dots, e_m)$ and $(f_1, \dots, f_n)$ be orthonormal bases for U and V, respectively. Then

$$\{e_a \otimes f_j \mid a \in \{1, \dots, m\},\ j \in \{1, \dots, n\}\} \tag{2.5}$$

is a basis for $\mathsf{U} \otimes \mathsf{V}$. Note that

$$\mathbb{G} \otimes \mathbb{H}(e_a \otimes f_j, e_b \otimes f_k) = \mathbb{G}(e_a, e_b)\mathbb{H}(f_j, f_k) = \delta_{ab}\delta_{jk},$$

which shows that $\mathbb{G} \otimes \mathbb{H}$ is indeed an inner product, as (2.5) is an orthonormal basis. ■

With $\mathbb{G}_{\pi_{\mathsf{E}}}$ a fibre metric on E and with $\mathbb{G}_{\mathsf{M}}$ be a Riemannian metric on M as above, let us denote by $\mathbb{G}_{\mathsf{M}}^{-1}$ the associated fibre metric on $\mathsf{T}^*\mathsf{M}$ defined by

$$\mathbb{G}_{\mathsf{M}}^{-1}(\alpha_x, \beta_x) = \mathbb{G}_{\mathsf{M}}(\mathbb{G}_{\mathsf{M}}^{\sharp}(\alpha_x), \mathbb{G}_{\mathsf{M}}^{\sharp}(\beta_x)).$$

In like manner, one has a fibre metric $\mathbb{G}_{\pi_{\mathsf{E}}}^{-1}$ on E^*. Then, by induction using the preceding lemma, we have a fibre metric in all tensor spaces associated with TM and E and their tensor products. We shall denote by $\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}}$ any of these various fibre metrics. In particular, we have a fibre metric $\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}}$ on $\mathrm{T}^j(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{E}$ induced by $\mathbb{G}_{\mathsf{M}}^{-1}$ and $\mathbb{G}_{\pi_{\mathsf{E}}}$. By restriction, this gives a fibre metric on $\mathrm{S}^j(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{E}$. We can thus define a fibre metric $\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}},m}$ on $\mathsf{J}^m\mathsf{E}$ given by

$$\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}},m}(j_m\xi(x), j_m\eta(x)) = \sum_{j=0}^{m} \mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}}\left(\frac{1}{j!}D^j_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}}(\xi)(x), \frac{1}{j!}D^j_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}}(\eta)(x)\right). \tag{2.6}$$

Associated to this inner product on fibres is the norm on fibres, which we denote by $\|\cdot\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}},m}}$. We shall use these fibre norms continually in our descriptions of our various topologies for real analytic vector bundles, cf. Section 6. The appearance of the factorials in the fibre metric (2.6) appears superfluous at this point. However, it is essential in order for the real analytic topology defined by our seminorms to be independent of the choices of ∇^{M}, $\nabla^{\pi_{\mathsf{E}}}$, $\mathbb{G}_{\mathsf{M}}$, and $\mathbb{G}_{\pi_{\mathsf{E}}}$, cf. Theorem 8.10.

The preceding constructions can be applied particularly to the tangent bundle of the total space of a vector bundle $\pi_{\mathsf{E}}\colon \mathsf{E} \to \mathsf{M}$. Indeed, given a Riemannian metric on M, a fibre metric on E, an affine connection on M, and a vector bundle connection in E, the constructions of Section 4.1 give a Riemannian metric on E, and this, along with its Levi-Civita connection, gives the data required to define fibre norms for the jet bundles $\mathsf{J}^m\mathsf{TE}$. A substantial amount of the work in the paper will be to consider lifts to E of objects on M. The continuity of operations like this requires us to relate the jet bundle decompositions of $\mathsf{J}^m\mathsf{TE}$ with those of $\mathsf{J}^m\mathsf{E}$ and $\mathsf{J}^m\mathsf{TM}$.

2.4. Seminorms for the real analytic topology. In this section we provide explicit seminorms for Martineau's topologies for $\Gamma^\omega(\mathsf{E})$. Throughout this section, we will work with a vector bundle $\pi_{\mathsf{E}}\colon \mathsf{E} \to \mathsf{M}$ and the data ∇^{M}, $\nabla^{\pi_{\mathsf{E}}}$, $\mathbb{G}_{\mathsf{M}}$, and $\mathbb{G}_{\pi_{\mathsf{E}}}$ that define the fibre metrics for jet bundles as per Section 2.3. To define seminorms for $\Gamma^\omega(\mathsf{E})$, let $\mathrm{c}_0(\mathbb{Z}_{\geq 0};\mathbb{R}_{>0})$ denote the space of sequences in $\mathbb{R}_{>0}$, indexed by $\mathbb{Z}_{\geq 0}$, and converging to zero. We shall denote a typical element of $\mathrm{c}_0(\mathbb{Z}_{\geq 0};\mathbb{R}_{>0})$ by $\boldsymbol{a} = (a_j)_{j\in\mathbb{Z}_{\geq 0}}$. Now, for $\mathcal{K} \subseteq \mathsf{M}$ and $\boldsymbol{a} \in \mathrm{c}_0(\mathbb{Z}_{\geq 0};\mathbb{R}_{>0})$, we define a seminorm $p^\omega_{\mathcal{K},\boldsymbol{a}}$ for $\Gamma^\omega(\mathsf{E})$ by

$$p^\omega_{\mathcal{K},\boldsymbol{a}}(\xi) = \sup\{a_0a_1\cdots a_m\|j_m\xi(x)\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}},m}} \mid x \in \mathcal{K},\ m \in \mathbb{Z}_{\geq 0}\}.$$

The family of seminorms $p^\omega_{\mathcal{K},\boldsymbol{a}}$, $\mathcal{K} \subseteq \mathsf{M}$ compact, $\boldsymbol{a} \in \mathrm{c}_0(\mathbb{Z}_{\geq 0};\mathbb{R}_{>0})$, defines a locally convex topology on $\Gamma^\omega(\mathsf{E})$ that we call the ***$\mathbf{C}^\omega$-topology***. As we have mentioned above, this topology is webbed, ultrabornological, nuclear, and Suslin, but is not metrisable.

While we are in the process of defining seminorms, let us also define seminorms for the set $\Gamma^\infty(\mathsf{E})$ of smooth sections. While we are primarily interested in the difficult real analytic case in this paper, it is useful and illustrative to, at times, make comparisons with the smooth case. In any case, the topology we consider for $\Gamma^\infty(\mathsf{E})$ is that of uniform convergence of derivatives on compact sets. A moment's thought will convince one that the appropriate seminorms are

$$p^\infty_{\mathcal{K},m}(\xi) = \sup\{\|j_m\xi(x)\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}},m}} \mid x \in \mathcal{K}\}$$

for $\mathcal{K} \subseteq \mathsf{M}$ compact and for $m \in \mathbb{Z}_{\geq 0}$. These seminorms define a Polish topology for $\Gamma^\infty(\mathsf{E})$ called the ***$\mathbf{C}^\infty$-topology***. We note that, for the smooth topology, the seminorms are defined for fixed order jets. As we shall indicate as we go along, it is this fact that leads to simplifications of the results in the paper when applied to the smooth case.

The following lemma, providing bounds for real analytic sections, is a global version of a well-known classical result [e.g., Krantz and Parks 2002, Proposition 2.2.10]. We refer to [Jafarpour and Lewis 2014, Lemma 2.6] for a proof.

2.3 Lemma: (Characterisation of real analytic sections) *Let $\pi_{\mathsf{E}}\colon \mathsf{E} \to \mathsf{M}$ be a real analytic vector bundle. For $\xi \in \Gamma^\infty(\mathsf{E})$, the following statements are equivalent:*

(i) $\xi \in \Gamma^\omega(\mathsf{E})$;

(ii) for $\mathcal{K} \subseteq \mathsf{M}$ compact, there exists $C, r \in \mathbb{R}_{>0}$ such that

$$p^\infty_{\mathcal{K},m}(\xi) \leq Cr^{-m}, \qquad m \in \mathbb{Z}_{\geq 0}.$$

3. Tensors on the total space of a vector bundle

Many of the geometric constructions we undertake in the paper, and estimates associated with these geometric constructions, involves tensors of various sorts defined on the total space of a vector bundle. In this section we present the classes of such tensors as arise in our presentation. We also define a number of algebraic operations on these tensors. Many of the constructions we see here will seem, on an initial reading, disconnected from the objectives of the paper. However, the constructions are essential in Section 4. This is not very encouraging, however, since the constructions and results of Section 4 themselves appear *non sequitur* to the objectives of the paper. It is only in the later sections of the paper that the relevance of all of these constructions will become apparent. For this reason, perhaps a good strategy would be to skip over this section and the next in a first reading, coming back to them when they are subsequently needed.

There is nothing particularly real analytic with the material in this section, so the smooth and real analytic cases are considered side-by-side.

3.1. Functions on vector bundles. Among the geometric constructions we will consider are those associated to a particular set of functions on a vector bundle.

3.1 Definition: (Fibre-linear functions) Let $r \in \{\infty, \omega\}$ and let $\pi_{\mathsf{E}} \colon \mathsf{E} \to \mathsf{M}$ be a vector bundle of class C^r. A function $F \in \mathrm{C}^r(\mathsf{E})$ is ***fibre-linear*** if, for each $x \in \mathsf{M}$, $F|\mathsf{E}_x$ is a linear function. We denote by $\mathrm{Lin}^r(\mathsf{E})$ the set of C^r-fibre-linear functions on E. •

Let us give some elementary properties of the sets of fibre-linear functions.

3.2 Lemma: (Properties of fibre-linear functions) *Let $r \in \{\infty, \omega\}$ and let $\pi_{\mathsf{E}} \colon \mathsf{E} \to \mathsf{M}$ be a vector bundle of class C^r. Then the following statements hold:*

(i) $\mathrm{Lin}^r(\mathsf{E})$ is a submodule of the $\mathrm{C}^r(\mathsf{M})$-module $\mathrm{C}^r(\mathsf{E})$;

(ii) for $F \in \mathrm{Lin}^r(\mathsf{E})$, there exists $\lambda_F \in \Gamma^r(\mathsf{E}^*)$ such that

$$F(e) = \langle \lambda_F \circ \pi_{\mathsf{E}}(e); e\rangle, \qquad e \in \mathsf{E},$$

and, moreover, the map $F \mapsto \lambda_F$ is an isomorphism of $\mathrm{C}^r(\mathsf{M})$-modules;

Proof: (i) Let $F \in \mathrm{Lin}^r(\mathsf{E})$ and $f \in \mathrm{C}^r(\mathsf{M})$. Then

$$f \cdot F(e) = (f \circ \pi_{\mathsf{E}}(e))F(e),$$

and so $f \cdot F$ is fibre-linear since a scalar multiple of a linear function is a linear function. Also, since the pointwise sum of linear functions is a linear function, we conclude that $\mathrm{Lin}^r(\mathsf{E})$ is indeed a submodule of $\mathrm{C}^r(\mathsf{E})$.

(ii) This merely follows by definition of the dual bundle E^*. ■

In a rather related manner, we can consider other classes of functions on vector bundles.

3.3 Definition: (Lifts and evaluations of one-forms and functions) Let $r \in \{\infty, \omega\}$ and let $\pi_{\mathsf{E}} \colon \mathsf{E} \to \mathsf{M}$ be a C^r-vector bundle.

(i) For $\lambda \in \Gamma^r(\mathsf{E}^*)$, the ***vertical evaluation*** of λ is $\lambda^{\mathrm{e}} \in \mathrm{Lin}^r(\mathsf{E})$ defined by $\lambda^{\mathrm{e}}(e_x) = \langle \lambda(x); e_x\rangle$.

(ii) For $f \in \mathrm{C}^r(\mathsf{M})$, the ***horizontal lift*** of f is the function $f^{\mathrm{h}} \in \mathrm{C}^r(\mathsf{E})$ defined by $f^{\mathrm{h}} = \pi_{\mathsf{E}}^* f$. •

3.2. Vector fields on vector bundles. Next we turn to vector fields on the total space of a vector bundle. As with our consideration of functions in the preceding section, we restrict attention to vector fields that interact nicely with the vector bundle structure.

We begin with the notion of the vertical lift of a section.

3.4 Definition: (Vertical lift of a section) Let $r \in \{\infty, \omega\}$ and let $\pi_{\mathsf{E}}\colon \mathsf{E} \to \mathsf{M}$ be a vector bundle of class C^r.

(i) For $e_x, e'_x \in \mathsf{E}_x$, we define the ***vertical lift*** of e'_x to e_x to be

$$\mathrm{vlft}(e_x, e'_x) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (e_x + te'_x).$$

(ii) Given a section $\xi \in \Gamma^r(\mathsf{E})$, we define the ***vertical lift*** of ξ to E to be the vector field

$$\xi^{\mathrm{v}}(e_x) = \mathrm{vlft}(e_x, \xi(x)). \qquad \bullet$$

Next we consider another sort of lift, this one requiring a connection $\nabla^{\pi_{\mathsf{E}}}$ in the vector bundle $\pi_{\mathsf{E}}\colon \mathsf{E} \to \mathsf{B}$. We let $\mathsf{VE} = \ker(T\pi_{\mathsf{E}})$ be the vertical subbundle. As mentioned in Section 1.2, the connection $\nabla^{\pi_{\mathsf{E}}}$ defines a complement HE to VE called the horizontal subbundle. We let $\mathrm{ver}, \mathrm{hor}\colon \mathsf{TE} \to \mathsf{TE}$ be the projections onto VE and HE.

3.5 Definition: (Horizontal lift of a vector field) Let $r \in \{\infty, \omega\}$, let $\pi_{\mathsf{E}}\colon \mathsf{E} \to \mathsf{M}$ be a vector bundle of class C^r, and let $\nabla^{\pi_{\mathsf{E}}}$ be a C^r-connection in E.

(i) For $e_x \in \mathsf{E}_x$ and $v_x \in \mathsf{T}_x\mathsf{M}$, the ***horizontal lift*** of v_x to e_x is the unique vector $\mathrm{hlft}(e_x, v_x) \in \mathsf{H}_{e_x}\mathsf{E}$ satisfying

$$T_{e_x}\pi_{\mathsf{E}}(\mathrm{hlft}(e_x, v_x)) = v_x.$$

(ii) For $X \in \Gamma^r(\mathsf{TM})$ on M, we denote by X^{h} the ***horizontal lift*** of X to E, this being the vector field $X^{\mathrm{h}} \in \Gamma^r(\mathsf{TE})$ satisfying

$$X^{\mathrm{h}}(e_x) = \mathrm{hlft}(e_x, X(x)). \qquad \bullet$$

Next we provide formulae for differentiating various sorts of functions with respect to various sorts of vector fields.

3.6 Lemma: (Differentiation of functions on vector bundles) *Let $r \in \{\infty, \omega\}$ and let $\pi_{\mathsf{E}}\colon \mathsf{E} \to \mathsf{M}$ a vector bundle of class C^r. Let $f \in \mathrm{C}^r(\mathsf{M})$, $\lambda \in \Gamma^r(\mathsf{E}^*)$, $X \in \Gamma^r(\mathsf{TM})$, and $\xi \in \Gamma^r(\mathsf{E})$,*

Then the following statements hold:

(i) $\mathscr{L}_{X^{\mathrm{h}}} f^{\mathrm{h}} = (\mathscr{L}_X f)^{\mathrm{h}}$;

(ii) $\mathscr{L}_{\xi^{\mathrm{v}}} f^{\mathrm{h}} = 0$;

(iii) $\mathscr{L}_{\xi^{\mathrm{v}}} \lambda^{\mathrm{e}} = \langle \lambda; \xi \rangle^{\mathrm{h}}$;

Additionally, let $\nabla^{\pi_{\mathsf{E}}}$ be a C^r-linear connection in $\pi_{\mathsf{E}}\colon \mathsf{E} \to \mathsf{M}$. Then

(iv) $\mathscr{L}_{X^{\mathrm{h}}} \lambda^{\mathrm{e}} = (\nabla^{\pi_{\mathsf{E}}}_X \lambda)^{\mathrm{e}}$.

Proof: (i) We compute

$$\begin{aligned}\mathscr{L}_{X^{\mathrm{h}}} f^{\mathrm{h}}(e) &= \langle \mathrm{d}(\pi_{\mathsf{E}}^* f)(e); X^{\mathrm{h}}(e)\rangle = \langle \mathrm{d}f \circ \pi_{\mathsf{E}}(e); T_e\pi_{\mathsf{E}}(X^{\mathrm{h}}(e))\rangle \\ &= \langle \mathrm{d}f \circ \pi_{\mathsf{E}}(e); X \circ \pi_{\mathsf{E}}(e)\rangle = (\mathscr{L}_X f)^{\mathrm{h}}(e).\end{aligned}$$

(ii) Since f^{h} is constant on fibres of π_{E} and ξ^{v} is tangent to fibres, we have

$$f^{\mathrm{h}}(e + t\xi \circ \pi_{\mathsf{E}}(e)) = f(e).$$

Differentiating with respect to t at $t = 0$ gives the result.

(iii) Here we compute

$$\begin{aligned}\mathscr{L}_{\xi^{\mathrm{v}}} \lambda^{\mathrm{e}}(e) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \langle \lambda(e + t\xi \circ \pi_{\mathsf{E}}(e)); e + t\xi \circ \pi_{\mathsf{E}}(e)\rangle \\ &= \langle \lambda \circ \pi_{\mathsf{E}}(e); \xi \circ \pi_{\mathsf{E}}(e)\rangle = \langle \lambda; \xi\rangle^{\mathrm{h}},\end{aligned}$$

so completing the proof.

(iv) Let $e \in \mathsf{E}$ and let $t \mapsto \gamma(t)$ be the integral curve for X satisfying $\gamma(0) = \pi_{\mathsf{E}}(e)$ and let $t \mapsto \gamma^{\mathrm{h}}(t)$ be the integral curve for X^{h} satisfying $\gamma^{\mathrm{h}}(0) = e$. Then $t \mapsto \gamma^{\mathrm{h}}(t)$ is the parallel translation of e along γ, and as such we have $\nabla^{\pi_{\mathsf{E}}}_{\gamma'(t)} \gamma^{\mathrm{h}}(t) = 0$. Then

$$\mathscr{L}_{X^{\mathrm{h}}} \lambda^{\mathrm{e}}(e) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \langle \lambda \circ \gamma(t); \gamma^{\mathrm{h}}(t)\rangle = \langle \nabla^{\pi_{\mathsf{E}}}_X \lambda \circ \pi_{\mathsf{E}}(e); e\rangle,$$

as claimed. ■

In Section 4 we shall have a great deal more to say about differentiation of objects on the total space of a vector bundle when one has more structure present than we use in the preceding result.

3.3. Linear mappings on vector bundles. Now we turn to an examination of linear maps associated to a vector bundle $\pi_{\mathsf{E}} \colon \mathsf{E} \to \mathsf{M}$. We shall consider vector bundle mappings of two sorts: (1) with values in the trivial line bundle $\mathbb{R}_{\mathsf{M}}$; (2) with values in E. The first sort of mappings are, of course, simply sections of the dual bundle, or linear functions of the sort studied in Section 3.1. Our interest here is in lifting such objects to the total space.

First we work with one-forms. If we have a connection $\nabla^{\pi_{\mathsf{E}}}$ in a vector bundle $\pi_{\mathsf{E}} \colon \mathsf{E} \to \mathsf{M}$, then this gives us a splitting $\mathsf{TE} = \mathsf{HE} \oplus \mathsf{VE}$, and hence a splitting $\mathsf{T}^*\mathsf{E} = \mathsf{H}^*\mathsf{E} \oplus \mathsf{V}^*\mathsf{E}$ with

$$\mathsf{H}^*\mathsf{E} = \mathrm{ann}(\mathsf{VE}), \quad \mathsf{V}^*\mathsf{E} = \mathrm{ann}(\mathsf{HE}).$$

Note that $\mathsf{H}^*_e\mathsf{E} = \mathrm{image}(T^*_e\pi_{\mathsf{E}})$.

3.7 Definition: (Lifts of one-forms and dual sections) Let $r \in \{\infty, \omega\}$ and let $\pi_{\mathsf{E}} \colon \mathsf{E} \to \mathsf{M}$ be a C^r-vector bundle.

(i) For $\alpha_x \in \mathsf{T}^*_x\mathsf{M}$ and $e_x \in \mathsf{E}_x$, the ***horizontal lift*** of α_x to e_x is $\mathrm{hlft}(e_x, \alpha_x) = T^*_{e_x}\pi_{\mathsf{E}}(\alpha_x)$.

(ii) The ***horizontal lift*** of $\alpha \in \Gamma^r(\mathsf{T}^*\mathsf{M})$ is $\alpha^{\mathrm{h}} = \pi_{\mathsf{E}}^*\alpha \in \Gamma^r(\mathsf{T}^*\mathsf{E})$.

Additionally, let $\nabla^{\pi_{\mathsf{E}}}$ be a connection in E.

(iii) For $\lambda_x \in \mathsf{E}_x^*$ and $e_x \in \mathsf{E}_x^*$, then the ***vertical lift*** of λ_x is the unique vector $\mathrm{vlft}(e_x, \lambda_x) \in \mathsf{V}_{e_x}^* \mathsf{E}$ satisfying

$$\langle \mathrm{vlft}(e_x, \lambda_x); \mathrm{vlft}(e_x, u_x) \rangle = \langle \lambda_x; u_x \rangle$$

for every $u_x \in \mathsf{E}_x$.

(iv) The ***vertical lift*** of $\lambda \in \Gamma^r(\mathsf{E}^*)$ is the one-form $\lambda^{\mathrm{v}} \in \Gamma^r(\mathsf{T}^*\mathsf{E})$ satisfying

$$\lambda^{\mathrm{v}}(e_x) = \mathrm{vlft}(e_x, \lambda(x)).$$ •

We also have natural ways of lifting homomorphisms of vector bundles.

3.8 Definition: (Vertical evaluation and vertical lift of an homomorphism) Let $r \in \{\infty, \omega\}$, and let $\pi_{\mathsf{E}} \colon \mathsf{E} \to \mathsf{M}$ and $\pi_{\mathsf{F}} \colon \mathsf{F} \to \mathsf{M}$ be C^r-vector bundles. For $L \in \Gamma^r(\mathsf{F} \otimes \mathsf{E}^*)$,

(i) the ***vertical evaluation*** of L is the section $L^{\mathrm{e}} \in \Gamma^r(\pi_{\mathsf{E}}^*\mathsf{F})$ defined by

$$L^{\mathrm{e}}(e_x) = (e_x, L(e_x)).$$

If, additionally, $\nabla^{\pi_{\mathsf{E}}}$ is a connection in E,

(ii) the ***vertical lift*** of L is the vector bundle homomorphism $L^{\mathrm{v}} \in \Gamma^r(\pi_{\mathsf{E}}^*\mathsf{F} \otimes \mathsf{T}^*\mathsf{E})$ defined by

$$L^{\mathrm{v}}(Z) = (e, L \circ \mathrm{ver}(Z))$$

for $Z \in \mathsf{T}_e\mathsf{E}$, noting that $\mathrm{ver}(Z) \in \mathsf{V}_e\mathsf{E} \simeq \mathsf{E}_{\pi_{\mathsf{E}}(e)}$. •

We shall be especially interested in two cases of the vector bundle F.

1. $\mathsf{F} = \mathbb{R}_{\mathsf{M}}$: In this case, $\mathsf{F} \otimes \mathsf{E}^* \simeq \mathsf{E}^*$, $\pi_{\mathsf{E}}^*\mathsf{F} \simeq \mathbb{R}_{\mathsf{E}}$, and $\pi_{\mathsf{E}}^*\mathsf{F} \otimes \mathsf{T}^*\mathsf{E} \simeq \mathsf{T}^*\mathsf{E}$. One can easily see that, if $\lambda \in \Gamma^r(\mathsf{E}^*)$, then the vertical evaluation as per Definition 3.8 agrees with that of Definition 3.3, and the vertical lift as per Definition 3.8 agrees with that of Definition 3.7.
2. $\mathsf{F} = \mathsf{E}$: In this case, $\mathsf{F} \otimes \mathsf{E}^* \simeq \mathrm{T}_1^1(\mathsf{E})$, i.e., the set of endomorphisms of E. We also have $\pi_{\mathsf{E}}^*\mathsf{F} \simeq \mathsf{VE}$ [Kolář, Michor, and Slovák 1993, §6.11]. Thus, for $L \in \Gamma^r(\mathrm{T}_1^1(\mathsf{E}))$, L^{e} is a VE-valued vector field. Also, L^{v} is a VE-valued endomorphism of TE.

Let us perform some analysis of the vertical evaluation and vertical lift of an homomorphism. First of all, for $e_1, e_2 \in \mathsf{E}_x$,

$$L^{\mathrm{e}}(e_1 + e_2) = (e_1, L(e_1)) + (e_2, L(e_2)) = L^{\mathrm{e}}(e_1) + L^{\mathrm{e}}(e_2),$$

where addition is with respect to the vector bundle structure

$$\begin{array}{ccc} \mathsf{E} & \xrightarrow{L^{\mathrm{e}}} & \pi_{\mathsf{E}}^*\mathsf{F} \\ {\scriptstyle \pi_{\mathsf{E}}}\downarrow & & \downarrow{\scriptstyle \pi_{\mathsf{E}}^*\pi_{\mathsf{F}}} \\ \mathsf{M} & \xrightarrow[Z]{} & \mathsf{F} \end{array}$$

where Z is the zero section. Thus L^{e} is a "linear" section over E. We define the vector bundle mapping

$$\begin{aligned} P_{\mathsf{E},\mathsf{F}} \colon \pi_{\mathsf{E}}^*\mathsf{F} \otimes \mathsf{V}^*\mathsf{E} &\to \pi_{\mathsf{E}}^*\mathsf{F} \\ L_e &\mapsto L_e(e) \end{aligned} \tag{3.1}$$

over id_{E}, noting that $e \in \mathsf{E}_{\pi_{\mathsf{E}}(e)} \simeq \mathsf{V}_e\mathsf{E}$. Then, given $A \in \Gamma^r(\pi_{\mathsf{E}}^*\mathsf{F} \otimes \mathsf{V}^*\mathsf{E})$, $P_{\mathsf{E},\mathsf{F}} \circ A$ is a section of $\pi_{\mathsf{E}}^*\mathsf{E}$. Moreover, $P_{\mathsf{E},\mathsf{F}} \circ L^{\mathrm{v}} = L^{\mathrm{e}}$ for $L \in \Gamma^r(\mathsf{F} \otimes \mathsf{E}^*)$.

We shall make use of these observations in Section 5.

Let us recast the preceding observations in a slightly different way. To start, note that, given $\lambda \in \Gamma^r(\mathsf{E}^*)$ and $\eta \in \Gamma^r(\mathsf{F})$, we have $\eta \otimes \lambda \in \Gamma^r(\mathsf{F} \otimes \mathsf{E}^*)$. The tensor product on the left can be thought of as being of $\mathrm{C}^r(\mathsf{M})$-modules.[1] Moreover, such sections of the bundle of endomorphisms locally generate the sections of the homomorphism bundle. Note that

$$(\eta \otimes \lambda)^{\mathrm{e}} = \xi^{\mathrm{v}} \otimes \lambda^{\mathrm{e}},$$

as is directly verified. In this case, since $\mathrm{C}^r(\mathsf{M})$ is a subring of $\mathrm{C}^r(\mathsf{E})$ (by pull-back), we can regard the tensor product as being of $\mathrm{C}^r(\mathsf{E})$-modules. Therefore,

$$L^{\mathrm{e}} \in \Gamma^r(\Gamma^r(\pi_{\mathsf{E}}^*\mathsf{F}) \otimes \mathrm{Lin}^r(\mathsf{E})).$$

Since $\mathrm{Lin}^r(\mathsf{E}) \subseteq \mathrm{C}^r(\mathsf{E})$, the tensor product is mere multiplication in this case.

A similar sort of analysis can be made for the vertical lift of an homomorphism. In this case, given $\lambda \in \Gamma^r(E^*)$ and $\eta \in \Gamma^r(\mathsf{F})$, we have $\xi \otimes \lambda \in \Gamma^r(\mathsf{F} \otimes \mathsf{E}^*)$, as in the preceding paragraph. In this case, the vertical lift satisfies

$$(\xi \otimes \lambda)^{\mathrm{v}} = \xi^{\mathrm{v}} \otimes \lambda^{\mathrm{v}}.$$

3.4. Tensors fields on vector bundles.

Next we discuss the extension of our lifts of functions, sections, and vector fields to higher-order tensors. The extension is to tensor powers of the pull-back $\pi_{\mathsf{E}}^*\mathsf{T}^*\mathsf{M}$ of the cotangent bundle to the total space of the vector bundle. Other sorts of lifts are possible, especially in the presence of a connection in the vector bundle. We restrict ourselves to the tensor powers of the pull-back of $\mathsf{T}^*\mathsf{M}$ since our interest is in jet bundles, and these tensor powers represent derivatives with respect to the base.

We make the following definitions.

3.9 Definition: (Lifts of tensors) Let $r \in \{\infty, \omega\}$, and let $\pi_{\mathsf{E}}\colon \mathsf{E} \to \mathsf{M}$ and $\pi_{\mathsf{F}}\colon \mathsf{F} \to \mathsf{M}$ be a C^r-vector bundles. Let $k \in \mathbb{Z}_{>0}$.

(i) For $A \in \Gamma^r(\mathrm{T}^k(\mathsf{T}^*\mathsf{M}))$, the ***horizontal lift*** of A is $A^{\mathrm{h}} \in \Gamma^r(\mathrm{T}^k(\mathsf{T}^*\mathsf{E}))$ defined by

$$A^{\mathrm{h}}(Z_1, \dots, Z_k) = A(T_e\pi_{\mathsf{E}}(Z_1), \dots, T_e\pi_{\mathsf{E}}(Z_k))$$

for $Z_1, \dots, Z_k \in \mathsf{T}_e\mathsf{E}$.[2]

[1] This corresponds to the well-known isomorphism

$$\Gamma^r(\mathsf{E}) \otimes_{\mathrm{C}^r(\mathsf{M})} \Gamma^r(\mathsf{F}) \simeq \Gamma^r(\mathsf{E} \otimes \mathsf{F})$$

of $\mathrm{C}^r(\mathsf{M})$-modules. While this isomorphism is well-known, it is commonly not correctly proved, as proofs are given that admit a direct translation to the holomorphic setting, where the assertion is generally false. A correct proof in the smooth case is given by Conlon [2001, Theorem 7.5.5]. His proof makes use (without saying this explicitly) of the Serre–Swan Theorem. Since the Serre–Swan Theorem is valid for vector bundles over smooth, real analytic, and Stein manifolds (see [Lewis 2023, Theorem 20]), Conlon's proof applies in these cases.

[2] Of course, this is nothing but the usual definition of pull-back, which we repeat for the sake of symmetry.

(ii) For $A \in \Gamma^r(\mathrm{T}^k(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{E})$, the ***vertical lift*** of A is $A^{\mathrm{v}} \in \Gamma^r(\mathrm{T}^k(\mathsf{T}^*\mathsf{E}) \otimes \mathsf{TE})$ defined by

$$A^{\mathrm{v}}(Z_1, \dots, Z_k) = \mathrm{vlft}(e, A(T_e\pi_{\mathsf{E}}(Z_1), \dots, T_e\pi_{\mathsf{E}}(Z_k))),$$

for $Z_1, \dots, Z_k \in \mathsf{T}_e\mathsf{E}$.

(iii) For $A \in \Gamma^r(\mathrm{T}^k(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{F} \otimes \mathsf{E}^*)$, the ***vertical evaluation*** of A is $A^{\mathrm{e}} \in \Gamma^r(\mathrm{T}^k(\mathsf{T}^*\mathsf{E}) \otimes \pi_{\mathsf{E}}^*\mathsf{F})$ defined by

$$A^{\mathrm{e}}(Z_1, \dots, Z_k) = (e, A(T_e\pi_{\mathsf{E}}(Z_1), \dots, T_e\pi_{\mathsf{E}}(Z_k))(e)),$$

for $Z_1, \dots, Z_k \in \mathsf{T}_e\mathsf{E}$.

Additionally, let $\nabla^{\pi_{\mathsf{E}}}$ be a connection in E.

(iv) For $A \in \Gamma^r(\mathrm{T}^k(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{TM})$, the ***horizontal lift*** of A is $A^{\mathrm{h}} \in \Gamma^r(\mathrm{T}^k(\mathsf{T}^*\mathsf{E}) \otimes \mathsf{TE})$ defined by

$$A^{\mathrm{h}}(Z_1, \dots, Z_k) = \mathrm{hlft}(e, A(T_e\pi_{\mathsf{E}}(Z_1), \dots, T_e\pi_{\mathsf{E}}(Z_k)))$$

for $Z_1, \dots, Z_k \in \mathsf{T}_e\mathsf{E}$.

(v) For $A \in \Gamma^r(\mathrm{T}^k(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{E}^*)$, the ***vertical lift*** of A is $A^{\mathrm{v}} \in \Gamma^r(\mathrm{T}^k(\mathsf{T}^*\mathsf{E}) \otimes \mathsf{T}^*\mathsf{E})$ defined by

$$A^{\mathrm{v}}(Z_1, \dots, Z_k) = \mathrm{vlft}(e, A(T_e\pi_{\mathsf{E}}(Z_1), \dots, T_e\pi_{\mathsf{E}}(Z_k)))$$

for $Z_1, \dots, Z_k \in \mathsf{T}_e\mathsf{E}$.

(vi) For $A \in \Gamma^r(\mathrm{T}^k(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{F} \otimes \mathsf{E}^*)$, the ***vertical lift*** of A is $A^{\mathrm{v}} \in \Gamma^r(\mathrm{T}^k(\mathsf{T}^*\mathsf{E}) \otimes \pi_{\mathsf{E}}^*\mathsf{F} \otimes \mathsf{T}^*\mathsf{E})$ defined by

$$A^{\mathrm{v}}(Z_1, \dots, Z_k)(Z) = (e, A(T_e\pi_{\mathsf{E}}(Z_1), \dots, T_e\pi_{\mathsf{E}}(Z_k))(\mathrm{ver}(Z))),$$

for $Z_1, \dots, Z_k, Z \in \mathsf{T}_e\mathsf{E}$. •

3.5. Tensor contractions. In our differentiation results of Section 4, we shall make use of certain generalisations of the contraction operator on tensors. What we have is a sort of "contraction and insertion" operation. We describe this here in the setting of linear algebra, since this is where it most naturally resides. The constructions can, of course, be extended to vector bundles by performing the vector space constructions on fibres.

Let V be a finite-dimensional $\mathbb{R}$-vector space, let $k \in \mathbb{Z}_{>0}$ and $l \in \mathbb{Z}_{\geq 0}$, and let $\alpha \in \mathrm{T}^k(\mathsf{V}^*)$ and $\beta \in \mathrm{T}^l(\mathsf{V}^*) \otimes \mathsf{V}$. For $j \in \{1, \dots, k\}$, define the ***jth insertion of β in α*** by $\mathrm{Ins}_j(\alpha, \beta) \in \mathrm{T}^{k+l-1}(\mathsf{V}^*)$ by

$$\mathrm{Ins}_j(\alpha, \beta)(v_1, \dots, v_{k+l-1}) = \alpha(v_1, \dots, v_{j-1}, \beta(v_j, v_{k+1}, \dots, v_{k+l-1}), v_{j+1}, \dots, v_k).$$

To be clear, when $l = 0$ we have

$$\mathrm{Ins}_j(\alpha, v)(v_1, \dots, v_{k-1}) = \alpha(v_1, \dots, v_{j-1}, v, v_j, \dots, v_{k-1}).$$

We will also find it helpful to consider tensor contraction when one of the arguments (the second is the one we care about) is fixed. Thus let $\beta \in \mathrm{T}^l(\mathsf{V}^*) \otimes \mathsf{V}$ and define $\mathrm{Ins}_{j,\beta}(\alpha) = \mathrm{Ins}_j(\alpha, \beta)$.

We shall also need notation for a specific sort of swapping of arguments of a tensor. Let $\alpha \in \mathrm{T}^k(\mathsf{V})$ and let $j_1, j_2 \in \{1, \dots, k\}$. We define

$$\mathrm{push}_{j_1,j_2}\alpha(v_1, \dots, v_k) = \begin{cases} \alpha(v_1, \dots, v_{j_1-1}, v_{j_1+1}, \dots, v_{j_2}, v_{j_1}, v_{j_2+1}, \dots, v_k), & j_1 \leq j_2, \\ \alpha(v_1, \dots, v_{j_2-1}, v_{j_1}, v_{j_2}, \dots, v_{j_1-1}, v_{j_1+1}, \dots, v_k), & j_1 > j_2. \end{cases}$$

The idea is that push_{j_1,j_2} drops v_{j_1} into the j_2-slot, and shifts the arguments to make room for this. The "insertion" and "push" mappings can be generalised in the obvious way to give $\mathrm{Ins}_j(A, \beta)$ and $\mathrm{push}_{j_1,j_2}(A)$ for $A \in \mathrm{T}^k(\mathsf{V}^*) \otimes \mathsf{U}$ and $\beta \in (\mathrm{T}^l(\mathsf{V}^*) \otimes \mathsf{V}) \otimes \mathsf{U}$ (resp. $A \in \mathsf{U} \otimes \mathrm{T}^k(\mathsf{V}^*)$ and $\beta \in \mathsf{U} \otimes (\mathrm{T}^l(\mathsf{V}^*) \otimes \mathsf{V})$), just by acting on the first (resp. second) component of the tensor product.

The final tensor construction we make is that of a linear tensor derivation. Given $A \in \mathrm{End}_{\mathbb{R}}(\mathsf{V})$, we define a derivation D_A of the tensor algebra $\oplus_{r,s \in \mathbb{Z}_{\geq 0}} \mathrm{T}^r_s(\mathsf{V})$ by $D_A(a) = 0$ for $a \in \mathrm{T}^0_0(\mathsf{V}) \simeq \mathbb{R}$, and $D_A(v) = A(v)$ for $v \in \mathrm{T}^1_0(\mathsf{V}) \simeq \mathsf{V}$. It then follows that $D_A(\alpha) = -A^*(\alpha)$ for $\alpha \in \mathsf{V}^*$. More generally, we have the following result which expresses a well-known formula [e.g., Nelson 1967, §3.4] in terms of our insertion operation.

3.10 Lemma: (Insertion and tensor derivation I) *Let V be a finite-dimensional $\mathbb{R}$-vector space, let $A \in \mathrm{End}_{\mathbb{R}}(\mathsf{V})$, let $r, s \in \mathbb{Z}_{>0}$, and let $T \in \mathrm{T}^r_s(\mathsf{V})$. Then*

$$D_A(T) = \sum_{j=1}^r \mathrm{Ins}_j(T, A^*) - \sum_{j=1}^s \mathrm{Ins}_{r+j}(T, A).$$

Proof:

$$\begin{aligned} D_A(T)&(\beta^1, \dots, \beta^r, u_1, \dots, u_s) \\ &= \sum_{j=1}^r v_1 \otimes \dots \otimes A(v_j) \otimes \dots v_r \otimes \alpha^1 \otimes \dots \otimes \alpha^s(\beta^1, \dots, \beta^r, u_1, \dots, u_s) \\ &\quad - \sum_{j=1}^s v_1 \otimes \dots \otimes v_r \otimes \alpha^1 \otimes \dots \otimes A^*(\alpha^j) \otimes \dots \otimes \alpha^s(\beta^1, \dots, \beta^r, u_1, \dots, u_s) \\ &= \sum_{j=1}^r \beta^1(v_1) \cdots \beta^j(A(v_j)) \cdots \beta^r(v_r)\alpha^1(u_1) \cdots \alpha^s(u_s) \\ &\quad - \sum_{j=1}^s \beta^1(v_1) \cdots \beta^r(v_r)\alpha^1(u_1) \cdots A^*(\alpha^j)(u_j) \cdots \alpha^s(u_s) \\ &= \sum_{j=1}^r \beta^1(v_1) \cdots A^*(\beta^j)(v_j) \cdots \beta^r(v_r)\alpha^1(u_1) \cdots \alpha^s(u_s) \\ &\quad - \sum_{j=1}^s \beta^1(v_1) \cdots \beta^r(v_r)\alpha^1(u_1) \cdots \alpha^j(A(u_j)) \cdots \alpha^s(u_s) \\ &= \sum_{j=1}^r T(\beta^1, \dots, A^*(\beta^j), \dots, \beta^r, u_1, \dots, u_s) \\ &\quad - \sum_{j=1}^s T(\beta^1, \dots, \beta^r, u_1, \dots, A(u_j), \dots, u_s). \end{aligned}$$

■

We shall make a minor extension of the preceding notion of a derivation associated to an endomorphism. Let $k, r, s \in \mathbb{Z}_{>0}$. Here we let $T \in \mathrm{T}^r_s(\mathsf{V})$ and $S \in \mathrm{T}^1_k(\mathsf{V})$. For $v_1, \dots, v_{k-1} \in \mathsf{V}$, we define $S_{(v_1,\dots,v_{k-1})} \in \mathrm{End}_{\mathbb{R}}(\mathsf{V})$ by

$$S_{(v_1,\dots,v_{k-1})}(v) = S(v, v_1, \dots, v_{k-1}).$$

Denote $S^* \in \mathrm{T}^1_{k-1}(\mathsf{V}) \otimes \mathsf{V}^*$ by

$$\langle S^*(\beta, v_1, \dots, v_{k-1}); v\rangle = \langle \beta; S(v, v_1, \dots, v_{k-1})\rangle$$

so that

$$S^*(\beta, v_1, \dots, v_{k-1}) = S^*_{(v_1,\dots,v_{k-1})}(\beta).$$

We then define $D_S(T) \in \mathrm{T}^r_{s+k-1}(\mathsf{V})$ by

$$D_S(T)(\beta^1, \dots, \beta^r, u_1, \dots, u_{s+k-1}) = D_{S_{(u_{s+1}\dots,u_{s+k-1})}}(T)(\beta^1, \dots, \beta^r, u_1, \dots, u_s). \tag{3.2}$$

The following elementary lemma gives a simpler formula for the previous constructions.

3.11 Lemma: (Insertion and tensor derivation II) *Let V be a finite-dimensional $\mathbb{R}$-vector space, let $k \in \mathbb{Z}_{>0}$, let $S \in \mathrm{T}^1_k(\mathsf{V})$, let $r, s \in \mathbb{Z}_{>0}$, and let $T \in \mathrm{T}^r_s(\mathsf{V})$. Then*

$$D_S(T) = \sum_{j=1}^r \mathrm{Ins}_j(T, S^*) - \sum_{j=1}^s \mathrm{Ins}_{r+j}(T, S).$$

Proof: We have

$$\begin{aligned}
D_S(T)(\beta^1, \dots,&\beta^r, u_1, \dots, u_{k+s-1}) \\
&= \sum_{j=1}^r \mathrm{Ins}_j(T, S^*_{(u_{s+1},\dots,u_{s+k-1})})(\beta^1, \dots, \beta^r, u_1, \dots, u_s) \\
&\quad - \sum_{j=1}^s \mathrm{Ins}_{r+j}(T, S_{(u_{s+1},\dots,u_{s+k-1})})(\beta^1, \dots, \beta^r, u_1, \dots, u_s) \\
&= \sum_{j=1}^r T(\beta^1, \dots, S^*_{(u_{s+1},\dots,u_{s+k-1})}(\beta_j), \dots, \beta^r, u_1, \dots, u_s) \\
&\quad - \sum_{j=1}^s T(\beta^1, \dots, \beta^r, u_1, \dots, S_{(u_{s+1},\dots,u_{s+k-1})}(u_j), \dots, u_s) \\
&= \sum_{j=1}^r \mathrm{Ins}_j(T, S^*)(\beta^1, \dots, \beta^r, u_1, \dots, u_{s+k-1}) \\
&\quad - \sum_{j=1}^r \mathrm{Ins}_{r+j}(T, S)(\beta^1, \dots, \beta^r, u_1, \dots, u_{s+k-1}),
\end{aligned}$$

as claimed. ■

Let us summarise this in the cases of interest. The cases of interest will be two in number. The first is when $S \in \mathrm{T}^1_2(\mathsf{V})$ and $T = T_0 \otimes v$ for $T_0 \in \mathrm{T}^k(\mathsf{V}^*)$ and $v \in \mathsf{V}$. In this case the preceding lemma gives

$$\begin{aligned} D_S(T)(v_1, \dots, v_{k+1}, \beta) &= \mathrm{Ins}_{k+1}(T_0 \otimes v, S^*)(v_1, \dots, v_{k+1}, \beta) - \sum_{j=1}^k \mathrm{Ins}_j(T_0 \otimes v, S)(v_1, \dots, v_{k+1}, \beta) \\ &= T_0(v_1, \dots, v_k)\langle \beta; S_{v_{k+1}}(v)\rangle - \langle \beta; v\rangle \sum_{j=1}^k \mathrm{Ins}_j(T_0, S)(v_1, \dots, v_{k+1}) \\ &= T_0(v_1, \dots, v_k)\langle \beta; S(v, v_{k+1})\rangle - \langle \beta; v\rangle \sum_{j=1}^k \mathrm{Ins}_j(T_0, S)(v_1, \dots, v_{k+1}). \end{aligned} \tag{3.3}$$

The second case we will consider is when $S \in \mathrm{T}^1_2(\mathsf{V})$ and $T = T_0 \otimes \alpha$ for $T_0 \in \mathrm{T}^k(\mathsf{V}^*)$ and $\alpha \in \mathsf{V}^*$. In this case we have

$$\begin{aligned} D_S(T)(v_1, \dots, v_{k+2}) &= -\mathrm{Ins}_{k+1}(T_0 \otimes \alpha, S)(v_1, \dots, v_{k+2}) - \sum_{j=1}^k \mathrm{Ins}_j(T_0 \otimes \alpha, S)(v_1, \dots, v_{k+2}) \\ &= -T_0(v_1, \dots, v_k)\alpha(S(v_{k+1}, v_{k+2})) - \langle \alpha; v_{k+2}\rangle \sum_{j=1}^k \mathrm{Ins}_j(T_0, S)(v_1, \dots, v_{k+1}). \end{aligned} \tag{3.4}$$

4. Differentiation of tensors on the total space of a vector bundle

In this section we establish some technical results for differentiation via connections of various objects—functions, vector fields, tensors—on vector bundles. These results will allow us to intrinsically perform the many calculations required to determine the recursive relations given in Section 5 between jets on M and jets on E for a vector bundle $\pi_{\mathsf{E}} : \mathsf{E} \to \mathsf{M}$. As with the constructions of the preceding section, the results in this section might seem *non sequitur* to the objectives of the paper. And, as with the results of the preceding section, perhaps a good strategy is to hurdle over this section until the results are subsequently needed.

As with the material in Section 3, there is nothing in this section that really separates the real analytic case from the smooth case, so the presentation treats both cases on an equal footing. What is true, however, is that the complications of the computations in this section are most useful in the real analytic setting of the paper. If one only wants to prove the continuity results in Section 9 in the smooth case, then simpler computations would suffice.

4.1. Vector bundles as Riemannian submersions. In this section we let $\pi_{\mathsf{E}} : \mathsf{E} \to \mathsf{M}$ be a vector bundle with $\pi_{\mathsf{TE}} : \mathsf{TE} \to \mathsf{E}$ its tangent bundle. We shall construct on E a Riemannian metric in a more or less natural way, using a Riemannian metric on M, an affine connection on M, a fibre metric on E, and a vector bundle connection in E. For the initial part of the construction, we do not require the affine connection on M to be the Levi-Civita connection, but we will only work with the case when it is, since there are useful formulae one can prove in this case. Let us indicate how one builds the Riemannian metric on E.

Let $r \in \{\infty, \omega\}$. We let $\pi_{\mathsf{E}}\colon \mathsf{E} \to \mathsf{M}$ be a vector bundle of class C^r and suppose that $\nabla^{\pi_{\mathsf{E}}}$ is a vector bundle connection on E, $\mathbb{G}_{\mathsf{M}}$ is a Riemannian metric on M, and $\mathbb{G}_{\pi_{\mathsf{E}}}$ is a fibre metric for E with all data of class C^r. The total space E can be equipped with a Riemannian metric via a natural adaptation of the Sasaki metric for tangent bundles [Sasaki 1958]. To define the inner product, we use the splitting determined by the connection to give the inner product on $\mathsf{T}_e\mathsf{E}$ by

$$\mathbb{G}_{\mathsf{E}}(w_1, w_2) = \mathbb{G}_{\mathsf{M}}(\mathrm{hor}(w_1), \mathrm{hor}(w_2)) + \mathbb{G}_{\pi_{\mathsf{E}}}(\mathrm{ver}(w_1), \mathrm{ver}(w_2)). \tag{4.1}$$

This then turns E into a Riemannian manifold. We denote by ∇^{E} the Levi-Civita connection associated with $\mathbb{G}_{\mathsf{E}}$. Since the connection giving the splitting is of class C^r if $\nabla^{\pi_{\mathsf{E}}}$ is of class C^r, the Riemannian metric $\mathbb{G}_{\mathsf{E}}$ and its Levi-Civita connection are of class C^r if $\mathbb{G}_{\mathsf{M}}$ and $\mathbb{G}_{\pi_{\mathsf{E}}}$ are of class C^r.

When we are working in this setting of Riemannian metrics and Levi-Civita connections on the total space of a vector bundle $\pi_{\mathsf{E}}\colon \mathsf{E} \to \mathsf{M}$, we shall denote by $\mathbb{G}_{\mathsf{M}}$ the Riemannian metric on M and by ∇^{M} its Levi-Civita connection.

We note that the choice of metric $\mathbb{G}_{\mathsf{E}}$ ensures that $\pi_{\mathsf{E}}\colon \mathsf{E} \to \mathsf{M}$ is a Riemannian submersion if we equip M with its Riemannian metric $\mathbb{G}_{\mathsf{M}}$ used to build $\mathbb{G}_{\mathsf{E}}$. Moreover, the fibres of π_{E} are totally geodesic submanifolds. There are a few constructions involving Riemannian submersions that will be helpful for us, and we review these here. Let us introduce some notation apropos to this. We do this in a general setting. Thus let $(\mathsf{F}, \mathbb{G}_{\mathsf{F}})$ and $(\mathsf{M}, \mathbb{G}_{\mathsf{M}})$ be Riemannian manifolds. Let $\pi\colon \mathsf{F} \to \mathsf{M}$ be a ***Riemannian submersion***, i.e., for each $y \in \mathsf{F}$,

$$\mathbb{G}_{\mathsf{M}}(T_y\pi(u), T_y\pi(v)) = \mathbb{G}_{\mathsf{F}}(u, v)$$

for every $u, v \in \mathsf{T}_y\mathsf{F}$ that are orthogonal to $\ker(T_y\pi)$. We let $\mathsf{VF} = \ker(T\pi)$ be the vertical subbundle with HF its $\mathbb{G}_{\mathsf{F}}$-orthogonal complement, which we call the horizontal subbundle. We let $\mathrm{ver}, \mathrm{hor}\colon \mathsf{TF} \to \mathsf{TF}$ be the projections onto VF and HF, just as we have done for vector bundles. For a vector field X on M, we denote by X^{h} the horizontal lift of X to F. This is the unique HF-valued vector field satisfying $T_y\pi(X^{\mathrm{h}}(y)) = X \circ \pi(y)$ for each $y \in \mathsf{F}$.

Given a submanifold of S of a Riemannian manifold $(\mathsf{M}, \mathbb{G}_{\mathsf{M}})$, S inherits the Riemannian metric $\mathbb{G}_{\mathsf{S}}$ obtained by pulling back $\mathbb{G}_{\mathsf{M}}$ by the inclusion $\iota_{\mathsf{S}}\colon \mathsf{S} \to \mathsf{M}$. The submanifold S is ***totally geodesic*** if every geodesic for $(\mathsf{S}, \mathbb{G}_{\mathsf{S}})$ is also a geodesic for $(\mathsf{M}, \mathbb{G}_{\mathsf{M}})$.

Following [O'Neill 1968], for a C^r-Riemannian submersion $\pi\colon \mathsf{F} \to \mathsf{N}$, there are two associated tensors that characterise the submersion. Specifically, we define

$$A_\pi, T_\pi \in \Gamma^r(\mathrm{T}^2(\mathsf{T}^*\mathsf{F}) \otimes \mathsf{TF})$$

by

$$\begin{aligned} A_\pi(\xi, \eta) &= \mathrm{ver}(\nabla^{\mathsf{F}}_{\mathrm{hor}(\xi)} \mathrm{hor}(\eta)) + \mathrm{hor}(\nabla^{\mathsf{F}}_{\mathrm{hor}(\xi)} \mathrm{ver}(\eta)), \\ T_\pi(\xi, \eta) &= \mathrm{hor}(\nabla^{\mathsf{F}}_{\mathrm{ver}(\xi)} \mathrm{ver}(\eta)) + \mathrm{ver}(\nabla^{\mathsf{F}}_{\mathrm{ver}(\xi)} \mathrm{hor}(\eta)) \end{aligned} \tag{4.2}$$

for $\xi, \eta \in \Gamma^1(\mathsf{TF})$. One can easily verify that A_π and T_π are indeed tensors as claimed. Since the fibres of π are submanifolds, we can define the ***vertical covariant derivative*** as the projection of the covariant derivative onto sections:

$$\nabla^{\mathrm{ver}}_U V = \mathrm{ver}(\nabla^{\mathsf{F}}_U V)$$

for vertical vector fields U and V.

With all this background, we have the following result with tells us how to covariantly differentiate vector fields on the total space of a vector bundle.

4.1 Lemma: (Covariant derivatives for Riemannian submersions) *Let $r \in \{\infty, \omega\}$. Let $(\mathsf{F}, \mathbb{G}_{\mathsf{F}})$ and $(\mathsf{M}, \mathbb{G}_{\mathsf{M}})$ be C^r-Riemannian manifolds with ∇^{F} and ∇^{M} the Levi-Civita connections. Let $\pi\colon \mathsf{F} \to \mathsf{M}$ be a Riemannian submersion. Let $X, Y \in \Gamma^r(\mathsf{TM})$ and let $U, V \in \Gamma^r(\mathsf{TF})$ be vertical vector fields. Then the following statements hold:*

(i) $\mathrm{hor}(\nabla^{\mathsf{F}}_{X^{\mathrm{h}}} Y^{\mathrm{h}}) = (\nabla^{\mathsf{M}}_X Y)^{\mathrm{h}}$;
(ii) $A_\pi(X^{\mathrm{h}}, Y^{\mathrm{h}}) = -\frac{1}{2}\,\mathrm{ver}([X^{\mathrm{h}}, Y^{\mathrm{h}}])$;
(iii) $\nabla^{\mathsf{F}}_U V = \nabla^{\mathrm{ver}}_U V + T_\pi(U, V)$;
(iv) $\nabla^{\mathsf{F}}_V X^{\mathrm{h}} = \mathrm{hor}(\nabla^{\mathsf{F}}_V X^{\mathrm{h}}) + T_\pi(V, X^{\mathrm{h}})$;
(v) $\nabla^{\mathsf{F}}_{X^{\mathrm{h}}} V = \mathrm{ver}(\nabla^{\mathsf{F}}_{X^{\mathrm{h}}} V) + A_\pi(X^{\mathrm{h}}, V)$;
(vi) $\nabla^{\mathsf{F}}_{X^{\mathrm{h}}} Y^{\mathrm{h}} = (\nabla^{\mathsf{M}}_X Y)^{\mathrm{h}} + A_\pi(X^{\mathrm{h}}, Y^{\mathrm{h}})$.
(vii) $\mathbb{G}_{\mathsf{F}}(\nabla^{\mathsf{F}}_V X^{\mathrm{h}}, Y^{\mathrm{h}}) = -\frac{1}{2}\mathbb{G}_{\mathsf{F}}(\mathrm{ver}([X^{\mathrm{h}}, Y^{\mathrm{h}}]), V) = \mathbb{G}_{\mathsf{F}}(\nabla^{\mathsf{F}}_V Y^{\mathrm{h}}, X^{\mathrm{h}})$.

Additionally, if the fibres of π are totally geodesic submanifolds of F, then the following statements hold:

(viii) $T_\pi = 0$;
(ix) $\nabla^{\mathrm{ver}}|\mathsf{F}_x$ *is the Levi-Civita connection for the submanifold Riemannian metric on* F_x;
(x) $\mathrm{ver}(\nabla^{\mathsf{F}}_{X^{\mathrm{h}}} V) = \mathrm{ver}([X^{\mathrm{h}}, V])$;
(xi) $\nabla^{\mathsf{F}}_V X^{\mathrm{h}}$ *is horizontal and* $\nabla^{\mathsf{F}}_V X^{\mathrm{h}} = A_\pi(X^{\mathrm{h}}, V)$.

Finally, if $\mathsf{F} = \mathsf{E}$ is the total space of a vector bundle and if $\mathbb{G}_{\mathsf{E}}$ is the Riemannian metric on E defined above, then the following additional statements hold for sections $\xi, \eta \in \Gamma^r(\mathsf{E})$:

(xii) $\nabla^{\mathsf{E}}_{\xi^{\mathrm{v}}} \eta^{\mathrm{v}} = 0$;
(xiii) $\mathrm{ver}(\nabla^{\mathsf{E}}_{X^{\mathrm{h}}} \xi^{\mathrm{v}}) = (\nabla^{\pi}_X \xi)^{\mathrm{v}}$.

Proof: We use the Koszul formula for the Levi-Civita connection:

$$2\mathbb{G}_{\mathsf{F}}(\nabla^{\mathsf{F}}_\xi \eta, \zeta) = \mathscr{L}_\xi(\mathbb{G}_{\mathsf{F}}(\eta, \zeta)) + \mathscr{L}_\eta(\mathbb{G}_{\mathsf{F}}(\xi, \zeta)) - \mathscr{L}_\zeta(\mathbb{G}(\xi, \eta)) \\ + \mathbb{G}_{\mathsf{F}}([\xi, \eta], \zeta) - \mathbb{G}_{\mathsf{F}}([\xi, \zeta], \eta) - \mathbb{G}_{\mathsf{F}}([\eta, \zeta], \xi) \quad (4.3)$$

for vector fields ξ, η, and ζ on F [Kobayashi and Nomizu 1963, Page 160]. We shall also use the formulae

$$\mathscr{L}_\zeta(\mathbb{G}_{\mathsf{F}}(\xi, \eta)) = \mathbb{G}_{\mathsf{F}}(\nabla^{\mathsf{F}}_\zeta \xi, \eta) + \mathbb{G}_{\mathsf{F}}(\xi, \nabla^{\mathsf{F}}_\zeta \eta) \quad (4.4)$$

(saying that the Levi-Civita connection is a metric connection) and

$$\nabla^{\mathsf{F}}_\xi \eta - \nabla^{\mathsf{F}}_\eta \xi = [\xi, \eta] \quad (4.5)$$

(saying that the Levi-Civita connection is torsion-free). Both of these formulae are determinable from the Koszul formula.

Let us make some preliminary computations. First, since X^{h} and Y^{h} are π-related to X and Y, we have that $[X^{\mathrm{h}}, Y^{\mathrm{h}}]$ is π-related to $[X, Y]$ [Abraham, Marsden, and Ratiu 1988, Proposition 4.2.25]. Thus

$$\operatorname{hor}([X^{\mathrm{h}}, Y^{\mathrm{h}}]) = [X, Y]^{\mathrm{h}}. \tag{4.6}$$

In like manner, since V is π-related to the zero vector field and X^{h} is π-related to X, $[V, X^{\mathrm{h}}]$ is π-related to the zero vector field. That is,

$$\operatorname{hor}([V, X^{\mathrm{h}}]) = 0. \tag{4.7}$$

Next, if f is a function on M, then

$$\mathscr{L}_{X^{\mathrm{h}}}(\pi^* f) = \langle \mathrm{d}(\pi^* f); X^{\mathrm{h}} \rangle = \langle \pi^* \mathrm{d}f; X^{\mathrm{h}} \rangle,$$

from which we deduce

$$\mathscr{L}_{X^{\mathrm{h}}}(\pi^* f)(y) = \langle \mathrm{d}f \circ \pi(y); X \circ \pi(y) \rangle, \qquad y \in \mathsf{F}. \tag{4.8}$$

We trivially have

$$\mathscr{L}_V(\pi^* f) = 0.$$

(i) One can use (4.3) with $\xi = X^{\mathrm{h}}$, $\eta = Y^{\mathrm{h}}$, and $\zeta = Z^{\mathrm{h}}$, and the formulae (4.6) and (4.8) to give

$$\mathbb{G}_{\mathsf{F}}(\nabla^{\mathsf{F}}_{X^{\mathrm{h}}} Y^{\mathrm{h}}, Z^{\mathrm{h}}) = \pi^* \mathbb{G}_{\mathsf{M}}(\nabla^{\mathsf{M}}_X Y, Z).$$

This shows that

$$\operatorname{hor}(\nabla^{\mathsf{F}}_{X^{\mathrm{h}}} Y^{\mathrm{h}}) = (\nabla^{\mathsf{M}}_X Y)^{\mathrm{h}}. \tag{4.9}$$

(ii) Now we use (4.3) with $\xi = X^{\mathrm{h}}$, $\eta = Y^{\mathrm{h}}$, and $\zeta = V$. We immediately have that the first three terms on the right in (4.3) are zero. By (4.7), the last two terms on the right in (4.3) are zero. Thus we have

$$2\mathbb{G}_{\mathsf{F}}(\nabla^{\mathsf{F}}_{X^{\mathrm{h}}} Y^{\mathrm{h}}, V) = \mathbb{G}_{\mathsf{F}}([X, Y]^{\mathrm{h}}, V),$$

and so

$$A_\pi(X^{\mathrm{h}}, Y^{\mathrm{h}}) = \operatorname{ver}(\nabla^{\mathsf{F}}_{X^{\mathrm{h}}} Y^{\mathrm{h}}) = \frac{1}{2} \operatorname{ver}([X, Y]^{\mathrm{h}}).$$

(iii) We have

$$\nabla^{\mathsf{F}}_U V = \operatorname{ver}(\nabla^{\mathsf{F}}_U V) + \operatorname{hor}(\nabla^{\mathsf{F}}_U V) = \nabla^{\mathrm{ver}}_U V + T_\pi(U, V),$$

as claimed.

(iv) We have

$$\nabla^{\mathsf{F}}_V X^{\mathrm{h}} = \operatorname{hor}(\nabla^{\mathsf{F}}_V X^{\mathrm{h}}) + \operatorname{ver}(\nabla^{\mathsf{F}}_V X^{\mathrm{h}}) = \operatorname{hor}(\nabla^{\mathsf{F}}_V X^{\mathrm{h}}) + T_\pi(V, X^{\mathrm{h}}),$$

as claimed.

(v) We have

$$\nabla^{\mathsf{F}}_{X^{\mathrm{h}}} V = \operatorname{ver}(\nabla^{\mathsf{F}}_{X^{\mathrm{h}}} V) + \operatorname{hor}(\nabla^{\mathsf{F}}_{X^{\mathrm{h}}} V) = \operatorname{ver}(\nabla^{\mathsf{F}}_{X^{\mathrm{h}}} V) + A_\pi(X^{\mathrm{h}}, V),$$

as claimed.

(vi) We have

$$\nabla^{\mathsf{F}}_{X^{\mathrm{h}}} Y^{\mathrm{h}} = \mathrm{hor}(\nabla^{\mathsf{F}}_{X^{\mathrm{h}}} Y^{\mathrm{h}}) + \mathrm{ver}(\nabla^{\mathsf{F}}_{X^{\mathrm{h}}} Y^{\mathrm{h}}) = (\nabla^{\mathsf{M}}_X Y)^{\mathrm{h}} + A_\pi(X^{\mathrm{h}}, Y^{\mathrm{h}}),$$

using part (i).

(vii) This is a direct computation using (4.5), (4.7), (4.4), and part (i):

$$\begin{aligned}\mathbb{G}_{\mathsf{F}}(\nabla^{\mathsf{F}}_V X^{\mathrm{h}}, Y^{\mathrm{h}}) &= \mathbb{G}_{\mathsf{F}}(\nabla^{\mathsf{F}}_{X^{\mathrm{h}}} V, Y^{\mathrm{h}}) + \mathbb{G}_{\mathsf{F}}([V, X^{\mathrm{h}}], Y^{\mathrm{h}}) \\ &= -\mathbb{G}_{\mathsf{F}}(\nabla^{\mathsf{F}}_{X^{\mathrm{h}}} Y^{\mathrm{h}}, V) \\ &= -\frac{1}{2}\mathbb{G}_{\mathsf{F}}([X^{\mathrm{h}}, Y^{\mathrm{h}}], V) = -\frac{1}{2}\mathbb{G}_{\mathsf{F}}(\mathrm{ver}([X^{\mathrm{h}}, Y^{\mathrm{h}}]), V).\end{aligned} \tag{4.10}$$

(viii) and (ix) These are properties of totally geodesic submanifolds, so we first prove the result for the following situation.

1 Sublemma: *Let $(\mathsf{M}, \mathbb{G}_{\mathsf{M}})$ be a Riemannian manifold and let $\mathsf{S} \subseteq \mathsf{M}$ be a submanifold. We let $\mathbb{G}_{\mathsf{S}} = \iota_{\mathsf{S}}^*\mathbb{G}$ be the induced Riemannian metric on S. We let ∇^{M} and ∇^{S} be the Levi-Civita connections. Then S is totally geodesic if and only if $\nabla^{\mathsf{M}}_X Y$ is tangent to S whenever $X, Y \in \Gamma^1(\mathsf{TM})$ are tangent to S.*

Proof: We let $\mathsf{NS} \subseteq \mathsf{TM}|\mathsf{S}$ be the normal bundle. We define the second fundamental form for S to be the section Π_{S} of $\mathrm{T}^2(\mathsf{TS}) \otimes \mathsf{NS}$ defined by

$$\Pi_{\mathsf{S}}(X, Y) = \mathrm{pr}_{\mathsf{NS}}(\nabla^{\mathsf{M}}_X Y)$$

for vector fields X and Y on M that are tangent to S, where $\mathrm{pr}_{\mathsf{NS}} \colon \mathsf{TM}|\mathsf{S} \to \mathsf{NS}$ is the orthogonal projection onto NS.

We claim that Π_{S} is symmetric. Indeed, by (4.5) we have

$$\Pi_{\mathsf{S}}(X, Y) - \Pi_{\mathsf{S}}(Y, X) = \mathrm{pr}_{\mathsf{NS}}([X, Y]) = 0$$

since $[X, Y]$ is tangent to S if X and Y are tangent to S.

Next we claim that $\mathrm{pr}_{\mathsf{TS}}(\nabla^{\mathsf{M}}_X Y) = \nabla^{\mathsf{S}}_X Y$ for vector fields X and Y that are tangent to S, where $\mathrm{pr}_{\mathsf{TS}} \colon \mathsf{TM}|\mathsf{S} \to \mathsf{TS}$ is the orthogonal projection. To prove this, we show that

$$(X, Y) \mapsto \mathrm{pr}_{\mathsf{TS}}(\nabla^{\mathsf{M}}_X Y),$$

when restricted to S, satisfies the defining conditions (4.4) and (4.5) for the Levi-Civita connection for $\mathbb{G}_{\mathsf{S}}$. Indeed, because $[X, Y]$ is tangent to S whenever X and Y are tangent to S, we determine that, when restricted to S,

$$\mathrm{pr}_{\mathsf{TS}}(\nabla^{\mathsf{M}}_X Y - \nabla^{\mathsf{M}}_Y X) = \mathrm{pr}_{\mathsf{TS}}([X, Y]) = [X, Y]$$

for all vector fields X and Y tangent to S. This shows that $(X, Y) \mapsto \mathrm{pr}_{\mathsf{TS}}(\nabla^{\mathsf{M}}_X Y)$ satisfies (4.5). Also, when we restrict to S, we have

$$\begin{aligned}\mathscr{L}_Z(\mathbb{G}_{\mathsf{S}}(X, Y)) = \mathscr{L}_Z(\mathbb{G}_{\mathsf{M}}(X, Y)) &= \mathbb{G}_{\mathsf{M}}(\nabla^{\mathsf{M}}_Z X, Y) + \mathbb{G}_{\mathsf{M}}(X, \nabla^{\mathsf{S}}_Z Y) \\ &= \mathbb{G}_{\mathsf{S}}(\mathrm{pr}_{\mathsf{TS}}(\nabla^{\mathsf{M}}_Z X), Y) + \mathbb{G}_{\mathsf{S}}(X, \mathrm{pr}_{\mathsf{TS}}(\nabla^{\mathsf{M}}_Z Y))\end{aligned}$$

for all vector fields X, Y, and Z that are tangent to S. This shows that $(X, Y) \mapsto \mathrm{pr}_{\mathsf{TS}}(\nabla^{\mathsf{M}}_X Y)$ satisfies (4.4).

Now we can prove the sublemma. First suppose that S is totally geodesic. Let $v_x \in \mathsf{TS}$ and let $t \mapsto \gamma(t)$ be a geodesic for ∇^{S} satisfying $\gamma'(0) = v_x$. Then γ is also a geodesic for ∇^{M}. Thus

$$\begin{aligned} 0 &= \nabla^{\mathsf{M}}_{\gamma'(t)}\gamma'(t) = \nabla^{\mathsf{S}}_{\gamma'(t)}\gamma'(t) \\ &= \mathrm{pr}_{\mathsf{TS}}(\nabla^{\mathsf{M}}_{\gamma'(t)}\gamma'(t)) \\ &= \mathrm{pr}_{\mathsf{TS}}(\nabla^{\mathsf{M}}_{\gamma'(t)}\gamma'(t)) + \mathrm{pr}_{\mathsf{NS}}(\nabla^{\mathsf{M}}_{\gamma'(t)}\gamma'(t)), \end{aligned}$$

from which we conclude, evaluating at $t = 0$, that $\Pi_{\mathsf{S}}(v_x, v_x) = 0$. Since Π_{S} is symmetric, $\Pi_{\mathsf{S}} = 0$. Thus

$$\nabla^{\mathsf{S}}_X Y = \mathrm{pr}_{\mathsf{TS}}(\nabla^{\mathsf{M}}_X Y) = \nabla^{\mathsf{M}}_X Y$$

for vector fields X and Y on M tangent to S. The converse, that S is totally geodesic if $\nabla^{\mathsf{M}}_X Y = \nabla^{\mathsf{S}}_X Y$ for all vector fields X and Y on M tangent to S, is clear. ▼

Given the sublemma, let $x \in \mathsf{M}$ and let $\mathsf{S} = \pi^{-1}(x)$ be the fibre. As we showed in the proof of the sublemma, if U and V are vertical vector fields (in particular, they are tangent to S), then

$$\nabla^{\mathsf{F}}_U V = \mathrm{ver}(\nabla^{\mathsf{F}}_U V) + T_\pi(U, V) = \nabla^{\mathsf{S}}_U V.$$

Matching vertical and horizontal parts on S gives

$$\nabla^{\mathrm{ver}}_U V = \nabla^{\mathsf{S}}_U V, \quad T_\pi(U, V) = 0,$$

as claimed.

(xi) It follows immediately from parts (iv) and (viii) that $\nabla^{\mathsf{F}}_V X^{\mathrm{h}}$ is horizontal. We also have

$$\nabla^{\mathsf{F}}_V X^{\mathrm{h}} = \mathrm{hor}(\nabla^{\mathsf{F}}_V X^{\mathrm{h}}) = \mathrm{hor}(\nabla^{\mathsf{F}}_{X^{\mathrm{h}}} V) + \mathrm{hor}([V, X^{\mathrm{h}}])$$

by (4.5). By part (v), the first term on the far right is $A_\pi(X^{\mathrm{h}}, V)$ and, by (4.7), the second term in the far right is zero.

(x) By (4.5), we have

$$\mathrm{ver}(\nabla^{\mathsf{F}}_{X^{\mathrm{h}}} V) = \mathrm{ver}(\nabla^{\mathsf{F}}_V X^{\mathrm{h}}) + \mathrm{ver}([X^{\mathrm{h}}, V]).$$

By part (xi) the first term on the right is zero.

(xii) We note here that the fibres of $\pi_{\mathsf{E}} \colon \mathsf{E} \to \mathsf{M}$ are vector spaces and the restriction of $\mathbb{G}_{\mathsf{E}}$ to E_x is just the constant Riemannian metric $\mathbb{G}_{\pi_{\mathsf{E}}}(x)$. Thus covariant derivatives on fibres are just ordinary derivatives. Now, since vertical lifts restricted to fibres are constant, their ordinary derivatives are zero, and this gives the assertion.

(xiii) Here, by part (x), we have

$$\mathrm{ver}(\nabla^{\mathsf{E}}_{X^{\mathrm{h}}} \xi^{\mathrm{v}}) = \mathrm{ver}([X^{\mathrm{h}}, \xi^{\mathrm{v}}]).$$

By (4.7), $[X^{\mathrm{h}}, \xi^{\mathrm{v}}]$ is vertical. By [Abraham, Marsden, and Ratiu 1988, Proposition 4.2.34], we have

$$[\xi^{\mathrm{v}}, X^{\mathrm{h}}] = \frac{1}{2}\,\frac{\mathrm{d}^2}{\mathrm{d}t^2}\bigg|_{t=0} \Phi^{X^{\mathrm{h}}}_{-t} \circ \Phi^{\xi^{\mathrm{v}}}_{-t} \circ \Phi^{X^{\mathrm{h}}}_t \circ \Phi^{\xi^{\mathrm{v}}}_t(e).$$

Using the fact that $\Phi_t^{\xi^{\mathrm{v}}}(e) = e + t\xi \circ \pi_{\mathsf{E}}(e)$ and that $\Phi_t^{X^{\mathrm{h}}}(e)$ is the parallel transport $t \mapsto \tau_t^{\gamma}$ along integral curve γ for X through $\pi_{\mathsf{E}}(e)$, we directly calculate

$$\Phi_{-t}^{X^{\mathrm{h}}} \circ \Phi_{-t}^{\xi^{\mathrm{v}}} \circ \Phi_t^{X^{\mathrm{h}}} \circ \Phi_t^{\xi^{\mathrm{v}}}(e) = e - t(\tau_t^{\gamma}(\xi \circ \gamma(t)) - \xi \circ \gamma(0)),$$

and from this, using the relationship between parallel transport [Kobayashi and Nomizu 1963, page 114] and covariant derivative, we have $[\xi^{\mathrm{v}}, X^{\mathrm{h}}] = -(\nabla_X^{\pi_{\mathsf{E}}} \xi)^{\mathrm{v}}$. ■

4.2. Derivatives of tensor contractions. In Section 3.5 we constructed a tensor contraction/insertion operator. Let us consider the derivative of this operation.

4.2 Lemma: (Covariant differential of insertion I) *Let $r \in \{\infty, \omega\}$. Let $\pi_{\mathsf{E}} \colon \mathsf{E} \to \mathsf{M}$ a vector bundle of class C^r, let $\nabla^{\pi_{\mathsf{E}}}$ be a C^r-vector bundle connection in E, let $k, l \in \mathbb{Z}_{>0}$, let $A \in \Gamma^r(\mathrm{T}^k(\mathsf{E}^*))$, and let $S \in \Gamma^r(\mathrm{T}^l(\mathsf{E}^*) \otimes \mathsf{E})$. For $j \in \{1, \dots, k\}$ we have*

$$\nabla^{\pi_{\mathsf{E}}}(\mathrm{Ins}_j(A, S)) = \mathrm{Ins}_j(\nabla^{\pi_{\mathsf{E}}} A, S) + \mathrm{Ins}_j(A, \nabla^{\pi_{\mathsf{E}}} S).$$

Proof: We let $\xi_a \in \Gamma^r(\mathsf{E})$, $a \in \{1, \dots, k+l-1\}$, and $X \in \Gamma^r(\mathsf{TM})$. We calculate

$$\begin{aligned}
\mathscr{L}_X(\mathrm{Ins}_j(A,S)(\xi_1,\dots,\xi_{k+l-1})) &= (\nabla_X^{\pi_{\mathsf{E}}}\mathrm{Ins}_j(A,S))(\xi_1,\dots,\xi_{k+l-1}) \\
&\quad + \sum_{a=1}^{k+l-1} \mathrm{Ins}_j(A,S)(\xi_1,\dots,\nabla_X^{\pi_{\mathsf{E}}}\xi_a,\dots,\xi_{k+l-1}) \\
&= (\nabla_X^{\pi_{\mathsf{E}}}\mathrm{Ins}_j(A,S))(\xi_1,\dots,\xi_{k+l-1}) \\
&\quad + \sum_{a=1}^{j-1} A(\xi_1,\dots,\nabla_X^{\pi_{\mathsf{E}}}\xi_a,\dots,\xi_{j-1},S(\xi_j,\xi_{k+1},\dots,\xi_{k+l-1}),\xi_{j+1},\dots,\xi_k) \\
&\quad + A(\xi_1,\dots,\xi_{j-1},S(\nabla_X^{\pi_{\mathsf{E}}}\xi_j,\xi_{k+1},\dots,\xi_{k+l-1}),\xi_{j+1},\dots,\xi_k) \\
&\quad + \sum_{a=j+1}^{k} A(\xi_1,\dots,\xi_{j-1},S(\xi_j,\xi_{k+1},\dots,\xi_{k+l-1}),\xi_{j+1},\dots,\nabla_X^{\pi_{\mathsf{E}}}\xi_a,\dots,\xi_k) \\
&\quad + \sum_{a=k+1}^{k+l-1} A(\xi_1,\dots,\xi_{j-1},S(\xi_j,\xi_{k+1},\dots,\nabla_X^{\pi_{\mathsf{E}}}\xi_a,\dots,\xi_{k+l-1}),\xi_{j+1},\dots,\xi_k).
\end{aligned}$$

We also calculate

$$\begin{aligned}
&\mathscr{L}_X(\mathrm{Ins}_j(A,S)(\xi_1,\dots,\xi_{k+l-1})) \\
&\quad = \mathscr{L}_X(A(\xi_1,\dots,\xi_{j-1},S(\xi_j,\xi_{k+1},\dots,\xi_{k+l-1}),\xi_{j+1},\dots,\xi_k)) \\
&\quad = (\nabla_X^{\pi_{\mathsf{E}}}A)(\xi_1,\dots,\xi_{j-1},S(\xi_j,\xi_{k+1},\dots,\xi_{k+l-1}),\xi_{j+1},\dots,\xi_k) \\
&\qquad + \sum_{a=1}^{j-1} A(\xi_1,\dots,\nabla_X^{\pi_{\mathsf{E}}}\xi_a,\dots,\xi_{j-1},S(\xi_j,\xi_{k+1},\dots,\xi_{k+l-1}),\xi_{j+1},\dots,\xi_k) \\
&\qquad + A(\xi_1,\dots,\xi_{j-1},(\nabla_X^{\pi_{\mathsf{E}}}S)(\xi_j,\xi_{k+1},\dots,\xi_{k+l-1}),\xi_{j+1},\dots,\xi_k) \\
&\qquad + A(\xi_1,\dots,\xi_{j-1},S(\nabla_X^{\pi_{\mathsf{E}}}\xi_j,\xi_{k+1},\dots,\xi_{k+l-1}),\xi_{j+1},\dots,\xi_k) \\
&\qquad + \sum_{a=j+1}^{k} A(\xi_1,\dots,\xi_{j-1},S(\xi_j,\xi_{k+1},\dots,\xi_{k+l-1}),\xi_{j+1},\dots,\nabla_X^{\pi_{\mathsf{E}}}\xi_a,\dots,\xi_k)
\end{aligned}$$

$$+\sum_{a=k+1}^{k+l-1} A(\xi_1,\dots,\xi_{j-1},S(\xi_j,\xi_{k+1},\dots,\nabla_X^{\pi_{\mathsf{E}}}\xi_a,\dots,\xi_{k+l-1}),\xi_{j+1},\dots,\xi_k).$$

Comparing the right-hand sides of the preceding calculations gives

$$\begin{aligned}(\nabla^{\pi_{\mathsf{E}}}\mathrm{Ins}_j(A,S))&(\xi_1,\dots,\xi_{k+l-1},X)\\ &=(\nabla^{\pi_{\mathsf{E}}}A)(\xi_1,\dots,\xi_{j-1},S(\xi_j,\xi_{k+1},\dots,\xi_{k+l-1}),\xi_{j+1},\dots,\xi_k,\xi_{k+l-1},X)\\ &\quad+A(\xi_1,\dots,\xi_{j-1},(\nabla^{\pi_{\mathsf{E}}}S)(\xi_j,\xi_{k+1},\dots,\xi_{k+l-1},X),\xi_{j+1},\dots,\xi_k)\\ &=\mathrm{Ins}_j(\nabla^{\pi_{\mathsf{E}}}A,S)(\xi_1,\dots,\xi_{k+l-1},X)+\mathrm{Ins}_j(A,\nabla^{\pi_{\mathsf{E}}}S)(\xi_1,\dots,\xi_{k+l-1},X),\end{aligned}$$

and this gives the result. ■

Using this result, we can easily compute the derivative for tensor insertion with one of the arguments fixed.

4.3 Lemma: (Covariant differential of tensor insertion II) *Let $r\in\{\infty,\omega\}$. Let $\pi_{\mathsf{E}}\colon\mathsf{E}\to\mathsf{M}$ a vector bundle of class C^r, let $\nabla^{\pi_{\mathsf{E}}}$ be a C^r-vector bundle connection in E, let $l\in\mathbb{Z}_{>0}$, and let $S\in\Gamma^r(\mathrm{T}^l(\mathsf{E}^*)\otimes\mathsf{E})$. Then, for $k\in\mathbb{Z}_{>0}$ and $j\in\{1,\dots,k\}$,*

$$(\nabla^{\pi_{\mathsf{E}}}\mathrm{Ins}_{S,j})(A)=\mathrm{Ins}_j(A,\nabla^{\pi_{\mathsf{E}}}S).$$

Proof: We have

$$\nabla^{\pi_{\mathsf{E}}}(\mathrm{Ins}_{S,j}(A))=(\nabla^{\pi_{\mathsf{E}}}\mathrm{Ins}_{S,j})(A)+\mathrm{Ins}_{S,j}(\nabla^{\pi_{\mathsf{E}}}A)$$

and

$$\nabla^{\pi_{\mathsf{E}}}(\mathrm{Ins}_j(A,S))=\mathrm{Ins}_j(\nabla^{\pi_{\mathsf{E}}}A,S)+\mathrm{Ins}_j(A,\nabla^{\pi_{\mathsf{E}}}S).$$

Comparing the equations, noting that $\mathrm{Ins}_{S,j}(\nabla^{\pi_{\mathsf{E}}}A)=\mathrm{Ins}_j(\nabla^{\pi_{\mathsf{E}}}A,S)$, the result follows. ■

Related to tensor contraction is the evaluation of a vector bundle mapping. We shall consider the derivative of this evaluation. In stating the result, we use a bit of tensor notation that we now introduce. Let V be a finite-dimensional $\mathbb{R}$-vector space and let $A\in\mathrm{T}^1_{k+1}(\mathsf{V}^*)$ and $B\in\mathrm{T}^1_l(\mathsf{V})$. We then denote by $A(B)\in\mathrm{T}^{k+l}(\mathsf{V}^*)$ the tensor defined by

$$A(B)(v_1,\dots,v_k,v_{k+1},\dots,v_{k+l})=A(v_1,\dots,v_k,B(v_{k+1},\dots,v_{k+l}))\qquad(4.11)$$

Thus $A(B)$ is shorthand for $\mathrm{Ins}_{k+1}(A,B)$. With this notation, we have the following result.

4.4 Lemma: (Leibniz Rule for tensor evaluation) *Let $\pi_{\mathsf{E}}\colon\mathsf{E}\to\mathsf{M}$ and $\pi_{\mathsf{F}}\colon\mathsf{F}\to\mathsf{M}$ be smooth vector bundles and let $\nabla^{\pi_{\mathsf{E}}}$ and $\nabla^{\pi_{\mathsf{F}}}$ be smooth vector bundle connections in E and F, respectively. Let ∇^{M} be an affine connection on M. Let $L\in\Gamma^\infty(\mathsf{F}\otimes\mathsf{E}^*)$. Then*

$$D^k_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{F}}}}(L\circ\xi)=\sum_{l=0}^k\binom{k}{l}\mathrm{Sym}_k\left(D^l_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{F}}\otimes\pi_{\mathsf{E}}}}(L)(D^{k-l}_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}}(\xi))\right),$$

for $\xi\in\Gamma^\infty(\mathsf{E})$.

Proof: First we claim that

$$\nabla^{\mathsf{M},\pi_{\mathsf{F}},k}(L\circ\xi)(X_1,\dots,X_k) = \sum_{l=0}^{k}\sum_{\sigma\in\mathfrak{S}_{l,k-l}}(\nabla^{\mathsf{M},\pi_{\mathsf{F}}\otimes\pi_{\mathsf{E}},l}L(X_{\sigma(1)},\dots,X_{\sigma(l)}))(\nabla^{\mathsf{M},\pi_{\mathsf{E}},k-l}\xi(X_{\sigma(l+1)},\dots,X_{\sigma(k)})), \quad (4.12)$$

for $\xi\in\Gamma^k(\mathsf{E})$. This clearly holds for $k=1$. So suppose it true for $k\geq 1$ and compute

$$\begin{aligned}
&\nabla^{\pi_{\mathsf{F}}}(\nabla^{\mathsf{M},\pi_{\mathsf{F}},k}(L\circ\xi)(X_1,\dots,X_k))(X_{k+1})\\
&\quad = \nabla^{\mathsf{M},\pi_{\mathsf{F}},k+1}(L\circ\xi)(X_1,\dots,X_k,X_{k+1}) + \sum_{j=1}^{k}\nabla^{\mathsf{M},\pi_{\mathsf{F}},k}(L\circ\xi)(X_1,\dots,\nabla^{\mathsf{M}}_{X_{k+1}}X_j,\dots,X_k)\\
&\quad = \nabla^{\mathsf{M},\pi_{\mathsf{F}},k+1}(L\circ\xi)(X_1,\dots,X_k,X_{k+1})\\
&\qquad + \sum_{j=1}^{m}\sum_{l=0}^{j-1}\sum_{\sigma\in\mathfrak{S}_{l,k-l}}(\nabla^{\mathsf{M},\pi_{\mathsf{F}}\otimes\pi_{\mathsf{E}},l}L(X_{\sigma(1)},\dots,X_{\sigma(l)}))\\
&\qquad\qquad (\nabla^{\mathsf{M},\pi_{\mathsf{E}},k-l}\xi(X_{\sigma(l+1)},\dots,\nabla^{\mathsf{M}}_{X_{k+1}}X_j,\dots,X_{\sigma(k)}))\\
&\qquad + \sum_{j=1}^{k}\sum_{l=j}^{k}\sum_{\sigma\in\mathfrak{S}_{l,k-l}}(\nabla^{\mathsf{M},\pi_{\mathsf{F}}\otimes\pi_{\mathsf{E}},l}L(X_{\sigma(1)},\dots,\nabla^{\mathsf{M}}_{X_{k+1}}X_j,\dots,X_{\sigma(l)}))\\
&\qquad\qquad (\nabla^{\mathsf{M},\pi_{\mathsf{E}},k-l}\xi(X_{\sigma(l+1)},\dots,X_{\sigma(k)})),
\end{aligned}$$

using the induction hypothesis. We also compute

$$\begin{aligned}
&\nabla^{\pi_{\mathsf{F}}}(\nabla^{\mathsf{M},\pi_{\mathsf{F}},k}(L\circ\xi)(X_1,\dots,X_k))(X_{k+1})\\
&\quad = \sum_{l=0}^{k}\sum_{\sigma\in\mathfrak{S}_{l,k-l}}(\nabla^{\mathsf{M},\pi_{\mathsf{F}}\otimes\pi_{\mathsf{E}},l+1}L)(X_{\sigma(1)},\dots,X_{\sigma(l)},X_{k+1})\\
&\qquad\qquad (\nabla^{\mathsf{M},\pi_{\mathsf{E}},k-l}\xi(X_{\sigma(l+1)},\dots,X_{\sigma(k)}))\\
&\qquad + \sum_{l=0}^{k}\sum_{\sigma\in\mathfrak{S}_{l,k-l}}\sum_{j=1}^{l}(\nabla^{\pi_{\mathsf{F}}\otimes\pi_{\mathsf{E}},l}L(X_{\sigma(l)},\dots,\nabla^{\mathsf{M}}_{X_{k+1}}X_{\sigma(j)},\dots,X_{\sigma(l)})\\
&\qquad\qquad (\nabla^{\mathsf{M},\pi_{\mathsf{E}},k-l}\xi(X_{\sigma(l+1)},\dots,X_{\sigma(k)}))\\
&\qquad + \sum_{l=0}^{k}\sum_{\sigma\in\mathfrak{S}_{l,k-l}}(\nabla^{\mathsf{M},\pi_{\mathsf{F}}\otimes\pi_{\mathsf{E}},l}L(X_{\sigma(1)},\dots,X_{\sigma(l)}))\\
&\qquad\qquad (\nabla^{\mathsf{M},\pi_{\mathsf{E}},k-l+1}\xi(X_{\sigma(l+1)},\dots,X_{\sigma(k)},X_{k+1}))\\
&\qquad + \sum_{l=0}^{k}\sum_{\sigma\in\mathfrak{S}_{l,k-l}}\sum_{j=l+1}^{k}(\nabla^{\mathsf{M},\pi_{\mathsf{F}}\otimes\pi_{\mathsf{E}},l}L(X_{\sigma(1)},\dots,X_{\sigma(l)}))
\end{aligned}$$

$$(\nabla^{\mathsf{M},\pi_{\mathsf{E}},k-l+1}\xi(X_{\sigma(l+1)},\dots,\nabla^{\mathsf{M}}_{X_{k+1}}X_{\sigma(j)},\dots,X_{\sigma(k)})).$$

Comparing the preceding two equations gives

$$\begin{aligned}
&\nabla^{\mathsf{M},\pi_{\mathsf{F}},k+1}(L\circ\xi)(X_1,\dots,X_k,X_{k+1})\\
&\quad=\sum_{l=0}^{k}\sum_{\sigma\in\mathfrak{S}_{l,k-l}}(\nabla^{\mathsf{M},\pi_{\mathsf{F}}\otimes\pi_{\mathsf{E}},l+1}L)(X_{\sigma(1)},\dots,X_{\sigma(l)},X_{k+1})\\
&\qquad\qquad(\nabla^{\mathsf{M},\pi_{\mathsf{E}},k-l}\xi(X_{\sigma(l+1)},\dots,X_{\sigma(k)}))\\
&\qquad+\sum_{l=0}^{k}\sum_{\sigma\in\mathfrak{S}_{l,k-l}}(\nabla^{\mathsf{M},\pi_{\mathsf{F}}\otimes\pi_{\mathsf{E}},l}L(X_{\sigma(1)},\dots,X_{\sigma(l)}))\\
&\qquad\qquad(\nabla^{\mathsf{M},\pi_{\mathsf{E}},k-l+1}\xi(X_{\sigma(l+1)},\dots,X_{\sigma(k)},X_{k+1}))\\
&\quad=\sum_{l=0}^{k+1}\sum_{\sigma\in\mathfrak{S}_{l,k+1-l}}(\nabla^{\mathsf{M},\pi_{\mathsf{F}}\otimes\pi_{\mathsf{E}},l}L(X_{\sigma(1)},\dots,X_{\sigma(l)}))\\
&\qquad\qquad(\nabla^{\mathsf{M},\pi_{\mathsf{E}},k+1-l}\xi(X_{\sigma(l+1)},\dots,X_{\sigma(k+1)})),
\end{aligned}$$

giving (4.12).

For $A\in\mathrm{T}^k(\mathsf{V}^*)$ and $\sigma\in\mathfrak{S}_k$, we use the notation

$$\sigma(A)(v_1,\dots,v_k)=A(v_{\sigma(1)},\dots,v_{\sigma(k)}).$$

For $\sigma\in\mathfrak{S}_k$, write $\sigma=\sigma_1\circ\sigma_2$ for $\sigma_1\in\mathfrak{S}_{k,l}$ and $\sigma_2\in\mathfrak{S}_{k|l}$. Now we compute

$$\begin{aligned}
D^k_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{F}}}}(L\circ\xi)&=\frac{1}{k!}\sum_{\sigma\in\mathfrak{S}_k}\sigma(\nabla^{\pi_{\mathsf{F}},k}(L\circ\xi))\\
&=\frac{1}{k!}\sum_{l=0}^{k}\sum_{\sigma\in\mathfrak{S}_k}\sum_{\sigma'\in\mathfrak{S}_{l,k-l}}\sigma'\circ\sigma(\nabla^{\pi_{\mathsf{F}}\otimes\pi_{\mathsf{E}},l}L(\nabla^{\pi_{\mathsf{E}},k-l}\xi))\\
&=\frac{1}{k!}\sum_{l=0}^{k}\sum_{\sigma'\in\mathfrak{S}_{l,k-l}}\sum_{\sigma_1\in\mathfrak{S}_{l,k-l}}\sum_{\sigma_2\in\mathfrak{S}_{l|k-l}}\sigma'\circ\sigma_1\circ\sigma_2(\nabla^{\pi_{\mathsf{F}}\otimes\pi_{\mathsf{E}},l}L(\nabla^{\pi_{\mathsf{E}},k-l}\xi))\\
&=\sum_{l=0}^{k}\sum_{\sigma'\in\mathfrak{S}_{l,k-l}}\sum_{\sigma_1\in\mathfrak{S}_{l,k-l}}\frac{l!(k-l)!}{k!}\sigma'\circ\sigma_1(D^l_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{F}}\otimes\pi_{\mathsf{E}}}}L(D^{k-l}_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}}(\xi)))\\
&=\sum_{l=0}^{k}\sum_{\sigma'\in\mathfrak{S}_{l,k-l}}\frac{l!(k-l)!}{k!}\sigma'\circ\left(\sum_{\sigma\in\mathfrak{S}_k}\frac{k!}{l!(k-l)!}\operatorname{Sym}_k(D^l_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{F}}\otimes\pi_{\mathsf{E}}}}L(D^{k-l}_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}}(\xi)))\right)\\
&=\sum_{l=0}^{k}\frac{k!}{l!(k-l)!}\left(\operatorname{Sym}_k(D^l_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{F}}\otimes\pi_{\mathsf{E}}}}L(D^{k-l}_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}}(\xi)))\right),
\end{aligned}$$

making reference to (1.1) in the penultimate step, and noting that $\operatorname{card}(\mathfrak{S}_{l,k-l})=\frac{k!}{l!(k-l)!}$. This is the desired result. ■

4.3. Derivatives of tensors on the total space of a vector bundle. In Definition 3.9 we gave definitions for a variety of lifts of tensor fields. Here we give formulae for differentiating these. We shall make ongoing and detailed use of the formulae we develop in this section, and decent notation is an integral part of arriving at useable expressions.

Let $r \in \{\infty, \omega\}$ and let $\pi_{\mathsf{E}} \colon \mathsf{E} \to \mathsf{M}$ be a C^r-vector bundle. We consider a C^r-affine connection ∇^{M} on M and a C^r-vector bundle connection $\nabla^{\pi_{\mathsf{E}}}$ in E. The connection ∇^{M} induces a covariant derivative for tensor fields $A \in \Gamma^r(\mathrm{T}^k_l(\mathsf{TM}))$ on M, $k, l \in \mathbb{Z}_{\geq 0}$. This covariant derivative we denote by ∇^{M}, dropping the particular k and l. Similarly, the connection $\nabla^{\pi_{\mathsf{E}}}$ induces a covariant derivative for sections $B \in \Gamma^\infty(\mathrm{T}^k_l(\mathsf{E}))$ of the tensor bundles associated with E, $k, l \in \mathbb{Z}_{\geq 0}$. This covariant derivative we denote by $\nabla^{\pi_{\mathsf{E}}}$, again dropping the particular k and l. We have already made use of these conventions, e.g., in Lemmata 4.2 and 4.3. We will also consider differentiation of sections of $\mathrm{T}^{k_1}_{l_1}(\mathsf{TM}) \otimes \mathrm{T}^{k_2}_{l_2}(\mathsf{E})$. Here we denote the covariant derivative by $\nabla^{\mathsf{M},\pi_{\mathsf{E}}}$. If we have another C^r-vector bundle $\pi_{\mathsf{F}} \colon \mathsf{F} \to \mathsf{M}$ with a C^r-affine connection $\nabla^{\pi_{\mathsf{F}}}$, then $\nabla^{\pi_{\mathsf{E}}}$ and $\nabla^{\pi_{\mathsf{F}}}$ induce a covariant derivative in $\mathrm{T}^{k_1}_{l_1}(\mathsf{E}) \otimes \mathrm{T}^{k_2}_{l_2}(\mathsf{F})$, and we denote this covariant derivative by $\nabla^{\pi_{\mathsf{F}} \otimes \pi_{\mathsf{E}}}$.

Another construction we need in this section concerns pull-back bundles. Let $r \in \{\infty, \omega\}$, let M and N be C^r-manifolds, let $\pi_{\mathsf{F}} \colon \mathsf{F} \to \mathsf{M}$ be a C^r-vector bundle, and let $\Phi \in \mathrm{C}^r(\mathsf{N}; \mathsf{M})$. We then have the pull-back bundle $\Phi^*\pi_{\mathsf{F}} \colon \Phi^*\mathsf{F} \to \mathsf{N}$, which is a vector bundle over N. Given a section η of F, we have a section $\Phi^*\eta$ of $\Phi^*\mathsf{F}$ defined by $\Phi^*\eta(y) = (y, \eta \circ \Phi(y))$. Given a C^r-vector bundle connection $\nabla^{\pi_{\mathsf{F}}}$ in F, we can define a C^r-connection $\Phi^*\nabla^{\pi_{\mathsf{F}}}$ in $\Phi^*\mathsf{F}$ by requiring that

$$\Phi^*\nabla^{\pi_{\mathsf{F}}}_Z \Phi^*\eta(y) = \Phi^*(\nabla^{\pi_{\mathsf{F}}}_{T_y\Phi(Z)}\eta)$$

for a C^∞-section η and for $Z \in \mathsf{T}_y\mathsf{N}$. Given an affine connection ∇^{N} on N, we then have an affine connection on $\mathrm{T}^k(\mathsf{T}^*\mathsf{N}) \otimes \Phi^*\mathsf{F}$ induced by tensor product by ∇^{N} and $\Phi^*\nabla^{\pi_{\mathsf{F}}}$. This connection we denote by $\nabla^{\mathsf{N},\Phi^*\pi_{\mathsf{F}}}$, consistent with our notation above. If we additionally have an injection $\psi \colon \Phi^*\mathsf{F} \to \mathsf{TN}$, then we have

$$\nabla^{\mathsf{N}}_Z(\psi \circ \Phi^*\eta) = \psi \circ (\Phi^*\nabla^{\pi_{\mathsf{F}}}_Z \Phi^*\eta) + B_\psi(\Phi^*\eta, Z)$$

for some tensor $B_\psi \in \mathrm{T}^1_2(\mathsf{TE})$.

A special case of the preceding paragraph is when $\Phi = \pi_{\mathsf{E}}$ for a vector bundle $\pi_{\mathsf{E}} \colon \mathsf{E} \to \mathsf{M}$ and $\mathsf{F} = \mathsf{E}$. In this case, $\pi_{\mathsf{E}}^*\xi = \xi^{\mathrm{v}}$ and $\pi^*_{\mathsf{E}}\mathsf{F} \simeq \mathsf{VE}$ and so we indeed have a natural inclusion of $\pi^*_{\mathsf{E}}\mathsf{F}$ in TE. Moreover, by Lemma 4.1(xiii),

$$\pi^*_{\mathsf{E}}\nabla^{\pi_{\mathsf{E}}}_Z \pi^*_{\mathsf{E}}\xi = (\nabla^{\pi_{\mathsf{E}}}_{T\pi_{\mathsf{E}}(Z)}\xi)^{\mathrm{v}},$$

and so

$$\nabla^{\mathsf{E}}_Z \pi^*_{\mathsf{E}}\xi = \pi^*_{\mathsf{E}}\nabla^{\pi_{\mathsf{E}}}_Z \pi^*_{\mathsf{E}}\xi + A_{\pi_{\mathsf{E}}}(Z, \xi^{\mathrm{v}}). \tag{4.13}$$

With the preceding, we can give formulae for differentiating tensors on vector bundles, rather mirroring what we did in Lemma 3.6 for functions.

4.5 Lemma: (Differentiation of lifted tensors on vector bundles) *Let $r \in \{\infty, \omega\}$. Let $\pi_{\mathsf{E}} \colon \mathsf{E} \to \mathsf{M}$ and $\pi_{\mathsf{F}} \colon \mathsf{F} \to \mathsf{M}$ be vector bundles of class C^r. Let $\mathbb{G}_{\mathsf{M}}$ be a C^r-Riemannian metric on M, let ∇^{M} be the Levi-Civita connection, let $\mathbb{G}_{\pi_{\mathsf{E}}}$ be a C^r-fibre metric on E, and let $\nabla^{\pi_{\mathsf{E}}}$ be a $\mathbb{G}_{\pi_{\mathsf{E}}}$-vector bundle connection of class C^r in E. Let $\nabla^{\pi_{\mathsf{F}}}$ be a C^r-vector bundle connection in F. Let $\mathbb{G}_{\mathsf{E}}$ be the associated C^r-Riemannian metric on E from (4.1). Define*

$$B_{\pi_{\mathsf{E}}} = \mathrm{push}_{1,2}\mathrm{Ins}_1(\mathrm{Ins}_2(A_{\pi_{\mathsf{E}}}, \mathrm{hor}), \mathrm{hor}) + \mathrm{Ins}_2(A_{\pi_{\mathsf{E}}}, \mathrm{ver}) + \mathrm{push}_{1,2}\mathrm{Ins}_2(A_{\pi_{\mathsf{E}}}, \mathrm{ver}),$$

where $A_{\pi_{\mathsf{E}}}$ is defined as in (4.2).

Then we have the following statements, recalling from (3.2) the derivation $D_{B_{\pi_{\mathsf{E}}}}$:

(i) *for $k \in \mathbb{Z}_{>0}$ and $A \in \Gamma^r(\mathrm{T}^k(\mathsf{T}^*\mathsf{M}))$, we have*

$$\nabla^{\mathsf{E}}(A^{\mathrm{h}}) = (\nabla^{\mathsf{M}} A)^{\mathrm{h}} + D_{B_{\pi_{\mathsf{E}}}}(A^{\mathrm{h}});$$

(ii) *for $A \in \Gamma^r(\mathrm{T}^k(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{E})$, we have*

$$\nabla^{\mathsf{E}}(A^{\mathrm{v}}) = (\nabla^{\mathsf{M},\pi_{\mathsf{E}}} A)^{\mathrm{v}} + D_{B_{\pi_{\mathsf{E}}}}(A^{\mathrm{v}});$$

(iii) *for $A \in \Gamma^r(\mathrm{T}^k(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{TM})$, we have*

$$\nabla^{\mathsf{E}}(A^{\mathrm{h}}) = (\nabla^{\mathsf{M}} A)^{\mathrm{h}} + D_{B_{\pi_{\mathsf{E}}}}(A^{\mathrm{h}});$$

(iv) *for $A \in \Gamma^r(\mathrm{T}^k(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{F} \otimes \mathsf{E}^*)$, we have*

$$\nabla^{\mathsf{E},\pi_{\mathsf{F}}}(A^{\mathrm{v}}) = (\nabla^{\mathsf{M},\pi_{\mathsf{E}}\otimes\pi_{\mathsf{F}}} A)^{\mathrm{v}} + D_{B_{\pi_{\mathsf{E}}}}(A^{\mathrm{v}});$$

(v) *for $A \in \Gamma^r(\mathrm{T}^k(\mathsf{T}^*\mathsf{M}) \otimes \mathrm{T}^1_1(\mathsf{E}))$, we have*

$$\nabla^{\mathsf{E}}(A^{\mathrm{v}}) = (\nabla^{\mathsf{M},\pi_{\mathsf{E}}} A)^{\mathrm{v}} + D_{B_{\pi_{\mathsf{E}}}}(A^{\mathrm{v}});$$

(vi) *for $A \in \Gamma^r(\mathrm{T}^k(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{F} \otimes \mathsf{E}^*)$, we have*

$$\nabla^{\mathsf{E},\pi_{\mathsf{F}}}(A^{\mathrm{e}}) = (\nabla^{\mathsf{M},\pi_{\mathsf{E}}\otimes\pi_{\mathsf{F}}} A)^{\mathrm{e}} + D_{B_{\pi_{\mathsf{E}}}}(A^{\mathrm{e}}) + A^{\mathrm{v}}.$$

(vii) *for $A \in \Gamma^r(\mathrm{T}^k(\mathsf{T}^*\mathsf{M}) \otimes \mathrm{T}^1_1(\mathsf{E}))$, we have*

$$\nabla^{\mathsf{E}}(A^{\mathrm{e}}) = (\nabla^{\mathsf{M},\pi_{\mathsf{E}}} A)^{\mathrm{e}} + D_{B_{\pi_{\mathsf{E}}}}(A^{\mathrm{e}}) + A^{\mathrm{v}}.$$

Proof: Before we begin the proof proper, let us justify a "without loss of generality" argument that we will make for the last four parts of the proof. The arguments all have to do with assuming that it is sufficient, when working with differential operators on spaces of tensor products, to work with pure tensor products. Let us be a little specific about this. Let $\pi_{\mathsf{E}}\colon \mathsf{E} \to \mathsf{M}$, $\pi_{\mathsf{F}}\colon \mathsf{F} \to \mathsf{M}$, and $\pi_{\mathsf{G}}\colon \mathsf{G} \to \mathsf{M}$ be C^r-vector bundles. Suppose that $\Delta_1, \Delta_2\colon \mathsf{J}^m(\mathsf{E} \otimes \mathsf{F}) \to \mathsf{G}$ are linear differential operators of order m. We wish to give conditions under which $\Delta_1 = \Delta_2$. Of course, this is equivalent to giving conditions under which, for a differential operator $\Delta\colon \mathsf{J}^m(\mathsf{E} \otimes \mathsf{F}) \to \mathsf{G}$, $\Delta = 0$. To do so, we claim that, without loss of generality, we can simply prove that $\Delta(j_m(\xi \otimes \eta)) = 0$ for all $\xi \in \Gamma^r(\mathsf{E})$ and $\eta \in \Gamma^r(\mathsf{E})$.

To prove this sufficiency, we state and prove a couple of sublemmata. The second is the one of interest to us, and the first is used to prove the second. Simpler versions of the first lemma are called Hadamard's Lemma, but we could not find a reference to the form we require.

1 Sublemma: *Let $r \in \{\infty, \omega\}$. Let $\mathcal{U} \subseteq \mathbb{R}^n$ be a neighbourhood of $\mathbf{0}$, let $\mathsf{S} \subseteq \mathbb{R}^n$ be the subspace*

$$\mathsf{S} = \{(x^1, \dots, x^n) \in \mathbb{R}^n \mid x^1 = \dots = x^s = 0\},$$

let $k \in \mathbb{Z}_{\geq 0}$, and let $f \in \mathrm{C}^r(\mathsf{B}(\epsilon, \mathbf{0}))$ satisfy $\boldsymbol{D}^j f(\boldsymbol{x}) = 0$ for all $j \in \{0, 1, \dots, k\}$ and $\boldsymbol{x} \in \mathsf{S} \cap \mathcal{U}$. Let $\mathrm{pr}_{\mathsf{S}} \colon \mathbb{R}^n \to \mathsf{S}$ be the natural projection onto the first s-components. Then there exists a neighbourhood $\mathcal{V} \subseteq \mathcal{U}$ of $\mathbf{0}$, $g_I \in \mathrm{C}^r(\mathcal{V})$, $I \in \mathbb{Z}^s_{>0}$, $|I| = k+1$, such that

$$f(\boldsymbol{x}) = \sum_{\substack{I \in \mathbb{Z}^s_{>0} \\ |I| = k+1}}^{n} g_I(\boldsymbol{x}) \, \mathrm{pr}_{\mathsf{S}}(\boldsymbol{x})^I, \qquad \boldsymbol{x} \in \mathcal{V}.$$

Proof: We prove the sublemma by induction on k. For $k = 0$, the hypothesis is that f vanishes on $\mathsf{S} \cap \mathcal{U}$. Let $\mathcal{W} \subseteq \mathsf{S}$ be a neighbourhood of $\mathbf{0}$ and let $\epsilon \in \mathbb{R}_{>0}$ be such that $\mathsf{B}(\epsilon, \boldsymbol{y}) \subseteq \mathcal{U}$ for all $\boldsymbol{x} \in \mathcal{W}$, possibly after shrinking $\mathcal{W}$. Let

$$\mathcal{V} = \bigcup_{\boldsymbol{x} \in \mathcal{W}} \mathsf{B}(\epsilon, \boldsymbol{x}).$$

Let $\boldsymbol{x} = (\boldsymbol{x}_1, \boldsymbol{x}_2) \in \mathcal{V}$ (with $\boldsymbol{x}_1 \in \mathsf{S}$) and define

$$\begin{aligned} \gamma_{\boldsymbol{x}} \colon [0,1] &\to \mathbb{R} \\ t &\mapsto f(\boldsymbol{x}_1, t\boldsymbol{x}_2). \end{aligned}$$

We calculate

$$\begin{aligned} f(\boldsymbol{x}) = f(\boldsymbol{x}_1, \boldsymbol{x}_2) &= f(\boldsymbol{x}_1, \boldsymbol{x}_2) - f(\boldsymbol{x}_1, \mathbf{0}) \\ &= \gamma_{\boldsymbol{x}}(1) - \gamma_{\boldsymbol{x}}(0) = \int_0^1 \gamma'_{\boldsymbol{x}}(t) \, \mathrm{d}t \\ &= \int_0^1 \sum_{j=1}^s x^j \frac{\partial f}{\partial x^j}((\boldsymbol{x}_1, t\boldsymbol{x}_2)) \, \mathrm{d}t = \sum_{j=1}^s x^j g_j(\boldsymbol{x}), \end{aligned}$$

where

$$g_j(\boldsymbol{x}) = \int_0^1 \frac{\partial f}{\partial x^j}(\boldsymbol{x}_1, t\boldsymbol{x}_2) \, \mathrm{d}t, \qquad j \in \{1, \dots, s\}.$$

It remains to show that the functions $g_1, \dots, g_s$ are of class C^r. By standard theorems on interchanging derivatives and integrals [Jost 2005, Theorem 16.11], we can conclude that $g_1, \dots, g_m$ are smooth when f is smooth. If the data are holomorphic, swapping integrals and derivatives allows us to conclude that $g_1, \dots, g_s$ are holomorphic when f is holomorphic, by verifying the Cauchy–Riemann equations. In the real analytic case, we can complexify to a complex neighbourhood of $\mathbf{0}$, and so conclude real analyticity by holomorphicity of the complexification.

As a standin for a full proof by induction, let us see how the case $k = 1$ follows from the case $k = 0$. The general inductive argument is the same, only with more notation.

We note that, for $\boldsymbol{x} \in \mathcal{V}$, we have

$$\frac{\partial f}{\partial x^k}(\boldsymbol{x}) = \begin{cases} g_k(\boldsymbol{x}) + \sum_{j=1}^s x^j \frac{\partial g_j}{\partial x^k}(\boldsymbol{x}), & k \in \{1, \dots, s\}, \\ \sum_{j=1}^s x^j \frac{\partial g_j}{\partial x^k}(\boldsymbol{x}), & k \in \{s+1, \dots, n\}. \end{cases}$$

Thus $\boldsymbol{D}f(\boldsymbol{x}) = 0$ for $\boldsymbol{x} \in \mathcal{S} \cap \mathcal{V}$ if and only if $g_1(\boldsymbol{x}) = \cdots = g_s(\boldsymbol{x}) = 0$. Thus one can apply the arguments from the first part of the proof to write

$$g_k(\boldsymbol{x}) = \sum_{j=1}^{s} x^j g_{kj}(\boldsymbol{x})$$

on a neighbourhood of $\boldsymbol{0}$. Thus

$$f(\boldsymbol{x}) = \sum_{j,k=1}^{s} x^k x^j g_{kj}(\boldsymbol{x}),$$

giving the desired form of f in this case. ▼

2 Sublemma: *Let $r \in \{\infty, \omega\}$, let $\pi_{\mathsf{E}} \colon \mathsf{E} \to \mathsf{M}$ be a C^r-vector bundle, and let $\mathsf{S} \subseteq \mathsf{M}$ be a closed C^r-submanifold. Let $k \in \mathbb{Z}_{\geq 0}$. Let $\mathcal{V}$ be a neighbourhood of S and let $\xi_{\mathsf{S}} \in \Gamma^r(\mathsf{E}|\mathcal{V})$. Then there exists $\xi \in \Gamma^r(\mathsf{E})$ such that $j_k\xi(x) = j_k\xi_{\mathsf{S}}(x)$.*

Proof: Let $\mathscr{Z}_{\mathsf{S}}^k$ be the sheaf of C^r-sections of E such that whose k-jet vanishes on S. We have the exact sequence

$$0 \longrightarrow \mathscr{Z}_{\mathsf{S}}^k \longrightarrow \mathscr{G}_{\mathsf{E}}^r \xrightarrow{\Psi} \mathscr{G}_{\mathsf{E}}^r / \mathscr{Z}_{\mathsf{S}}^k \longrightarrow 0$$

Note that the stalk of the quotient sheaf at $x \in \mathsf{S}$ consists of germs of sections whose k-jets agree on S.

Now, if $x \notin \mathsf{S}$, then there is a neighbourhood $\mathcal{U}$ of x such that $\mathcal{V} \cap \mathsf{S} = \varnothing$, and so $\mathscr{Z}_{\mathsf{S}}^k(\mathcal{U}) = \mathscr{G}_{\mathsf{E}}^r(\mathcal{U})$. That $\mathscr{Z}_{\mathsf{S}}^k$ is locally finitely generated at x then follows since $\mathscr{G}_{\mathsf{E}}^r$ is locally finitely generated. If $x \in \mathsf{S}$, choose a submanifold chart $(\mathcal{U}, \phi)$ for S about x so that

$$\mathsf{S} \cap \mathcal{U} = \{y \in \mathcal{U} \mid \ \phi(y) = (0, \dots, 0, x^{s+1}, \dots, x^n)\}.$$

Then the k-jet of a function f on $\mathcal{U}$ vanishes on S if and only if it is a $\mathrm{C}^r(\mathcal{U})$-linear combination of polynomial functions in $x^1, \dots, x^s$ of degree $k+1$; this follows by the previous sublemma. Thus, if $\xi_1, \dots, \xi_m$ is a local basis of sections of E about x, then the (finite) set of products of these sections with the polynomial functions in $x^1, \dots, x^s$ of degree at least $k+1$ generates $\Gamma^r(\mathsf{E}|\mathcal{U})$ as a $\mathrm{C}^r(\mathcal{U})$-module. This shows that $\mathscr{Z}_{\mathsf{S}}^k$ is locally finitely generated about x. This shows that $\mathscr{Z}_{\mathsf{S}}^k$ is coherent in the case $r = \omega$.

Cartan's Theorem B [Cartan 1957, Proposition 6] shows, in the case $r = \omega$, that Ψ is surjective on global sections. The case of $r = \infty$ follows in a similar way, using the fact that positive cohomology groups for sheave of modules of smooth functions vanish ([Wells Jr. 2008, Proposition 3.11], along with [Wells Jr. 2008, Examples 3.4(d, e)] and [Wells Jr. 2008, Proposition 3.5]). This implies that there exists $\xi \in \Gamma^r(\mathsf{E})$ such that, for each $x \in \mathsf{S}$, $[\xi]_x = [\xi_{\mathsf{S}}]_x$. This, however, means precisely that $j_k\xi(x) = j_k\xi_{\mathsf{S}}(x)$ for each $x \in \mathsf{S}$. ▼

Now suppose that we have proved that $\Delta(j_m(\xi \otimes \eta)) = 0$ for all $\xi \in \Gamma^r(\mathsf{E})$ and $\eta \in \Gamma^r(\mathsf{E})$. Let $x \in \mathsf{X}$ and let $\alpha \in \mathsf{T}_x^{*m}\mathsf{M}$. Let $u \in \mathsf{E}_x$ and $v \in \mathsf{F}_x$. By the previous sublemma, there exists $f \in \mathrm{C}^r(\mathsf{M})$ such that $j_m f(x) = \alpha$. Then, keeping in mind the identification (1.5),

$$\Delta(\alpha \otimes (u \otimes v)) = \Delta(j_m(f(\xi \otimes \eta))) = \Delta(j_m((f\xi) \otimes \eta)) = 0.$$

Since every element of $\mathsf{J}^m_x\mathsf{E}$ is a finite linear combination of terms of the form $\alpha\otimes(u\otimes v)$ for $\alpha\in\mathsf{T}^{*m}_x\mathsf{M}$, $u\in\mathsf{E}_x$, and $v\in\mathsf{F}_x$, we conclude that $\Delta(j_mA)(x)=0$ for every $A\in\Gamma^r(\mathsf{E}\otimes\mathsf{F})$.

Now we proceed with the proof.

(i) We have

$$\mathscr{L}_{Z_{k+1}}(A^{\mathrm{h}}(Z_1,\dots,Z_k))=(\nabla^{\mathsf{E}}_{Z_{k+1}}A^{\mathrm{h}})(Z_1,\dots,Z_k)+\sum_{j=1}^k A^{\mathrm{h}}(Z_1,\dots,\nabla^{\mathsf{E}}_{Z_{k+1}}Z_j,\dots,Z_k).$$

We consider four cases.

1. $Z_j=X_j^{\mathrm{h}}$, $j\in\{1,\dots,k+1\}$: Here we have

$$\mathscr{L}_{X^{\mathrm{h}}_{k+1}}(A^{\mathrm{h}}(X_1^{\mathrm{h}},\dots,X_k^{\mathrm{h}}))=(\mathscr{L}_{X_{k+1}}(A(X_1,\dots,X_k)))^{\mathrm{h}}$$

(by Lemma 3.6(i)) and

$$A^{\mathrm{h}}(X_1^{\mathrm{h}},\dots,\nabla^{\mathsf{E}}_{X^{\mathrm{h}}_{k+1}}X_j^{\mathrm{h}},\dots,X_k^{\mathrm{h}})=(A(X_1,\dots,\nabla^{\mathsf{M}}_{X_{k+1}}X_j,\dots,X_k))^{\mathrm{h}}$$

(by Lemma 4.1(i)). Thus we conclude that

$$\nabla^{\mathsf{E}}A^{\mathrm{h}}(X_1^{\mathrm{h}},\dots,X_{k+1}^{\mathrm{h}})=((\nabla^{\mathsf{M}}A)(X_1,\dots,X_{k+1}))^{\mathrm{h}}.$$

2. $Z_j=X_j^{\mathrm{h}}$, $j\in\{1,\dots,k\}$, $Z_{k+1}=\xi^{\mathrm{v}}_{k+1}$: Here we calculate

$$\mathscr{L}_{\xi^{\mathrm{v}}_{k+1}}(A^{\mathrm{h}}(X_1^{\mathrm{h}},\dots,X_k^{\mathrm{h}}))=\mathscr{L}_{\xi^{\mathrm{v}}_{k+1}}(A(X_1,\dots,X_k))^{\mathrm{h}}=0$$

(using the definition of A^{h} and Lemma 3.6(ii)) and

$$A^{\mathrm{h}}(X_1^{\mathrm{h}},\dots,\nabla^{\mathsf{E}}_{\xi^{\mathrm{v}}_{k+1}}X_j^{\mathrm{h}},\dots,X_k^{\mathrm{h}})=A^{\mathrm{h}}(X_1^{\mathrm{h}},\dots,A_{\pi_{\mathsf{E}}}(X_j^{\mathrm{h}},\xi^{\mathrm{v}}_{k+1}),\dots,X_k^{\mathrm{h}})$$

(using Lemma 4.1(xi)). Thus we conclude that

$$\nabla^{\mathsf{E}}A^{\mathrm{h}}(X_1^{\mathrm{h}},\dots,X_k^{\mathrm{h}},\xi^{\mathrm{v}}_{k+1})=-\sum_{j=1}^k A^{\mathrm{h}}(X_1^{\mathrm{h}},\dots,A_{\pi_{\mathsf{E}}}(X_j^{\mathrm{h}},\xi^{\mathrm{v}}_{k+1}),\dots,X_k^{\mathrm{h}}).$$

3. $Z_j=\xi_j^{\mathrm{v}}$ for some $j\in\{1,\dots,k\}$, $Z_{k+1}=X^{\mathrm{h}}_{k+1}$: We calculate

$$\mathscr{L}_{X^{\mathrm{h}}_{k+1}}(A^{\mathrm{h}}(Z_1,\dots,\xi_j^{\mathrm{v}},Z_k))=0$$

(by definition of A^{h}) and

$$A^{\mathrm{h}}(Z_1,\dots,\nabla^{\mathsf{E}}_{X^{\mathrm{h}}_{k+1}}\xi_j^{\mathrm{v}},\dots,Z_k)=A^{\mathrm{h}}(Z_1,\dots,A_{\pi_{\mathsf{E}}}(X^{\mathrm{h}}_{k+1},\xi_j^{\mathrm{v}}),\dots,Z_k)$$

(by Lemma 4.1(v)). Thus

$$\nabla^{\mathsf{E}}A^{\mathrm{h}}(Z_1,\dots,\xi_j^{\mathrm{v}},\dots,Z_k,X^{\mathrm{h}}_{k+1})=-A^{\mathrm{h}}(Z_1,\dots,A_{\pi_{\mathsf{E}}}(X^{\mathrm{h}}_{k+1},\xi_j^{\mathrm{v}}),\dots,Z_k).$$

4. $Z_j = \xi_j^{\mathrm{v}}$ for some $j \in \{1, \dots, k\}$, $Z_{k+1} = \xi_{k+1}^{\mathrm{v}}$: We have

$$\mathscr{L}_{\xi_{k+1}^{\mathrm{v}}}(A^{\mathrm{h}}(Z_1, \dots, \xi_j^{\mathrm{v}}, \dots, Z_k)) = 0$$

(by definition of A^{h}) and

$$A^{\mathrm{h}}(Z_1, \dots, \nabla^{\mathsf{E}}_{\xi_{k+1}^{\mathrm{v}}} \xi_j^{\mathrm{v}}, \dots, Z_k) = 0$$

(by Lemma 4.1(iii)). Thus

$$\nabla^{\mathsf{E}} A^{\mathrm{h}}(Z_1, \dots, \xi_j^{\mathrm{v}}, \dots, Z_k, \xi_{k+1}^{\mathrm{v}}) = 0.$$

Putting this all together, and keeping in mind that $A_{\pi_{\mathsf{E}}}$ is vertical when both arguments are vertical, we have

$$\begin{aligned}\nabla^{\mathsf{E}} A^{\mathrm{h}}(Z_1, \dots, Z_{k+1}) = (\nabla^{\mathsf{M}} A)^{\mathrm{h}}(Z_1, \dots, Z_{k+1})& \\ - \sum_{j=1}^{k} A^{\mathrm{h}}(Z_1, \dots, A_{\pi_{\mathsf{E}}}(\mathrm{hor}(Z_j), \mathrm{ver}(Z_{k+1})), \dots, Z_k)& \\ - \sum_{j=1}^{k} A^{\mathrm{h}}(Z_1, \dots, A_{\pi_{\mathsf{E}}}(\mathrm{hor}(Z_{k+1}), \mathrm{ver}(Z_j)), \dots, Z_k).&\end{aligned}$$

Now we note that

$$\begin{aligned}B_{\pi_{\mathsf{E}}}(Z_j, Z_{k+1}) &= A_{\pi_{\mathsf{E}}}(\mathrm{hor}(Z_{k+1}), \mathrm{hor}(Z_j)) + A_{\pi_{\mathsf{E}}}(Z_j, \mathrm{ver}(Z_{k+1})) + A_{\pi_{\mathsf{E}}}(Z_{k+1}, \mathrm{ver}(Z_j)) \\ &= A_{\pi_{\mathsf{E}}}(\mathrm{hor}(Z_j), \mathrm{ver}(Z_{k+1})) + A_{\pi_{\mathsf{E}}}(\mathrm{hor}(Z_{k+1}), \mathrm{ver}(Z_j)) + \text{something vertical},\end{aligned}$$

using Lemma 4.1(ii) and the definition of $A_{\pi_{\mathsf{E}}}$. Thus

$$\nabla^{\mathsf{E}} A^{\mathrm{h}} = (\nabla^{\mathsf{M}} A)^{\mathrm{h}} - \sum_{j=1}^{k} \mathrm{Ins}_j(A^{\mathrm{h}}, B_{\pi_{\mathsf{E}}}),$$

which gives this part of the lemma by Lemma 3.11.

(ii) First we compute, for $Z \in \Gamma^r(\mathsf{TE})$,

$$\begin{aligned}\nabla^{\mathsf{E}}_Z \xi^{\mathrm{v}} = \nabla^{\mathsf{E}}_{\mathrm{hor}(Z)} \xi^{\mathrm{v}} + \nabla^{\mathsf{E}}_{\mathrm{ver}(Z)} \xi^{\mathrm{v}} &= (\nabla^{\mathsf{M},\pi_{\mathsf{E}}}_{T\pi_{\mathsf{E}}(Z)} \xi)^{\mathrm{v}} + A_{\pi_{\mathsf{E}}}(T\pi_{\mathsf{E}}(Z), \xi^{\mathrm{v}}) \\ &= (\nabla^{\mathsf{M},\pi_{\mathsf{E}}}_{T\pi_{\mathsf{E}}(Z)} \xi)^{\mathrm{v}} + A_{\pi_{\mathsf{E}}}(Z, \xi^{\mathrm{v}})\end{aligned}$$

using Lemma 4.1(iii), (v), and (xiii), and the definition of $A_{\pi_{\mathsf{E}}}$. If we note that

$$B_{\pi_{\mathsf{E}}}(\xi^{\mathrm{v}}, Z) = A_{\pi_{\mathsf{E}}}(\mathrm{hor}(Z), \mathrm{hor}(\xi^{\mathrm{v}})) + A_{\pi_{\mathsf{E}}}(\xi^{\mathrm{v}}, \mathrm{ver}(Z)) + A_{\pi_{\mathsf{E}}}(Z, \mathrm{ver}(\xi^{\mathrm{v}})) = A_{\pi_{\mathsf{E}}}(Z, \xi^{\mathrm{v}})$$

(using the definition of $A_{\pi_{\mathsf{E}}}$), we have

$$\nabla^{\mathsf{E}}_Z \xi^{\mathrm{v}} = (\nabla^{\mathsf{M},\pi_{\mathsf{E}}}_{T\pi_{\mathsf{E}}(Z)} \xi)^{\mathrm{v}} + B_{\pi_{\mathsf{E}}}(\xi^{\mathrm{v}}, Z).$$

Now, it suffices to prove this part of the lemma for $A = A_0^{\mathrm{h}} \otimes \xi^{\mathrm{v}}$ for $A_0 \in \Gamma^r(\mathrm{T}^k(\mathsf{T}^*\mathsf{M}))$ and $\xi \in \Gamma^r(\mathsf{E})$. For $Z \in \Gamma^r(\mathsf{TE})$, we have

$$\begin{aligned}\nabla^{\mathsf{E}}_Z(A_0^{\mathrm{h}} \otimes \xi^{\mathrm{v}}) &= (\nabla^{\mathsf{E}}_Z A_0^{\mathrm{h}}) \otimes \xi^{\mathrm{v}} + (A_0^{\mathrm{h}}) \otimes \nabla^{\mathsf{E}}_Z \xi^{\mathrm{v}} \\ &= (\nabla^{\mathsf{M}}_{T\pi_{\mathsf{E}}(Z)} A_0)^{\mathrm{h}} \otimes \xi^{\mathrm{v}} - \sum_{j=1}^k \mathrm{Ins}_j(A_0^{\mathrm{h}}, B_{\pi_{\mathsf{E}},Z}) \otimes \xi^{\mathrm{v}} \\ &\quad + A_0^{\mathrm{h}} \otimes (\nabla^{\pi_{\mathsf{E}}}_{T\pi_{\mathsf{E}}(Z)} \xi)^{\mathrm{v}} + A_0^{\mathrm{h}} \otimes B_{\pi_{\mathsf{E}}}(\xi^{\mathrm{v}}, Z).\end{aligned}$$

We have

$$\begin{aligned}(\nabla^{\mathsf{M}}_{T\pi_{\mathsf{E}}(Z)} A_0)^{\mathrm{h}} \otimes \xi^{\mathrm{v}} + A_0^{\mathrm{h}} \otimes (\nabla^{\pi_{\mathsf{E}}}_{T\pi_{\mathsf{E}}(Z)} \xi)^{\mathrm{v}} &= ((\nabla^{\mathsf{M}}_{T\pi_{\mathsf{E}}(Z)} A_0) \otimes \xi)^{\mathrm{v}} + (A_0^{\mathrm{h}} \otimes (\nabla^{\pi_{\mathsf{E}}}_{T\pi_{\mathsf{E}}(Z)} \xi))^{\mathrm{v}} \\ &= (\nabla^{\mathsf{M},\pi_{\mathsf{E}}}_{T\pi_{\mathsf{E}}(Z)} (A_0 \otimes \xi))^{\mathrm{v}}.\end{aligned}$$

Thus, by (3.3) and the first part of the lemma, we have

$$A_0^{\mathrm{h}} \otimes B_{\pi_{\mathsf{E}}}(\xi^{\mathrm{v}}, Z) - \sum_{j=1}^k \mathrm{Ins}_j(A_0^{\mathrm{h}}, B_{\pi_{\mathsf{E}},Z}) \otimes \xi^{\mathrm{v}} = D_{B_{\pi_{\mathsf{E}},Z}}(A_0^{\mathrm{h}} \otimes \xi^{\mathrm{v}}).$$

Assembling the preceding three computations gives this part of the lemma.

(iii) First note that

$$\nabla^{\mathsf{E}}_Z X^{\mathrm{h}} = \nabla^{\mathsf{E}}_{\mathrm{hor}(Z)} X^{\mathrm{h}} + \nabla^{\mathsf{E}}_{\mathrm{ver}(Z)} X^{\mathrm{h}} = (\nabla^{\mathsf{M}}_{T\pi_{\mathsf{E}}(Z)} X)^{\mathrm{h}} + A_{\pi_{\mathsf{E}}}(\mathrm{hor}(Z), X^{\mathrm{h}}) + A_{\pi_{\mathsf{E}}}(X^{\mathrm{h}}, \mathrm{ver}(Z))$$

using Lemma 4.1(xi). Now we have

$$\begin{aligned}B_{\pi_{\mathsf{E}}}(X^{\mathrm{h}}, Z) &= A_{\pi_{\mathsf{E}}}(\mathrm{hor}(Z), X^{\mathrm{h}}) + A_{\pi_{\mathsf{E}}}(X^{\mathrm{h}}, \mathrm{ver}(Z)) + A_{\pi_{\mathsf{E}}}(Z, \mathrm{ver}(X^{\mathrm{h}})) \\ &= A_{\pi_{\mathsf{E}}}(\mathrm{hor}(Z), X^{\mathrm{h}}) + A_{\pi_{\mathsf{E}}}(X^{\mathrm{h}}, \mathrm{ver}(Z)).\end{aligned}$$

Thus we have

$$\nabla^{\mathsf{E}}_Z X^{\mathrm{h}} = (\nabla^{\mathsf{M}}_{T\pi_{\mathsf{E}}(Z)} X)^{\mathrm{h}} + B_{\pi_{\mathsf{E}}}(X^{\mathrm{h}}, Z).$$

Now it suffices to prove this part of the lemma for $A = A_0^{\mathrm{h}} \otimes X^{\mathrm{h}}$ for $A_0 \in \Gamma^r(\mathrm{T}^k(\mathsf{T}^*\mathsf{M}))$ and $X \in \Gamma^r(\mathsf{TM})$. In this case we calculate, for $Z \in \Gamma^r(\mathsf{TE})$,

$$\begin{aligned}\nabla^{\mathsf{E}}_Z(A_0^{\mathrm{h}} \otimes X^{\mathrm{h}}) &= (\nabla^{\mathsf{E}}_Z A_0^{\mathrm{h}}) \otimes X^{\mathrm{h}} + A_0^{\mathrm{h}} \otimes \nabla^{\mathsf{E}}_Z X^{\mathrm{h}} \\ &= (\nabla^{\mathsf{M}}_{T\pi_{\mathsf{E}}(Z)} A_0)^{\mathrm{h}} \otimes X^{\mathrm{h}} - \sum_{j=1}^k \mathrm{Ins}_j(A_0^{\mathrm{h}}, B_{\pi_{\mathsf{E}},Z}) \otimes X^{\mathrm{h}} \\ &\quad + A_0^{\mathrm{h}} \otimes (\nabla^{\mathsf{M}}_{T\pi_{\mathsf{E}}(Z)} X)^{\mathrm{h}} + A_0^{\mathrm{h}} \otimes B_{\pi_{\mathsf{E}}}(X^{\mathrm{h}}, \mathrm{ver}(Z)).\end{aligned}$$

We have

$$(\nabla^{\mathsf{M}}_{T\pi_{\mathsf{E}}(Z)} A_0)^{\mathrm{h}} \otimes X^{\mathrm{h}} + A_0^{\mathrm{h}} \otimes (\nabla^{\mathsf{M}}_{T\pi_{\mathsf{E}}(Z)} X)^{\mathrm{h}} = (\nabla^{\mathsf{M}}_{T\pi_{\mathsf{E}}(Z)} (A_0 \otimes X))^{\mathrm{h}}.$$

We also have, by (3.3) and the first part of the lemma,

$$A_0^{\mathrm{h}} \otimes B_{\pi_{\mathsf{E}}}(X^{\mathrm{h}}, \mathrm{ver}(Z)) - \sum_{j=1}^k \mathrm{Ins}_j(A_0^{\mathrm{h}}, B_{\pi_{\mathsf{E}},Z}) \otimes X^{\mathrm{h}} = D_{(B_{\pi_{\mathsf{E}}})_Z}(A_0^{\mathrm{h}} \otimes X^{\mathrm{h}}).$$

Putting the above computations together gives this part of the lemma.

(iv) First we need to compute $\nabla^{\mathsf{E}}\lambda^{\mathrm{v}}$. We do this by using the formula

$$\mathscr{L}_{Z_1}\langle\lambda^{\mathrm{v}}; Z_2\rangle = \langle\nabla^{\mathsf{E}}_{Z_1}\lambda^{\mathrm{v}}; Z_2\rangle + \langle\lambda^{\mathrm{v}}; \nabla^{\mathsf{E}}_{Z_1}Z_2\rangle$$

in four cases.

1. $Z_1 = X_1^{\mathrm{h}}$ and $Z_2 = X_2^{\mathrm{h}}$: Here we have

$$\mathscr{L}_{X_1^{\mathrm{h}}}\langle\lambda^{\mathrm{v}}; X_2^{\mathrm{h}}\rangle = 0$$

and

$$\langle\lambda^{\mathrm{v}}; \nabla^{\mathsf{E}}_{X_1^{\mathrm{h}}}X_2^{\mathrm{h}}\rangle = \langle\lambda^{\mathrm{v}}; A_{\pi_{\mathsf{E}}}(X_1^{\mathrm{h}}, X_2^{\mathrm{h}})\rangle$$

(by Lemma 4.1(vi)) giving

$$\langle\nabla^{\mathsf{E}}_{X_1^{\mathrm{h}}}\lambda^{\mathrm{v}}; X_2^{\mathrm{h}}\rangle = -\langle\lambda^{\mathrm{v}}; A_{\pi_{\mathsf{E}}}(X_1^{\mathrm{h}}, X_2^{\mathrm{h}})\rangle = \langle\lambda^{\mathrm{v}}; A_{\pi_{\mathsf{E}}}(X_2^{\mathrm{h}}, X_1^{\mathrm{h}})\rangle$$

(by Lemma 4.1(ii)). Thus we have

$$\langle\nabla^{\mathsf{E}}_{X_1^{\mathrm{h}}}\lambda^{\mathrm{v}}; X_2^{\mathrm{h}}\rangle = \langle A^*_{\pi_{\mathsf{E}}}(\lambda^{\mathrm{v}}, X_1^{\mathrm{h}}); X_2^{\mathrm{h}}\rangle.$$

2. $Z_1 = X^{\mathrm{h}}$ and $Z_2 = \xi^{\mathrm{v}}$: We compute

$$\mathscr{L}_{X^{\mathrm{h}}}\langle\lambda^{\mathrm{v}}; \xi^{\mathrm{v}}\rangle = (\mathscr{L}_X\langle\lambda; \xi\rangle)^{\mathrm{h}}$$

(by Lemma 3.6(i)) and

$$\langle\lambda^{\mathrm{v}}; \nabla^{\mathsf{E}}_{X^{\mathrm{h}}}\xi^{\mathrm{v}}\rangle = \langle\lambda^{\mathrm{v}}; (\nabla^{\pi_{\mathsf{E}}}_X\xi)^{\mathrm{v}}\rangle = \langle\lambda; \nabla^{\pi_{\mathsf{E}}}_X\xi\rangle^{\mathrm{h}}$$

(by Lemma 4.1(xiii)). Thus

$$\langle\nabla^{\mathsf{E}}_{X^{\mathrm{h}}}\lambda^{\mathrm{v}}; \xi^{\mathrm{v}}\rangle = (\mathscr{L}_X\langle\lambda; \xi\rangle)^{\mathrm{h}} - \langle\lambda; \nabla^{\pi_{\mathsf{E}}}_X\xi\rangle^{\mathrm{h}}$$

or

$$\langle\nabla^{\mathsf{E}}_{X^{\mathrm{h}}}\lambda^{\mathrm{v}}; \xi^{\mathrm{v}}\rangle = \langle(\nabla^{\pi_{\mathsf{E}}}_X\lambda)^{\mathrm{v}}; \xi^{\mathrm{v}}\rangle.$$

3. $Z_1 = \xi^{\mathrm{v}}$ and $Z_2 = X^{\mathrm{h}}$: In this case we compute

$$\mathscr{L}_{\xi^{\mathrm{v}}}\langle\lambda^{\mathrm{v}}; X^{\mathrm{h}}\rangle = 0$$

and

$$\langle\lambda^{\mathrm{v}}; \nabla^{\mathsf{E}}_{\xi^{\mathrm{v}}}X^{\mathrm{h}}\rangle = 0$$

(by Lemma 4.1(xi)) giving

$$\langle\nabla^{\mathsf{E}}_{\xi^{\mathrm{v}}}\lambda^{\mathrm{v}}; X^{\mathrm{h}}\rangle = 0.$$

4. $Z_1 = \xi_1^{\mathrm{v}}$ and $Z_2 = \xi_2^{\mathrm{v}}$: We have

$$\mathscr{L}_{\xi_1^{\mathrm{v}}}\langle\lambda^{\mathrm{v}}; \xi_2^{\mathrm{v}}\rangle = \mathscr{L}_{\xi_1^{\mathrm{v}}}\langle\lambda; \xi_2\rangle^{\mathrm{h}} = 0$$

(by Lemma 3.6(ii)) and

$$\langle\lambda^{\mathrm{v}}; \nabla^{\mathsf{E}}_{\xi_1^{\mathrm{v}}}\xi_2^{\mathrm{v}}\rangle = 0$$

(using Lemma 4.1(xii)). This gives

$$\langle\nabla^{\mathsf{E}}_{\xi_1^{\mathrm{v}}}\lambda^{\mathrm{v}}; \xi_2^{\mathrm{v}}\rangle = 0.$$

Putting the above together,

$$\nabla^{\mathsf{E}}_Z \lambda^{\mathrm{v}} = (\nabla^{\pi_{\mathsf{E}}}_{T\pi_{\mathsf{E}}(Z)} \lambda)^{\mathrm{v}} + \mathrm{hor}(A^*_{\pi_{\mathsf{E}}}(\lambda^{\mathrm{v}}, \mathrm{hor}(Z))).$$

Now we note that

$$\begin{aligned}\langle B^*_{\pi_{\mathsf{E}}}(\lambda^{\mathrm{v}}, Z_1); Z_2\rangle &= \langle \lambda^{\mathrm{v}}; B_{\pi_{\mathsf{E}}}(Z_2, Z_1)\rangle \\ &= \langle \lambda^{\mathrm{v}}; A_{\pi_{\mathsf{E}}}(\mathrm{hor}(Z_1), \mathrm{hor}(Z_2))\rangle + \langle \lambda^{\mathrm{v}}; A_{\pi_{\mathsf{E}}}(Z_1, \mathrm{ver}(Z_2))\rangle \\ &\quad + \langle \lambda^{\mathrm{v}}; A_{\pi_{\mathsf{E}}}(Z_2, \mathrm{ver}(Z_1))\rangle \\ &= -\langle \lambda^{\mathrm{v}}; A_{\pi_{\mathsf{E}}}(\mathrm{hor}(Z_2), \mathrm{hor}(Z_1))\rangle \\ &= -\langle A^*_{\pi_{\mathsf{E}}}(\lambda^{\mathrm{v}}, \mathrm{hor}(Z_1)); \mathrm{hor}(Z_2)\rangle,\end{aligned}$$

using Lemma 4.1(xi). Thus

$$\nabla^{\mathsf{E}}_Z \lambda^{\mathrm{v}} = (\nabla^{\pi_{\mathsf{E}}}_{T\pi_{\mathsf{E}}(Z)} \lambda)^{\mathrm{v}} - B^*_{\pi_{\mathsf{E}}}(\lambda^{\mathrm{v}}, Z).$$

Now, it suffices to prove this part of the lemma for $A = A_0 \otimes \lambda \otimes \eta$ for $A_0 \in \Gamma^r(\mathrm{T}^k(\mathsf{T}^*\mathsf{M}))$, $\lambda \in \Gamma^r(\mathsf{E}^*)$, and $\eta \in \Gamma^r(\mathsf{F})$. Here we calculate, for $Z \in \Gamma^r(\mathsf{TE})$,

$$\begin{aligned}\nabla^{\mathsf{E},\pi_{\mathsf{F}}}_Z (A^{\mathrm{h}}_0 \otimes \lambda^{\mathrm{v}} \otimes \pi^*_{\mathsf{E}}\eta) &= (\nabla^{\mathsf{E}}_Z A^{\mathrm{h}}_0) \otimes \lambda^{\mathrm{v}} \otimes \pi^*_{\mathsf{E}}\eta + A^{\mathrm{h}}_0 \otimes \nabla^{\mathsf{E}}_Z \lambda^{\mathrm{v}} \otimes \eta + A^{\mathrm{h}}_0 \otimes \lambda^{\mathrm{v}} + \pi^*_{\mathsf{E}} \nabla^{\pi_{\mathsf{F}}}_Z \pi^*_{\mathsf{E}}\eta \\ &= (\nabla^{\mathsf{M}}_{T\pi_{\mathsf{E}}(Z)} A_0)^{\mathrm{h}} \otimes \lambda^{\mathrm{v}} \otimes \pi^*_{\mathsf{E}}\eta - \sum_{j=1}^k \mathrm{Ins}_j(A^{\mathrm{h}}_0, B_{\pi_{\mathsf{E}},Z}) \otimes \lambda^{\mathrm{v}} \otimes \pi^*_{\mathsf{E}}\eta \\ &\quad + A^{\mathrm{h}}_0 \otimes (\nabla^{\pi_{\mathsf{E}}}_{T\pi_{\mathsf{E}}(Z)} \lambda)^{\mathrm{v}} \otimes \eta - A^{\mathrm{h}}_0 \otimes B^*_{\pi_{\mathsf{E}}}(\lambda^{\mathrm{v}}, Z) + A^{\mathrm{h}}_0 \otimes \lambda^{\mathrm{v}} \otimes \pi^*_{\mathsf{E}}(\nabla^{\pi_{\mathsf{F}}}_{T\pi_{\mathsf{E}}(Z)} \eta).\end{aligned}$$

We have

$$\begin{aligned}&(\nabla^{\mathsf{M}}_{T\pi_{\mathsf{E}}(Z)} A_0)^{\mathrm{h}} \otimes \lambda^{\mathrm{v}} + A^{\mathrm{h}}_0 \otimes (\nabla^{\pi_{\mathsf{E}}}_{T\pi_{\mathsf{E}}(Z)} \lambda)^{\mathrm{v}} + A^{\mathrm{h}}_0 \otimes \lambda^{\mathrm{v}} \otimes \pi^*_{\mathsf{E}}(\nabla^{\pi_{\mathsf{F}}}_{T\pi_{\mathsf{E}}(Z)} \eta) \\ &\qquad = (\nabla^{\mathsf{M}}_{T\pi_{\mathsf{E}}(Z)} A_0 \otimes \lambda \otimes \eta)^{\mathrm{v}} + (A_0 \otimes \nabla^{\pi_{\mathsf{E}}}_{T\pi_{\mathsf{E}}(Z)} \lambda \otimes \eta)^{\mathrm{v}} + (A_0 \otimes \lambda^{\mathrm{v}} \otimes \nabla^{\pi_{\mathsf{F}}}_{T\pi_{\mathsf{E}}(Z)} \eta)^{\mathrm{v}} \\ &\qquad = (\nabla^{\pi_{\mathsf{E}} \otimes \pi_{\mathsf{F}}}_{T\pi_{\mathsf{E}}(Z)} (A_0 \otimes \lambda \otimes \eta))^{\mathrm{v}}\end{aligned}$$

and, by (3.4) and the first part of the lemma,

$$-A^{\mathrm{h}}_0 \otimes B^*_{\pi_{\mathsf{E}}}(\lambda^{\mathrm{v}}, Z) \otimes \pi^*_{\mathsf{E}}\eta - \sum_{j=1}^k \mathrm{Ins}_j(A^{\mathrm{h}}_0, B_{\pi_{\mathsf{E}},Z}) \otimes \lambda^{\mathrm{v}} \otimes \pi^*_{\mathsf{E}}\eta = D_{B_{\pi_{\mathsf{E}},Z}}(A^{\mathrm{h}}_0 \otimes \lambda^{\mathrm{v}} \otimes \pi^*_{\mathsf{E}}\eta).$$

Assembling the preceding computations gives this part of the lemma.

(v) This is a slight modification of the preceding part of the proof, taking the formula (4.13) into account.

(vi) By Lemma 3.6(iii) and (iv) we have

$$\nabla^{\mathsf{E}}_Z \lambda^{\mathrm{e}} = \mathscr{L}_Z \lambda^{\mathrm{e}} = (\nabla^{\pi_{\mathsf{E}}}_{T\pi_{\mathsf{E}}(Z)} \lambda)^{\mathrm{e}} + \langle \lambda^{\mathrm{v}}; Z\rangle.$$

With the constructions following Definition 3.8 in mind, we work with $A = A^{\mathrm{h}}_0 \otimes \lambda^{\mathrm{e}} \otimes \pi^*_{\mathsf{E}}\eta$ for $A_0 \in \Gamma^r(\mathrm{T}^k(\mathsf{T}^*\mathsf{M}))$, $\lambda \in \Gamma^r(\mathsf{E}^*)$, and $\eta \in \Gamma^r(\mathsf{F})$. If we keep in mind that λ^{e} is a function, then we can simply write $A = A_0 \otimes (\lambda^{\mathrm{e}}\pi_{\mathsf{E}}^*\eta)$. We now calculate

$$\begin{aligned}\nabla_Z^{\mathsf{E},\pi_{\mathsf{F}}}(A_0^{\mathrm{h}}\otimes\lambda^{\mathrm{e}}\otimes\pi_{\mathsf{E}}^*\eta) &= (\nabla_Z^{\mathsf{E}}A_0^{\mathrm{h}})\otimes\lambda^{\mathrm{e}}\otimes\pi_{\mathsf{E}}^*\eta + A_0^{\mathrm{h}}\otimes(\nabla_Z^{\mathsf{E}}\lambda^{\mathrm{e}})\otimes\pi_{\mathsf{E}}^*\eta + A_0^{\mathrm{h}}\otimes\lambda^{\mathrm{e}}\otimes(\pi_{\mathsf{E}}^*\nabla_Z^{\pi_{\mathsf{F}}}\pi_{\mathsf{E}}^*\eta)\\ &= (\nabla^{\mathsf{M}}_{T\pi_{\mathsf{E}}(Z)}A_0)^{\mathrm{h}}\otimes\lambda^{\mathrm{e}}\otimes\pi_{\mathsf{E}}^*\eta - \sum_{j=1}^k \mathrm{Ins}_j(A_0^{\mathrm{h}},B_{\pi_{\mathsf{E}}})\otimes\lambda^{\mathrm{e}}\otimes\pi_{\mathsf{E}}^*\eta\\ &\quad + A_0^{\mathrm{h}}\otimes(\nabla^{\pi_{\mathsf{E}}}_{T\pi_{\mathsf{E}}(Z)}\lambda)^{\mathrm{e}}\otimes\pi_{\mathsf{E}}^*\eta + A_0^{\mathrm{h}}\otimes(\lambda^{\mathrm{v}}(Z))\otimes\pi_{\mathsf{E}}^*\eta\\ &\quad + A_0^{\mathrm{h}}\otimes\lambda^{\mathrm{e}}\otimes\pi_{\mathsf{E}}^*(\nabla^{\pi_{\mathsf{F}}}_{T\pi_{\mathsf{E}}(Z)}\eta) + A_0^{\mathrm{h}}\otimes\lambda^{\mathrm{e}}\otimes B_{\pi_{\mathsf{E}}}(\pi_{\mathsf{E}}^*\eta,Z).\end{aligned}$$

We have

$$\begin{aligned}&(\nabla^{\mathsf{M}}_{T\pi_{\mathsf{E}}(Z)}A_0)^{\mathrm{h}}\otimes\lambda^{\mathrm{e}}\otimes\pi_{\mathsf{E}}^*\eta + A_0^{\mathrm{h}}\otimes(\nabla^{\pi_{\mathsf{E}}}_{T\pi_{\mathsf{E}}(Z)}\lambda)^{\mathrm{e}}\otimes\pi_{\mathsf{E}}^*\eta + A_0^{\mathrm{h}}\otimes\lambda^{\mathrm{e}}\otimes\pi_{\mathsf{E}}^*(\nabla^{\pi_{\mathsf{F}}}_{T\pi_{\mathsf{E}}(Z)}\eta)\\ &\qquad = (\nabla^{\mathsf{M}}_{T\pi_{\mathsf{E}}(Z)}A_0\otimes\lambda\otimes\eta)^{\mathrm{e}} + (A_0\otimes\nabla^{\pi_{\mathsf{E}}}_{T\pi_{\mathsf{E}}(Z)}\lambda\otimes\eta)^{\mathrm{e}} + (A_0\otimes\lambda\otimes\nabla^{\pi_{\mathsf{F}}}_{T\pi_{\mathsf{E}}(Z)}\eta)^{\mathrm{e}}\\ &\qquad = (\nabla^{\mathsf{M},\pi_{\mathsf{E}}\otimes\pi_{\mathsf{F}}}_{T\pi_{\mathsf{E}}(Z)}(A_0\otimes\lambda\otimes\eta))^{\mathrm{e}}.\end{aligned}$$

Next we note that

$$A_0^{\mathrm{h}}\otimes\lambda^{\mathrm{e}}\otimes B_{\pi_{\mathsf{E}}}(\pi_{\mathsf{E}}^*\eta,Z) - \sum_{j=1}^k \mathrm{Ins}_j(A_0^{\mathrm{h}},B_{\pi_{\mathsf{E}}})\otimes\lambda^{\mathrm{e}}\otimes\pi_{\mathsf{E}}^*\eta = D_{B_{\pi_{\mathsf{E}}},Z}(A_0\otimes(\lambda^{\mathrm{e}}\pi_{\mathsf{E}}^*\eta)),$$

keeping in mind that λ^{e} is a function, so the tensor products with λ^{e} are just multiplication. Again making reference to the constructions following Definition 3.8, we have

$$A_0^{\mathrm{h}}\otimes\lambda^{\mathrm{v}}\otimes\pi_{\mathsf{E}}^*\eta = (A_0\otimes\lambda\otimes\eta)^{\mathrm{v}},$$

and the lemma follows by combining the preceding three formulae.

(vii) This is a slight modification of the preceding part of the proof, taking the formula (4.13) into account. ■

4.4. Prolongation. In our geometric setting, differentiation means "prolongation" by taking jets. In this section, we illustrate how our decompositions of Section 2.2 interact with prolongation.

We let $r \in \{\infty,\omega\}$ and let $\pi_{\mathsf{E}}\colon \mathsf{E}\to\mathsf{M}$ be a C^r-vector bundle with $\mathsf{J}^m\mathsf{E}$, $m\in\mathbb{Z}_{\geq 0}$, its jet bundles. We suppose that we have a C^r-affine connection ∇^{M} on M and a C^r-vector bundle connection $\nabla^{\pi_{\mathsf{E}}}$ in E. Because we have the decomposition

$$\mathsf{J}^m\mathsf{E} \simeq \bigoplus_{j=0}^m (\mathrm{S}^j(\mathsf{T}^*\mathsf{M})\otimes\mathsf{E})$$

by Lemma 2.1, it follows that the vector bundle $\mathsf{J}^m\mathsf{E}$ has a C^r-connection that we denote by ∇^{π_m}. Explicitly,

$$\nabla^{\pi_m}_X j_m\xi = (S^m_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}})^{-1}(\nabla^{\pi_{\mathsf{E}}}_X\xi, \nabla^{\mathsf{M},\pi_{\mathsf{E}}}_X D^1_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}}(\xi),\dots,\nabla^{\mathsf{M},\pi_{\mathsf{E}}}_X D^m_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}}(\xi)).$$

Therefore, the construction of Lemma 2.1 can be applied to $\mathsf{J}^m\mathsf{E}$, and all that remains to sort out is notation.

To this end, for $k,m\in\mathbb{Z}_{\geq 0}$, let us denote by $\nabla^{\mathsf{M},\pi_m}$ the connection in $\mathrm{T}^k(\mathsf{T}^*\mathsf{M})\oplus\mathsf{J}^m\mathsf{E}$ induced, via tensor product, by the connections ∇^{M} and ∇^{π_m}. Then, for $\xi\in\Gamma^r(\mathsf{E})$, denote

$$\nabla^{\mathsf{M},\pi_m,k}j_m\xi=\underbrace{\nabla^{\mathsf{M},\pi_m}\cdots(\nabla^{\mathsf{M},\pi_m}}_{k-1\ \text{times}}(\nabla^{\pi_m}j_m\xi))\in\Gamma^r(\mathrm{T}^k(\mathsf{T}^*\mathsf{M})\otimes\mathsf{J}^m\mathsf{E}).$$

We also denote

$$D^k_{\nabla^{\mathsf{M}},\nabla^{\pi_m}}(j_m\xi)=\mathrm{Sym}_k\otimes\mathrm{id}_{\mathsf{J}^m\mathsf{E}}(\nabla^{\mathsf{M},\pi_m,k}j_m\xi)\in\Gamma^r(\mathrm{S}^k(\mathsf{T}^*\mathsf{M})\otimes\mathsf{J}^m\mathsf{E}).$$

This can be refined further by explicitly decomposing $\mathsf{J}^m\mathsf{E}$, so let us provide the notation for making this refinement. For $m,k\in\mathbb{Z}_{\geq 0}$ and for $A\in\Gamma^r(\mathrm{T}^m(\mathsf{T}^*\mathsf{M})\otimes\mathsf{E})$, we denote

$$\nabla^{\mathsf{M},\pi_{\mathsf{E}},k}A=\underbrace{\nabla^{\mathsf{M},\pi_{\mathsf{E}}}\cdots\nabla^{\mathsf{M},\pi_{\mathsf{E}}}}_{k\ \text{times}}A\in\Gamma^r(\mathrm{T}^{m+k}(\mathsf{T}^*\mathsf{M})\otimes\mathsf{E})$$

and

$$D^{k,m}_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}}(A)=\mathrm{Sym}_k\otimes\mathrm{id}_{\mathrm{T}^m(\mathsf{T}^*\mathsf{M})\otimes\mathsf{E}}(\nabla^{\mathsf{M},\pi_{\mathsf{E}},k}A)\in\Gamma^r(\mathrm{S}^k(\mathsf{T}^*\mathsf{M})\otimes\mathrm{T}^m(\mathsf{T}^*\mathsf{M})\otimes\mathsf{E}).$$

Note that, if $A\in\Gamma^r(\mathrm{S}^m(\mathsf{T}^*\mathsf{M})\otimes\mathsf{E})$, then

$$D^{k,m}_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}}(A)\in\Gamma^r(\mathrm{S}^k(\mathsf{T}^*\mathsf{M})\otimes\mathrm{S}^m(\mathsf{T}^*\mathsf{M})\otimes\mathsf{E}).$$

An immediate consequence of Lemma 2.1 is then the following result.

4.6 Lemma: (Decompositions of jet bundles of jet bundles) *The maps*

$$\begin{aligned}S^k_{\nabla^{\mathsf{M}},\nabla^{\pi_m}}\colon\mathsf{J}^k\mathsf{J}^m\mathsf{E}&\to\bigoplus_{j=0}^k(\mathrm{S}^j(\mathsf{T}^*\mathsf{M})\otimes\mathsf{J}^m\mathsf{E})\\ j_kj_m\xi(x)&\mapsto(j_m\xi(x),D^1_{\nabla^{\mathsf{M}},\nabla^{\pi_m}}(j_m\xi)(x),\dots,D^k_{\nabla^{\mathsf{M}},\nabla^{\pi_m}}(j_m\xi)(x))\end{aligned}$$

and

$$S^{k,m}_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}}\colon\mathsf{J}^k\mathsf{J}^m\mathsf{E}\to\bigoplus_{j=0}^k\left(\mathrm{S}^j(\mathsf{T}^*\mathsf{M})\otimes\left(\bigoplus_{l=0}^m\mathrm{S}^l(\mathsf{T}^*\mathsf{M})\otimes\mathsf{E}\right)\right)$$

defined by

$$\begin{aligned}j_kj_m\xi(x)\mapsto((\xi(x),D^1_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}}(\xi)(x),\dots,D^m_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}}(\xi)(x)),&\\ (D^{1,0}_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}}\xi(x),D^{1,1}_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}}\circ D^1_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}}(\xi)(x),\dots,D^{1,m}_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}}\circ D^m_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}}(\xi)(x)),\dots,&\\ (D^{k,0}_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}}\xi(x),D^{k,1}_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}}\circ D^1_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}}(\xi)(x),\dots,D^{k,m}_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}}\circ D^m_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}}(\xi)(x)))&\end{aligned}$$

are isomorphisms of vector bundles, and, for each $k\in\mathbb{Z}_{>0}$, the diagrams

$$\begin{array}{ccc}\mathsf{J}^{k+1}\mathsf{J}^m\mathsf{E} & \xrightarrow{S^{k+1}_{\nabla^{\mathsf{M}},\nabla^{\pi_m}}} & \bigoplus_{j=0}^{k+1}(\mathrm{S}^j(\mathsf{T}^*\mathsf{M})\otimes\mathsf{J}^m\mathsf{E})\\ {\scriptstyle(\pi_m)^{k+1}_k}\downarrow & & \downarrow{\scriptstyle\mathrm{pr}^{k+1}_k}\\ \mathsf{J}^k\mathsf{J}^m\mathsf{E} & \xrightarrow[S^k_{\nabla^{\mathsf{M}},\nabla^{\pi_m}}]{} & \bigoplus_{j=0}^{k}(\mathrm{S}^j(\mathsf{T}^*\mathsf{M})\otimes\mathsf{J}^m\mathsf{E})\end{array}$$

and

$$\begin{array}{ccc}
\mathsf{J}^{k+1}\mathsf{J}^m\mathsf{E} & \xrightarrow{S^{k+1,m}_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}}} & \bigoplus_{j=0}^{k+1}\left(\mathrm{S}^j(\mathsf{T}^*\mathsf{M})\otimes\left(\bigoplus_{l=0}^m \mathrm{S}^l(\mathsf{T}^*\mathsf{M})\otimes\mathsf{E}\right)\right) \\
\downarrow{\scriptstyle (\pi_m)^{k+1}_k} & & \downarrow{\scriptstyle \mathrm{pr}^{k+1}_k} \\
\mathsf{J}^{k}\mathsf{J}^m\mathsf{E} & \xrightarrow[S^{k,m}_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}}]{} & \bigoplus_{j=0}^{k}\left(\mathrm{S}^j(\mathsf{T}^*\mathsf{M})\otimes\left(\bigoplus_{l=0}^m \mathrm{S}^l(\mathsf{T}^*\mathsf{M})\otimes\mathsf{E}\right)\right)
\end{array}$$

commute, where pr^{k+1}_k *are the obvious projections, stripping off the last component of the direct sum.*

Now we recall [Saunders 1989, Definition 6.2.25] the inclusion, for $k, m \in \mathbb{Z}_{\geq 0}$,

$$\begin{aligned}\pi_{k,m} \colon \mathsf{J}^{k+m}\mathsf{E} &\to \mathsf{J}^k\mathsf{J}^m\mathsf{E}\\ j_{m+k}\xi(x) &\mapsto j_k j_m\xi(x).\end{aligned}$$

Let us understand this mapping using our decompositions of jet bundles. Note that, for a $\mathbb{R}$-vector space V and for $r, s \in \mathbb{Z}_{\geq 0}$, we have an inclusion,

$$\Delta_{r,s} \colon \mathrm{S}^{r+s}(\mathsf{V}^*) \to \mathrm{S}^r(\mathsf{V}^*) \otimes \mathrm{S}^s(\mathsf{V}^*). \tag{4.14}$$

Let us give an explicit formula for these inclusions.

4.7 Lemma: (Inclusions for symmetric tensors) *For a finite-dimensional* $\mathbb{R}$*-vector space* V *and for* $r, s \in \mathbb{Z}_{\geq 0}$*,*

$$\Delta_{r,s} = (\mathrm{Sym}_r \otimes \mathrm{Sym}_s) \circ \iota_{r,s},$$

where

$$\iota_{r,s} \colon \mathrm{S}^{r+s}(\mathsf{V}^*) \to \mathrm{T}^{r+s}(\mathsf{V}^*) = \mathrm{T}^r(\mathsf{V}^*) \otimes \mathrm{T}^s(\mathsf{V}^*)$$

is the inclusion.

Proof: We note that $\mathrm{Sym}_{r+s} \circ \Delta_{r,s} = \mathrm{id}_{\mathrm{S}^{r+s}(\mathsf{V}^*)}$, simply since $\Delta_{r,s}$ is the inclusion and Sym_{r+s} is orthogonal projection onto $\mathrm{S}^{r+s}(\mathsf{V}^*) \subseteq \mathrm{T}^{r+s}(\mathsf{V}^*)$; see Sublemma 1 from the proof of Lemma 7.5. Thus it will suffice to show that

$$\mathrm{Sym}_{r+s} \circ (\mathrm{Sym}_r \otimes \mathrm{Sym}_s) \circ \iota_{r,s} = \mathrm{id}_{\mathrm{S}^{r+s}(\mathsf{V}^*)}.$$

For $A \in \mathrm{S}^{r+s}(\mathsf{V}^*)$ we have

$$\begin{aligned}\mathrm{Sym}_r \otimes \mathrm{Sym}_s(A)(v_1, \dots, v_s) &= \frac{1}{r!s!}\sum_{\sigma_1 \in \mathfrak{S}_r}\sum_{\sigma_2 \in \mathfrak{S}_s} A(v_{\sigma_1(1)}, \dots, v_{\sigma_1(r)}, v_{r+\sigma_2(1)}, \dots, v_{r+\sigma_2(s)})\\ &= A(v_1, \dots, v_r, v_{r+1}, \dots, v_{r+s}),\end{aligned}$$

since A is symmetric, and so symmetric on the first r and last s entries. Since $\mathrm{Sym}_{r+s}(A) = A$, our claim follows, and so does the lemma. ■

Now, given a C^r-vector bundle $\pi_{\mathsf{E}}\colon \mathsf{E}\to\mathsf{M}$, the preceding mapping then induces by tensor product mappings

$$\Delta_{r,s}\otimes\mathrm{id}_{\mathsf{E}}\colon \mathrm{S}^{r+s}(\mathsf{T}^*\mathsf{M})\otimes\mathsf{E}\to\mathrm{S}^r(\mathsf{T}^*\mathsf{M})\otimes\mathrm{S}^s(\mathsf{T}^*\mathsf{M})\otimes\mathsf{E},$$

and so the mapping

$$\widehat{\Delta}_{k,m}\pi_{\mathsf{E}}\colon \bigoplus_{r=0}^{k+m}\mathrm{S}^r(\mathsf{T}^*\mathsf{M})\otimes\mathsf{E}\to\bigoplus_{j=0}^{k}\mathrm{S}^j(\mathsf{T}^*\mathsf{M})\otimes\left(\bigoplus_{l=0}^{m}\mathrm{S}^l(\mathsf{T}^*\mathsf{M})\otimes\mathsf{E}\right).$$

Explicitly,

$$\begin{aligned}\widehat{\Delta}_{k,m}&\pi_{\mathsf{E}}(A_0,\dots,A_{k+m})\\&=((\Delta_{0,0}(A_0),\Delta_{0,1}(A_1),\dots,\Delta_{0,m}(A_m)),(\Delta_{1,1}(A_1),\Delta_{1,2}(A_2),\dots,\Delta_{1,m}(A_{m+1})),\\&\qquad\dots,(\Delta_{k,0}(A_k),\Delta_{k,1}(A_{k+1}),\dots,\Delta_{k,m}(A_{k+m}))),\end{aligned}\tag{4.15}$$

where $A_r\in\mathrm{S}^r(\mathsf{T}^*\mathsf{E})\otimes\mathsf{E}$, $r\in\{0,1,\dots,k+m\}$, and where we abbreviate $\Delta_{j,l}\otimes\mathrm{id}_{\mathsf{E}}$ by $\Delta_{j,l}$ in an attempt to achieve concision.

We now have the following result.

4.8 Lemma: (Decomposition of prolongation of jet bundles) *For $r\in\{\infty,\omega\}$ and a C^r-vector bundle $\pi_{\mathsf{E}}\colon\mathsf{E}\to\mathsf{M}$ and for $k,m\in\mathbb{Z}_{\ge0}$, the following diagram commutes*

$$\begin{array}{ccc}\mathsf{J}^{k+m}\mathsf{E} & \xrightarrow{\ \pi_{k,m}\ } & \mathsf{J}^k\mathsf{J}^m\mathsf{E}\\ \Big\downarrow{\scriptstyle S^{k+m}_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}}} & & \Big\downarrow{\scriptstyle S^{k,m}_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}}}\\ \bigoplus_{r=0}^{k+m}\mathrm{S}^r(\mathsf{T}^*\mathsf{M})\otimes\mathsf{E} & \xrightarrow[\widehat{\Delta}_{k,m}\pi_{\mathsf{E}}]{} & \bigoplus_{j=0}^{k}\mathrm{S}^j(\mathsf{T}^*\mathsf{M})\otimes\left(\bigoplus_{l=0}^{m}\mathrm{S}^l(\mathsf{T}^*\mathsf{M})\otimes\mathsf{E}\right)\end{array}$$

Proof: We note that, by definition of the symbols involved,

$$\nabla^{\mathsf{M},\pi_m,k}(\nabla^{\mathsf{M},\pi,m}\xi)=\nabla^{\mathsf{M},\pi,m+k}\xi,\qquad k,m\in\mathbb{Z}_{\ge0},\ \xi\in\Gamma^r(\mathsf{E}).$$

By Lemma 4.7, we have

$$\begin{aligned}D^{k+m}_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}}\xi&=\mathrm{Sym}_{k+m}\otimes\mathrm{id}_{\mathsf{E}}(\nabla^{\mathsf{M},\pi_{\mathsf{E}},k+m}\xi)\\&=\Delta_{k,m}\otimes\mathrm{id}_{\mathsf{E}}(\nabla^{\mathsf{M},\pi_{\mathsf{E}},k+m}\xi)\\&=\Delta_{k,m}\otimes\mathrm{id}_{\mathsf{E}}(\nabla^{\mathsf{M},\pi_m,k}(\nabla^{\mathsf{M},\pi_{\mathsf{E}},m}\xi))\\&=\mathrm{Sym}_k\otimes\mathrm{Sym}_m\otimes\mathrm{id}_{\mathsf{E}}(\nabla^{\mathsf{M},\pi_m,k}(\nabla^{\mathsf{M},\pi_{\mathsf{E}},m}\xi))\\&=D^{k,m}_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}}(D^m_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}}\xi).\end{aligned}$$

Using this observation, and the definition of the mappings $S^{k+m}_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}}$ and $S^{k,m}_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}}$, the lemma follows by a straightforward computation involving mere notation. ■

5. Isomorphisms defined by lifts and pull-backs

Let $r \in \{\infty, \omega\}$ and let $\pi_{\mathsf{E}} \colon \mathsf{E} \to \mathsf{M}$ be a C^r-vector bundle. In this section we carefully study isomorphisms that arise from lifts of objects on M to objects on E, of the sorts introduced in Sections 3.1, 3.2, and 3.3. In particular, we shall see that jets of geometric objects can be decomposed (as in Section 2.2) before or after lifting. We wish here to relate these two sorts of decompositions for all of the lifts we consider in this work. This makes use of our constructions of Section 4 to give explicit decompositions for jets of certain sections of certain jet bundles on the total space of a vector bundle. Indeed, it is the results in this section that provide the motivation for the rather intricate constructions of Section 4. For these constructions, we additionally suppose that we have a Riemannian metric $\mathbb{G}_{\mathsf{M}}$ on M and a fibre metric $\mathbb{G}_{\pi_{\mathsf{E}}}$ on E. We suppose that ∇^{M} is the Levi-Civita connection for $\mathbb{G}_{\mathsf{M}}$. This data gives rise to a Riemannian metric $\mathbb{G}_{\mathsf{E}}$ on E with its Levi-Civita connection ∇^{E}. We break the discussion into eight cases, corresponding to the seven parts of Lemma 4.5, along with a construction for pull-backs of functions. The constructions, statements, and proofs are somewhat repetitive, so we do not give proofs that are essentially identical to previous proofs. While the results are similar, they are not the same, so we must go through all of the cases. There is probably a "meta" result here, but it would take a small journey in itself to setup the framework for this. For our purposes, we stick to a treatment that is concrete at the cost of being dull.

In this section, as in the previous two sections, we shall state results on an equal footing for the smooth and real analytic cases. However, the detailed recursion formulae we give are not really necessary if one wants to prove the continuity results of Section 9 in the smooth case. Thus one should really regard the results of this section as being particular to the real analytic setting.

In any event, throughout this section we let $r \in \{\infty, \omega\}$.

5.1. Isomorphisms for horizontal lifts of functions. Here we consider the horizontal lift mapping

$$\mathrm{C}^r(\mathsf{M}) \ni f \mapsto \pi_{\mathsf{E}}^* f \in \mathrm{C}^r(\mathsf{E}).$$

We wish to relate the decomposition associated with the jets of f to those associated with the jets of $\pi_{\mathsf{E}}^* f$. Associated with this, let us denote by $\mathsf{P}^{*m}\mathsf{E}$ the subbundle of $\mathbb{R}_{\mathsf{E}} \oplus \mathsf{T}^{*m}\mathsf{E}$ defined by

$$\mathsf{P}^{*m}_e\mathsf{E} = \{j_m(\pi_{\mathsf{E}}^* f)(e) \mid\ f \in \mathrm{C}^m(\mathsf{M})\}.$$

Following Lemma 2.1, our constructions have to do with iterated covariant differentials. The basis of all of our formulae will be a formula for iterated covariant differentials of horizontal lifts of functions on M. Thus we let $f \in \mathrm{C}^\infty(\mathsf{M})$ and consider

$$\nabla^{\mathsf{E},m}\pi_{\mathsf{E}}^* f \triangleq \underbrace{\nabla^{\mathsf{E}} \cdots \nabla^{\mathsf{E}}}_{m \text{ times}} \pi_{\mathsf{E}}^* f, \qquad m \in \mathbb{Z}_{>0}.$$

We state the first two lemmata that we will use. We recall from Lemma 4.5 the definition of $B_{\pi_{\mathsf{E}}}$.

5.1 Lemma: (Iterated covariant differentials of horizontal lifts of functions I) *Let $r \in \{\infty, \omega\}$ and let $\pi_{\mathsf{E}} \colon \mathsf{E} \to \mathsf{M}$ be a C^r-vector bundle, with the data prescribed in Section 4.1 to define the Riemannian metric $\mathbb{G}_{\mathsf{E}}$ on E. For $m \in \mathbb{Z}_{\geq 0}$, there exist C^r-vector bundle mappings*

$$(A^m_s, \mathrm{id}_{\mathsf{E}}) \in \mathrm{VB}^r(\mathrm{T}^s(\pi^*_{\mathsf{E}}\mathsf{T}^*\mathsf{M}); \mathrm{T}^m(\mathsf{T}^*\mathsf{E})), \qquad s \in \{0, 1, \dots, m\},$$

such that

$$\nabla^{\mathsf{E},m}\pi^*_{\mathsf{E}} f = \sum_{s=0}^{m} A^m_s(\pi^*_{\mathsf{E}}\nabla^{\mathsf{M},s} f)$$

for all $f \in \mathrm{C}^m(\mathsf{M})$. Moreover, the vector bundle mappings $A^m_0, A^m_1, \dots, A^m_m$ satisfy the recursion relations prescribed by

$$A^0_0(\beta_0) = \beta_0, \;\; A^1_1(\beta_1) = \beta_1, \;\; A^1_0 = 0,$$

and

$$\begin{aligned}
&A^{m+1}_{m+1}(\beta_{m+1}) = \beta_{m+1},\\
&\quad A^{m+1}_s(\beta_s) = (\nabla^{\mathsf{E}} A^m_s)(\beta_s) + A^m_{s-1} \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\beta_s) - \sum_{j=1}^{s} A^m_s \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\mathrm{Ins}_j(\beta_s, B_{\pi_{\mathsf{E}}})), \;\; s \in \{1, \dots, m\},\\
&\quad A^{m+1}_0(\beta_0) = (\nabla^{\mathsf{E}} A^m_0)(\beta_0),
\end{aligned}$$

where $\beta_s \in \mathrm{T}^s(\pi^*_{\mathsf{E}}\mathsf{T}^*\mathsf{M})$, $s \in \{0, 1, \dots, m\}$.

Proof: The assertion clearly holds for the initial conditions of the recursion, simply because

$$\pi^* f = \pi^* f, \quad \mathrm{d}(\pi^* f) = \pi^*\mathrm{d}f + 0f.$$

So suppose it true for $m \in \mathbb{Z}_{>0}$. Thus

$$\nabla^{\mathsf{E},m}\pi^*_{\mathsf{E}} f = \sum_{s=0}^{m} A^m_s(\pi^*_{\mathsf{E}}\nabla^{\mathsf{M},s} f),$$

where the vector bundle mappings A^a_s, $a \in \{0, 1, \dots, m\}$, $s \in \{0, 1, \dots, a\}$, satisfy the recursion relations from the statement of the lemma. Then

$$\begin{aligned}
\nabla^{\mathsf{E},m+1}\pi^*_{\mathsf{E}} f = &\sum_{s=0}^{m}(\nabla^{\mathsf{E}} A^m_s)(\pi^*_{\mathsf{E}}\nabla^{\mathsf{M},s} f) + \sum_{s=0}^{m} A^m_s \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\nabla^{\mathsf{E}}\pi^*_{\mathsf{E}}\nabla^{\mathsf{M},s} f)\\
= &\sum_{s=0}^{m}(\nabla^{\mathsf{E}} A^m_s)(\pi^*_{\mathsf{E}}\nabla^{\mathsf{M},s} f) + \sum_{s=0}^{m} A^m_s \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\pi^*_{\mathsf{E}}\nabla^{\mathsf{M},s+1} f)\\
&- \sum_{s=0}^{m}\sum_{j=1}^{s} A^m_s \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\mathrm{Ins}_j(\pi^*_{\mathsf{E}}\nabla^{\mathsf{M},s} f, B_{\pi_{\mathsf{E}}}))\\
= &\,\pi^*_{\mathsf{E}}\nabla^{\mathsf{M},m+1} f + \sum_{s=1}^{m}\Bigg((\nabla^{\mathsf{E}} A^m_s)(\pi^*_{\mathsf{E}}\nabla^{\mathsf{M},s} f) + A^m_{s-1} \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\pi^*_{\mathsf{E}}\nabla^{\mathsf{M},s} f)\\
&- \sum_{j=1}^{s} A^m_s \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\mathrm{Ins}_j(\pi^*_{\mathsf{E}}\nabla^{\mathsf{M},s} f, B_{\pi_{\mathsf{E}}}))\Bigg) + (\nabla^{\mathsf{E}} A^m_0)(\pi^*_{\mathsf{E}} f)
\end{aligned}$$

by Lemma 4.5(i). From this, the lemma follows. ■

We shall also need to "invert" the relationship of the preceding lemma.

5.2 Lemma: (Iterated covariant differentials of horizontal lifts of functions II) *Let $r \in \{\infty, \omega\}$ and let $\pi_{\mathsf{E}} \colon \mathsf{E} \to \mathsf{M}$ be a C^r-vector bundle, with the data prescribed in Section 4.1 to define the Riemannian metric $\mathbb{G}_{\mathsf{E}}$ on E. For $m \in \mathbb{Z}_{\geq 0}$, there exist C^r-vector bundle mappings*

$$(B^m_s, \mathrm{id}_{\mathsf{E}}) \in \mathrm{VB}^r(\mathrm{T}^s(\mathsf{T}^*\mathsf{E}); \mathrm{T}^m(\pi_{\mathsf{E}}^*\mathsf{T}^*\mathsf{M})), \qquad s \in \{0, 1, \dots, m\},$$

such that

$$\pi_{\mathsf{E}}^*\nabla^{\mathsf{M},m} f = \sum_{s=0}^m B^m_s(\nabla^{\mathsf{E},s}\pi_{\mathsf{E}}^* f)$$

for all $f \in \mathrm{C}^m(\mathsf{M})$. Moreover, the vector bundle mappings $B^m_0, B^m_1, \dots, B^m_m$ satisfy the recursion relations prescribed by

$$B^0_0(\alpha_0) = \alpha_0, \ \ B^1_1(\alpha_1) = \alpha_1, \ \ B^1_0 = 0,$$

and

$$B^{m+1}_{m+1}(\alpha_{m+1}) = \alpha_{m+1},$$
$$B^{m+1}_s(\alpha_s) = (\nabla^{\mathsf{E}} B^m_s)(\alpha_s) + B^m_{s-1} \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\alpha_s) + \sum_{j=1}^m \mathrm{Ins}_j(B^m_s(\alpha_s), B_{\pi_{\mathsf{E}}}), \ \ s \in \{1, \dots, m\},$$
$$B^{m+1}_0(\alpha_0) = (\nabla^{\mathsf{E}} B^m_0)(\alpha_0) + \sum_{j=1}^m \mathrm{Ins}_j(B^m_0(\alpha_0), B_{\pi_{\mathsf{E}}}),$$

where $\alpha_s \in \mathrm{T}^s(\mathsf{T}^*\mathsf{E})$, $s \in \{0, 1, \dots, m\}$.

Proof: The assertion clearly holds for the initial conditions for the recursion since

$$\pi^* f = \pi^* f, \quad \pi^*(\mathrm{d}f) = \mathrm{d}(\pi^* f) + 0f.$$

So suppose it true for $m \in \mathbb{Z}_{>0}$. Thus

$$\pi_{\mathsf{E}}^*\nabla^{\mathsf{M},m} f = \sum_{s=0}^m B^m_s(\nabla^{\mathsf{E},s}\pi_{\mathsf{E}}^* f), \tag{5.1}$$

where the vector bundle mappings B^a_s, $a \in \{0, 1, \dots, m\}$, $s \in \{0, 1, \dots, a\}$, satisfy the recursion relations from the statement of the lemma. Then, by Lemma 4.5(i), we can work on the left-hand side of (5.1) to give

$$\begin{aligned}\nabla^{\mathsf{E}}\pi_{\mathsf{E}}^*\nabla^{\mathsf{M},m} f &= \pi_{\mathsf{E}}^*\nabla^{\mathsf{M},m+1} f - \sum_{j=1}^m \mathrm{Ins}_j(\pi_{\mathsf{E}}^*\nabla^{\mathsf{M},m} f, B_{\pi_{\mathsf{E}}}) \\ &= \pi_{\mathsf{E}}^*\nabla^{\mathsf{M},m+1} f - \sum_{s=0}^m \sum_{j=1}^m \mathrm{Ins}_j(B^m_s(\nabla^{\mathsf{E},s}\pi_{\mathsf{E}}^* f), B_{\pi_{\mathsf{E}}}).\end{aligned}$$

Working on the right-hand side of (5.1) gives

$$\nabla^{\mathsf{E}}\pi_{\mathsf{E}}^*\nabla^{\mathsf{M},m} f = \sum_{s=0}^m \nabla^{\mathsf{E}} B^m_s(\nabla^{\mathsf{E},s}\pi_{\mathsf{E}}^* f) + \sum_{s=0}^m B^m_s \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\nabla^{\mathsf{E},s+1}\pi_{\mathsf{E}}^* f).$$

Combining the preceding two equations gives

$$\begin{aligned}\pi_{\mathsf{E}}^*\nabla^{\mathsf{M},m+1}f = &\sum_{s=0}^{m}\nabla^{\mathsf{E}}B_s^m(\nabla^{\mathsf{E},s}\pi_{\mathsf{E}}^*f) + \sum_{s=0}^{m}B_s^m\otimes\mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\nabla^{\mathsf{E},s+1}\pi_{\mathsf{E}}^*f)\\ &+\sum_{s=0}^{m}\sum_{j=1}^{m}\mathrm{Ins}_j(B_s^m(\nabla^{\mathsf{E},s}\pi_{\mathsf{E}}^*f),B_{\pi_{\mathsf{E}}})\\ =&\ \nabla^{\mathsf{E},m+1}\pi_{\mathsf{E}}^*f + \sum_{s=1}^{m}\Bigg(\nabla^{\mathsf{E}}B_s^m(\nabla^{\mathsf{E},s}\pi_{\mathsf{E}}^*f) + B_{s-1}^m\otimes\mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\nabla^{\mathsf{E},s}\pi_{\mathsf{E}}^*f)\\ &+\sum_{j=1}^{m}\mathrm{Ins}_j(B_s^m(\nabla^{\mathsf{E},s}\pi_{\mathsf{E}}^*f),B_{\pi_{\mathsf{E}}})\Bigg) + \nabla^{\mathsf{E}}B_0^m(\pi_{\mathsf{E}}^*f) + \sum_{j=1}^{m}\mathrm{Ins}_j(B_0^m(\pi_{\mathsf{E}}^*f),B_{\pi_{\mathsf{E}}}),\end{aligned}$$

and the lemma follows from this. ■

Next we turn to symmetrised versions of the preceding lemmata. We show that the preceding two lemmata induce corresponding mappings between symmetric tensors.

5.3 Lemma: (Iterated symmetrised covariant differentials of horizontal lifts of functions I) *Let $r\in\{\infty,\omega\}$ and let $\pi_{\mathsf{E}}\colon\mathsf{E}\to\mathsf{M}$ be a C^r-vector bundle, with the data prescribed in Section 4.1 to define the Riemannian metric $\mathbb{G}_{\mathsf{E}}$ on E. For $m\in\mathbb{Z}_{\geq 0}$, there exist C^r-vector bundle mappings*

$$(\widehat{A}_s^m,\mathrm{id}_{\mathsf{E}})\in\mathrm{VB}^r(\mathrm{S}^s(\pi_{\mathsf{E}}^*\mathsf{T}^*\mathsf{M});\mathrm{S}^m(\mathsf{T}^*\mathsf{E})),\qquad s\in\{0,1,\dots,m\},$$

such that

$$\mathrm{Sym}_m\circ\nabla^{\mathsf{E},m}\pi_{\mathsf{E}}^*f = \sum_{s=0}^{m}\widehat{A}_s^m(\mathrm{Sym}_s\circ\pi_{\mathsf{E}}^*\nabla^{\mathsf{M},s}f)$$

for all $f\in\mathrm{C}^m(\mathsf{M})$.

Proof: We define $A^m\colon\mathrm{T}^{\leq m}(\pi_{\mathsf{E}}^*\mathsf{T}^*\mathsf{M})\to\mathrm{T}^{\leq m}(\mathsf{T}^*\mathsf{E})$ by

$$A^m(\pi_{\mathsf{E}}^*f,\pi_{\mathsf{E}}^*\nabla^{\mathsf{M}}f,\dots,\pi_{\mathsf{E}}^*\nabla^{\mathsf{M},m}f) = \left(A_0^0(\pi_{\mathsf{E}}^*f),\sum_{s=0}^{1}A_s^1(\pi_{\mathsf{E}}^*\nabla^{\mathsf{M},s}f),\dots,\sum_{s=0}^{m}A_s^m(\pi_{\mathsf{E}}^*\nabla^{\mathsf{M},s}f)\right).$$

Let us organise the mappings we require into the following diagram:

$$\begin{array}{ccccc}\mathrm{T}^{\leq m}(\pi_{\mathsf{E}}^*\mathsf{T}^*\mathsf{M}) & \xrightarrow{\mathrm{Sym}_{\leq m}} & \mathrm{S}^{\leq m}(\pi_{\mathsf{E}}^*\mathsf{T}^*\mathsf{M}) & \xrightarrow{S_{\nabla^{\mathsf{M}}}^m} & \pi_{\mathsf{E}}^*(\mathbb{R}_{\mathsf{M}}\oplus\mathsf{T}^{*m}\mathsf{M})\\ \Big\downarrow{\scriptstyle A^m} & & \Big\downarrow{\scriptstyle\widehat{A}^m} & & \Big\downarrow{\scriptstyle\mathrm{id}_{\mathbb{R}}\oplus j_m\pi_{\mathsf{E}}}\\ \mathrm{T}^{\leq m}(\mathsf{T}^*\mathsf{E}) & \xrightarrow{\mathrm{Sym}_{\leq m}} & \mathrm{S}^{\leq m}(\mathsf{T}^*\mathsf{E}) & \xrightarrow{S_{\nabla^{\mathsf{E}}}^m} & \mathbb{R}_{\mathsf{E}}\oplus\mathsf{T}^{*m}\mathsf{E}\end{array}\qquad(5.2)$$

Here $\widehat{A}^m$ is defined so that the right square commutes. We shall show that the left square also commutes. Indeed,

$$\begin{aligned}&\widehat{A}^m\circ\mathrm{Sym}_{\leq m}(\pi_{\mathsf{E}}^*f,\pi_{\mathsf{E}}^*\nabla^{\mathsf{M}}f,\dots,\pi_{\mathsf{E}}^*\nabla^{\mathsf{M},m}f)\\ &\qquad= (S_{\nabla^{\mathsf{E}}}^m)^{-1}\circ(\mathrm{id}_{\mathbb{R}}\oplus j_m\pi_{\mathsf{E}})\circ S_{\nabla^{\mathsf{M}}}^m\circ\mathrm{Sym}_{\leq m}(\pi_{\mathsf{E}}^*f,\pi_{\mathsf{E}}^*\nabla^{\mathsf{M}}f,\dots,\pi_{\mathsf{E}}^*\nabla^{\mathsf{M},m}f)\\ &\qquad= \mathrm{Sym}_{\leq m}(\pi_{\mathsf{E}}^*f,\nabla^{\mathsf{E}}\pi_{\mathsf{E}}^*f,\dots,\nabla^{\mathsf{E},m}\pi_{\mathsf{E}}^*f)\\ &\qquad= \mathrm{Sym}_{\leq m}\circ A^m(\pi_{\mathsf{E}}^*f,\pi_{\mathsf{E}}^*\nabla^{\mathsf{M}}f,\dots,\pi_{\mathsf{E}}^*\nabla^{\mathsf{M},m}f).\end{aligned}$$

Thus the diagram (5.2) commutes. Now we have

$$\begin{aligned}\widehat{A}^m \circ \mathrm{Sym}_{\le m}(\pi_{\mathsf{E}}^* f, \pi_{\mathsf{E}}^* \nabla^{\mathsf{M}} f, \dots, \pi_{\mathsf{E}}^* \nabla^{\mathsf{M},m} f) \\ = \left(\mathrm{Sym}_1 \circ A_0^0(\pi_{\mathsf{E}}^* f), \sum_{s=0}^{1} \mathrm{Sym}_2 \circ A_s^1(\pi_{\mathsf{E}}^* \nabla^{\mathsf{M},s} f), \dots, \sum_{s=0}^{m} \mathrm{Sym}_m \circ A_s^m(\pi_{\mathsf{E}}^* \nabla^{\mathsf{M},s} f)\right).\end{aligned}$$

Thus, if we define

$$\widehat{A}_s^m(\mathrm{Sym}_s \circ \pi_{\mathsf{E}}^* \nabla^{\mathsf{M},s} f) = \mathrm{Sym}_m \circ A_s^m(\pi_{\mathsf{E}}^* \nabla^{\mathsf{M},s} f), \tag{5.3}$$

then we have

$$\mathrm{Sym}_m \circ \nabla^{\mathsf{E},m} \pi_{\mathsf{E}}^* f = \sum_{s=0}^{m} \widehat{A}_s^m(\mathrm{Sym}_s \circ \pi_{\mathsf{E}}^* \nabla^{\mathsf{M},s} f),$$

as desired. ■

Next we consider the "inverse" of the preceding lemma.

5.4 Lemma: (Iterated symmetrised covariant differentials of horizontal lifts of functions II) *Let $r \in \{\infty, \omega\}$ and let $\pi_{\mathsf{E}} \colon \mathsf{E} \to \mathsf{M}$ be a C^r-vector bundle, with the data prescribed in Section 4.1 to define the Riemannian metric $\mathbb{G}_{\mathsf{E}}$ on E. For $m \in \mathbb{Z}_{\ge 0}$, there exist C^r-vector bundle mappings*

$$(\widehat{B}_s^m, \mathrm{id}_{\mathsf{E}}) \in \mathrm{VB}^r(\mathrm{S}^s(\mathsf{T}^*\mathsf{E}); \mathrm{S}^m(\pi_{\mathsf{E}}^* \mathsf{T}^*\mathsf{M})), \qquad s \in \{0, 1, \dots, m\},$$

such that

$$\mathrm{Sym}_m \circ \pi_{\mathsf{E}}^* \nabla^{\mathsf{M},m} f = \sum_{s=0}^{m} \widehat{B}_s^m(\mathrm{Sym}_s \circ \nabla^{\mathsf{E},s} \pi_{\mathsf{E}}^* f)$$

for all $f \in \mathrm{C}^m(\mathsf{M})$.

Proof: We define $B^m \colon \mathrm{T}^{\le m}(\mathsf{T}^*\mathsf{E}) \to \mathrm{T}^{\le m}(\pi_{\mathsf{E}}^* \mathsf{T}^*\mathsf{M})$ by requiring that

$$B^m(\pi_{\mathsf{E}}^* f, \dots, \nabla^{\mathsf{E},m} \pi_{\mathsf{E}}^* f) = \left(B_0^0(\pi_{\mathsf{E}}^* f), \sum_{s=0}^{1} B_s^1(\nabla^{\mathsf{E},s} \pi_{\mathsf{E}}^* f), \dots, \sum_{s=0}^{m} B_s^m(\nabla^{\mathsf{E},m} \pi_{\mathsf{E}}^* f)\right),$$

as in Lemma 5.2. Note that the mapping

$$\mathrm{id}_{\mathbb{R}} \oplus j_m \pi_{\mathsf{E}} \colon \pi_{\mathsf{E}}^*(\mathbb{R}_{\mathsf{M}} \oplus \mathsf{T}^{*m}\mathsf{M}) \to \mathsf{P}^{*m}\mathsf{E}$$

is well-defined and a vector bundle isomorphism. Let us organise the mappings we require into the following diagram:

$$\begin{array}{ccccc} \mathrm{T}^{\le m}(\mathsf{T}^*\mathsf{E}) & \xrightarrow{\mathrm{Sym}_{\le m}} & \mathrm{S}^{\le m}(\mathsf{T}^*\mathsf{E}) & \xrightarrow{S^m_{\nabla^{\mathsf{E}}}} & \mathsf{P}^{*m}\mathsf{E} \\ \downarrow{\scriptstyle B^m} & & \downarrow{\scriptstyle \widehat{B}^m} & & \uparrow{\scriptstyle \mathrm{id}_{\mathbb{R}} \oplus j_m \pi_{\mathsf{E}}} \\ \mathrm{T}^{\le m}(\pi_{\mathsf{E}}^* \mathsf{T}^*\mathsf{M}) & \xrightarrow{\mathrm{Sym}_{\le m}} & \mathrm{S}^{\le m}(\pi_{\mathsf{E}}^* \mathsf{T}^*\mathsf{M}) & \xrightarrow{S^m_{\nabla^{\mathsf{M}}}} & \pi_{\mathsf{E}}^*(\mathbb{R}_{\mathsf{M}} \oplus \mathsf{T}^{*m}\mathsf{M}) \end{array} \tag{5.4}$$

Here $\widehat{B}^m$ is defined so that the right square commutes. We shall show that the left square also commutes. Indeed,

$$\begin{aligned}\widehat{B}^m \circ \mathrm{Sym}_{\leq m}(\pi_{\mathsf{E}}^* f, \nabla^{\mathsf{E}} \pi_{\mathsf{E}}^* f, \dots, \nabla^{\mathsf{E},m} \pi_{\mathsf{E}}^* f)& \\ &= (S^m_{\nabla^{\mathsf{M}}})^{-1} \circ (\mathrm{id}_{\mathbb{R}} \oplus j_m \pi_{\mathsf{E}})^{-1} \circ S^m_{\nabla^{\mathsf{E}}} \circ \mathrm{Sym}_{\leq m}(\pi_{\mathsf{E}}^* f, \nabla^{\mathsf{E}} \pi_{\mathsf{E}}^* f, \dots, \nabla^{\mathsf{E},m} \pi_{\mathsf{E}}^* f) \\ &= \mathrm{Sym}_{\leq m}(\pi_{\mathsf{E}}^* f, \pi_{\mathsf{E}}^* \nabla^{\mathsf{M}} f, \dots, \pi_{\mathsf{E}}^* \nabla^{\mathsf{M},m} f) \\ &= \mathrm{Sym}_{\leq m} \circ B^m(\pi_{\mathsf{E}}^* f, \nabla^{\mathsf{E}} \pi_{\mathsf{E}}^* f, \dots, \nabla^{\mathsf{E},m} \pi_{\mathsf{E}}^* f).\end{aligned}$$

Thus the diagram (5.4) commutes. Thus, if we define $\hat{B}^m_s$ so as to satisfy

$$\widehat{B}^m_s(\mathrm{Sym}_s \circ \nabla^{\mathsf{E},s} \pi_{\mathsf{E}}^* f) = \mathrm{Sym}_m \circ B^m_s(\nabla^{\mathsf{E},s} \pi_{\mathsf{E}}^* f),$$

then we have

$$\mathrm{Sym}_m \circ \pi_{\mathsf{E}}^* \nabla^{\mathsf{M},m} f = \sum_{s=0}^{m} \widehat{B}^m_s(\mathrm{Sym}_s \circ \nabla^{\mathsf{E},s} \pi_{\mathsf{E}}^* f),$$

as desired. ■

The following lemma provides two decompositions of $\mathsf{P}^{*m}\mathsf{E}$, one "downstairs" and one "upstairs," and the relationship between them. The assertion simply results from an examination of the preceding four lemmata.

5.5 Lemma: (Decomposition of jets of horizontal lifts of functions) *Let $r \in \{\infty, \omega\}$ and let $\pi_{\mathsf{E}} \colon \mathsf{E} \to \mathsf{M}$ be a C^r-vector bundle, with the data prescribed in Section 4.1 to define the Riemannian metric $\mathbb{G}_{\mathsf{E}}$ on E. Then there exist C^r-vector bundle mappings*

$$A^m_{\nabla^{\mathsf{E}}} \in \mathrm{VB}^r(\mathsf{P}^{*m}\mathsf{E}; \mathrm{S}^{\leq m}(\pi_{\mathsf{E}}^* \mathsf{T}^*\mathsf{M})), \quad B^m_{\nabla^{\mathsf{E}}} \in \mathrm{VB}^r(\mathsf{P}^{*m}\mathsf{E}; \mathrm{S}^{\leq m}(\mathsf{T}^*\mathsf{E})),$$

defined by

$$\begin{aligned} A^m_{\nabla^{\mathsf{E}}}(j_m(\pi_{\mathsf{E}}^* f)(e)) &= \mathrm{Sym}_{\leq m}(\pi_{\mathsf{E}}^* f(e), \pi_{\mathsf{E}}^* \nabla^{\mathsf{M}} f(e), \dots, \pi_{\mathsf{E}}^* \nabla^{\mathsf{M},m} f(e)), \\ B^m_{\nabla^{\mathsf{E}}}(j_m(\pi_{\mathsf{E}}^* f)(e)) &= \mathrm{Sym}_{\leq m}(\pi_{\mathsf{E}}^* f(e), \nabla^{\mathsf{E}} \pi_{\mathsf{E}}^* f(e), \dots, \nabla^{\mathsf{E},m} \pi_{\mathsf{E}}^* f(e)). \end{aligned}$$

Moreover, $A^m_{\nabla^{\mathsf{E}}}$ is an isomorphism, $B^m_{\nabla^{\mathsf{E}}}$ is injective, and

$$\begin{aligned} B^m_{\nabla^{\mathsf{E}}} \circ (A^m_{\nabla^{\mathsf{E}}})^{-1} \circ (\mathrm{Sym}_{\leq m}(\pi_{\mathsf{E}}^* f(e), \pi_{\mathsf{E}}^* \nabla^{\mathsf{M}} f(e), \dots, \pi_{\mathsf{E}}^* \nabla^{\mathsf{M},m} f(e))& \\ = \left(A^0_0(\pi_{\mathsf{E}}^* f(e)), \sum_{s=0}^{1} \widehat{A}^1_s(\mathrm{Sym}_s \circ \pi_{\mathsf{E}}^* \nabla^{\mathsf{M},s} f(e)), \dots, \sum_{s=0}^{m} \widehat{A}^m_s(\mathrm{Sym}_s \circ \pi_{\mathsf{E}}^* \nabla^{\mathsf{M},s} f(e)) \right)& \end{aligned}$$

and

$$\begin{aligned} A^m_{\nabla^{\mathsf{E}}} \circ (B^m_{\nabla^{\mathsf{E}}})^{-1} \circ \mathrm{Sym}_{\leq m}(\pi_{\mathsf{E}}^* f(e), \nabla^{\mathsf{E}} \pi_{\mathsf{E}}^* f(e), \dots, \nabla^{\mathsf{E},m} \pi_{\mathsf{E}}^* f(e))& \\ = \left(B^0_0(\pi_{\mathsf{E}}^* f(e)), \sum_{s=0}^{1} \widehat{B}^1_s(\mathrm{Sym}_s \circ \nabla^{\mathsf{E},s} \pi_{\mathsf{E}}^* f(e)), \dots, \sum_{s=0}^{m} \widehat{B}^m_s(\mathrm{Sym}_s \circ \nabla^{\mathsf{E},s} \pi_{\mathsf{E}}^* f(e)) \right),& \end{aligned}$$

where the vector bundle mappings $\widehat{A}^m_s$ and $\widehat{B}^m_s$, $s \in \{0, 1, \dots, m\}$, are as in Lemmata 5.3 and 5.4.

5.2. Isomorphisms for vertical lifts of sections. Next we consider vertical lifts of sections, i.e., the mapping

$$\Gamma^r(\mathsf{E}) \ni \xi \mapsto \xi^{\mathrm{v}} \in \Gamma^r(\mathsf{TE}).$$

We wish to relate the decomposition of the jets of ξ with those of ξ^{v}. Associated with this, we denote

$$\mathsf{V}_e^{*m}\mathsf{E} = \{j_m\xi^{\mathrm{v}}(e) \mid \xi \in \Gamma^m(\mathsf{E})\}.$$

By (1.5), we have

$$\mathsf{V}_e^{*m}\mathsf{E} \simeq \mathsf{P}_e^{*m}\mathsf{E} \otimes \mathsf{V}_e\mathsf{E}.$$

As with the constructions of the preceding section, we wish to use Lemma 2.1 to provide a decomposition of $\mathsf{V}^{*m}\mathsf{E}$, and to do so we need to understand the covariant derivatives

$$\nabla^{\mathsf{E},m}\xi^{\mathrm{v}} \triangleq \underbrace{\nabla^{\mathsf{E}} \cdots \nabla^{\mathsf{E}}}_{m \text{ times}}\xi^{\mathrm{v}}, \qquad m \in \mathbb{Z}_{\geq 0}.$$

In our development, we shall use the notation used in the preceding section in a slightly different, but similar, context. This seems reasonable since we have to do more or less the same thing six times, and using six different pieces of notation will be excessively burdensome.

The first result we give is the following.

5.6 Lemma: (Iterated covariant differentials of vertical lifts of sections I) *Let $r \in \{\infty, \omega\}$ and let $\pi_{\mathsf{E}} \colon \mathsf{E} \to \mathsf{M}$ be a C^r-vector bundle, with the data prescribed in Section 4.1 to define the Riemannian metric $\mathbb{G}_{\mathsf{E}}$ on E. For $m \in \mathbb{Z}_{\geq 0}$, there exist C^r-vector bundle mappings*

$$(A_s^m, \mathrm{id}_{\mathsf{E}}) \in \mathrm{VB}^r(\mathrm{T}^s(\pi_{\mathsf{E}}^*\mathsf{T}^*\mathsf{M}) \otimes \mathsf{VE}; \mathrm{T}^m(\mathsf{T}^*\mathsf{E}) \otimes \mathsf{VE}), \qquad s \in \{0, 1, \dots, m\},$$

such that

$$\nabla^{\mathsf{E},m}\xi^{\mathrm{v}} = \sum_{s=0}^{m} A_s^m((\nabla^{\mathsf{M},\pi_{\mathsf{E}},s}\xi)^{\mathrm{v}})$$

for all $\xi \in \Gamma^m(\mathsf{E})$. Moreover, the vector bundle mappings $A_0^m, A_1^m, \dots, A_m^m$ satisfy the recursion relations prescribed by $A_0^0(\beta_0) = \beta_0$ and

$$\begin{aligned}
A_{m+1}^{m+1}(\beta_{m+1}) &= \beta_{m+1},\\
A_s^{m+1}(\beta_s) &= (\nabla^{\mathsf{E}}A_s^m)(\beta_s) + A_{s-1}^m \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\beta_s) - \sum_{j=1}^{s} A_s^m \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\mathrm{Ins}_j(\beta_s, B_{\pi_{\mathsf{E}}}))\\
&\quad + A_s^m \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\mathrm{Ins}_{s+1}(\beta_s, B_{\pi_{\mathsf{E}}}^*)), \quad s \in \{1, \dots, m\},\\
A_0^{m+1}(\beta_0) &= (\nabla^{\mathsf{E}}A_0^m)(\beta_0) + A_0^m \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\mathrm{Ins}_1(\beta_0, B_{\pi_{\mathsf{E}}}^*)),
\end{aligned}$$

where $\beta_s \in \mathrm{T}^s(\pi_{\mathsf{E}}^*\mathsf{T}^*\mathsf{M}) \otimes \mathsf{VE}$, $s \in \{0, 1, \dots, m+1\}$.

Proof: The assertion clearly holds for $m = 0$, so suppose it true for $m \in \mathbb{Z}_{>0}$. Thus

$$\nabla^{\mathsf{E},m}\xi^{\mathrm{v}} = \sum_{s=0}^{m} A_s^m((\nabla^{\mathsf{M},\pi_{\mathsf{E}},s}\xi)^{\mathrm{v}}),$$

where the vector bundle mappings A^a_s, $a \in \{0,1,\dots,m\}$, $s \in \{0,1,\dots,a\}$, satisfy the recursion relations from the statement of the lemma. Then

$$\begin{aligned}
\nabla^{\mathsf{E},m+1}\xi^{\mathrm{v}} &= \sum_{s=0}^{m}(\nabla^{\mathsf{E}}A^m_s)((\nabla^{\mathsf{M},\pi_{\mathsf{E}},s}\xi)^{\mathrm{v}}) + \sum_{s=0}^{m} A^m_s \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\nabla^{\mathsf{E}}(\nabla^{\mathsf{M},\pi_{\mathsf{E}},s}\xi)^{\mathrm{v}}) \\
&= \sum_{s=0}^{m}(\nabla^{\mathsf{E}}A^m_s)((\nabla^{\mathsf{M},\pi_{\mathsf{E}},s}\xi)^{\mathrm{v}}) + \sum_{s=0}^{m} A^m_s \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}((\nabla^{\mathsf{M},\pi_{\mathsf{E}},s+1}\xi)^{\mathrm{v}}) \\
&\quad - \sum_{s=1}^{m}\sum_{j=1}^{s} A^m_s \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\mathrm{Ins}_j((\nabla^{\mathsf{M},\pi_{\mathsf{E}},s}\xi)^{\mathrm{v}}, B_{\pi_{\mathsf{E}}})) \\
&\quad + \sum_{s=1}^{m} A^m_s \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\mathrm{Ins}_{s+1}((\nabla^{\mathsf{M},\pi_{\mathsf{E}},s}\xi)^{\mathrm{v}}, B^*_{\pi_{\mathsf{E}}})) + A^m_0 \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\mathrm{Ins}_1(\xi^{\mathrm{v}}, B^*_{\pi_{\mathsf{E}}})) \\
&= (\nabla^{\mathsf{M},\pi_{\mathsf{E}},m+1}\xi)^{\mathrm{v}} + \sum_{s=1}^{m}\Bigg((\nabla^{\mathsf{E}}A^m_s)((\nabla^{\mathsf{M},\pi_{\mathsf{E}},s}\xi)^{\mathrm{v}}) + A^m_{s-1} \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}((\nabla^{\mathsf{M},\pi_{\mathsf{E}},s}\xi)^{\mathrm{v}}) \\
&\quad - \sum_{j=1}^{s} A^m_s \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\mathrm{Ins}_j((\nabla^{\mathsf{M},\pi_{\mathsf{E}},s}\xi)^{\mathrm{v}}, B_{\pi_{\mathsf{E}}})) + A^m_s \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\mathrm{Ins}_{s+1}((\nabla^{\mathsf{M},\pi_{\mathsf{E}},s}\xi)^{\mathrm{v}}, B^*_{\pi_{\mathsf{E}}}))\Bigg) \\
&\quad + (\nabla^{\mathsf{E}}A^m_0)(\xi^{\mathrm{v}}) + A^m_0 \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\mathrm{Ins}_1(\xi^{\mathrm{v}}, B^*_{\pi_{\mathsf{E}}}))
\end{aligned}$$

by Lemma 4.5(ii). From this, the lemma follows. ■

Now we "invert" the constructions from the preceding lemma.

5.7 Lemma: (Iterated covariant differentials of vertical lifts of sections II) *Let $r \in \{\infty,\omega\}$ and let $\pi_{\mathsf{E}}\colon \mathsf{E} \to \mathsf{M}$ be a C^r-vector bundle, with the data prescribed in Section 4.1 to define the Riemannian metric $\mathbb{G}_{\mathsf{E}}$ on E. For $m \in \mathbb{Z}_{\ge 0}$, there exist C^r-vector bundle mappings*

$$(B^m_s, \mathrm{id}_{\mathsf{E}}) \in \mathrm{VB}^r(\mathrm{T}^m(\mathsf{T}^*\mathsf{E}) \otimes \mathsf{VE}; \mathrm{T}^m(\pi^*_{\mathsf{E}}\mathsf{T}^*\mathsf{M}) \otimes \mathsf{VE}), \qquad s \in \{0,1,\dots,m\},$$

such that

$$(\nabla^{\mathsf{M},\pi_{\mathsf{E}},m}\xi)^{\mathrm{v}} = \sum_{s=0}^{m} B^m_s(\nabla^{\mathsf{E},s}\xi^{\mathrm{v}})$$

for all $\xi \in \Gamma^m(\mathsf{E})$. Moreover, the vector bundle mappings $B^m_0, B^m_1, \dots, B^m_m$ satisfy the recursion relations prescribed by $B^0_0(\alpha_0) = \alpha_0$ and

$$\begin{aligned}
B^{m+1}_{m+1}(\alpha_{m+1}) &= \alpha_{m+1}, \\
B^{m+1}_s(\alpha_s) &= (\nabla^{\mathsf{E}}B^m_s)(\alpha_s) + B^m_{s-1} \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\alpha_s) + \sum_{j=1}^{m} \mathrm{Ins}_j(B^m_s(\alpha_s), B_{\pi_{\mathsf{E}}}) \\
&\qquad - \mathrm{Ins}_{m+1}(B^m_s(\alpha_s), B^*_{\pi_{\mathsf{E}}}), \quad s \in \{1,\dots,m\}, \\
B^{m+1}_0(\alpha_0) &= (\nabla^{\mathsf{E}}B^m_0)(\alpha_0) + \sum_{j=1}^{m} \mathrm{Ins}_j(B^m_0(\alpha_0), B_{\pi_{\mathsf{E}}}) - \mathrm{Ins}_{m+1}(B^m_0(\alpha_0), B^*_{\pi_{\mathsf{E}}}),
\end{aligned}$$

where $\alpha_s \in \mathrm{T}^s(\mathsf{T}^*\mathsf{E}) \otimes \mathsf{VE}$, $s \in \{0,1,\dots,m+1\}$.

Proof: The assertion clearly holds for $m = 0$, so suppose it true for $m \in \mathbb{Z}_{>0}$. Thus

$$(\nabla^{\mathsf{M},\pi_{\mathsf{E}},m}\xi)^{\mathrm{v}} = \sum_{s=0}^{m} B_s^m(\nabla^{\mathsf{E},s}\xi^{\mathrm{v}}), \tag{5.5}$$

where the vector bundle mappings B_s^a, $a \in \{0, 1, \ldots, m\}$, $s \in \{0, 1, \ldots, a\}$, satisfy the recursion relations from the statement of the lemma. Then, by Lemma 4.5(ii), we can work on the left-hand side of (5.5) to give

$$\begin{aligned}\nabla^{\mathsf{E}}(\nabla^{\mathsf{M},\pi_{\mathsf{E}},m}\xi)^{\mathrm{v}} &= (\nabla^{\mathsf{M},\pi_{\mathsf{E}},m+1}\xi)^{\mathrm{v}} - \sum_{j=1}^{m}\mathrm{Ins}_j((\nabla^{\mathsf{M},\pi_{\mathsf{E}},m}\xi)^{\mathrm{v}}, B_{\pi_{\mathsf{E}}}) + \mathrm{Ins}_{m+1}((\nabla^{\mathsf{M},\pi_{\mathsf{E}},m}\xi)^{\mathrm{v}}, B^*_{\pi_{\mathsf{E}}})\\ &= (\nabla^{\mathsf{M},\pi_{\mathsf{E}},m+1}\xi)^{\mathrm{v}} - \sum_{s=0}^{m}\sum_{j=1}^{m}\mathrm{Ins}_j(B_s^m(\nabla^{\mathsf{E},s}\xi^{\mathrm{v}}), B_{\pi_{\mathsf{E}}}) + \sum_{s=0}^{m}\mathrm{Ins}_{m+1}(B_s^m(\nabla^{\mathsf{E},s}\xi^{\mathrm{v}}), B^*_{\pi_{\mathsf{E}}}).\end{aligned}$$

Working on the right-hand side of (5.5) gives

$$\nabla^{\mathsf{E}}(\nabla^{\mathsf{M},\pi_{\mathsf{E}},m}\xi)^{\mathrm{v}} = \sum_{s=0}^{m}\nabla^{\mathsf{E}}B_s^m(\nabla^{\mathsf{E},s}\xi^{\mathrm{v}}) + \sum_{s=0}^{m} B_s^m \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\nabla^{\mathsf{E},s+1}\xi^{\mathrm{v}}).$$

Combining the preceding two equations gives

$$\begin{aligned}\nabla^{\mathsf{M},\pi_{\mathsf{E}},m+1}\xi^{\mathrm{v}} = &\sum_{s=0}^{m}\nabla^{\mathsf{E}}B_s^m(\nabla^{\mathsf{E},s}\xi^{\mathrm{v}}) + \sum_{s=0}^{m} B_s^m \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\nabla^{\mathsf{E},s+1}\xi^{\mathrm{v}})\\ &+ \sum_{s=0}^{m}\sum_{j=1}^{m}\mathrm{Ins}_j(B_s^m(\nabla^{\mathsf{E},s}\xi^{\mathrm{v}}), B_{\pi_{\mathsf{E}}}) - \mathrm{Ins}_{m+1}((\nabla^{\mathsf{M},\pi_{\mathsf{E}},m}\xi)^{\mathrm{v}}, B^*_{\pi_{\mathsf{E}}})\\ = \nabla^{\mathsf{E},m+1}\xi^{\mathrm{v}} &+ \sum_{s=1}^{m}\Bigg(\nabla^{\mathsf{E}}B_s^m(\nabla^{\mathsf{E},s}\xi^{\mathrm{v}}) + B_{s-1}^m \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\nabla^{\mathsf{E},s}\xi^{\mathrm{v}})\\ &+ \sum_{j=1}^{m}\mathrm{Ins}_j(B_s^m(\nabla^{\mathsf{E},s}\xi^{\mathrm{v}}), B_{\pi_{\mathsf{E}}}) - \mathrm{Ins}_{m+1}(B_s^m(\nabla^{\mathsf{E},s}\xi^{\mathrm{v}}), B^*_{\pi_{\mathsf{E}}})\Bigg)\\ &+ \nabla^{\mathsf{E}}B_0^m(\xi^{\mathrm{v}}) + \sum_{j=1}^{m}\mathrm{Ins}_j(B_0^m(\xi^{\mathrm{v}}), B_{\pi_{\mathsf{E}}}) - \mathrm{Ins}_{m+1}(B_0^m(\xi^{\mathrm{v}}), B^*_{\pi_{\mathsf{E}}}),\end{aligned}$$

and the lemma follows from this. ■

Next we turn to symmetrised versions of the preceding lemmata. We show that the preceding two lemmata induce corresponding mappings between symmetric tensors.

5.8 Lemma: (Iterated symmetrised covariant differentials of vertical lifts of sections I) *Let $r \in \{\infty, \omega\}$ and let $\pi_{\mathsf{E}}\colon \mathsf{E} \to \mathsf{M}$ be a C^r-vector bundle, with the data prescribed in Section 4.1 to define the Riemannian metric $\mathbb{G}_{\mathsf{E}}$ on E. For $m \in \mathbb{Z}_{\geq 0}$, there exist C^r-vector bundle mappings*

$$(\widehat{A}_s^m, \mathrm{id}_{\mathsf{E}}) \in \mathrm{VB}^r(\mathrm{S}^s(\pi_{\mathsf{E}}^*\mathsf{T}^*\mathsf{M}) \otimes \mathsf{VE}; \mathrm{S}^m(\mathsf{T}^*\mathsf{E}) \otimes \mathsf{VE}), \qquad s \in \{0, 1, \ldots, m\},$$

such that

$$(\mathrm{Sym}_m \otimes \mathrm{id}_{\mathsf{VE}}) \circ \nabla^{\mathsf{E},m}\xi^{\mathrm{v}} = \sum_{s=0}^{m} \widehat{A}^m_s((\mathrm{Sym}_s \otimes \mathrm{id}_{\mathsf{VE}}) \circ (\nabla^{\mathsf{M},\pi_{\mathsf{E}},s}\xi)^{\mathrm{v}})$$

for all $\xi \in \Gamma^m(\mathsf{E})$*.*

Proof: The proof follows very similarly to that of Lemma 5.3, but taking the tensor product of everything with VE. We shall present the complete construction here, but will not repeat it for similar proofs that follow.

We define $A^m\colon \mathrm{T}^{\leq m}(\pi_{\mathsf{E}}^*\mathsf{T}^*\mathsf{M}) \otimes \mathsf{VE} \to \mathrm{T}^{\leq m}(\mathsf{T}^*\mathsf{E}) \otimes \mathsf{VE}$ by

$$A^m(\xi^{\mathrm{v}}, (\nabla^{\pi_{\mathsf{E}}}\xi)^{\mathrm{v}}, \dots, (\nabla^{\mathsf{M},\pi_{\mathsf{E}},m}\xi)^{\mathrm{v}}) = \left(A^0_0(\xi^{\mathrm{v}}), \sum_{s=0}^{1} A^1_s((\nabla^{\mathsf{M},\pi_{\mathsf{E}},s}\xi)^{\mathrm{v}}), \dots, \sum_{s=0}^{m} A^m_s((\nabla^{\mathsf{M},\pi_{\mathsf{E}},s}\xi)^{\mathrm{v}})\right)$$

Let us organise the mappings we require into the following diagram:

$$\begin{array}{ccccc}
\mathrm{T}^{\leq m}(\pi_{\mathsf{E}}^*\mathsf{T}^*\mathsf{M}) \otimes \mathsf{VE} & \xrightarrow{\mathrm{Sym}_{\leq m} \otimes \mathrm{id}_{\mathsf{VE}}} & \mathrm{S}^{\leq m}(\pi_{\mathsf{E}}^*\mathsf{T}^*\mathsf{M}) \otimes \mathsf{VE} & \xrightarrow{S^m_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}} \otimes \mathrm{id}_{\mathsf{VE}}} & \pi_{\mathsf{E}}^*(\mathbb{R}_{\mathsf{M}} \oplus \mathsf{T}^{*m}\mathsf{M}) \otimes \mathsf{VE} \\
\downarrow{\scriptstyle A^m} & & \downarrow{\scriptstyle \widehat{A}^m} & & \downarrow{\scriptstyle (\mathrm{id}_{\mathbb{R}} \oplus j_m\pi_{\mathsf{E}}) \otimes \mathrm{id}_{\mathsf{VE}}} \\
\mathrm{T}^{\leq m}(\mathsf{T}^*\mathsf{E}) \otimes \mathsf{VE} & \xrightarrow{\mathrm{Sym}_{\leq m} \otimes \mathrm{id}_{\mathsf{VE}}} & \mathrm{S}^{\leq m}(\mathsf{T}^*\mathsf{E}) \otimes \mathsf{VE} & \xrightarrow{S^m_{\nabla^{\mathsf{E}}} \otimes \mathrm{id}_{\mathsf{VE}}} & (\mathbb{R}_{\mathsf{M}} \oplus \mathsf{T}^{*m}\mathsf{E}) \otimes \mathsf{VE}
\end{array} \tag{5.6}$$

Here $\widehat{A}^m$ is defined so that the right square commutes. We shall show that the left square also commutes. Indeed,

$$\begin{aligned}
&\widehat{A}^m \circ \mathrm{Sym}_{\leq m} \otimes \mathrm{id}_{\mathsf{VE}}(\xi^{\mathrm{v}}, (\nabla^{\pi_{\mathsf{E}}}\xi)^{\mathrm{v}}, \dots, (\nabla^{\mathsf{M},\pi_{\mathsf{E}},m}\xi)^{\mathrm{v}}) \\
&\quad = (S^m_{\nabla^{\mathsf{E}}} \otimes \mathrm{id}_{\mathsf{VE}})^{-1} \circ ((\mathrm{id}_{\mathbb{R}} \otimes j_m\pi_{\mathsf{E}}) \otimes \mathrm{id}_{\mathsf{VE}}) \circ (S^m_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}} \otimes \mathrm{id}_{\mathsf{VE}}) \\
&\qquad \circ (\mathrm{Sym}_{\leq m} \otimes \mathrm{id}_{\mathsf{VE}})(\xi^{\mathrm{v}}, (\nabla^{\pi_{\mathsf{E}}}\xi)^{\mathrm{v}}, \dots, (\nabla^{\mathsf{M},\pi_{\mathsf{E}},m}\xi)^{\mathrm{v}}) \\
&\quad = \mathrm{Sym}_{\leq m} \otimes \mathrm{id}_{\mathsf{VE}}(\xi^{\mathrm{v}}, \nabla^{\mathsf{E}}\xi^{\mathrm{v}}, \dots, \nabla^{\mathsf{E},m}\xi^{\mathrm{v}}) \\
&\quad = (\mathrm{Sym}_{\leq m} \otimes \mathrm{id}_{\mathsf{VE}}) \circ A^m(\xi^{\mathrm{v}}, (\nabla^{\pi_{\mathsf{E}}}\xi)^{\mathrm{v}}, \dots, (\nabla^{\mathsf{M},\pi_{\mathsf{E}},m}\xi)^{\mathrm{v}}).
\end{aligned}$$

Thus the diagram (5.6) commutes. Thus, if we define

$$\widehat{A}^m_s((\mathrm{Sym}_s \otimes \mathrm{id}_{\mathsf{VE}}) \circ (\nabla^{\mathsf{M},\pi_{\mathsf{E}},s}\xi)^{\mathrm{v}}) = (\mathrm{Sym}_m \otimes \mathrm{id}_{\mathsf{VE}}) \circ A^m_s((\nabla^{\mathsf{M},\pi_{\mathsf{E}},s}\xi)^{\mathrm{v}}),$$

then we have

$$(\mathrm{Sym}_m \otimes \mathrm{id}_{\mathsf{VE}}) \circ \nabla^{\mathsf{E},m}\xi^{\mathrm{v}} = \sum_{s=0}^{m} \widehat{A}^m_s((\mathrm{Sym}_s \otimes \mathrm{id}_{\mathsf{VE}}) \circ (\nabla^{\mathsf{M},\pi_{\mathsf{E}},s}\xi)^{\mathrm{v}}),$$

as desired. ■

The preceding lemma gives rise to an "inverse," which we state in the following lemma.

5.9 Lemma: (Iterated symmetrised covariant differentials of vertical lifts of sections II) *Let $r \in \{\infty, \omega\}$ and let $\pi_{\mathsf{E}} \colon \mathsf{E} \to \mathsf{M}$ be a C^r-vector bundle, with the data prescribed in Section 4.1 to define the Riemannian metric $\mathbb{G}_{\mathsf{E}}$ on E. For $m \in \mathbb{Z}_{\geq 0}$, there exist C^r-vector bundle mappings*

$$(\widehat{B}^m_s, \mathrm{id}_{\mathsf{E}}) \in \mathrm{VB}^r(\mathrm{S}^s(\mathsf{T}^*\mathsf{E}) \otimes \mathsf{VE}; \mathrm{S}^m(\pi_{\mathsf{E}}^*\mathsf{T}^*\mathsf{M}) \otimes \mathsf{VE}), \qquad s \in \{0, 1, \dots, m\},$$

such that

$$(\mathrm{Sym}_m \otimes_{\mathrm{id}_{\mathsf{VE}}}) \circ (\nabla^{\mathsf{M},\pi_{\mathsf{E}},m}\xi)^{\mathrm{v}} = \sum_{s=0}^m \widehat{B}^m_s((\mathrm{Sym}_s \otimes \mathrm{id}_{\mathsf{VE}}) \circ \nabla^{\mathsf{E},s}\xi^{\mathrm{v}})$$

for all $\xi \in \Gamma^m(\mathsf{E})$.

Proof: This follows along the lines of Lemma 5.4 in the same manner as Lemma 5.8 follows from Lemma 5.3, by taking tensor products with VE. ■

We can put together the previous four lemmata into the following decomposition result, which is to be regarded as the main result of this section.

5.10 Lemma: (Decomposition of jets of vertical lifts of sections) *Let $r \in \{\infty, \omega\}$ and let $\pi_{\mathsf{E}} \colon \mathsf{E} \to \mathsf{M}$ be a C^r-vector bundle, with the data prescribed in Section 4.1 to define the Riemannian metric $\mathbb{G}_{\mathsf{E}}$ on E. Then there exist C^r-vector bundle mappings*

$$A^m_{\nabla^{\mathsf{E}}} \in \mathrm{VB}^r(\mathsf{P}^{*m}\mathsf{E} \otimes \mathsf{VE}; \mathrm{S}^{\leq m}(\pi_{\mathsf{E}}^*\mathsf{T}^*\mathsf{M}) \otimes \mathsf{VE}), \quad B^m_{\nabla^{\mathsf{E}}} \in \mathrm{VB}^r(\mathsf{P}^{*m}\mathsf{E} \otimes \mathsf{VE}; \mathrm{S}^{\leq m}(\mathsf{T}^*\mathsf{E}) \otimes \mathsf{VE}),$$

defined by

$$\begin{aligned} A^m_{\nabla^{\mathsf{E}}}(j_m(\xi^{\mathrm{v}})(e)) &= \mathrm{Sym}_{\leq m} \otimes \mathrm{id}_{\mathsf{VE}}(\xi^{\mathrm{v}}(e), (\nabla^{\pi_{\mathsf{E}}}\xi)^{\mathrm{v}}(e), \dots, (\nabla^{\mathsf{M},\pi_{\mathsf{E}},m}\xi)^{\mathrm{v}}(e)), \\ B^m_{\nabla^{\mathsf{E}}}(j_m(\xi^{\mathrm{v}})(e)) &= \mathrm{Sym}_{\leq m} \otimes \mathrm{id}_{\mathsf{VE}}(\xi^{\mathrm{v}}(e), \nabla^{\mathsf{E}}\xi^{\mathrm{v}}(e), \dots, \nabla^{\mathsf{E},m}\xi^{\mathrm{v}}(e)). \end{aligned}$$

Moreover, $A^m_{\nabla^{\mathsf{E}}}$ is an isomorphism, $B^m_{\nabla^{\mathsf{E}}}$ is injective, and

$$\begin{aligned} &B^m_{\nabla^{\mathsf{E}}} \circ (A^m_{\nabla^{\mathsf{E}}})^{-1} \circ (\mathrm{Sym}_{\leq m} \otimes \mathrm{id}_{\mathsf{VE}})(\xi^{\mathrm{v}}(e), (\nabla^{\pi_{\mathsf{E}}}\xi)^{\mathrm{v}}(e), \dots, (\nabla^{\mathsf{M},\pi_{\mathsf{E}},m}\xi)^{\mathrm{v}}(e)) \\ &= \left(\xi^{\mathrm{v}}(e), \sum_{s=0}^1 \widehat{A}^1_s((\mathrm{Sym}_s \otimes \mathrm{id}_{\mathsf{VE}}) \circ (\nabla^{\mathsf{M},\pi_{\mathsf{E}},s}\xi)^{\mathrm{v}}(e)), \dots, \sum_{s=0}^m \widehat{A}^m_s((\mathrm{Sym}_s \otimes \mathrm{id}_{\mathsf{VE}}) \circ (\nabla^{\mathsf{M},\pi_{\mathsf{E}},s}\xi)^{\mathrm{v}}(e))\right) \end{aligned}$$

and

$$\begin{aligned} &A^m_{\nabla^{\mathsf{E}}} \circ (B^m_{\nabla^{\mathsf{E}}})^{-1} \circ (\mathrm{Sym}_{\leq m} \otimes \mathrm{id}_{\mathsf{VE}})(\xi^{\mathrm{v}}(e), \nabla^{\mathsf{E}}\xi^{\mathrm{v}}(e), \dots, \nabla^{\mathsf{E},m}\xi^{\mathrm{v}}(e)) \\ &\qquad = \left(\xi^{\mathrm{v}}(e), \sum_{s=0}^1 \widehat{B}^1_s((\mathrm{Sym}_s \otimes \mathrm{id}_{\mathsf{VE}}) \circ \nabla^{\mathsf{E},s}\xi^{\mathrm{v}}(e)), \dots, \sum_{s=0}^m \widehat{B}^m_s((\mathrm{Sym}_s \otimes \mathrm{id}_{\mathsf{VE}}) \circ \nabla^{\mathsf{E},s}\xi^{\mathrm{v}}(e))\right), \end{aligned}$$

where the vector bundle mappings $\widehat{A}^m_s$ and $\widehat{B}^m_s$, $s \in \{0, 1, \dots, m\}$, are as in Lemmata 5.8 and 5.9.

5.3. Isomorphisms for horizontal lifts of vector fields. Next we consider horizontal lifts of vector fields via the mapping

$$\Gamma^r(\mathsf{TM}) \ni X \mapsto X^{\mathrm{h}} \in \Gamma^r(\mathsf{TE}).$$

We wish to relate the decomposition of the jets of X with the jets of X^{h}. Associated with this, we denote

$$\mathsf{H}^{*m}_e\mathsf{E} = \{j_m X^{\mathrm{h}}(e) \mid\ X \in \Gamma^m(\mathsf{TM})\}.$$

By (1.5), we have

$$\mathsf{H}^{*m}_e\mathsf{E} \simeq \mathsf{P}^{*m}_e\mathsf{E} \otimes \mathsf{H}_e\mathsf{E}.$$

As with the constructions of the preceding sections, we wish to use Lemma 2.1 to provide a decomposition of $\mathsf{H}^{*m}\mathsf{E}$, and to do so we need to understand the covariant derivatives

$$\nabla^{\mathsf{E},m} X^{\mathrm{h}} \triangleq \underbrace{\nabla^{\mathsf{E}} \cdots \nabla^{\mathsf{E}}}_{m \text{ times}} X^{\mathrm{h}}, \qquad m \in \mathbb{Z}_{\geq 0}.$$

In this section we omit proofs, since proofs follow along entirely similar lines to those of the preceding section.

The first result we give is the following.

5.11 Lemma: (Iterated covariant differentials of horizontal lifts of vector fields I) *Let $r \in \{\infty, \omega\}$ and let $\pi_{\mathsf{E}}\colon \mathsf{E} \to \mathsf{M}$ be a C^r-vector bundle, with the data prescribed in Section 4.1 to define the Riemannian metric $\mathbb{G}_{\mathsf{E}}$ on E. For $m \in \mathbb{Z}_{\geq 0}$, there exist C^r-vector bundle mappings*

$$(A^m_s, \mathrm{id}_{\mathsf{E}}) \in \mathrm{VB}^r(\mathrm{T}^s(\pi^*_{\mathsf{E}}\mathsf{T}^*\mathsf{M}) \otimes \mathsf{HE}; \mathrm{T}^m(\mathsf{T}^*\mathsf{E}) \otimes \mathsf{HE}), \qquad s \in \{0, 1, \dots, m\},$$

such that

$$\nabla^{\mathsf{E},m} X^{\mathrm{h}} = \sum_{s=0}^{m} A^m_s((\nabla^{\mathsf{M},s} X)^{\mathrm{h}})$$

for all $X \in \Gamma^m(\mathsf{TM})$. Moreover, the vector bundle mappings $A^m_0, A^m_1, \dots, A^m_m$ satisfy the recursion relations prescribed by $A^0_0(\beta_0) = \beta_0$ and

$$\begin{aligned} A^{m+1}_{m+1}(\beta_{m+1}) &= \beta_{m+1}, \\ A^{m+1}_s(\beta_s) &= (\nabla^{\mathsf{E}} A^m_s)(\beta_s) + A^m_{s-1} \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\beta_s) - \sum_{j=1}^{s} A^m_s \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\mathrm{Ins}_j(\beta_s, B_{\pi_{\mathsf{E}}})) \\ &\quad + A^m_s \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\mathrm{Ins}_{s+1}(\beta_s, B^*_{\pi_{\mathsf{E}}})), \quad s \in \{1, \dots, m\}, \\ A^{m+1}_0(\beta_0) &= (\nabla^{\mathsf{E}} A^m_0)(\beta_0) + A^m_0 \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\mathrm{Ins}_1(\beta_0, B^*_{\pi_{\mathsf{E}}})), \end{aligned}$$

where $\beta_s \in \mathrm{T}^s(\pi^*_{\mathsf{E}}\mathsf{T}^*\mathsf{M}) \otimes \mathsf{HE}$, $s \in \{0, 1, \dots, m+1\}$.

Proof: This follows in the same manner as Lemma 5.6, making use of Lemma 4.5(iii). ■

The following lemma “inverts” the relations from the preceding one.

5.12 Lemma: (Iterated covariant differentials of horizontal lifts of vector fields II) *Let $r \in \{\infty, \omega\}$ and let $\pi_{\mathsf{E}} \colon \mathsf{E} \to \mathsf{M}$ be a C^r-vector bundle, with the data prescribed in Section 4.1 to define the Riemannian metric $\mathbb{G}_{\mathsf{E}}$ on E. For $m \in \mathbb{Z}_{\geq 0}$, there exist C^r-vector bundle mappings*

$$(B^m_s, \mathrm{id}_{\mathsf{E}}) \in \mathrm{VB}^r(\mathrm{T}^s(\mathsf{T}^*\mathsf{E}) \otimes \mathsf{HE}; \mathrm{T}^m(\pi^*_{\mathsf{E}}\mathsf{T}^*\mathsf{M}) \otimes \mathsf{HE}), \qquad s \in \{0, 1, \dots, m\},$$

such that

$$(\nabla^{\mathsf{M},m} X)^{\mathrm{h}} = \sum_{s=0}^{m} B^m_s(\nabla^{\mathsf{E},s} X^{\mathrm{h}})$$

for all $X \in \Gamma^m(\mathsf{TM})$. Moreover, the vector bundle mappings $B^m_0, B^m_1, \dots, B^m_m$ satisfy the recursion relations prescribed by $B^0_0(\alpha_0) = \alpha_0$ and

$$\begin{aligned}
B^{m+1}_{m+1}(\alpha_{m+1}) &= \alpha_{m+1},\\
B^{m+1}_s(\alpha_s) &= (\nabla^{\mathsf{E}} B^m_s)(\alpha_s) + B^m_{s-1} \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\alpha_s) + \sum_{j=1}^{m} \mathrm{Ins}_j(B^m_s(\alpha_s), B_{\pi_{\mathsf{E}}})\\
&\qquad - \mathrm{Ins}_{m+1}(B^m_s(\alpha_s), B^*_{\pi_{\mathsf{E}}}), \quad s \in \{1, \dots, m\},\\
B^{m+1}_0(\alpha_0) &= (\nabla^{\mathsf{E}} B^m_0)(\alpha_0) + \sum_{j=1}^{m} \mathrm{Ins}_j(B^m_0(\alpha_0), B_{\pi_{\mathsf{E}}}) - \mathrm{Ins}_{m+1}(B^m_0(\alpha_0), B^*_{\pi_{\mathsf{E}}}),
\end{aligned}$$

where $\alpha_s \in \mathrm{T}^s(\mathsf{T}^*\mathsf{E}) \otimes \mathsf{HE}$, $s \in \{0, 1, \dots, m+1\}$.

Proof: This follows in the same manner as Lemma 5.7, making use of Lemma 4.5(iii). ■

Now we can give the symmetrised versions of the preceding lemmata.

5.13 Lemma: (Iterated symmetrised covariant differentials of horizontal lifts of vector fields I) *Let $r \in \{\infty, \omega\}$ and let $\pi_{\mathsf{E}} \colon \mathsf{E} \to \mathsf{M}$ be a C^r-vector bundle, with the data prescribed in Section 4.1 to define the Riemannian metric $\mathbb{G}_{\mathsf{E}}$ on E. For $m \in \mathbb{Z}_{\geq 0}$, there exist C^r-vector bundle mappings*

$$(\widehat{A}^m_s, \mathrm{id}_{\mathsf{E}}) \in \mathrm{VB}^r(\mathrm{S}^s(\pi^*_{\mathsf{E}}\mathsf{T}^*\mathsf{M}) \otimes \mathsf{HE}; \mathrm{S}^m(\mathsf{T}^*\mathsf{E}) \otimes \mathsf{HE}), \qquad s \in \{0, 1, \dots, m\},$$

such that

$$(\mathrm{Sym}_m \otimes \mathrm{id}_{\mathsf{HE}}) \circ \nabla^{\mathsf{E},m} X^{\mathrm{h}} = \sum_{s=0}^{m} \widehat{A}^m_s((\mathrm{Sym}_s \otimes \mathrm{id}_{\mathsf{HE}}) \circ (\nabla^{\mathsf{M},s} X)^{\mathrm{h}})$$

for all $X \in \Gamma^m(\mathsf{TM})$.

Proof: This follows along the lines of Lemma 5.3 in the same manner as Lemma 5.8 follows from Lemma 5.3, by taking tensor products with HE. ■

5.14 Lemma: (Iterated symmetrised covariant differentials of horizontal lifts of vector fields II) *Let $r \in \{\infty, \omega\}$ and let $\pi_{\mathsf{E}} \colon \mathsf{E} \to \mathsf{M}$ be a C^r-vector bundle, with the data prescribed in Section 4.1 to define the Riemannian metric $\mathbb{G}_{\mathsf{E}}$ on E. For $m \in \mathbb{Z}_{\geq 0}$, there exist C^r-vector bundle mappings*

$$(\widehat{B}^m_s, \mathrm{id}_{\mathsf{E}}) \in \mathrm{VB}^r(\mathrm{S}^s(\mathsf{T}^*\mathsf{E}) \otimes \mathsf{HE}; \mathrm{S}^m(\pi_{\mathsf{E}}^*\mathsf{T}^*\mathsf{M}) \otimes \mathsf{HE}), \qquad s \in \{0, 1, \dots, m\},$$

such that

$$(\mathrm{Sym}_m \otimes_{\mathrm{id}_{\mathsf{HE}}}) \circ (\nabla^{\mathsf{M},m} X)^{\mathrm{h}} = \sum_{s=0}^{m} \widehat{B}^m_s((\mathrm{Sym}_s \otimes \mathrm{id}_{\mathsf{HE}}) \circ \nabla^{\mathsf{E},s} X^{\mathrm{h}})$$

for all $X \in \Gamma^m(\mathsf{TM})$.

Proof: This follows along the lines of Lemma 5.4 in the same manner as Lemma 5.8 follows from Lemma 5.3, by taking tensor products with HE. ■

We can put together the previous four lemmata into the following decomposition result, which is to be regarded as the main result of this section.

5.15 Lemma: (Decomposition of jets of horizontal lifts of vector fields) *Let $r \in \{\infty, \omega\}$ and let $\pi_{\mathsf{E}} \colon \mathsf{E} \to \mathsf{M}$ be a C^r-vector bundle, with the data prescribed in Section 4.1 to define the Riemannian metric $\mathbb{G}_{\mathsf{E}}$ on E. Then there exist C^r-vector bundle mappings*

$$A^m_{\nabla^{\mathsf{E}}} \in \mathrm{VB}^r(\mathsf{P}^{*m}\mathsf{E} \otimes \mathsf{HE}; \mathrm{S}^{\leq m}(\pi_{\mathsf{E}}^*\mathsf{T}^*\mathsf{M}) \otimes \mathsf{HE}), \quad B^m_{\nabla^{\mathsf{E}}} \in \mathrm{VB}^r(\mathsf{P}^{*m}\mathsf{E} \otimes \mathsf{HE}; \mathrm{S}^{\leq m}(\mathsf{T}^*\mathsf{E}) \otimes \mathsf{HE}),$$

defined by

$$\begin{aligned} A^m_{\nabla^{\mathsf{E}}}(j_m(X^{\mathrm{h}})(e)) &= \mathrm{Sym}_{\leq m} \otimes \mathrm{id}_{\mathsf{HE}}(X^{\mathrm{h}}(e), (\nabla^{\mathsf{M}} X)^{\mathrm{h}}(e), \dots, (\nabla^{\mathsf{M},m} X)^{\mathrm{h}}(e)), \\ B^m_{\nabla^{\mathsf{E}}}(j_m(X^{\mathrm{h}})(e)) &= \mathrm{Sym}_{\leq m} \otimes \mathrm{id}_{\mathsf{HE}}(X^{\mathrm{h}}(e), \nabla^{\mathsf{E}} X^{\mathrm{h}}(e), \dots, \nabla^{\mathsf{E},m} X^{\mathrm{h}}(e)). \end{aligned}$$

Moreover, $A^m_{\nabla^{\mathsf{E}}}$ is an isomorphism, $B^m_{\nabla^{\mathsf{E}}}$ is injective, and

$$\begin{aligned} & B^m_{\nabla^{\mathsf{E}}} \circ (A^m_{\nabla^{\mathsf{E}}})^{-1} \circ (\mathrm{Sym}_{\leq m} \otimes_{\mathrm{id}_{\mathsf{HE}}})(X^{\mathrm{h}}(e), (\nabla^{\mathsf{M}} X)^{\mathrm{h}}(e), \dots, (\nabla^{\mathsf{M},m} X)^{\mathrm{h}}(e)) \\ &\quad = \left(X^{\mathrm{h}}(e), \sum_{s=0}^{1} \widehat{A}^1_s((\mathrm{Sym}_s \otimes \mathrm{id}_{\mathsf{HE}}) \circ (\nabla^{\mathsf{M},s} X)(e)), \dots, \sum_{s=0}^{m} \widehat{A}^m_s((\mathrm{Sym}_s \otimes \mathrm{id}_{\mathsf{HE}}) \circ (\nabla^{\mathsf{M},s} X)^{\mathrm{h}}(e)) \right) \end{aligned}$$

and

$$\begin{aligned} & A^m_{\nabla^{\mathsf{E}}} \circ (B^m_{\nabla^{\mathsf{E}}})^{-1} \circ (\mathrm{Sym}_{\leq m} \otimes \mathrm{id}_{\mathsf{HE}})(X^{\mathrm{h}}(e), \nabla^{\mathsf{E}} X^{\mathrm{h}}(e), \dots, \nabla^{\mathsf{E},m} X^{\mathrm{h}}(e)) \\ &\quad = \left(X^{\mathrm{h}}(e), \sum_{s=0}^{1} \widehat{B}^1_s((\mathrm{Sym}_s \otimes \mathrm{id}_{\mathsf{HE}}) \circ \nabla^{\mathsf{E},s} X^{\mathrm{h}}(e)), \dots, \sum_{s=0}^{m} \widehat{B}^m_s((\mathrm{Sym}_s \otimes \mathrm{id}_{\mathsf{HE}}) \circ \nabla^{\mathsf{E},s} X^{\mathrm{h}}(e)) \right), \end{aligned}$$

where the vector bundle mappings $\widehat{A}^m_s$ and $\widehat{B}^m_s$, $s \in \{0, 1, \dots, m\}$, are as in Lemmata 5.13 and 5.14.

5.4. Isomorphisms for vertical lifts of dual sections. Next we consider vertical lifts of sections of the dual bundle, i.e., the mapping defined by

$$\Gamma^r(\mathsf{E}^*) \ni \lambda \mapsto \lambda^{\mathrm{v}} \in \Gamma^r(\mathsf{T}^*\mathsf{E}).$$

Our objective is to relate the decomposition of the jets of λ with the decomposition of the jets of λ^{v}. To do this, we denote

$$\mathsf{F}^{*m}_e\mathsf{E} = \{j_m\lambda^{\mathrm{v}}(e) \mid \lambda \in \Gamma^m(\mathsf{E}^*)\}.$$

By (1.5), we have

$$\mathsf{F}^{*m}_e\mathsf{E} \simeq \mathsf{P}^{*m}_e\mathsf{E} \otimes \mathsf{V}^*_e\mathsf{E}.$$

As with the constructions of the preceding sections, we wish to use Lemma 2.1 to provide a decomposition of $\mathsf{F}^{*m}\mathsf{E}$, and to do so we need to understand the covariant derivatives

$$\nabla^{\mathsf{E},m}\lambda^{\mathrm{v}} \triangleq \underbrace{\nabla^{\mathsf{E}}\cdots\nabla^{\mathsf{E}}}_{m\ \text{times}}\lambda^{\mathrm{v}}, \qquad m \in \mathbb{Z}_{\geq 0}.$$

In this section we omit proofs, since proofs follow along entirely similar lines to those of the preceding section.

The first result we give is the following.

5.16 Lemma: (Iterated covariant differentials of vertical lifts of dual sections I) *Let $r \in \{\infty, \omega\}$ and let $\pi_{\mathsf{E}}\colon \mathsf{E} \to \mathsf{M}$ be a C^r-vector bundle, with the data prescribed in Section 4.1 to define the Riemannian metric $\mathbb{G}_{\mathsf{E}}$ on E. For $m \in \mathbb{Z}_{\geq 0}$, there exist C^r-vector bundle mappings*

$$(A^m_s, \mathrm{id}_{\mathsf{E}}) \in \mathrm{VB}^r(\mathrm{T}^s(\pi^*_{\mathsf{E}}\mathsf{T}^*\mathsf{M}) \otimes \mathsf{V}^*\mathsf{E}; \mathrm{T}^m(\mathsf{T}^*\mathsf{E}) \otimes \mathsf{V}^*\mathsf{E}), \qquad s \in \{0, 1, \dots, m\},$$

such that

$$\nabla^{\mathsf{E},m}\lambda^{\mathrm{v}} = \sum_{s=0}^{m} A^m_s((\nabla^{\mathsf{M},\pi_{\mathsf{E}},s}\lambda)^{\mathrm{v}})$$

for all $\lambda \in \Gamma^m(\mathsf{E}^*)$. Moreover, the vector bundle mappings $A^m_0, A^m_1, \dots, A^m_m$ satisfy the recursion relations prescribed by $A^0_0(\beta_0) = \beta_0$ and

$$\begin{aligned} &A^{m+1}_{m+1}(\beta_{m+1}) = \beta_{m+1},\\ &\quad A^{m+1}_s(\beta_s) = (\nabla^{\mathsf{E}}A^m_s)(\beta_s) + A^m_{s-1} \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\beta_s) - \sum_{j=1}^{s} A^m_s \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\mathrm{Ins}_j(\beta_s, B_{\pi_{\mathsf{E}}})),\ \ s \in \{1, \dots, m\},\\ &\quad A^{m+1}_0(\beta_0) = (\nabla^{\mathsf{E}}A^m_0)(\beta_0) - A^m_0 \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\mathrm{Ins}_1(\beta_0, B_{\pi_{\mathsf{E}}})), \end{aligned}$$

where $\beta_s \in \mathrm{T}^s(\pi^*_{\mathsf{E}}\mathsf{T}^*\mathsf{M}) \otimes \mathsf{V}^*\mathsf{E}$, $s \in \{0, 1, \dots, m+1\}$.

Proof: This follows in the same manner as Lemma 5.6, making use of Lemma 4.5(iv). ■

The “inverse” of the preceding lemma is as follows.

5.17 Lemma: (Iterated covariant differentials of vertical lifts of dual sections II) *Let* $r \in \{\infty, \omega\}$ *and let* $\pi_{\mathsf{E}}\colon \mathsf{E} \to \mathsf{M}$ *be a* C^r*-vector bundle, with the data prescribed in Section 4.1 to define the Riemannian metric* $\mathbb{G}_{\mathsf{E}}$ *on* E*. For* $m \in \mathbb{Z}_{\geq 0}$*, there exist* C^r*-vector bundle mappings*

$$(B^m_s, \mathrm{id}_{\mathsf{E}}) \in \mathrm{VB}^r(\mathrm{T}^s(\mathsf{T}^*\mathsf{E}) \otimes \mathsf{V}^*\mathsf{E}; \mathrm{T}^m(\pi^*_{\mathsf{E}}\mathsf{T}^*\mathsf{M}) \otimes \mathsf{V}^*\mathsf{E}), \qquad s \in \{0, 1, \dots, m\},$$

such that

$$(\nabla^{\mathsf{M},\pi_{\mathsf{E}},m}\lambda)^{\mathrm{v}} = \sum_{s=0}^{m} B^m_s(\nabla^{\mathsf{E},s}\lambda^{\mathrm{v}})$$

for all $\lambda \in \Gamma^m(\mathsf{E}^*)$*. Moreover, the vector bundle mappings* $B^m_0, B^m_1, \dots, B^m_m$ *satisfy the recursion relations prescribed by* $B^0_0(\alpha_0) = \alpha_0$ *and*

$$\begin{aligned}
&B^{m+1}_{m+1}(\alpha_{m+1}) = \alpha_{m+1},\\
&\quad B^{m+1}_s(\alpha_s) = (\nabla^{\mathsf{E}} B^m_s)(\alpha_s) + B^m_{s-1} \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\alpha_s) + \sum_{j=1}^{m} \mathrm{Ins}_j(B^m_s(\alpha_s), B_{\pi_{\mathsf{E}}}), \ \ s \in \{1, \dots, m\},\\
&\quad B^{m+1}_0(\alpha_0) = (\nabla^{\mathsf{E}} B^m_0)(\alpha_0) + \sum_{j=1}^{m+1} \mathrm{Ins}_j(B^m_0(\alpha_0), B_{\pi_{\mathsf{E}}}),
\end{aligned}$$

where $\alpha_s \in \mathrm{T}^s(\mathsf{T}^*\mathsf{E}) \otimes \mathsf{V}^*\mathsf{E}$*,* $s \in \{0, 1, \dots, m+1\}$*.*

Proof: This follows in the same manner as Lemma 5.7, making use of Lemma 4.5(iv). ■

Next we turn to symmetrised versions of the preceding lemmata. We show that the preceding two lemmata induce corresponding mappings between symmetric tensors.

5.18 Lemma: (Iterated symmetrised covariant differentials of vertical lifts of dual sections I) *Let* $r \in \{\infty, \omega\}$ *and let* $\pi_{\mathsf{E}}\colon \mathsf{E} \to \mathsf{M}$ *be a* C^r*-vector bundle, with the data prescribed in Section 4.1 to define the Riemannian metric* $\mathbb{G}_{\mathsf{E}}$ *on* E*. For* $m \in \mathbb{Z}_{\geq 0}$*, there exist* C^r*-vector bundle mappings*

$$(\widehat{A}^m_s, \mathrm{id}_{\mathsf{E}}) \in \mathrm{VB}^r(\mathrm{S}^s(\pi^*_{\mathsf{E}}\mathsf{T}^*\mathsf{M}) \otimes \mathsf{V}^*\mathsf{E}; \mathrm{S}^m(\mathsf{T}^*\mathsf{E}) \otimes \mathsf{V}^*\mathsf{E}), \qquad s \in \{0, 1, \dots, m\},$$

such that

$$(\mathrm{Sym}_m \otimes \mathrm{id}_{\mathsf{V}^*\mathsf{E}}) \circ \nabla^{\mathsf{E},m}\lambda^{\mathrm{v}} = \sum_{s=0}^{m} \widehat{A}^m_s((\mathrm{Sym}_s \otimes \mathrm{id}_{\mathsf{V}^*\mathsf{E}}) \circ (\nabla^{\mathsf{M},\pi_{\mathsf{E}},s}\lambda)^{\mathrm{v}})$$

for all $\lambda \in \Gamma^m(\mathsf{E}^*)$*.*

Proof: This follows along the lines of Lemma 5.3 in the same manner as Lemma 5.8 follows from Lemma 5.3, by taking tensor products with $\mathsf{V}^*\mathsf{E}$. ■

The preceding lemma gives rise to an "inverse," which we state in the following lemma.

5.19 Lemma: (Iterated symmetrised covariant differentials of vertical lifts of dual sections II) *Let $r \in \{\infty, \omega\}$ and let $\pi_{\mathsf{E}} \colon \mathsf{E} \to \mathsf{M}$ be a C^r-vector bundle, with the data prescribed in Section 4.1 to define the Riemannian metric $\mathbb{G}_{\mathsf{E}}$ on E. For $m \in \mathbb{Z}_{\geq 0}$, there exist C^r-vector bundle mappings*

$$(\widehat{B}^m_s, \mathrm{id}_{\mathsf{E}}) \in \mathrm{VB}^r(\mathrm{S}^s(\mathsf{T}^*\mathsf{E}) \otimes \mathsf{V}^*\mathsf{E}; \mathrm{S}^m(\pi^*_{\mathsf{E}}\mathsf{T}^*\mathsf{M}) \otimes \mathsf{V}^*\mathsf{E}), \qquad s \in \{0, 1, \dots, m\},$$

such that

$$(\mathrm{Sym}_m \otimes_{\mathrm{id}_{\mathsf{V}^*\mathsf{E}}}) \circ (\nabla^{\mathsf{M},\pi_{\mathsf{E}},m}\lambda)^{\mathrm{v}} = \sum_{s=0}^{m} \widehat{B}^m_s((\mathrm{Sym}_s \otimes \mathrm{id}_{\mathsf{V}^*\mathsf{E}}) \circ \nabla^{\mathsf{E},s}\lambda^{\mathrm{v}})$$

for all $\lambda \in \Gamma^m(\mathsf{E}^*)$.

Proof: This follows along the lines of Lemma 5.4 in the same manner as Lemma 5.8 follows from Lemma 5.3, by taking tensor products with $\mathsf{V}^*\mathsf{E}$. ■

We can put together the previous four lemmata into the following decomposition result, which is to be regarded as the main result of this section.

5.20 Lemma: (Decomposition of jets of vertical lifts of dual sections) *Let $r \in \{\infty, \omega\}$ and let $\pi_{\mathsf{E}} \colon \mathsf{E} \to \mathsf{M}$ be a C^r-vector bundle, with the data prescribed in Section 4.1 to define the Riemannian metric $\mathbb{G}_{\mathsf{E}}$ on E. Then there exist C^r-vector bundle mappings*

$$\begin{aligned} A^m_{\nabla^{\mathsf{E}}} &\in \mathrm{VB}^r(\mathsf{P}^{*m}\mathsf{E} \otimes \mathsf{V}^*\mathsf{E}; \mathrm{S}^{\leq m}(\pi^*_{\mathsf{E}}\mathsf{T}^*\mathsf{M}) \otimes \mathsf{V}^*\mathsf{E}), \\ B^m_{\nabla^{\mathsf{E}}} &\in \mathrm{VB}^r(\mathsf{P}^{*m}\mathsf{E} \otimes \mathsf{V}^*\mathsf{E}; \mathrm{S}^{\leq m}(\mathsf{T}^*\mathsf{E}) \otimes \mathsf{V}^*\mathsf{E}), \end{aligned}$$

defined by

$$\begin{aligned} A^m_{\nabla^{\mathsf{E}}}(j_m(\lambda^{\mathrm{v}})(e)) &= \mathrm{Sym}_{\leq m} \otimes \mathrm{id}_{\mathsf{V}^*\mathsf{E}}(\lambda^{\mathrm{v}}(e), (\nabla^{\pi_{\mathsf{E}}}\lambda)^{\mathrm{v}}(e), \dots, (\nabla^{\mathsf{M},\pi_{\mathsf{E}},m}\lambda)^{\mathrm{v}}(e)), \\ B^m_{\nabla^{\mathsf{E}}}(j_m(\lambda^{\mathrm{v}})(e)) &= \mathrm{Sym}_{\leq m} \otimes \mathrm{id}_{\mathsf{V}^*\mathsf{E}}(\lambda^{\mathrm{v}}(e), \nabla^{\mathsf{E}}\lambda^{\mathrm{v}}(e), \dots, \nabla^{\mathsf{E},m}\lambda^{\mathrm{v}}(e)). \end{aligned}$$

Moreover, $A^m_{\nabla^{\mathsf{E}}}$ is an isomorphism, $B^m_{\nabla^{\mathsf{E}}}$ is injective, and

$$B^m_{\nabla^{\mathsf{E}}} \circ (A^m_{\nabla^{\mathsf{E}}})^{-1} \circ (\mathrm{Sym}_{\leq m} \otimes \mathrm{id}_{\mathsf{V}^*\mathsf{E}})(\lambda^{\mathrm{v}}(e), (\nabla^{\pi_{\mathsf{E}}}\lambda)^{\mathrm{v}}(e), \dots, (\nabla^{\mathsf{M},\pi_{\mathsf{E}},m}\lambda)^{\mathrm{v}}(e)) = \\ \left(\lambda^{\mathrm{v}}(e), \sum_{s=0}^{1} \widehat{A}^1_s((\mathrm{Sym}_s \otimes \mathrm{id}_{\mathsf{V}^*\mathsf{E}}) \circ (\nabla^{\mathsf{M},\pi_{\mathsf{E}},s}\lambda)^{\mathrm{v}}(e)), \dots, \sum_{s=0}^{m} \widehat{A}^m_s((\mathrm{Sym}_s \otimes \mathrm{id}_{\mathsf{V}^*\mathsf{E}}) \circ (\nabla^{\mathsf{M},\pi_{\mathsf{E}},s}\lambda)^{\mathrm{v}}(e))\right)$$

and

$$\begin{aligned} &A^m_{\nabla^{\mathsf{E}}} \circ (B^m_{\nabla^{\mathsf{E}}})^{-1} \circ (\mathrm{Sym}_{\leq m} \otimes \mathrm{id}_{\mathsf{V}^*\mathsf{E}})(\lambda^{\mathrm{v}}(e), \nabla^{\mathsf{E}}\lambda^{\mathrm{v}}(e), \dots, \nabla^{\mathsf{E},m}\lambda^{\mathrm{v}}(e)) \\ &\quad = \left(\lambda^{\mathrm{v}}(e), \sum_{s=0}^{1} \widehat{B}^1_s((\mathrm{Sym}_s \otimes \mathrm{id}_{\mathsf{V}^*\mathsf{E}}) \circ \nabla^{\mathsf{E},s}\lambda^{\mathrm{v}}(e)), \dots, \sum_{s=0}^{m} \widehat{B}^m_s((\mathrm{Sym}_s \otimes \mathrm{id}_{\mathsf{V}^*\mathsf{E}}) \circ \nabla^{\mathsf{E},s}\lambda^{\mathrm{v}}(e))\right), \end{aligned}$$

where the vector bundle mappings $\widehat{A}^m_s$ and $\widehat{B}^m_s$, $s \in \{0, 1, \dots, m\}$, are as in Lemmata 5.18 and 5.19.

5.5. Isomorphisms for vertical lifts of endomorphisms. Next we consider vertical lifts of endomorphisms defined by the mapping

$$\Gamma^r(\mathrm{T}^1_1(\mathsf{E})) \ni L \mapsto L^{\mathrm{v}} \in \Gamma^r(\mathrm{T}^1_1(\mathsf{TE})).$$

We wish to relate the decomposition of the jets of L with those of L^{v}. Associated with this, we denote

$$\mathsf{L}^{*m}_e\mathsf{E} = \{j_m L^{\mathrm{v}}(e) \mid\ L \in \Gamma^m(\mathrm{T}^1_1(\mathsf{E}))\}.$$

By (1.5), we have

$$\mathsf{L}^{*m}_e\mathsf{E} \simeq \mathsf{P}^{*m}_e\mathsf{E} \otimes \mathrm{T}^1_1(\mathsf{V}_e\mathsf{E}).$$

As with the constructions of the preceding sections, we wish to use Lemma 2.1 to provide a decomposition of $\mathsf{L}^{*m}\mathsf{E}$, and to do so we need to understand the covariant derivatives

$$\nabla^{\mathsf{E},m} L^{\mathrm{v}} \triangleq \underbrace{\nabla^{\mathsf{E}} \cdots \nabla^{\mathsf{E}}}_{m \text{ times}} L^{\mathrm{v}}, \qquad m \in \mathbb{Z}_{\geq 0}.$$

In this section we omit proofs, since proofs follow along entirely similar lines to those of the preceding section.

The first result we give is the following.

5.21 Lemma: (Iterated covariant differentials of vertical lifts of endomorphisms I) *Let $r \in \{\infty, \omega\}$ and let $\pi_{\mathsf{E}} \colon \mathsf{E} \to \mathsf{M}$ be a C^r-vector bundle, with the data prescribed in Section 4.1 to define the Riemannian metric $\mathbb{G}_{\mathsf{E}}$ on E. For $m \in \mathbb{Z}_{\geq 0}$, there exist C^r-vector bundle mappings*

$$(A^m_s, \mathrm{id}_{\mathsf{E}}) \in \mathrm{VB}^r(\mathrm{T}^s(\pi^*_{\mathsf{E}}\mathsf{T}^*\mathsf{M}) \otimes \mathrm{T}^1_1(\mathsf{VE}); \mathrm{T}^m(\mathsf{T}^*\mathsf{E}) \otimes \mathrm{T}^1_1(\mathsf{VE})), \qquad s \in \{0, 1, \dots, m\},$$

such that

$$\nabla^{\mathsf{E},m} L^{\mathrm{v}} = \sum_{s=0}^{m} A^m_s((\nabla^{\mathsf{M},\pi_{\mathsf{E}},s} L)^{\mathrm{v}})$$

for all $L \in \Gamma^m(\mathrm{T}^1_1(\mathsf{E}))$. Moreover, the vector bundle mappings $A^m_0, A^m_1, \dots, A^m_m$ satisfy the recursion relations prescribed by $A^0_0(\beta_0) = \beta_0$ and

$$\begin{aligned}
A^{m+1}_{m+1}(\beta_{m+1}) &= \beta_{m+1},\\
A^{m+1}_s(\beta_s) &= (\nabla^{\mathsf{E}} A^m_s)(\beta_s) + A^m_{s-1} \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\beta_s) - \sum_{j=1}^{s} A^m_s \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\mathrm{Ins}_j(\beta_s, B_{\pi_{\mathsf{E}}}))\\
&\quad + A^m_s \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\mathrm{Ins}_{s+1}(\beta_s, B^*_{\pi_{\mathsf{E}}})), \ \ s \in \{1, \dots, m\},\\
A^{m+1}_0(\beta_0) &= (\nabla^{\mathsf{E}} A^m_0)(\beta_0) - A^m_0 \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\mathrm{Ins}_1(\beta_0, B_{\pi_{\mathsf{E}}})) + A^m_0 \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\mathrm{Ins}_2(\beta_0, B^*_{\pi_{\mathsf{E}}})),
\end{aligned}$$

where $\beta_s \in \mathrm{T}^s(\pi^*_{\mathsf{E}}\mathsf{T}^*\mathsf{M}) \otimes \mathrm{T}^1_1(\mathsf{VE})$, $s \in \{0, 1, \dots, m+1\}$.

Proof: This follows in the same manner as Lemma 5.6, making use of Lemma 4.5(v). ■

The "inverse" of the preceding lemma is as follows.

5.22 Lemma: (Iterated covariant differentials of vertical lifts of endomorphisms II) *Let $r \in \{\infty, \omega\}$ and let $\pi_{\mathsf{E}}\colon \mathsf{E} \to \mathsf{M}$ be a C^r-vector bundle, with the data prescribed in Section 4.1 to define the Riemannian metric $\mathbb{G}_{\mathsf{E}}$ on E. For $m \in \mathbb{Z}_{\geq 0}$, there exist C^r-vector bundle mappings*

$$(B^m_s, \mathrm{id}_{\mathsf{E}}) \in \mathrm{VB}^r(\mathrm{T}^s(\mathsf{T}^*\mathsf{E}) \otimes \mathrm{T}^1_1(\mathsf{VE}); \mathrm{T}^m(\pi^*_{\mathsf{E}}\mathsf{T}^*\mathsf{M}) \otimes \mathrm{T}^1_1(\mathsf{VE})), \qquad s \in \{0, 1, \dots, m\},$$

such that

$$(\nabla^{\mathsf{M},\pi_{\mathsf{E}},m} L)^{\mathrm{v}} = \sum_{s=0}^{m} B^m_s(\nabla^{\mathsf{E},s} L^{\mathrm{v}})$$

for all $L \in \Gamma^m(\mathrm{T}^1_1(\mathsf{E}))$. Moreover, the vector bundle mappings $B^m_0, B^m_1, \dots, B^m_m$ satisfy the recursion relations prescribed by $B^0_0(\alpha_0) = \alpha_0$ and

$$\begin{aligned}
B^{m+1}_{m+1}&(\alpha_{m+1}) = \alpha_{m+1},\\
&B^{m+1}_s(\alpha_s) = (\nabla^{\mathsf{E}} B^m_s)(\alpha_s) + B^m_{s-1} \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\alpha_s) + \sum_{j=1}^{m} \mathrm{Ins}_j(B^m_s(\alpha_s), B_{\pi_{\mathsf{E}}})\\
&\qquad\qquad - \mathrm{Ins}_{m+1}(B^m_s(\alpha_s), B^*_{\pi_{\mathsf{E}}}), \quad s \in \{1, \dots, m\},\\
&B^{m+1}_0(\alpha_0) = (\nabla^{\mathsf{E}} B^m_0)(\alpha_0) + \sum_{j=1}^{m} \mathrm{Ins}_j(B^m_0(\alpha_0), B_{\pi_{\mathsf{E}}}) - \mathrm{Ins}_{m+1}(B^m_0(\alpha_0), B_{\pi_{\mathsf{E}}}),
\end{aligned}$$

where $\alpha_s \in \mathrm{T}^s(\mathsf{T}^*\mathsf{E}) \otimes \mathrm{T}^1_1(\mathsf{VE})$, $s \in \{0, 1, \dots, m+1\}$.

Proof: This follows in the same manner as Lemma 5.7, making use of Lemma 4.5(v). ■

Next we turn to symmetrised versions of the preceding lemmata. We show that the preceding two lemmata induce corresponding mappings between symmetric tensors.

5.23 Lemma: (Iterated symmetrised covariant differentials of vertical lifts of endomorphisms I) *Let $r \in \{\infty, \omega\}$ and let $\pi_{\mathsf{E}}\colon \mathsf{E} \to \mathsf{M}$ be a C^r-vector bundle, with the data prescribed in Section 4.1 to define the Riemannian metric $\mathbb{G}_{\mathsf{E}}$ on E. For $m \in \mathbb{Z}_{\geq 0}$, there exist C^r-vector bundle mappings*

$$(\widehat{A}^m_s, \mathrm{id}_{\mathsf{E}}) \in \mathrm{VB}^r(\mathrm{S}^s(\pi^*_{\mathsf{E}}\mathsf{T}^*\mathsf{M}) \otimes \mathrm{T}^1_1(\mathsf{VE}); \mathrm{S}^m(\mathsf{T}^*\mathsf{E}) \otimes \mathrm{T}^1_1(\mathsf{VE})), \qquad s \in \{0, 1, \dots, m\},$$

such that

$$(\mathrm{Sym}_m \otimes \mathrm{id}_{\mathrm{T}^1_1(\mathsf{VE})}) \circ \nabla^{\mathsf{E},m} L^{\mathrm{v}} = \sum_{s=0}^{m} \widehat{A}^m_s((\mathrm{Sym}_s \otimes \mathrm{id}_{\mathrm{T}^1_1(\mathsf{VE})}) \circ (\nabla^{\mathsf{M},\pi_{\mathsf{E}},s} L)^{\mathrm{v}})$$

for all $L \in \Gamma^m(\mathrm{T}^1_1(\mathsf{E}))$.

Proof: This follows along the lines of Lemma 5.3 in the same manner as Lemma 5.8 follows from Lemma 5.3, by taking tensor products with $\mathrm{T}^1_1(\mathsf{VE})$. ■

The preceding lemma gives rise to an "inverse," which we state in the following lemma.

5.24 Lemma: (Iterated symmetrised covariant differentials of vertical lifts of endomorphisms II) *Let $r \in \{\infty, \omega\}$ and let $\pi_{\mathsf{E}} \colon \mathsf{E} \to \mathsf{M}$ be a C^r-vector bundle, with the data prescribed in Section 4.1 to define the Riemannian metric $\mathbb{G}_{\mathsf{E}}$ on E. For $m \in \mathbb{Z}_{\geq 0}$, there exist C^r-vector bundle mappings*

$$(\widehat{B}^m_s, \mathrm{id}_{\mathsf{E}}) \in \mathrm{VB}^r(\mathrm{S}^s(\mathsf{T}^*\mathsf{E}) \otimes \mathrm{T}^1_1(\mathsf{VE}); \mathrm{S}^m(\pi^*_{\mathsf{E}}\mathsf{T}^*\mathsf{M}) \otimes \mathrm{T}^1_1(\mathsf{VE})), \qquad s \in \{0, 1, \dots, m\},$$

such that

$$(\mathrm{Sym}_m \otimes \mathrm{id}_{\mathrm{T}^1_1(\mathsf{VE})}) \circ (\nabla^{\mathsf{M},\pi_{\mathsf{E}},m} L)^{\mathrm{v}} = \sum_{s=0}^{m} \widehat{B}^m_s((\mathrm{Sym}_s \otimes \mathrm{id}_{\mathrm{T}^1_1(\mathsf{VE})}) \circ \nabla^{\mathsf{E},s} L^{\mathrm{v}})$$

for all $L \in \Gamma^m(\mathrm{T}^1_1(\mathsf{E}))$.

Proof: This follows along the lines of Lemma 5.4 in the same manner as Lemma 5.8 follows from Lemma 5.3, by taking tensor products with $\mathrm{T}^1_1(\mathsf{VE})$. ■

We can put together the previous four lemmata into the following decomposition result, which is to be regarded as the main result of this section.

5.25 Lemma: (Decomposition of jets of vertical lifts of endomorphisms) *Let $r \in \{\infty, \omega\}$ and let $\pi_{\mathsf{E}} \colon \mathsf{E} \to \mathsf{M}$ be a C^r-vector bundle, with the data prescribed in Section 4.1 to define the Riemannian metric $\mathbb{G}_{\mathsf{E}}$ on E. Then there exist C^r-vector bundle mappings*

$$A^m_{\nabla^{\mathsf{E}}} \in \mathrm{VB}^r(\mathsf{P}^{*m}\mathsf{E} \otimes \mathrm{T}^1_1(\mathsf{VE}); \mathrm{S}^{\leq m}(\pi^*_{\mathsf{E}}\mathsf{T}^*\mathsf{M}) \otimes \mathrm{T}^1_1(\mathsf{VE})),$$
$$B^m_{\nabla^{\mathsf{E}}} \in \mathrm{VB}^r(\mathsf{P}^{*m}\mathsf{E} \otimes \mathrm{T}^1_1(\mathsf{VE}); \mathrm{S}^{\leq m}(\mathsf{T}^*\mathsf{E}) \otimes \mathrm{T}^1_1(\mathsf{VE})),$$

defined by

$$A^m_{\nabla^{\mathsf{E}}}(j_m(L^{\mathrm{v}})(e)) = \mathrm{Sym}_{\leq m} \otimes \mathrm{id}_{\mathrm{T}^1_1(\mathsf{VE})}(L^{\mathrm{v}}(e), (\nabla^{\pi_{\mathsf{E}}} L)^{\mathrm{v}}(e), \dots, (\nabla^{\mathsf{M},\pi_{\mathsf{E}},m} L)^{\mathrm{v}}(e)),$$
$$B^m_{\nabla^{\mathsf{E}}}(j_m(L^{\mathrm{v}})(e)) = \mathrm{Sym}_{\leq m} \otimes \mathrm{id}_{\mathrm{T}^1_1(\mathsf{VE})}(L^{\mathrm{v}}(e), \nabla^{\mathsf{E}} L^{\mathrm{v}}(e), \dots, \nabla^{\mathsf{E},m} L^{\mathrm{v}}(e)).$$

Moreover, $A^m_{\nabla^{\mathsf{E}}}$ is an isomorphism, $B^m_{\nabla^{\mathsf{E}}}$ is injective, and

$$\begin{aligned}
B^m_{\nabla^{\mathsf{E}}} &\circ (A^m_{\nabla^{\mathsf{E}}})^{-1} \circ (\mathrm{Sym}_{\leq m} \otimes \mathrm{id}_{\mathrm{T}^1_1(\mathsf{VE})})(L^{\mathrm{v}}(e), (\nabla^{\pi_{\mathsf{E}}} L)^{\mathrm{v}}(e), \dots, (\nabla^{\mathsf{M},\pi_{\mathsf{E}},m} L)^{\mathrm{v}}(e)) \\
&= \Bigg(L^{\mathrm{v}}(e), \sum_{s=0}^{1} \widehat{A}^1_s((\mathrm{Sym}_s \otimes \mathrm{id}_{\mathrm{T}^1_1(\mathsf{VE})}) \circ (\nabla^{\mathsf{M},\pi_{\mathsf{E}},s} L)^{\mathrm{v}}(e)), \dots, \\
&\qquad \sum_{s=0}^{m} \widehat{A}^m_s((\mathrm{Sym}_s \otimes \mathrm{id}_{\mathrm{T}^1_1(\mathsf{VE})}) \circ (\nabla^{\mathsf{M},\pi_{\mathsf{E}},s} L)^{\mathrm{v}}(e))\Bigg)
\end{aligned}$$

and

$$\begin{aligned}
&A^m_{\nabla^{\mathsf{E}}} \circ (B^m_{\nabla^{\mathsf{E}}})^{-1} \circ (\mathrm{Sym}_{\leq m} \otimes \mathrm{id}_{\mathrm{T}^1_1(\mathsf{VE})})(L^{\mathrm{v}}(e), \nabla^{\mathsf{E}} L^{\mathrm{v}}(e), \dots, \nabla^{\mathsf{E},m} L^{\mathrm{v}}(e)) \\
&= \Bigg(L^{\mathrm{v}}(e), \sum_{s=0}^{1} \widehat{B}^1_s((\mathrm{Sym}_s \otimes \mathrm{id}_{\mathrm{T}^1_1(\mathsf{VE})}) \circ \nabla^{\mathsf{E},s} L^{\mathrm{v}}(e)), \dots, \sum_{s=0}^{m} \widehat{B}^m_s((\mathrm{Sym}_s \otimes \mathrm{id}_{\mathrm{T}^1_1(\mathsf{VE})}) \circ \nabla^{\mathsf{E},s} L^{\mathrm{v}}(e))\Bigg),
\end{aligned}$$

where the vector bundle mappings $\widehat{A}^m_s$ and $\widehat{B}^m_s$, $s \in \{0, 1, \dots, m\}$, are as in Lemmata 5.23 and 5.24.

5.6. Isomorphisms for vertical evaluations of dual sections. Next we consider vertical evaluations of endomorphisms given by the mapping

$$\Gamma^r(\mathsf{E}^*) \ni \lambda \mapsto \lambda^{\mathrm{e}} \in \mathrm{C}^r(\mathsf{E}).$$

To study the relationship between the decomposition of the jets of λ with those of the jets of λ^{e}, we denote

$$\mathsf{D}^{*m}_e\mathsf{E} = \{j_m\lambda^{\mathrm{e}}(e) \mid \lambda \in \Gamma^m(\mathsf{E}^*)\}.$$

By (1.5), we have

$$\mathsf{D}^{*m}_e\mathsf{E} \subseteq \mathsf{P}^{*m}_e\mathsf{E}.$$

As we shall see, one can be a little more explicit about the nature of $\mathsf{D}^{*m}_e\mathsf{E}$, and see that

$$\mathsf{D}^{*m}_e\mathsf{E} \simeq (\mathsf{P}^{*m}_e\mathsf{E} \otimes \mathsf{V}^*\mathsf{E}) \oplus (\mathsf{P}^{*\mathsf{E}}_{m-1} \otimes \mathsf{V}^*\mathsf{E}).$$

However, this sort of isomorphism is too cumbersome to make explicit. As with the constructions of the preceding sections, we wish to use Lemma 2.1 to provide a decomposition of $\mathsf{D}^{*m}\mathsf{E}$, and to do so we need to understand the covariant derivatives

$$\nabla^{\mathsf{E},m}\lambda^{\mathrm{e}} \triangleq \underbrace{\nabla^{\mathsf{E}}\cdots\nabla^{\mathsf{E}}}_{m \text{ times}}\lambda^{\mathrm{e}}, \qquad m \in \mathbb{Z}_{\geq 0}.$$

The results in this section have a slightly different character than in the preceding sections. We will not give the complete proofs, but will note that they are very similar to the complete proofs given in the next section.

Our first result is then the following.

5.26 Lemma: (Iterated covariant differentials of vertical evaluations of dual sections I) *Let $r \in \{\infty, \omega\}$ and let $\pi_{\mathsf{E}}\colon \mathsf{E} \to \mathsf{M}$ be a C^r-vector bundle, with the data prescribed in Section 4.1 to define the Riemannian metric $\mathbb{G}_{\mathsf{E}}$ on E. For $m \in \mathbb{Z}_{\geq 0}$, there exist C^r-vector bundle mappings*

$$(A^m_s, \mathrm{id}_{\mathsf{E}}) \in \mathrm{VB}^r(\mathrm{T}^s(\pi^*_{\mathsf{E}}\mathsf{T}^*\mathsf{M}); \mathrm{T}^m(\mathsf{T}^*\mathsf{E})), \qquad s \in \{0, 1, \dots, m\},$$

and

$$(C^m_s, \mathrm{id}_{\mathsf{E}}) \in \mathrm{VB}^r(\mathrm{T}^s(\pi^*_{\mathsf{E}}\mathsf{T}^*\mathsf{M}) \otimes \mathsf{V}^*\mathsf{E}; \mathrm{T}^{m-1}(\mathsf{T}^*\mathsf{E}) \otimes \mathsf{V}^*\mathsf{E}), \qquad s \in \{0, 1, \dots, m-1\},$$

such that

$$\nabla^{\mathsf{E},m}\lambda^{\mathrm{e}} = \sum_{s=0}^{m} A^m_s((\nabla^{\mathsf{M},\pi_{\mathsf{E}},s}\lambda)^{\mathrm{e}}) + \sum_{s=0}^{m-1} C^m_s((\nabla^{\mathsf{M},\pi_{\mathsf{E}},s}\lambda)^{\mathrm{v}})$$ [3]

for all $\lambda \in \Gamma^m(\mathsf{E}^*)$. Moreover, the vector bundle mappings $A^m_0, A^m_1, \dots, A^m_m$ and $C^m_0, C^m_1, \dots, C^m_{m-1}$ satisfy the recursion relations prescribed by

$$A^0_0(\beta_0) = \beta_0, \;\; A^1_1(\beta_1) = \beta_1, \;\; A^1_0(\beta_0) = \mathrm{Ins}_1(\beta_0, B_{\pi_{\mathsf{E}}}), \;\; C^1_0(\gamma_0) = \gamma_0,$$

[3] Here we regard $\mathsf{V}^*\mathsf{E}$ as a subbundle of $\mathsf{T}^*\mathsf{E}$.

and, for $m \geq 2$,

$$\begin{aligned}
A^{m+1}_{m+1}(\beta_{m+1}) &= \beta_{m+1}\\
A^{m+1}_{m}(\beta_m) &= A^m_{m-1}\otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\beta_m) - \sum_{j=1}^m \mathrm{Ins}_j(\beta_m, B_{\pi_{\mathsf{E}}})\\
A^{m+1}_{s}(\beta_s) &= (\nabla^{\mathsf{E}}A^m_s)(\beta_m) + A^m_{s-1}\otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\beta_s) - \sum_{j=1}^s A^m_s\otimes\mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\mathrm{Ins}_j(\beta_s, B_{\pi_{\mathsf{E}}})),\\
&\qquad s\in\{1,\dots,m-1\},\\
A^{m+1}_0(\beta_0) &= (\nabla^{\mathsf{E}}A^m_0)(\beta_0)
\end{aligned}$$

and

$$\begin{aligned}
C^{m+1}_m(\gamma_m) &= C^m_{m-1}\otimes\mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\gamma_m) + \gamma_m\\
C^{m+1}_s(\gamma_s) &= A^m_s\otimes\mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\gamma_s) + (\nabla^{\mathsf{E}}C^m_s)(\gamma_s) + C^m_{s-1}\otimes\mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\gamma_s)\\
&\qquad - \sum_{j=1}^{s+1} C^m_s\otimes\mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\mathrm{Ins}_j(\gamma_s, B_{\pi_{\mathsf{E}}})),\quad s\in\{1,\dots,m-1\},\\
C^{m+1}_0(\gamma_0) &= A^m_0\otimes\mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\gamma_0) + (\nabla^{\mathsf{E}}C^m_0)(\gamma_0) - C^m_0\otimes\mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\mathrm{Ins}_1(\gamma_0, B_{\pi_{\mathsf{E}}})),
\end{aligned}$$

where $\beta_s \in \mathrm{T}^s(\pi^*_{\mathsf{E}}\mathsf{T}^*\mathsf{M})$, $s\in\{0,1,\dots,m+1\}$, and $\gamma_s\in\mathrm{T}^s(\pi^*_{\mathsf{E}}\mathsf{T}^*\mathsf{M})\otimes\mathsf{V}^*\mathsf{E}$, $s\in\{0,1,\dots,m-1\}$.

Proof: This follows in the same manner as Lemma 5.31 below, making use of Lemma 4.5(vi). ■

Now we "invert" the constructions from the preceding lemma.

5.27 Lemma: (Iterated covariant differentials of vertical evaluations of dual sections II) *Let $r\in\{\infty,\omega\}$ and let $\pi_{\mathsf{E}}\colon \mathsf{E}\to\mathsf{M}$ be a C^r-vector bundle, with the data prescribed in Section 4.1 to define the Riemannian metric $\mathbb{G}_{\mathsf{E}}$ on E. For $m\in\mathbb{Z}_{\geq 0}$, there exist C^r-vector bundle mappings*

$$(B^m_s, \mathrm{id}_{\mathsf{E}}) \in \mathrm{VB}^r(\mathrm{T}^s(\mathsf{T}^*\mathsf{E}); \mathrm{T}^m(\pi^*_{\mathsf{E}}\mathsf{T}^*\mathsf{M})),\qquad s\in\{0,1,\dots,m\},$$

and

$$(D^m_s, \mathrm{id}_{\mathsf{E}}) \in \mathrm{VB}^r(\mathrm{T}^s(\mathsf{T}^*\mathsf{E})\otimes\mathsf{V}^*\mathsf{E}; \mathrm{T}^{m-1}(\pi^*_{\mathsf{E}}\mathsf{T}^*\mathsf{M})\otimes\mathsf{V}^*\mathsf{E}),\qquad s\in\{0,1,\dots,m-1\},$$

such that

$$(\nabla^{\mathsf{M},\pi_{\mathsf{E}},m}\lambda)^{\mathrm{e}} = \sum_{s=0}^m B^m_s(\nabla^{\mathsf{E},s}\lambda^{\mathrm{e}}) + \sum_{s=0}^{m-1} D^m_s(\nabla^{\mathsf{E},s}\lambda^{\mathrm{v}})$$

for all $\lambda\in\Gamma^m(\mathsf{E}^*)$. Moreover, the vector bundle mappings $B^m_0, B^m_1,\dots,B^m_m$ and

$D_0^m, D_1^m, \dots, D_{m-1}^m$ *satisfy the recursion relations prescribed by* $B_0^0(\alpha_0) = \alpha_0$*,* $D_0^1(\gamma_0) = \gamma_0$*,*

$$\begin{aligned}
B_{m+1}^{m+1}(\alpha_{m+1}) &= \alpha_{m+1} \\
B_m^{m+1}(\alpha_m) &= B_{m-1}^m \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\alpha_m) + \sum_{j=1}^m \mathrm{Ins}_j(\alpha_m, B_{\pi_{\mathsf{E}}}) - \mathrm{Ins}_{m+1}(\alpha_m, B_{\pi_{\mathsf{E}}}^*) \\
B_s^{m+1} &= (\nabla^{\mathsf{E}} B_s^m)(\alpha_s) + B_{s-1}^m \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\alpha_s) + \sum_{j=1}^m \mathrm{Ins}_j(B_s^m(\alpha_s), B_{\pi_{\mathsf{E}}}) \\
&\quad - \mathrm{Ins}_{m+1}(B_s^m(\alpha_s), B_{\pi_{\mathsf{E}}}^*), \quad s \in \{1, \dots, m-1\}, \\
B_0^{m+1}(\alpha_0) &= (\nabla^{\mathsf{E}} B_0^m)(\alpha_0) + \sum_{j=1}^m \mathrm{Ins}_j(B_0^m(\alpha_0), B_{\pi_{\mathsf{E}}}) - \mathrm{Ins}_{m+1}(B_0^m(\alpha_0), B_{\pi_{\mathsf{E}}}^*)
\end{aligned}$$

and

$$\begin{aligned}
D_m^{m+1}(\gamma_m) &= D_{m-1}^m \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\gamma_m) - \gamma_m \\
D_s^m(\gamma_s) &= (\nabla^{\mathsf{E}} D_s^m)(\gamma_s) + D_{s-1}^m \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\gamma_s) - \overline{B}_s^m(\gamma_s), \quad s \in \{1, \dots, m-1\}, \\
D_0^{m+1} &= (\nabla^{\mathsf{E}} D_0^m)(\gamma_0) - \overline{B}_0^m(\gamma_0)
\end{aligned}$$

for $\alpha_s \in \mathrm{T}^s(\mathsf{T}^*\mathsf{E})$*,* $s \in \{0, 1, \dots, m\}$*, and* $\gamma_s \in \mathrm{T}^s(\mathsf{T}^*\mathsf{E}) \otimes \mathsf{V}^*\mathsf{E}$*,* $s \in \{0, 1, \dots, m-1\}$*, and where*

$$(\overline{B}_s^m, \mathrm{id}_{\mathsf{E}}) \in \mathrm{VB}^r(\mathrm{T}^s(\mathsf{T}^*\mathsf{E}) \otimes \mathsf{V}^*\mathsf{E}; \mathrm{T}^m(\pi_{\mathsf{E}}^*\mathsf{T}^*\mathsf{M}) \otimes \mathsf{V}^*\mathsf{E}), \qquad s \in \{0, 1, \dots, m\},$$

are the vector bundle mappings from Lemma 5.17.

Proof: This follows in the same manner as Lemma 5.32 below, making use of Lemma 4.5(vi). ■

Next we turn to symmetrised versions of the preceding lemmata. We show that the preceding two lemmata induce corresponding mappings between symmetric tensors.

5.28 Lemma: (Iterated symmetrised covariant differentials of vertical evaluations of dual sections I) *Let* $r \in \{\infty, \omega\}$ *and let* $\pi_{\mathsf{E}} \colon \mathsf{E} \to \mathsf{M}$ *be a* C^r*-vector bundle, with the data prescribed in Section 4.1 to define the Riemannian metric* $\mathbb{G}_{\mathsf{E}}$ *on* E*. For* $m \in \mathbb{Z}_{\geq 0}$*, there exist* C^r*-vector bundle mappings*

$$(\widehat{A}_s^m, \mathrm{id}_{\mathsf{E}}) \in \mathrm{VB}^r(\mathrm{S}^s(\pi_{\mathsf{E}}^*\mathsf{T}^*\mathsf{M}); \mathrm{S}^m(\mathsf{T}^*\mathsf{E})), \qquad s \in \{0, 1, \dots, m\},$$

and

$$(\widehat{C}_s^m, \mathrm{id}_{\mathsf{E}}) \in \mathrm{VB}^r(\mathrm{S}^s(\pi_{\mathsf{E}}^*\mathsf{T}^*\mathsf{M}) \otimes \mathsf{V}^*\mathsf{E}; \mathrm{S}^m(\mathsf{T}^*\mathsf{E})), \qquad s \in \{0, 1, \dots, m-1\},$$

such that

$$\mathrm{Sym}_m \circ \nabla^{\mathsf{E},m} \lambda^{\mathrm{e}} = \sum_{s=0}^m \widehat{A}_s^m(\mathrm{Sym}_s \circ (\nabla^{\mathsf{M},\pi_{\mathsf{E}},s}\lambda)^{\mathrm{e}}) + \sum_{s=0}^{m-1} \widehat{C}_s^m((\mathrm{Sym}_s \otimes \mathrm{id}_{\mathsf{V}^*\mathsf{E}}) \circ (\nabla^{\mathsf{M},\pi_{\mathsf{E}},s}\lambda)^{\mathrm{v}})$$

for all $\lambda \in \Gamma^m(\mathsf{E}^*)$*.*

Proof: The proof here follows along the lines of Lemma 5.33 below. ■

The preceding lemma gives rise to an "inverse," which we state in the following lemma.

5.29 Lemma: (Iterated symmetrised covariant differentials of vertical evaluations of dual sections II) *Let $r \in \{\infty, \omega\}$ and let $\pi_{\mathsf{E}} \colon \mathsf{E} \to \mathsf{M}$ be a C^r-vector bundle, with the data prescribed in Section 4.1 to define the Riemannian metric $\mathbb{G}_{\mathsf{E}}$ on E. For $m \in \mathbb{Z}_{\geq 0}$, there exist C^r-vector bundle mappings*

$$(\widehat{B}^m_s, \mathrm{id}_{\mathsf{E}}) \in \mathrm{VB}^r(\mathrm{S}^s(\mathsf{T}^*\mathsf{E}); \mathrm{S}^m(\pi^*_{\mathsf{E}}\mathsf{T}^*\mathsf{M})), \qquad s \in \{0, 1, \dots, m\},$$

and

$$(\widehat{D}^m_s, \mathrm{id}_{\mathsf{E}}) \in \mathrm{VB}^r(\mathrm{S}^s(\mathsf{T}^*\mathsf{E}) \otimes \mathsf{V}^*\mathsf{E}; \mathrm{S}^m(\pi^*_{\mathsf{E}}\mathsf{T}^*\mathsf{M})), \qquad s \in \{0, 1, \dots, m-1\},$$

such that

$$\mathrm{Sym}_m \circ (\nabla^{\mathsf{M}, \pi_{\mathsf{E}}, m}\lambda)^{\mathrm{e}} = \sum_{s=0}^{m} \widehat{B}^m_s(\mathrm{Sym}_s \circ \nabla^{\mathsf{E}, s}\lambda^{\mathrm{e}}) + \sum_{s=0}^{m-1} \widehat{D}^m_s((\mathrm{Sym}_s \otimes \mathrm{id}_{\mathsf{V}^*\mathsf{E}}) \circ \nabla^{\mathsf{E}, s}\lambda^{\mathrm{v}})$$

for all $\lambda \in \Gamma^m(\mathsf{E}^*)$.

Proof: The proof here follows along the lines of Lemma 5.33 below. ■

We can put together the previous four lemmata, along with Lemma 5.25, into the following decomposition result, which is to be regarded as the main result of this section.

5.30 Lemma: (Decomposition of jets of vertical evaluations of dual sections) *Let $r \in \{\infty, \omega\}$ and let $\pi_{\mathsf{E}} \colon \mathsf{E} \to \mathsf{M}$ be a C^r-vector bundle, with the data prescribed in Section 4.1 to define the Riemannian metric $\mathbb{G}_{\mathsf{E}}$ on E. Then there exist C^r-vector bundle mappings*

$$A^m_{\nabla^{\mathsf{E}}} \in \mathrm{VB}^r(\mathsf{D}^{*m}\mathsf{E}; \mathrm{S}^{\leq m}(\pi^*_{\mathsf{E}}\mathsf{T}^*\mathsf{M})), \quad B^m_{\nabla^{\mathsf{E}}} \in \mathrm{VB}^r(\mathsf{D}^{*m}\mathsf{E}; \mathrm{S}^{\leq m}(\mathsf{T}^*\mathsf{E}))$$

defined by

$$\begin{aligned} A^m_{\nabla^{\mathsf{E}}}(j_m(\lambda^{\mathrm{e}})(e)) &= \mathrm{Sym}_{\leq m}(\lambda^{\mathrm{e}}(e), (\nabla^{\pi_{\mathsf{E}}}\lambda)^{\mathrm{e}}(e), \dots, (\nabla^{\mathsf{M}, \pi_{\mathsf{E}}, m}\lambda)^{\mathrm{e}}(e)), \\ B^m_{\nabla^{\mathsf{E}}}(j_m(\lambda^{\mathrm{e}})(e)) &= \mathrm{Sym}_{\leq m}(\lambda^{\mathrm{e}}(e), \nabla^{\mathsf{E}}\lambda^{\mathrm{e}}(e), \dots, \nabla^{\mathsf{E}, m}\lambda^{\mathrm{e}}(e)). \end{aligned}$$

Moreover, $A^m_{\nabla^{\mathsf{E}}}$ and $B^m_{\nabla^{\mathsf{E}}}$ are injective, and

$$\begin{aligned} B^m_{\nabla^{\mathsf{E}}} &\circ (A^m_{\nabla^{\mathsf{E}}})^{-1} \circ \mathrm{Sym}_{\leq m}(\lambda^{\mathrm{e}}(e), (\nabla^{\pi_{\mathsf{E}}}\lambda)^{\mathrm{e}}(e), \dots, (\nabla^{\mathsf{M}, \pi_{\mathsf{E}}, m}\lambda)^{\mathrm{e}}(e)) \\ &= \left(\lambda^{\mathrm{e}}(e), \sum_{s=0}^{1} \widehat{A}^1_s(\mathrm{Sym}_s \circ (\nabla^{\mathsf{M}, \pi_{\mathsf{E}}, s}\lambda)^{\mathrm{e}}(e)), \dots, \sum_{s=0}^{m} \widehat{A}^m_s(\mathrm{Sym}_s \circ (\nabla^{\mathsf{M}, \pi_{\mathsf{E}}, s}\lambda)^{\mathrm{e}}(e))\right) \\ &\quad + \Bigg(0, \lambda^{\mathrm{v}}(e), \sum_{s=0}^{1} \widehat{C}^2_s((\mathrm{Sym}_s \otimes \mathrm{id}_{\mathsf{V}^*\mathsf{E}}) \circ (\nabla^{\mathsf{M}, \pi_{\mathsf{E}}, s}\lambda)^{\mathrm{v}}(e)), \dots, \\ &\qquad \sum_{s=0}^{m-1} \widehat{C}^m_s((\mathrm{Sym}_s \otimes \mathrm{id}_{\mathsf{V}^*\mathsf{E}}) \circ (\nabla^{\mathsf{M}, \pi_{\mathsf{E}}, s}\lambda)^{\mathrm{v}}(e))\Bigg) \end{aligned}$$

and

$$\begin{aligned} A^m_{\nabla^{\mathsf{E}}} &\circ (B^m_{\nabla^{\mathsf{E}}})^{-1} \circ \mathrm{Sym}_{\leq m}(\lambda^{\mathrm{e}}(e), \nabla^{\mathsf{E}}\lambda^{\mathrm{e}}(e), \dots, \nabla^{\mathsf{E}, m}\lambda^{\mathrm{e}}(e)) \\ &= \left(\lambda^{\mathrm{e}}(e), \sum_{s=0}^{1} \widehat{B}^1_s(\mathrm{Sym}_s \circ \nabla^{\mathsf{E}, s}\lambda^{\mathrm{e}}(e)), \dots, \sum_{s=0}^{m} \widehat{B}^m_s(\mathrm{Sym}_s \circ \nabla^{\mathsf{E}, s}\lambda^{\mathrm{e}}(e))\right) \\ &+ \left(0, \lambda^{\mathrm{v}}(e), \sum_{s=0}^{1} \widehat{D}^2_s((\mathrm{Sym}_s \otimes \mathrm{id}_{\mathsf{V}^*\mathsf{E}}) \circ \nabla^{\mathsf{E}, s}\lambda^{\mathrm{v}}(e)), \dots, \sum_{s=0}^{m-1} \widehat{D}^m_s((\mathrm{Sym}_s \otimes \mathrm{id}_{\mathsf{V}^*\mathsf{E}}) \circ \nabla^{\mathsf{E}, s}\lambda^{\mathrm{v}}(e))\right), \end{aligned}$$

where the vector bundle mappings $\widehat{A}^m_s$ *and* $\widehat{B}^m_s$, $s \in \{0, 1, \dots, m\}$, *and* $\widehat{C}^m_s$ *and* $\widehat{D}^m_s$, $s \in \{0, 1, \dots, m-1\}$, *are as in Lemmata 5.28 and 5.29.*

5.7. Isomorphisms for vertical evaluations of endomorphisms. Next we consider vertical evaluations of endomorphisms, i.e., the mapping given by

$$\Gamma^r(\mathrm{T}^1_1(\mathsf{E})) \ni L \mapsto L^{\mathrm{e}} \in \Gamma^r(\mathsf{TE}).$$

To study the relationship between the decomposition of jets of L with those of L^{e}, we denote

$$\mathsf{C}^{*m}_e\mathsf{E} = \{j_m L^{\mathrm{e}}(e) \mid\ L \in \Gamma^m(\mathrm{T}^1_1(\mathsf{E}))\}.$$

By (1.5), we have

$$\mathsf{C}^{*m}_e\mathsf{E} \subseteq \mathsf{P}^{*m}_e\mathsf{E} \otimes \mathsf{V}_e\mathsf{E}.$$

As we shall see, one can be a little more explicit about the nature of $\mathsf{D}^{*m}_e\mathsf{E}$, and see that

$$\mathsf{C}^{*m}_e\mathsf{E} \simeq (\mathsf{P}^{*m}_e\mathsf{E} \otimes \mathsf{V}^*\mathsf{E} \otimes \mathsf{V}_e\mathsf{E}) \oplus (\mathsf{P}^{*\mathsf{E}}_{m-1} \otimes \mathsf{V}^*\mathsf{E} \otimes \mathsf{V}_e\mathsf{E}).$$

However, this sort of isomorphism is too cumbersome to make explicit. As with the constructions of the preceding sections, we wish to use Lemma 2.1 to provide a decomposition of $\mathsf{C}^{*m}\mathsf{E}$, and to do so we need to understand the covariant derivatives

$$\nabla^{\mathsf{E},m} L^{\mathrm{e}} \triangleq \underbrace{\nabla^{\mathsf{E}} \cdots \nabla^{\mathsf{E}}}_{m \text{ times}} L^{\mathrm{e}}, \qquad m \in \mathbb{Z}_{\geq 0}.$$

The results in this section have a slightly different character than in the preceding sections, so we provide complete proofs.

The first result we give is the following.

5.31 Lemma: (Iterated covariant differentials of vertical evaluations of endomorphisms I) *Let* $r \in \{\infty, \omega\}$ *and let* $\pi_{\mathsf{E}} \colon \mathsf{E} \to \mathsf{M}$ *be a* C^r*-vector bundle, with the data prescribed in Section 4.1 to define the Riemannian metric* $\mathbb{G}_{\mathsf{E}}$ *on* E*. For* $m \in \mathbb{Z}_{\geq 0}$*, there exist* C^r*-vector bundle mappings*

$$(A^m_s, \mathrm{id}_{\mathsf{E}}) \in \mathrm{VB}^r(\mathrm{T}^s(\pi^*_{\mathsf{E}}\mathsf{T}^*\mathsf{M}) \otimes \mathsf{VE}; \mathrm{T}^m(\mathsf{T}^*\mathsf{E}) \otimes \mathsf{VE}), \qquad s \in \{0, 1, \dots, m\},$$

and

$$(C^m_s, \mathrm{id}_{\mathsf{E}}) \in \mathrm{VB}^r(\mathrm{T}^s(\pi^*_{\mathsf{E}}\mathsf{T}^*\mathsf{M}) \otimes \mathrm{T}^1_1(\mathsf{VE}); \mathrm{T}^{m-1}(\mathsf{T}^*\mathsf{E}) \otimes \mathrm{T}^1_1(\mathsf{VE})), \qquad s \in \{0, 1, \dots, m-1\},$$

such that

$$\nabla^{\mathsf{E},m} L^{\mathrm{e}} = \sum_{s=0}^{m} A^m_s((\nabla^{\mathsf{M},\pi_{\mathsf{E}},s} L)^{\mathrm{e}}) + \sum_{s=0}^{m-1} C^m_s((\nabla^{\mathsf{M},\pi_{\mathsf{E}},s} L)^{\mathrm{v}})$$ [4]

[4] Here we regard $\mathrm{T}^1_1(\mathsf{VE})$ as a subbundle of $\mathsf{T}^*\mathsf{E} \otimes \mathsf{VE}$ by the mapping

$$\mathrm{T}^1_1(\mathsf{VE}) \ni A \mapsto A \circ \mathrm{ver} \in \mathsf{T}^*\mathsf{E} \otimes \mathsf{VE}.$$

for all $L \in \Gamma^m(\mathrm{T}^1_1(\mathsf{E}))$. *Moreover, the vector bundle mappings* $A^m_0, A^m_1, \dots, A^m_m$ *and* $C^m_0, C^m_1, \dots, C^m_{m-1}$ *satisfy the recursion relations prescribed by*

$$A^0_0(\beta_0) = \beta_0,\ \ A^1_1(\beta_1) = \beta_1,\ \ A^1_0(\beta_0) = \mathrm{Ins}_1(\beta_0, B_{\pi_{\mathsf{E}}}),\ \ C^1_0(\gamma_0) = \gamma_0,$$

and, for $m \geq 2$,

$$\begin{aligned}
A^{m+1}_{m+1}(\beta_{m+1}) &= \beta_{m+1}\\
A^{m+1}_m(\beta_m) &= A^m_{m-1} \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\beta_m) - \sum_{j=1}^m \mathrm{Ins}_j(\beta_m, B_{\pi_{\mathsf{E}}}) + \mathrm{Ins}_{m+1}(\beta_m, B^*_{\pi_{\mathsf{E}}})\\
A^{m+1}_s(\beta_s) &= (\nabla^{\mathsf{E}} A^m_s)(\beta_m) + A^m_{s-1} \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\beta_s) - \sum_{j=1}^s A^m_s \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\mathrm{Ins}_j(\beta_s, B_{\pi_{\mathsf{E}}}))\\
&\quad + A^m_s \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\mathrm{Ins}_{s+1}(\beta_s, B^*_{\pi_{\mathsf{E}}})),\ \ s \in \{1, \dots, m-1\},\\
A^{m+1}_0(\beta_0) &= (\nabla^{\mathsf{E}} A^m_0)(\beta_0) - A^m_0 \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\mathrm{Ins}_1(\beta_0, B^*_{\pi_{\mathsf{E}}}))
\end{aligned}$$

and

$$\begin{aligned}
C^{m+1}_m(\gamma_m) &= C^m_{m-1} \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\gamma_m) + \gamma_m\\
C^{m+1}_s(\gamma_s) &= A^m_s \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\gamma_s) + (\nabla^{\mathsf{E}} C^m_s)(\gamma_s) + C^m_{s-1} \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\gamma_s)\\
&\quad - \sum_{j=1}^{s+1} C^m_s \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\mathrm{Ins}_j(\gamma_s, B_{\pi_{\mathsf{E}}})) + C^m_s \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\mathrm{Ins}_{s+1}(\gamma_s, B^*_{\pi_{\mathsf{E}}})),\ \ s \in \{1, \dots, m-1\},\\
C^{m+1}_0(\gamma_0) &= A^m_0 \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\gamma_0) + (\nabla^{\mathsf{E}} C^m_0)(\gamma_0) - C^m_0 \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\mathrm{Ins}_1(\gamma_0, B_{\pi_{\mathsf{E}}}))\\
&\quad + C^m_0 \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\mathrm{Ins}_2(\gamma_0, B^*_{\pi_{\mathsf{E}}})),
\end{aligned}$$

where $\beta_s \in \mathrm{T}^s(\pi^*_{\mathsf{E}}\mathsf{T}^*\mathsf{M}) \otimes \mathsf{VE}$, $s \in \{0, 1, \dots, m\}$, *and* $\gamma_s \in \mathrm{T}^s(\pi^*_{\mathsf{E}}\mathsf{T}^*\mathsf{M}) \otimes \mathrm{T}^1_1(\mathsf{VE})$, $s \in \{0, 1, \dots, m-1\}$.

Proof: The assertion is clearly true for $m = 0$ and, for $m = 1$, we have

$$\nabla^{\mathsf{E}} L^{\mathrm{e}} = (\nabla^{\pi_{\mathsf{E}}} L)^{\mathrm{e}} + \mathrm{Ins}_1(L, B_{\pi_{\mathsf{E}}}) + L^{\mathrm{v}}$$

by Lemma 4.5(vii), which gives the result for $m = 1$. Thus suppose the result true for $m \geq 2$ so that

$$\nabla^{\mathsf{E},m} L^{\mathrm{e}} = \sum_{s=0}^m A^m_s((\nabla^{\mathsf{M},\pi_{\mathsf{E}},s} L)^{\mathrm{e}}) + \sum_{s=0}^{m-1} C^m_s((\nabla^{\mathsf{M},\pi_{\mathsf{E}},s} L)^{\mathrm{v}})$$

for vector bundle mappings A^m_s and C^m_s satisfying the stated recursion relations. We then compute

$$\begin{aligned}
\nabla^{\mathsf{E},m+1} L^{\mathrm{e}} = &\sum_{s=0}^m (\nabla^{\mathsf{E}} A^m_s)((\nabla^{\mathsf{M},\pi_{\mathsf{E}},s} L)^{\mathrm{e}}) + \sum_{s=0}^m A^m_s \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\nabla^{\mathsf{E}}(\nabla^{\mathsf{M},\pi_{\mathsf{E}},s} L)^{\mathrm{e}})\\
&+ \sum_{s=0}^{m-1} (\nabla^{\mathsf{E}} C^m_s)((\nabla^{\mathsf{M},\pi_{\mathsf{E}},s} L)^{\mathrm{v}}) + \sum_{s=0}^{m-1} C^m_s \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\nabla^{\mathsf{E}}(\nabla^{\mathsf{M},\pi_{\mathsf{E}},s} L)^{\mathrm{v}})\\
= &\sum_{s=0}^m (\nabla^{\mathsf{E}} A^m_s)((\nabla^{\mathsf{M},\pi_{\mathsf{E}},s} L)^{\mathrm{e}}) + \sum_{s=0}^m A^m_s \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}((\nabla^{\mathsf{M},\pi_{\mathsf{E}},s+1} L)^{\mathrm{e}})
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{s=1}^{m} \sum_{j=1}^{s} A_s^m \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\mathrm{Ins}_j((\nabla^{\mathsf{M},\pi_{\mathsf{E}},s} L)^{\mathrm{e}}, B_{\pi_{\mathsf{E}}}))\\
&+ \sum_{s=0}^{m} A_s^m \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\mathrm{Ins}_{s+1}((\nabla^{\mathsf{M},\pi_{\mathsf{E}},s} L)^{\mathrm{e}}, B_{\pi_{\mathsf{E}}}^*)) + \sum_{s=0}^{m} A_s^m \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}((\nabla^{\mathsf{M},\pi_{\mathsf{E}},s} L)^{\mathrm{v}})\\
&+ \sum_{s=0}^{m-1} (\nabla^{\mathsf{E}} C_s^m)((\nabla^{\mathsf{M},\pi_{\mathsf{E}},s} L)^{\mathrm{v}}) + \sum_{s=0}^{m-1} C_s^m \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}((\nabla^{\mathsf{M},\pi_{\mathsf{E}},s+1} L)^{\mathrm{v}})\\
&- \sum_{s=1}^{m-1} \sum_{j=1}^{s} C_s^m \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\mathrm{Ins}_j((\nabla^{\mathsf{M},\pi_{\mathsf{E}},s} L)^{\mathrm{v}}, B_{\pi_{\mathsf{E}}}))\\
&+ \sum_{s=0}^{m-1} C_s^m \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\mathrm{Ins}_{s+1}((\nabla^{\mathsf{M},\pi_{\mathsf{E}},s} L)^{\mathrm{v}}, B_{\pi_{\mathsf{E}}}^*))\\
= {}& (\nabla^{\mathsf{M},\pi_{\mathsf{E}},m+1} L)^{\mathrm{e}} + \Bigg(A_{m-1}^m \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}((\nabla^{\mathsf{M},\pi_{\mathsf{E}},m} L)^{\mathrm{e}}) - \sum_{j=1}^{m} \mathrm{Ins}_j((\nabla^{\mathsf{M},\pi_{\mathsf{E}},m} L)^{\mathrm{e}}, B_{\pi_{\mathsf{E}}})\\
&+ \mathrm{Ins}_{m+1}((\nabla^{\mathsf{M},\pi_{\mathsf{E}},m} L)^{\mathrm{e}}, B_{\pi_{\mathsf{E}}}^*) + (\nabla^{\mathsf{M},\pi_{\mathsf{E}},m} L)^{\mathrm{v}} + C_{m-1}^m \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}((\nabla^{\mathsf{M},\pi_{\mathsf{E}},m} L)^{\mathrm{v}}) \Bigg)\\
&+ \Bigg(\sum_{s=1}^{m-1} (\nabla^{\mathsf{E}} A_s^m)((\nabla^{\mathsf{M},\pi_{\mathsf{E}},s} L)^{\mathrm{e}}) + \sum_{s=1}^{m-1} A_{s-1}^m \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}((\nabla^{\mathsf{M},\pi_{\mathsf{E}},s} L)^{\mathrm{e}})\\
&- \sum_{s=1}^{m-1} \sum_{j=1}^{s} A_s^m \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\mathrm{Ins}_j((\nabla^{\mathsf{M},\pi_{\mathsf{E}},s} L)^{\mathrm{e}}), B_{\pi_{\mathsf{E}}})\\
&+ \sum_{s=1}^{m-1} A_s^m \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\mathrm{Ins}_{s+1}((\nabla^{\mathsf{M},\pi_{\mathsf{E}},s} L)^{\mathrm{e}}), B_{\pi_{\mathsf{E}}}^*) + \sum_{s=1}^{m-1} A_s^m \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}((\nabla^{\mathsf{M},\pi_{\mathsf{E}},s} L)^{\mathrm{v}})\\
&+ \sum_{s=1}^{m-1} (\nabla^{\mathsf{E}} C_s^m)((\nabla^{\mathsf{M},\pi_{\mathsf{E}},s} L)^{\mathrm{v}}) + \sum_{s=1}^{m-1} C_{s-1}^m \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}((\nabla^{\mathsf{M},\pi_{\mathsf{E}},s} L)^{\mathrm{v}})\\
&- \sum_{s=0}^{m-1} \sum_{j=1}^{s} C_s^m \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\mathrm{Ins}_j((\nabla^{\mathsf{M},\pi_{\mathsf{E}},s} L)^{\mathrm{v}}, B_{\pi_{\mathsf{E}}}))\\
&+ \sum_{s=1}^{m-1} C_s^m \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\mathrm{Ins}_{s+1}((\nabla^{\mathsf{M},\pi_{\mathsf{E}},s} L)^{\mathrm{v}}, B_{\pi_{\mathsf{E}}}^*)) \Bigg)\\
&+ (\nabla^{\mathsf{E}} A_0^m)(L^{\mathrm{e}}) + A_0^m \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\mathrm{Ins}_1(L^{\mathrm{e}}, B_{\pi_{\mathsf{E}}}^*)) + A_0^m \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}(L^{\mathrm{v}})\\
&+ (\nabla^{\mathsf{E}} C_0^m)(L^{\mathrm{v}}) - C_0^m \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\mathrm{Ins}_1(L^{\mathrm{v}}, B_{\pi_{\mathsf{E}}})) + C_0^m \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\mathrm{Ins}_2(L^{\mathrm{v}}, B_{\pi_{\mathsf{E}}}^*)).
\end{aligned}
$$

From these calculations, the lemma follows. ■

Now we "invert" the constructions from the preceding lemma.

5.32 Lemma: (Iterated covariant differentials of vertical evaluations of endomorphisms II) *Let $r \in \{\infty, \omega\}$ and let $\pi_{\mathsf{E}} \colon \mathsf{E} \to \mathsf{M}$ be a C^r-vector bundle, with the data prescribed in Section 4.1 to define the Riemannian metric $\mathbb{G}_{\mathsf{E}}$ on E. For $m \in \mathbb{Z}_{\geq 0}$, there exist*

C^r-vector bundle mappings

$$(B^m_s, \mathrm{id}_{\mathsf{E}}) \in \mathrm{VB}^r(\mathrm{T}^s(\mathsf{T}^*\mathsf{E}) \otimes \mathsf{VE}; \mathrm{T}^m(\pi^*_{\mathsf{E}}\mathsf{T}^*\mathsf{M}) \otimes \mathsf{VE}), \qquad s \in \{0, 1, \dots, m\},$$

and

$$(D^m_s, \mathrm{id}_{\mathsf{E}}) \in \mathrm{VB}^r(\mathrm{T}^s(\mathsf{T}^*\mathsf{E}) \otimes \mathrm{T}^1_1(\mathsf{VE}); \mathrm{T}^{m-1}(\pi^*_{\mathsf{E}}\mathsf{T}^*\mathsf{M}) \otimes \mathrm{T}^1_1(\mathsf{VE})), \qquad s \in \{0, 1, \dots, m-1\},$$

such that

$$(\nabla^{\mathsf{M},\pi_{\mathsf{E}},m} L)^{\mathrm{e}} = \sum_{s=0}^{m} B^m_s(\nabla^{\mathsf{E},s} L^{\mathrm{e}}) + \sum_{s=0}^{m-1} D^m_s(\nabla^{\mathsf{E},s} L^{\mathrm{v}})$$

for all $L \in \Gamma^m(\mathrm{T}^1_1(\mathsf{E}))$. Moreover, the vector bundle mappings $B^m_0, B^m_1, \dots, B^m_m$ and $D^m_0, D^m_1, \dots, D^m_{m-1}$ satisfy the recursion relations prescribed by $B^0_0(\alpha_0) = \alpha_0$, $D^1_0(\gamma_0) = \gamma_0$,

$$\begin{aligned}
B^{m+1}_{m+1}(\alpha_{m+1}) &= \alpha_{m+1} \\
B^{m+1}_m(\alpha_m) &= B^m_{m-1} \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\alpha_m) + \sum_{j=1}^{m} \mathrm{Ins}_j(\alpha_m, B_{\pi_{\mathsf{E}}}) - \mathrm{Ins}_{m+1}(\alpha, B^*_{\pi_{\mathsf{E}}}) \\
B^{m+1}_s &= (\nabla^{\mathsf{E}} B^m_s)(\alpha_s) + B^m_{s-1} \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\alpha_s) + \sum_{j=1}^{m} \mathrm{Ins}_j(B^m_s(\alpha_s), B_{\pi_{\mathsf{E}}}) \\
&\quad - \mathrm{Ins}_{m+1}(B^m_s(\alpha_s), B^*_{\pi_{\mathsf{E}}}), \quad s \in \{1, \dots, m-1\}, \\
B^{m+1}_0(\alpha_0) &= (\nabla^{\mathsf{E}} B^m_0)(\alpha_0) + \sum_{j=1}^{m} \mathrm{Ins}_j(B^m_0(\alpha_0), B_{\pi_{\mathsf{E}}}) - \mathrm{Ins}_{m+1}(B^m_0(\alpha_0), B^*_{\pi_{\mathsf{E}}})
\end{aligned}$$

and

$$\begin{aligned}
D^{m+1}_m(\gamma_m) &= D^m_{m-1} \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\gamma_m) - \gamma_m \\
D^m_s(\gamma_s) &= (\nabla^{\mathsf{E}} D^m_s)(\gamma_s) + D^m_{s-1} \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\gamma_s) - \overline{B}^m_s(\gamma_s), \quad s \in \{1, \dots, m-1\}, \\
D^{m+1}_0 &= (\nabla^{\mathsf{E}} D^m_0)(\gamma_0) - \overline{B}^m_0(\gamma_0)
\end{aligned}$$

for $\alpha_s \in \mathrm{T}^s(\mathsf{T}^*\mathsf{E} \otimes \mathsf{VE})$, $s \in \{0, 1, \dots, m+1\}$, and $\gamma_s \in \mathrm{T}^s(\mathsf{T}^*\mathsf{E}) \otimes \mathrm{T}^1_1(\mathsf{VE})$, $s \in \{0, 1, \dots, m\}$, and where

$$(\overline{B}^m_s, \mathrm{id}_{\mathsf{E}}) \in \mathrm{VB}^r(\mathrm{T}^s(\mathsf{T}^*\mathsf{E}) \otimes \mathrm{T}^1_1(\mathsf{VE}); \mathrm{T}^m(\pi^*_{\mathsf{E}}\mathsf{T}^*\mathsf{M}) \otimes \mathrm{T}^1_1(\mathsf{VE})), \qquad s \in \{0, 1, \dots, m\},$$

are the vector bundle mappings from Lemma 5.22.

Proof: The assertion is clearly true for $m = 0$, so suppose it true for $m \in \mathbb{Z}_{>0}$. Thus

$$(\nabla^{\mathsf{M},\pi_{\mathsf{E}},m} L)^{\mathrm{e}} = \sum_{s=0}^{m} B^m_s(\nabla^{\mathsf{E},s} L^{\mathrm{e}}) + \sum_{s=0}^{m-1} D^m_s((\nabla^{\mathsf{E},s} L)^{\mathrm{v}}). \tag{5.7}$$

Working on the left-hand side of this equation, using Lemma 4.5(vii), we have

$$\begin{aligned}\nabla^{\mathsf{E}}(\nabla^{\mathsf{M},\pi_{\mathsf{E}},m}L)^{\mathrm{e}} &= (\nabla^{\mathsf{M},\pi_{\mathsf{E}},m+1}L)^{\mathrm{e}} - \sum_{j=1}^{m}\mathrm{Ins}_j((\nabla^{\mathsf{M},\pi_{\mathsf{E}},m}L)^{\mathrm{e}}, B_{\pi_{\mathsf{E}}}) \\ &\quad + \mathrm{Ins}_{m+1}((\nabla^{\mathsf{M},\pi_{\mathsf{E}},m}L)^{\mathrm{e}}, B^*_{\pi_{\mathsf{E}}}) + (\nabla^{\mathsf{M},\pi_{\mathsf{E}},m}L)^{\mathrm{v}} \\ &= (\nabla^{\mathsf{M},\pi_{\mathsf{E}},m+1}L)^{\mathrm{e}} - \sum_{s=0}^{m}\sum_{j=1}^{m}\mathrm{Ins}_j(B^m_s(\nabla^{\mathsf{E},s}L^{\mathrm{e}}), B_{\pi_{\mathsf{E}}}) \\ &\quad + \sum_{s=0}^{m}\mathrm{Ins}_{m+1}(B^m_s(\nabla^{\mathsf{E},s}L^{\mathrm{e}}), B^*_{\pi_{\mathsf{E}}}) + \sum_{s=0}^{m}\overline{B}^m_s(\nabla^{\mathsf{E},s}L^{\mathrm{v}}).\end{aligned}$$

Working on the right-hand side of (5.7),

$$\begin{aligned}\nabla^{\mathsf{E}}(\nabla^{\mathsf{M},\pi_{\mathsf{E}},m}L)^{\mathrm{e}} &= \sum_{s=0}^{m}(\nabla^{\mathsf{E}}B^m_s)(\nabla^{\mathsf{E},s}L^{\mathrm{e}}) + \sum_{s=0}^{m}B^m_s\otimes\mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\nabla^{\mathsf{E},s+1}L^{\mathrm{e}}) \\ &\quad + \sum_{s=0}^{m-1}(\nabla^{\mathsf{E}}D^m_s)(\nabla^{\mathsf{E},s}L^{\mathrm{v}}) + \sum_{s=0}^{m-1}D^m_s\otimes\mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\nabla^{\mathsf{E},s+1}L^{\mathrm{v}}).\end{aligned}$$

Combining the preceding two computations,

$$\begin{aligned}(\nabla^{\mathsf{M},\pi_{\mathsf{E}},m+1}L)^{\mathrm{e}} &= \sum_{s=0}^{m}(\nabla^{\mathsf{E}}B^m_s)(\nabla^{\mathsf{E},s}L^{\mathrm{e}}) + \sum_{s=0}^{m}B^m_s\otimes\mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\nabla^{\mathsf{E},s+1}L^{\mathrm{e}}) \\ &\quad + \sum_{s=0}^{m-1}(\nabla^{\mathsf{E}}D^m_s)(\nabla^{\mathsf{E},s}L^{\mathrm{v}}) + \sum_{s=0}^{m-1}D^m_s\otimes\mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\nabla^{\mathsf{E},s+1}L^{\mathrm{v}}) \\ &\quad + \sum_{s=0}^{m}\sum_{j=1}^{m}\mathrm{Ins}_j(B^m_s(\nabla^{\mathsf{E},s}L^{\mathrm{e}}), B_{\pi_{\mathsf{E}}}) - \sum_{s=1}^{m}\mathrm{Ins}_{m+1}(B^m_s(\nabla^{\mathsf{E},s}L^{\mathrm{e}}), B^*_{\pi_{\mathsf{E}}}) \\ &\quad - \sum_{s=0}^{m}\overline{B}^m_s(\nabla^{\mathsf{E},s}L^{\mathrm{v}}) \\ &= \nabla^{\mathsf{E},m+1}L^{\mathrm{e}} + \Bigg(B^m_{m-1}\otimes\mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\nabla^{\mathsf{E},m}L^{\mathrm{e}}) + D^m_{m-1}\otimes\mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\nabla^{\mathsf{E},m}L^{\mathrm{v}}) \\ &\quad + \sum_{j=1}^{m}\mathrm{Ins}_j(\nabla^{\mathsf{E},m}L^{\mathrm{e}}, B_{\pi_{\mathsf{E}}}) - \mathrm{Ins}_{m+1}(\nabla^{\mathsf{E},m}L^{\mathrm{e}}, B^*_{\pi_{\mathsf{E}}}) - (\nabla^{\mathsf{E},m}L^{\mathrm{v}})\Bigg) \\ &\quad + \Bigg(\sum_{s=1}^{m-1}(\nabla^{\mathsf{E}}B^m_s)(\nabla^{\mathsf{E},s}L^{\mathrm{e}}) + \sum_{s=1}^{m-1}B^m_{s-1}\otimes\mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\nabla^{\mathsf{E},s}L^{\mathrm{e}}) + \sum_{s=1}^{m-1}(\nabla^{\mathsf{E}}D^m_s)(\nabla^{\mathsf{E},s}L^{\mathrm{v}}) \\ &\quad + \sum_{s=1}^{m-1}D^m_{s-1}\otimes\mathrm{id}_{\mathsf{T}^*\mathsf{E}}(\nabla^{\mathsf{E},s}L^{\mathrm{v}}) + \sum_{s=1}^{m-1}\sum_{j=1}^{m}\mathrm{Ins}_j(B^m_s(\nabla^{\mathsf{E},s}L^{\mathrm{e}}), B_{\pi_{\mathsf{E}}}) \\ &\quad - \sum_{s=1}^{m-1}\mathrm{Ins}_{m+1}(B^m_s(\nabla^{\mathsf{E},s}L^{\mathrm{e}}), B^*_{\pi_{\mathsf{E}}}) - \sum_{s=1}^{m-1}\overline{B}^m_s(\nabla^{\mathsf{E},s}L^{\mathrm{v}})\Bigg) \\ &\quad + \Bigg((\nabla^{\mathsf{E}}B^m_0)(L^{\mathrm{e}}) + (\nabla^{\mathsf{E}}D^m_0)(L^{\mathrm{v}}) + \sum_{j=1}^{m}\mathrm{Ins}_j(B^m_0(L^{\mathrm{e}}), B_{\pi_{\mathsf{E}}})\end{aligned}$$

$$- \operatorname{Ins}_{m+1}(B_0^m(L^{\mathrm{e}}), B^*_{\pi_{\mathsf{E}}}) - \overline{B}_0^m(L^{\mathrm{v}})\Bigg).$$

The lemma follows from these computations. ■

Next we turn to symmetrised versions of the preceding lemmata. We show that the preceding two lemmata induce corresponding mappings between symmetric tensors.

5.33 Lemma: (Iterated symmetrised covariant differentials of vertical evaluations of endomorphisms I) *Let $r \in \{\infty, \omega\}$ and let $\pi_{\mathsf{E}} \colon \mathsf{E} \to \mathsf{M}$ be a C^r-vector bundle, with the data prescribed in Section 4.1 to define the Riemannian metric $\mathbb{G}_{\mathsf{E}}$ on E. For $m \in \mathbb{Z}_{\geq 0}$, there exist C^r-vector bundle mappings*

$$(\widehat{A}^m_s, \mathrm{id}_{\mathsf{E}}) \in \mathrm{VB}^r(\mathrm{S}^s(\pi^*_{\mathsf{E}}\mathsf{T}^*\mathsf{M}) \otimes \mathsf{VE}; \mathrm{S}^m(\mathsf{T}^*\mathsf{E}) \otimes \mathsf{VE}), \qquad s \in \{0, 1, \dots, m\},$$

and

$$(\widehat{C}^m_s, \mathrm{id}_{\mathsf{E}}) \in \mathrm{VB}^r(\mathrm{S}^s(\pi^*_{\mathsf{E}}\mathsf{T}^*\mathsf{M}) \otimes \mathrm{T}^1_1(\mathsf{VE}); \mathrm{S}^m(\mathsf{T}^*\mathsf{E}) \otimes \mathsf{VE}), \qquad s \in \{0, 1, \dots, m-1\},$$

such that

$$\begin{aligned}(\mathrm{Sym}_m \otimes \mathrm{id}_{\mathsf{VE}}) \circ \nabla^{\mathsf{E},m} L^{\mathrm{e}} \\ = \sum_{s=0}^m \widehat{A}^m_s((\mathrm{Sym}_s \otimes \mathrm{id}_{\mathsf{VE}}) \circ (\nabla^{\mathsf{M},\pi_{\mathsf{E}},s} L)^{\mathrm{e}} + \sum_{s=0}^{m-1} \widehat{C}^m_s((\mathrm{Sym}_s \otimes \mathrm{id}_{\mathrm{T}^1_1(\mathsf{VE})}) \circ (\nabla^{\mathsf{M},\pi_{\mathsf{E}},s} L)^{\mathrm{v}})\end{aligned}$$

for all $L \in \Gamma^m(\mathrm{T}^1_1(\mathsf{E}))$.

Proof: Following along the lines of the proof of Lemma 5.8, we define $\widehat{A}^m_s$ by requiring that

$$\widehat{A}^m_s((\mathrm{Sym}_s \otimes \mathrm{id}_{\mathsf{VE}}) \circ (\nabla^{\mathsf{M},\pi_{\mathsf{E}},s} L)^{\mathrm{e}}) = (\mathrm{Sym}_m \otimes \mathrm{id}_{\mathsf{VE}}) \circ A^m_s((\nabla^{\mathsf{M},\pi_{\mathsf{E}},s} L)^{\mathrm{e}}),$$

and $\widehat{C}^m_s$ by requiring that

$$\widehat{C}^m_s((\mathrm{Sym}_s \otimes \mathrm{id}_{\mathrm{T}^1_1(\mathsf{VE})}) \circ (\nabla^{\mathsf{M},\pi_{\mathsf{E}},s} L)^{\mathrm{e}}) = (\mathrm{Sym}_m \otimes \mathrm{id}_{\mathsf{VE}}) \circ C^m_s((\nabla^{\mathsf{M},\pi_{\mathsf{E}},s} L)^{\mathrm{e}}).$$

That this definition of $\widehat{A}^m_s$ makes sense follows exactly as in the proof of Lemma 5.8. Let us see how the same arguments also apply to the definition of $\widehat{C}^m_s$.

For $m \in \mathbb{Z}_{>0}$, we define $C^m \colon \mathrm{T}^{\leq m-1}(\pi^*_{\mathsf{E}}\mathsf{T}^*\mathsf{M}) \otimes \mathrm{T}^1_1(\mathsf{VE}) \to \mathrm{T}^{\leq m}(\mathsf{T}^*\mathsf{E}) \otimes \mathsf{VE}$ by

$$\begin{aligned}C^m(L^{\mathrm{v}}, (\nabla^{\pi_{\mathsf{E}}} L)^{\mathrm{v}}, \dots, (\nabla^{\mathsf{M},\pi_{\mathsf{E}},m-1} L)^{\mathrm{v}}) \\ = \left(C^1_0(L^{\mathrm{v}}), \sum_{s=0}^1 C^2_s((\nabla^{\mathsf{M},\pi_{\mathsf{E}},s} L)^{\mathrm{v}}), \dots, \sum_{s=0}^{m-1} C^m_s((\nabla^{\mathsf{M},\pi_{\mathsf{E}},s} L)^{\mathrm{v}})\right),\end{aligned}$$

making the identification of $\mathrm{T}^1_1(\mathsf{VE})$ with a subspace of $\mathsf{T}^*\mathsf{E} \otimes \mathsf{VE}$ as in the footnote from Lemma 5.31. Note that we have a natural mapping

$$\mathsf{T}^*\mathsf{E} \otimes \mathsf{T}^{*m-1}\mathsf{E} \to \mathsf{T}^{*m}\mathsf{E}$$

cf. [Saunders 1989, Theorem 6.2.9]. This then induces a mapping

$$P_m\colon (\mathbb{R}_{\mathsf{E}} \oplus \mathsf{T}^{*m-1}\mathsf{E}) \otimes \mathrm{T}^1_1(\mathsf{VE}) \to (\mathbb{R}_{\mathsf{E}} \otimes \mathsf{T}^{*m}\mathsf{E}) \otimes \mathsf{VE}.$$

Now define

$$\widehat{P}_m\colon \pi^*_{\mathsf{E}}(\mathbb{R}_{\mathsf{M}} \oplus \mathsf{T}^{*m-1}\mathsf{M}) \otimes \mathrm{T}^1_1(\mathsf{VE}) \to (\mathbb{R}_{\mathsf{E}} \oplus \mathsf{T}^{*m}\mathsf{E}) \otimes \mathsf{VE}$$

by

$$\widehat{P}_m = P_m \circ ((\mathrm{id}_{\mathbb{R}} \oplus j_{m-1}\pi_{\mathsf{E}}) \otimes \mathrm{id}_{\mathrm{T}^1_1(\mathsf{VE})}),$$

noting that

$$\mathrm{id}_{\mathbb{R}} \oplus j_{m-1}\pi_{\mathsf{E}}\colon \pi^*_{\mathsf{E}}(\mathbb{R}_{\mathsf{M}}\mathsf{T}^{*m-1}\mathsf{M}) \to \mathbb{R}_{\mathsf{E}} \oplus \mathsf{T}^{*m-1}\mathsf{E}$$

is injective. Also define

$$Q_m\colon \mathrm{S}^{\le m-1}(\mathsf{T}^*\mathsf{E}) \otimes \mathrm{T}^1_1(\mathsf{VE}) \to \mathrm{S}^m(\mathsf{T}^*\mathsf{E}) \otimes \mathsf{VE}$$

by

$$\begin{aligned} Q_m(A_0 \otimes \alpha_0 \otimes u_0, \dots, A_{m-1} \otimes \alpha_{m-1} \otimes u_{m-1}) \\ = (\mathrm{Sym}_1(A_0 \otimes \alpha_0) \otimes u_0, \dots, \mathrm{Sym}_m(A_{m-1} \otimes \alpha_{m-1}) \otimes u_{m-1}). \end{aligned}$$

Note that the diagram

$$\begin{array}{ccc} \mathrm{S}^{\le m-1}(\mathsf{T}^*\mathsf{E}) \otimes \mathrm{T}^1_1(\mathsf{VE}) & \xrightarrow{S^{m-1}_{\nabla^{\mathsf{E}}} \otimes \mathrm{id}_{\mathrm{T}^1_1(\mathsf{VE})}} & (\mathbb{R}_{\mathsf{E}} \oplus \mathsf{T}^{*m-1}\mathsf{E}) \otimes \mathrm{T}^1_1(\mathsf{VE}) \\ {\scriptstyle Q_m}\downarrow & & \downarrow{\scriptstyle P_m} \\ \mathrm{S}^{\le m}(\mathsf{T}^*\mathsf{E}) \otimes \mathsf{VE} & \xrightarrow[S^m_{\nabla^{\mathsf{E}}} \otimes \mathrm{id}_{\mathsf{VE}}]{} & (\mathbb{R}_{\mathsf{E}} \oplus \mathsf{T}^{*m}\mathsf{E}) \otimes \mathsf{VE} \end{array}$$

commutes. We also define

$$\widehat{Q}_m = Q_m \circ (\pi^*_{m-1} \otimes \mathrm{id}_{\mathrm{T}^1_1(\mathsf{VE})}),$$

where

$$\pi^*_{m-1}\colon \mathrm{S}^{\le m-1}(\pi^*_{\mathsf{E}}\mathsf{T}^*\mathsf{M}) \to \mathrm{S}^{\le m-1}(\mathsf{T}^*\mathsf{E})$$

is the inclusion. Note that the diagram

$$\begin{array}{ccc} \mathrm{S}^{\le m-1}(\pi^*_{\mathsf{E}}\mathsf{T}^*\mathsf{M}) & \xrightarrow{\pi^*_{m-1}} & \mathrm{S}^{\le m-1}(\mathsf{T}^*\mathsf{E}) \\ {\scriptstyle S^{m-1}_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}}}\downarrow & & \downarrow{\scriptstyle S^{m-1}_{\nabla^{\mathsf{E}}}} \\ \pi^*_{\mathsf{E}}(\mathbb{R}_{\mathsf{M}} \oplus \mathsf{T}^{*m-1}\mathsf{M}) & \xrightarrow[\mathrm{id}_{\mathbb{R}} \oplus j_{m-1}\pi_{\mathsf{E}}]{} & \mathbb{R}_{\mathsf{E}} \oplus \mathsf{T}^{*m-1}\mathsf{E} \end{array}$$

commutes.

Let us organise the mappings we require into the following diagram:

$$\begin{array}{ccccc} \mathrm{T}^{\le m-1}(\pi^*_{\mathsf{E}}\mathsf{T}^*\mathsf{M}) \otimes \mathrm{T}^1_1(\mathsf{VE}) & \xrightarrow{\mathrm{Sym}_{\le m-1} \otimes \mathrm{id}_{\mathrm{T}^1_1(\mathsf{VE})}} & \mathrm{S}^{\le m-1}(\pi^*_{\mathsf{E}}\mathsf{T}^*\mathsf{M}) \otimes \mathrm{T}^1_1(\mathsf{VE}) & \xrightarrow{S^{m-1}_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}} \otimes \mathrm{id}_{\mathrm{T}^1_1(\mathsf{VE})}} & \pi^*_{\mathsf{E}}(\mathbb{R}_{\mathsf{M}} \oplus \mathsf{T}^{*m-1}\mathsf{M}) \otimes \mathrm{T}^1_1(\mathsf{VE}) \\ {\scriptstyle C^m}\downarrow & & \downarrow{\scriptstyle \widehat{C}^m} & & \downarrow{\scriptstyle \widehat{P}_m} \\ \mathrm{T}^{\le m}(\mathsf{T}^*\mathsf{E}) \otimes \mathsf{VE} & \xrightarrow{\mathrm{Sym}_{\le m} \otimes \mathrm{id}_{\mathsf{VE}}} & \mathrm{S}^{\le m}(\mathsf{T}^*\mathsf{E}) \otimes \mathsf{VE} & \xrightarrow{S^m_{\nabla^{\mathsf{E}}} \otimes \mathrm{id}_{\mathsf{VE}}} & (\mathbb{R}_{\mathsf{E}} \oplus \mathsf{T}^{*m}\mathsf{E}) \otimes \mathsf{VE} \end{array} \tag{5.8}$$

Here $\widehat{C}^m$ is defined so that the right square commutes, which is possible since the horizontal arrows in the right square are isomorphisms. We shall show that the left square also commutes. Indeed,

$$\begin{aligned}\widehat{C}^m \circ (\mathrm{Sym}_{\le m-1} \otimes \mathrm{id}_{\mathrm{T}^1_1(\mathsf{VE})})(L^{\mathrm{v}}, (\nabla^{\pi_{\mathsf{E}}} L)^{\mathrm{v}}, \dots, (\nabla^{\mathsf{M},\pi_{\mathsf{E}},m} L)^{\mathrm{v}}) \\ = (S^m_{\nabla^{\mathsf{E}}} \otimes \mathrm{id}_{\mathsf{VE}})^{-1} \circ \widehat{P}_m \circ (S^{m-1}_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}} \otimes \mathrm{id}_{\mathrm{T}^1_1(\mathsf{VE})}) \\ \circ (\mathrm{Sym}_{\le m-1} \otimes \mathrm{id}_{\mathrm{T}^1_1(\mathsf{VE})})(L^{\mathrm{v}}, (\nabla^{\pi_{\mathsf{E}}} L)^{\mathrm{v}}, \dots, (\nabla^{\mathsf{M},\pi_{\mathsf{E}},m} L^{\mathrm{v}}) \\ = (\mathrm{Sym}_{\le m-1} \otimes \mathrm{id}_{\mathsf{VE}})(L^{\mathrm{v}}, \nabla^{\mathsf{E}} L^{\mathrm{v}}, \dots, \nabla^{\mathsf{E},m} L^{\mathrm{v}}) \\ = (\mathrm{Sym}_{\le m} \otimes \mathrm{id}_{\mathsf{VE}}) \circ C^m(L^{\mathrm{v}}, (\nabla^{\pi_{\mathsf{E}}} L)^{\mathrm{v}}, \dots, (\nabla^{\mathsf{M},\pi_{\mathsf{E}},m} L)^{\mathrm{v}}).\end{aligned}$$

Thus the diagram (5.8) commutes. Thus, if we define

$$\widehat{C}^m_s((\mathrm{Sym}_s \otimes \mathrm{id}_{\mathrm{T}^1_1(\mathsf{VE})}) \circ (\nabla^{\mathsf{M},\pi_{\mathsf{E}},s} L)^{\mathrm{v}}) = (\mathrm{Sym}_m \otimes \mathrm{id}_{\mathsf{VE}}) \circ C^m_s((\nabla^{\mathsf{M},\pi_{\mathsf{E}},s} L)^{\mathrm{v}}),$$

then we have

$$(\mathrm{Sym}_m \otimes \mathrm{id}_{\mathsf{VE}}) \circ \nabla^{\mathsf{E},m} L^{\mathrm{e}} = \sum_{s=0}^{m} \widehat{C}^m_s((\mathrm{Sym}_s \otimes \mathrm{id}_{\mathrm{T}^1_1(\mathsf{VE})}) \circ (\nabla^{\mathsf{M},\pi_{\mathsf{E}},s} L)^{\mathrm{v}}),$$

as desired. ■

The preceding lemma gives rise to an "inverse," which we state in the following lemma.

5.34 Lemma: *Let $r \in \{\infty, \omega\}$ and let $\pi_{\mathsf{E}} \colon \mathsf{E} \to \mathsf{M}$ be a C^r-vector bundle, with the data prescribed in Section 4.1 to define the Riemannian metric $\mathbb{G}_{\mathsf{E}}$ on E. For $m \in \mathbb{Z}_{\ge 0}$, there exist C^r-vector bundle mappings*

$$(\widehat{B}^m_s, \mathrm{id}_{\mathsf{E}}) \in \mathrm{VB}^r(\mathrm{S}^s(\mathsf{T}^*\mathsf{E}) \otimes \mathsf{VE}; \mathrm{S}^m(\pi^*_{\mathsf{E}}\mathsf{T}^*\mathsf{M}) \otimes \mathsf{VE}), \qquad s \in \{0, 1, \dots, m\},$$

and

$$(\widehat{D}^m_s, \mathrm{id}_{\mathsf{E}}) \in \mathrm{VB}^r(\mathrm{S}^s(\mathsf{T}^*\mathsf{E}) \otimes \mathrm{T}^1_1(\mathsf{VE}); \mathrm{S}^m(\pi^*_{\mathsf{E}}\mathsf{T}^*\mathsf{M}) \otimes \mathsf{VE}), \qquad s \in \{0, 1, \dots, m-1\},$$

such that

$$\begin{aligned}(\mathrm{Sym}_m \otimes_{\mathrm{id}_{\mathsf{VE}}}) \circ (\nabla^{\mathsf{M},\pi_{\mathsf{E}},m} L)^{\mathrm{e}} \\ = \sum_{s=0}^{m} \widehat{B}^m_s((\mathrm{Sym}_s \otimes \mathrm{id}_{\mathsf{VE}}) \circ \nabla^{\mathsf{E},s} L^{\mathrm{e}}) + \sum_{s=0}^{m-1} \widehat{D}^m_s((\mathrm{Sym}_s \otimes \mathrm{id}_{\mathrm{T}^1_1(\mathsf{VE})}) \circ \nabla^{\mathsf{E},s} L^{\mathrm{v}})\end{aligned}$$

for all $L \in \Gamma^m(\mathrm{T}^1_1(\mathsf{E}))$.

Proof: Following along the lines of the proof of Lemma 5.8, we define $\widehat{B}^m_s$ by requiring that

$$\widehat{B}^m_s((\mathrm{Sym}_s \otimes \mathrm{id}_{\mathsf{VE}}) \circ \nabla^{\mathsf{E},s} L^{\mathrm{e}} = (\mathrm{Sym}_m \otimes \mathrm{id}_{\mathsf{VE}}) \circ B^m_s(\nabla^{\mathsf{E},s} L^{\mathrm{e}}),$$

and $\widehat{C}^m_s$ by requiring that

$$\widehat{C}^m_s((\mathrm{Sym}_s \otimes \mathrm{id}_{\mathrm{T}^1_1(\mathsf{VE})}) \circ \nabla^{\mathsf{E},s} L^{\mathrm{v}}) = (\mathrm{Sym}_m \otimes \mathrm{id}_{\mathsf{VE}}) \circ C^m_s(\nabla^{\mathsf{E},s} L^{\mathrm{v}}).$$

That these definitions make sense follows along the same lines as the proof of Lemma 5.33. ■

We can put together the previous four lemmata into the following decomposition result, which is to be regarded as the main result of this section.

5.35 Lemma: (Decomposition of jets of vertical evaluations of endomorphisms) *Let $r \in \{\infty, \omega\}$ and let $\pi_{\mathsf{E}} \colon \mathsf{E} \to \mathsf{M}$ be a C^r-vector bundle, with the data prescribed in Section 4.1 to define the Riemannian metric $\mathbb{G}_{\mathsf{E}}$ on E. Then there exist C^r-vector bundle mappings*

$$A^m_{\nabla^{\mathsf{E}}} \in \mathrm{VB}^r(\mathsf{C}^{*m}\mathsf{E}; \mathrm{S}^{\leq m}(\pi_{\mathsf{E}}^*\mathsf{T}^*\mathsf{M}) \otimes \mathsf{VE}), \quad B^m_{\nabla^{\mathsf{E}}} \in \mathrm{VB}^r(\mathsf{C}^{*m}\mathsf{E}; \mathrm{S}^{\leq m}(\mathsf{T}^*\mathsf{E}) \otimes \mathsf{VE})$$

defined by

$$\begin{aligned} A^m_{\nabla^{\mathsf{E}}}(j_m(L^{\mathrm{e}})(e)) &= \mathrm{Sym}_{\leq m} \otimes \mathrm{id}_{\mathsf{VE}}(L^{\mathrm{e}}(e), (\nabla^{\pi_{\mathsf{E}}} L)^{\mathrm{e}}(e), \dots, (\nabla^{\mathsf{M},\pi_{\mathsf{E}},m} L)^{\mathrm{e}}(e)), \\ B^m_{\nabla^{\mathsf{E}}}(j_m(L^{\mathrm{e}})(e)) &= \mathrm{Sym}_{\leq m} \otimes \mathrm{id}_{\mathsf{VE}}(L^{\mathrm{e}}(e), \nabla^{\mathsf{E}} L^{\mathrm{e}}(e), \dots, \nabla^{\mathsf{E},m} L^{\mathrm{e}}(e)). \end{aligned}$$

Moreover, $A^m_{\nabla^{\mathsf{E}}}$ and $B^m_{\nabla^{\mathsf{E}}}$ are injective, and

$$\begin{aligned} B^m_{\nabla^{\mathsf{E}}} \circ (A^m_{\nabla^{\mathsf{E}}})^{-1} &\circ (\mathrm{Sym}_{\leq m} \otimes \mathrm{id}_{\mathsf{VE}})(L^{\mathrm{e}}(e), (\nabla^{\pi_{\mathsf{E}}} L)^{\mathrm{e}}(e), \dots, (\nabla^{\mathsf{M},\pi_{\mathsf{E}},m} L)^{\mathrm{e}}(e)) \\ &= \Bigg(L^{\mathrm{e}}(e), \sum_{s=0}^{1} \widehat{A}^1_s((\mathrm{Sym}_s \otimes \mathrm{id}_{\mathsf{VE}}) \circ (\nabla^{\mathsf{M},\pi_{\mathsf{E}},s} L)^{\mathrm{e}}(e)), \dots, \\ &\qquad \sum_{s=0}^{m} \widehat{A}^m_s((\mathrm{Sym}_s \otimes \mathrm{id}_{\mathsf{VE}}) \circ (\nabla^{\mathsf{M},\pi_{\mathsf{E}},s} L)^{\mathrm{e}}(e)) \Bigg) \\ &\quad + \Bigg(0, L^{\mathrm{v}}(e), \sum_{s=0}^{1} \widehat{C}^2_s((\mathrm{Sym}_s \otimes \mathrm{id}_{\mathrm{T}^1_1(\mathsf{VE})}) \circ (\nabla^{\mathsf{M},\pi_{\mathsf{E}},s} L)^{\mathrm{v}}(e)), \dots, \\ &\qquad \sum_{s=0}^{m-1} \widehat{C}^m_s((\mathrm{Sym}_s \otimes \mathrm{id}_{\mathrm{T}^1_1(\mathsf{VE})}) \circ (\nabla^{\mathsf{M},\pi_{\mathsf{E}},s} L)^{\mathrm{v}}(e)) \Bigg) \end{aligned}$$

and

$$\begin{aligned} A^m_{\nabla^{\mathsf{E}}} \circ (B^m_{\nabla^{\mathsf{E}}})^{-1} &\circ (\mathrm{Sym}_{\leq m} \otimes \mathrm{id}_{\mathsf{VE}})(L^{\mathrm{e}}(e), \nabla^{\mathsf{E}} L^{\mathrm{e}}(e), \dots, \nabla^{\mathsf{E},m} L^{\mathrm{e}}(e)) \\ &= \Bigg(L^{\mathrm{e}}(e), \sum_{s=0}^{1} \widehat{B}^1_s((\mathrm{Sym}_s \otimes \mathrm{id}_{\mathsf{VE}}) \circ \nabla^{\mathsf{E},s} L^{\mathrm{e}}(e)), \dots, \sum_{s=0}^{m} \widehat{B}^m_s((\mathrm{Sym}_s \otimes \mathrm{id}_{\mathsf{VE}}) \circ \nabla^{\mathsf{E},s} L^{\mathrm{e}}(e)) \Bigg) \\ &\quad + \Bigg(0, L^{\mathrm{v}}(e), \sum_{s=0}^{1} \widehat{D}^2_s((\mathrm{Sym}_s \otimes \mathrm{id}_{\mathrm{T}^1_1(\mathsf{VE})}) \circ \nabla^{\mathsf{E},s} L^{\mathrm{v}}(e)), \dots, \\ &\qquad \sum_{s=0}^{m-1} \widehat{D}^m_s((\mathrm{Sym}_s \otimes \mathrm{id}_{\mathrm{T}^1_1(\mathsf{VE})}) \circ \nabla^{\mathsf{E},s} L^{\mathrm{v}}(e)) \Bigg), \end{aligned}$$

where the vector bundle mappings $\widehat{A}^m_s$ and $\widehat{B}^m_s$, $s \in \{0, 1, \dots, m\}$, and $\widehat{C}^m_s$ and $\widehat{D}^m_s$, $s \in \{0, 1, \dots, m-1\}$, are as in Lemmata 5.33 and 5.34.

5.8. Isomorphisms for pull-backs of functions. Next we generalise the presentation of Section 5.1 from the pull-back of a vector bundle projection to the pull-back by a general mapping. The development here is a little different from the preceding sections, so we first have a little bit of setting up to do. For C^r-manifolds M and N, and for $\Phi \in \mathrm{C}^r(\mathsf{M}; \mathsf{N})$, we consider the mapping

$$\mathrm{C}^r(\mathsf{N}) \ni f \mapsto \Phi^* f \in \mathrm{C}^r(\mathsf{M}).$$

We wish to compare the decomposition of jets of f with those of Φ^*f, and to do so we consider the subbundle $\mathsf{T}^{*m}_{\Phi}\mathsf{M}$ of $\mathsf{T}^{*m}\mathsf{M}$ defined by

$$\mathsf{T}^{*m}_{\Phi,x}\mathsf{M} = \{j_m(\Phi^*f)(x) \mid f \in \mathrm{C}^m(\mathsf{N})\}.$$

Following Lemma 2.1, we shall give a formula for iterated covariant differentials of pull-backs of functions on N. To do this, we let ∇^{M} and ∇^{N} be affine connections on M and N, respectively. We note that we have the vector bundle connection $\Phi^*\nabla^{\mathsf{N}}$ in the vector bundle $\Phi^*\mathsf{TN}$ over M. Explicitly,

$$(\Phi^*\nabla^{\mathsf{N}}_X\Phi^*Y)(x) = (x, \nabla^{\mathsf{N}}_{T_x\Phi(X(x))}Y).$$

Following our usual mild notational abuse, we shall also denote by $\Phi^*\nabla^{\mathsf{N}}$ the connection in the dual bundle $(\Phi^*\mathsf{TN})^* \simeq \Phi^*\mathsf{T}^*\mathsf{N}$. We have a natural mapping

$$\begin{aligned} \widehat{\Phi}\colon \mathsf{TM} &\to \Phi^*\mathsf{TN} \\ v_x &\mapsto (x, T_x\Phi(v_x)). \end{aligned}$$

This mapping induces a mappings on sections which we denote by the same symbol; thus we have the mapping

$$\widehat{\Phi}\colon \Gamma^\infty(\mathsf{TM}) \to \Gamma^\infty(\Phi^*\mathsf{TN}).$$

The following lemma gives an important tensor for our analysis.

5.36 Lemma: (Tensor for pull-back connection) *Let $r \in \{\infty, \omega\}$. Let M and N be C^r-manifolds and let ∇^{M} and ∇^{N} be C^r-affine connections on M and N, respectively. Let $\Phi \in \mathrm{C}^r(\mathsf{M};\mathsf{N})$. Then there exists $A_\Phi \in \Gamma^r(\mathrm{T}^2(\mathsf{T}^*\mathsf{M}) \otimes \Phi^*\mathsf{TN})$ such that, for $x \in \mathsf{M}$,*

$$\widehat{\Phi}(\nabla^{\mathsf{M}}_X Y)(x) - \Phi^*\nabla^{\mathsf{N}}_X\widehat{\Phi}(Y)(x) = A_\Phi(X(x), Y(x))$$

for $X, Y \in \Gamma^\infty(\mathsf{TM})$.

Proof: Let $K^{\mathsf{M}}\colon \mathsf{TTM} \to \mathsf{TM}$ and $K^{\mathsf{N}}\colon \mathsf{TTN} \to \mathsf{TN}$ be the connectors for ∇^{M} and ∇^{N} so that

$$\nabla^{\mathsf{M}}_X Y = K^{\mathsf{M}} \circ TY \circ X, \qquad X, Y \in \Gamma^\infty(\mathsf{TM}),$$

and

$$\nabla^{\mathsf{N}}_U V = K^{\mathsf{N}} \circ TV \circ U, \qquad U, V \in \Gamma^\infty(\mathsf{TN}).$$

We, moreover, have

$$\widehat{\Phi}(\nabla^{\mathsf{M}}_X Y) = T\Phi \circ K^{\mathsf{M}} \circ TY \circ X, \qquad X, Y \in \Gamma^\infty(\mathsf{TM}),$$

and

$$\Phi^*\nabla^{\mathsf{N}}_X\widehat{\Phi}(Y) = K^{\mathsf{N}} \circ T(T\Phi \circ Y) \circ X, \qquad X, Y \in \Gamma^\infty(\mathsf{TM})$$

[Michor 2008, §10.12]. In preparation to use these formulae, we have the following results.

1 Sublemma: *If* $\pi_{\mathsf{E}}\colon \mathsf{E}\to\mathsf{M}$ *is a smooth vector bundle, if* $\xi\in\Gamma^\infty(\mathsf{E})$*, and if* $f\in\mathrm{C}^\infty(\mathsf{M})$*, then*

$$T_x(f\xi)(v_x)=f(x)T_x\xi(v_x)+\langle\mathrm{d}f(x);v_x\rangle\xi^{\mathrm{v}}(x).$$

Proof: Let $\nabla^{\pi_{\mathsf{E}}}$ be a linear connection in the vector bundle E which gives the decomposition $\mathsf{TE}=\mathsf{HE}\oplus\mathsf{VE}$. Let hor and ver be the horizontal and vertical projections. Let $v_x\in\mathsf{T}_x\mathsf{M}$ and let $\gamma\colon I\to\mathsf{M}$ be a smooth curve for which $\gamma'(0)=v_x$. Denote $\Xi(t)=f\circ\gamma(t)\xi\circ\gamma(t)$ the corresponding curve in E. Then

$$\mathrm{hor}(\Xi'(t))=\mathrm{hlft}(f\circ\gamma(t)\xi\circ\gamma(t),\gamma'(t)),\quad \mathrm{ver}(\Xi'(t))=\mathrm{vlft}(f\circ\gamma(t)\xi\circ\gamma(t),\nabla^{\pi_{\mathsf{E}}}_{\gamma'(t)}\Xi(t)).$$

We now have

$$\nabla^{\pi_{\mathsf{E}}}_{\gamma'(t)}\Xi(t)=f\circ\gamma(t)\nabla^{\pi_{\mathsf{E}}}_{\gamma'(t)}\xi\circ\gamma(t)+\langle\mathrm{d}f\circ\gamma(t);\gamma'(t)\rangle\xi\circ\gamma(t).$$

Thus

$$\begin{aligned}T_x(f\xi)(v_x)=&\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}f\circ\gamma(t))\xi\circ\gamma(t)\\&=f(x)\,\mathrm{hlft}(f(x)\xi(x),v_x)+\mathrm{vlft}(f(x)\xi(x),f(x)\nabla^{\pi_{\mathsf{E}}}_{v_x}\xi+\langle\mathrm{d}f(x);v_x\rangle\xi(x))\\&=f(x)\Xi'(0)+\langle\mathrm{d}f(x);v_x\rangle\xi(x)=f(x)T_x\xi(v_x)+\langle\mathrm{d}f(x);v_x\rangle\xi(x)),\end{aligned}$$

as claimed. ▼

2 Sublemma: *If* M *and* N *are smooth manifolds, if* $\Phi\in\mathrm{C}^\infty(\mathsf{M};\mathsf{N})$*, and if* $X\in\Gamma^\infty(\mathsf{TM})$*, then*

$$TT\Phi\circ X^{\mathrm{v}}(v_x)=\mathrm{vlft}(T_x\Phi(v_x),T_x\Phi(X(x))).$$

Proof: We have

$$\begin{aligned}TT\Phi\circ X^{\mathrm{v}}(v_x)=&\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}T_x\Phi(v_x+tX(x))\\=&\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}(T_x\Phi(v_x)+tT_x\Phi(X(x)))\\=&\ \mathrm{vlft}(T_x\Phi(v_x),T_x\Phi(X(x))),\end{aligned}$$

as claimed. ▼

We now directly compute, using Sublemma 1,

$$\begin{aligned}\widehat{\Phi}(\nabla^{\mathsf{M}}_X fY)(x)&=T\Phi\circ K^{\mathsf{M}}\circ T(fY)\circ X(x)\\&=f(x)T\Phi\circ K^{\mathsf{M}}\circ TY\circ X(x)+\langle\mathrm{d}f(x);X(x)\rangle T\Phi\circ K^{\mathsf{M}}\circ Y^{\mathrm{v}}\circ X(x)\\&=f(x)\widehat{\Phi}(\nabla^{\mathsf{M}}_X Y)(x)+\langle\mathrm{d}f(x);X(x)\rangle T\Phi\circ X(x),\end{aligned}$$

noting that K^{M} is a left-inverse for vertical lift. We also directly compute, using both of the sublemmata above,

$$\begin{aligned}\Phi^*\nabla^{\mathsf{N}}_X\widehat{fY}(x)&=K^{\mathsf{N}}\circ TT\Phi\circ T(fY)\circ X(x)\\&=f(x)K^{\mathsf{N}}\circ T(T\Phi\circ Y)\circ X(x)+\langle\mathrm{d}f(x);X(x)\rangle K^{\mathsf{N}}\circ TT\Phi\circ Y^{\mathrm{v}}\circ X(x)\\&=f(x)\Phi^*\nabla^{\mathsf{N}}_X\widehat{Y}(x)+\langle\mathrm{d}f(x);X(x)\rangle K^{\mathsf{N}}(\mathrm{vlft}(T_x\Phi(X(x)),T_x\Phi(X(x))))\\&=f(x)\Phi^*\nabla^{\mathsf{N}}_X\widehat{Y}(x)+\langle\mathrm{d}f(x);X(x)\rangle T\Phi\circ Y\circ X(x),\end{aligned}$$

again noting that K^{N} is the left-inverse for the vertical lift. Combining the preceding two computations gives the tensoriality of

$$(X,Y) \mapsto \widehat{\Phi}(\nabla^{\mathsf{M}}_X Y)(x) - \Phi^*\nabla^{\mathsf{N}}_X \widehat{\Phi}(Y)(x),$$

and so gives $A_\Phi \in \Gamma^r(\mathrm{T}^2(\mathsf{TM}) \otimes \Phi^*\mathsf{TN})$ satisfying the assertion of the lemma. ■

Note that, if $A \in \Gamma^\infty(\mathrm{T}^k(\mathsf{T}^*\mathsf{N}))$, then Φ^*A denotes the pull-back of A to $\Gamma^\infty(\mathrm{T}^k(\mathsf{T}^*\mathsf{M}))$ and also the section of the tensor bundle $\mathrm{T}^k(\Phi^*\mathsf{T}^*\mathsf{N})$. Let $x \in \mathsf{T}_x\mathsf{M}$, let $v_1,\dots,v_k \in \mathsf{T}_x\mathsf{M}$, and denote $u_j = \mathsf{T}_x\Phi(v_j)$, $j \in \{1,\dots,k\}$. Note that

$$\begin{aligned}\Phi^*A((x,u_1),\dots,(x,u_k)) = A(u_1,\dots,u_k) &= A(T_x\Phi(v_1),\dots,T_x\Phi(v_k))\\ &= \Phi^*A(v_1,\dots,v_k),\end{aligned} \tag{5.9}$$

where we are using the two interpretations of the symbol Φ^*A.

With the above as background, we can now understand the iterated covariant derivatives

$$\nabla^{\mathsf{M},m}\Phi^*f = \underbrace{\nabla^{\mathsf{M}}\cdots\nabla^{\mathsf{M}}}_{m \text{ times}}\Phi^*f, \qquad m \in \mathbb{Z}_{>0},$$

and

$$\nabla^{\mathsf{N},m}f = \underbrace{\nabla^{\mathsf{N}}\cdots\nabla^{\mathsf{N}}}_{m \text{ times}}f, \qquad m \in \mathbb{Z}_{>0},$$

for $f \in \mathrm{C}^\infty(\mathsf{N})$. The following lemma gives the first part of this development, playing the rôle of Lemma 4.5 in this case.

5.37 Lemma: (Differentiation of pull-backs of covariant tensors) *Let $r \in \{\infty,\omega\}$. Let M and N be C^r-manifolds with C^r-affine connections ∇^{M} and ∇^{N}, respectively. Define $B_\Phi = \mathrm{push}_{1,2}A_\Phi$ with A_Φ as in Lemma 5.36. Then, for $k \in \mathbb{Z}_{>0}$ and $A \in \Gamma^r(\mathrm{T}^k(\mathsf{T}^*\mathsf{N}))$,*

$$\nabla^{\mathsf{M}}\Phi^*A = \Phi^*\nabla^{\mathsf{N}}A + D_{B_\Phi}(\Phi^*A).$$

Proof: Let $x \in \mathsf{M}$. Let $X_1,\dots,X_k \in \Gamma^\infty(\mathsf{TM})$. For $X_{k+1} \in \Gamma^\infty(\mathsf{TM})$, we have

$$\begin{aligned}&\mathscr{L}_{X_{k+1}}(\Phi^*A(X_1,\dots,X_k))\\ &\qquad = (\nabla^{\mathsf{M}}_{X_{k+1}}\Phi^*A)(X_1,\dots,X_k) + \sum_{j=1}^k \Phi^*A(X_1,\dots,\nabla^{\mathsf{M}}_{X_{k+1}}X_j,\dots,X_k)\end{aligned}$$

and

$$\begin{aligned}\mathscr{L}_{X_{k+1}}(\Phi^*A(\widehat{\Phi}(X_1),\dots,\widehat{\Phi}(X_k))) &= (\Phi^*\nabla^{\mathsf{N}}_{X_{k+1}}\Phi^*A)(\widehat{\Phi}(X_1),\dots,\widehat{\Phi}(X_k))\\ &+ \sum_{j=1}^k \Phi^*A(\widehat{\Phi}(X_1),\dots,\Phi^*\nabla^{\mathsf{N}}_{X_{k+1}}\widehat{\Phi}(X_j),\dots,\widehat{\Phi}(X_k)),\end{aligned}$$

using the two interpretations of Φ^*A. By (5.9) we have, in the above expressions,

$$\Phi^*A(X_1,\dots,X_k) = \Phi^*A(\widehat{\Phi}(X_1),\dots,\widehat{\Phi}(X_k)).$$

By (5.9) again, we have

$$\Phi^*A(X_1,\dots,\nabla^{\mathsf{M}}_{X_{k+1}}X_j,\dots,X_k)=\Phi^*A(\widehat{\Phi}(X_1),\dots,\widehat{\Phi}(\nabla^{\mathsf{M}}_{X_{k+1}}X_j),\dots,\widehat{\Phi}(X_k)).$$

Also note that

$$\begin{aligned}(\Phi^*\nabla^{\mathsf{N}}_{X_{k+1}}\Phi^*A)(\widehat{\Phi}(X_1),\dots,\widehat{\Phi}(X_k))(x)&=(\Phi^*\nabla^{\mathsf{N}}_{X_{k+1}}\Phi^*A)(T_x\Phi(X_1(x)),\dots,T_x\Phi(X_k(x)))\\&=\nabla^{\mathsf{N}}_{T_x\Phi(X_{k+1}(x))}A(T_x\Phi(X_1(x)),\dots,T_x\Phi(X_{k+1}(x)))\\&=\nabla^{\mathsf{N}}A(T_x\Phi(X_1(x)),\dots,T_x\Phi(X_{k+1}(x)))\\&=\Phi^*\nabla^{\mathsf{N}}A(X_1,\dots,X_{k+1})(x).\end{aligned}$$

Combining the above gives

$$\begin{aligned}&\nabla^{\mathsf{M}}\Phi^*A(X_1,\dots,X_{k+1})\\&\quad=\Phi^*\nabla^{\mathsf{N}}A(X_1,\dots,X_{k+1})\\&\qquad+\sum_{j=1}^{k}\Phi^*A(\widehat{\Phi}(X_1),\dots,\Phi^*\nabla^{\mathsf{N}}_{X_{k+1}}\widehat{\Phi}(X_j)-\widehat{\Phi}(\nabla^{\mathsf{M}}_{X_{k+1}}X_j),\dots,\widehat{\Phi}(X_k))\\&\quad=\Phi^*\nabla^{\mathsf{N}}A(X_1,\dots,X_{k+1})-\sum_{j=1}^{k}\Phi^*A(\widehat{\Phi}(X_1),\dots,A_\Phi(X_{k+1},X_j),\dots,\widehat{\Phi}(X_k)).\end{aligned}$$

Thus

$$\nabla^{\mathsf{M}}\Phi^*A=\Phi^*\nabla^{\mathsf{N}}A-\sum_{j=1}^{k}\mathrm{Ins}_j(\Phi^*A,B_\Phi),$$

giving the result by Lemma 3.11. ■

We now have the following lemma, the first of two regarding iterated covariant differentials.

5.38 Lemma: (Iterated covariant differentials of pull-backs of functions I) *Let $r\in\{\infty,\omega\}$ and let M and N be C^r-manifolds with C^r-affine connections ∇^{M} and ∇^{N}, respectively. For $m\in\mathbb{Z}_{\geq 0}$, there exist C^r-vector bundle mappings*

$$(A^m_s,\mathrm{id}_{\mathsf{M}})\in\mathrm{VB}^r(\mathrm{T}^s(\Phi^*\mathsf{T}^*\mathsf{N});\mathrm{T}^m(\mathsf{T}^*\mathsf{M})),\qquad s\in\{0,1,\dots,m\},$$

such that

$$\nabla^{\mathsf{M},m}\Phi^*f=\sum_{s=0}^{m}A^m_s(\Phi^*\nabla^{\mathsf{N},s}f)$$

for all $f\in\mathrm{C}^m(\mathsf{N})$. Moreover, the vector bundle mappings $A^m_0,A^m_1,\dots,A^m_m$ satisfy the recursion relations prescribed by

$$A^0_0(\beta_0)=\beta_0,\ \ A^1_1(\beta_1)=\beta_1,\ \ A^1_0=0,$$

and

$$\begin{aligned}&A^{m+1}_{m+1}(\beta_{m+1})=\beta_{m+1},\\&\quad A^{m+1}_s(\beta_s)=(\nabla^{\mathsf{M}}A^m_s)(\beta_s)+A^m_{s-1}\otimes\mathrm{id}_{\mathsf{T}^*\mathsf{M}}(\beta_s)-\sum_{j=1}^{s}A^m_s\otimes\mathrm{id}_{\mathsf{T}^*\mathsf{M}}(\mathrm{Ins}_j(\beta_s,B_\Phi)),\ \ s\in\{1,\dots,m\},\\&\quad A^{m+1}_0(\beta_0)=(\nabla^{\mathsf{M}}A^m_0)(\beta_0),\end{aligned}$$

where $\beta_s\in\mathrm{T}^s(\Phi^*\mathsf{T}^*\mathsf{N})$, $s\in\{0,1,\dots,m\}$.

Proof: The assertion clearly holds for the initial conditions of the recursion, simply because

$$\Phi^* f = \Phi^* f, \quad \mathrm{d}(\Phi^* f) = \Phi^* \mathrm{d}f + 0f.$$

So suppose that it holds for $m \in \mathbb{Z}_{>0}$. Thus

$$\nabla^{\mathsf{M},m}\Phi^* f = \sum_{s=0}^{m} A_s^m(\Phi^*\nabla^{\mathsf{N},s} f),$$

where the vector bundle mappings A_s^a, $a \in \{0,1,\dots,m\}$, $s \in \{0,1,\dots,a\}$, satisfy the stated recursion relations. Then

$$\begin{aligned}
\nabla^{\mathsf{M},m+1}\Phi^* f = {} & \sum_{s=0}^{m}(\nabla^{\mathsf{M}} A_s^m)(\Phi^*\nabla^{\mathsf{N},s} f) + \sum_{s=0}^{m} A_s^m \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{M}}(\nabla^{\mathsf{M}}\Phi^*\nabla^{\mathsf{N},s} f) \\
= {} & \sum_{s=0}^{m}(\nabla^{\mathsf{M}} A_s^m)(\Phi^*\nabla^{\mathsf{N},s} f) + \sum_{s=0}^{m} A_s^m \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{M}}(\Phi^*\nabla^{\mathsf{N},s+1} f) \\
& - \sum_{s=0}^{m}\sum_{j=1}^{s} A_s^m \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{M}} \operatorname{Ins}_j(\Phi^*\nabla^{\mathsf{N},s} f, B_\Phi) \\
= {} & \Phi^*\nabla^{\mathsf{N},m+1} f + \sum_{s=1}^{m}\Bigg((\nabla^{\mathsf{M}} A_s^m)(\Phi^*\nabla^{\mathsf{N},s} f) + A_{s-1}^m \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{M}}(\Phi^*\nabla^{\mathsf{N},s} f) \\
& - \sum_{j=1}^{s} A_s^m \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{M}}(\operatorname{Ins}_j(\Phi^*\nabla^{\mathsf{N},s} f, B_\Phi))\Bigg) + (\nabla^{\mathsf{M}} A_0^m)(\Phi^* f)
\end{aligned}$$

by Lemma 5.37. From this the lemma follows. ■

We shall also need to “invert” the relationship of the preceding lemma.

5.39 Lemma: (Iterated covariant differentials of pull-backs of functions II) *Let $r \in \{\infty,\omega\}$ and let M and N be C^r-manifolds with C^r-affine connections ∇^{M} and ∇^{N}, respectively. For $m \in \mathbb{Z}_{\geq 0}$, there exist C^r-vector bundle mappings*

$$(B_s^m, \mathrm{id}_{\mathsf{M}}) \in \mathrm{VB}^r(\mathrm{T}^s(\mathsf{T}^*\mathsf{M}); \mathrm{T}^m(\Phi^*\mathsf{T}^*\mathsf{N})), \qquad s \in \{0,1,\dots,m\},$$

such that

$$\Phi^*\nabla^{\mathsf{N},m} f = \sum_{s=0}^{m} B_s^m(\nabla^{\mathsf{M},s}\Phi^* f)$$

for all $f \in \mathrm{C}^m(\mathsf{N})$. Moreover, the vector bundle mappings $B_0^m, B_1^m, \dots, B_m^m$ satisfy the recursion relations prescribed by

$$B_0^0(\alpha_0) = \alpha_0, \;\; B_1^1(\alpha_1) = \alpha_1, \;\; B_0^1 = 0,$$

and

$$\begin{aligned}
& B_{m+1}^{m+1}(\alpha_{m+1}) = \alpha_{m+1}, \\
& \quad B_s^{m+1}(\alpha_s) = (\nabla^{\mathsf{M}} B_s^m)(\alpha_s) + B_{s-1}^m \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{M}}(\alpha_s) + \sum_{j=1}^{m} \operatorname{Ins}_j(B_s^m(\alpha_s), B_\Phi), \;\; s \in \{1,\dots,m\}, \\
& \quad B_0^{m+1}(\alpha_0) = (\nabla^{\mathsf{M}} B_0^m)(\alpha_0) + \sum_{j=1}^{m} \operatorname{Ins}_j(B_0^m(\alpha_0), B_\Phi),
\end{aligned}$$

where $\alpha_s \in \mathrm{T}^s(\mathsf{T}^*\mathsf{M})$*,* $s \in \{0, 1, \dots, m\}$*.*

Proof: The assertion clearly holds for the initial conditions for the recursion because

$$\Phi^* f = \Phi^* f, \quad \Phi^*(\mathrm{d}f) = \mathrm{d}(\Phi^* f) + 0f.$$

So suppose it true for $m \in \mathbb{Z}_{>0}$. Thus

$$\Phi^* \nabla^{\mathsf{N},m} f = \sum_{s=0}^{m} B_s^m(\nabla^{\mathsf{M},s}\Phi^* f), \tag{5.10}$$

where the vector bundle mappings B_s^a, $a \in \{0, 1, \dots, m\}$, $s \in \{0, 1, \dots, a\}$, satisfy the recursion relations from the statement of the lemma. Then, by Lemma 5.37, we can work on the left-hand side of (5.10) to give

$$\begin{aligned}\nabla^{\mathsf{M}}\Phi^*\nabla^{\mathsf{N},m} f &= \Phi^*\nabla^{\mathsf{N},m+1} f - \sum_{j=1}^{m} \mathrm{Ins}_j(\Phi^*\nabla^{\mathsf{N},m} f, B_\Phi) \\ &= \Phi^*\nabla^{\mathsf{N},m+1} f - \sum_{s=0}^{m}\sum_{j=1}^{m} \mathrm{Ins}_j(B_s^m(\nabla^{\mathsf{M},s}\Phi^* f), B_\Phi).\end{aligned}$$

Working on the right-hand side of (5.1) gives

$$\nabla^{\mathsf{M}}\Phi^*\nabla^{\mathsf{N},m} f = \sum_{s=0}^{m} \nabla^{\mathsf{M}} B_s^m(\nabla^{\mathsf{M},s}\Phi^* f) + \sum_{s=0}^{m} B_s^m \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{M}}(\nabla^{\mathsf{M},s+1}\Phi^* f).$$

Combining the preceding two equations gives

$$\begin{aligned}\Phi^*\nabla^{\mathsf{N},m+1} f = &\sum_{s=0}^{m} \nabla^{\mathsf{M}} B_s^m(\nabla^{\mathsf{M},s}\Phi^* f) + \sum_{s=0}^{m} B_s^m \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{M}}(\nabla^{\mathsf{M},s+1}\Phi^* f) \\ &+ \sum_{s=0}^{m}\sum_{j=1}^{m} \mathrm{Ins}_j(B_s^m(\nabla^{\mathsf{M},s}\Phi^* f), B_\Phi) \\ = &\ \nabla^{\mathsf{M},m+1}\Phi^* f + \sum_{s=1}^{m}\Bigg(\nabla^{\mathsf{M}} B_s^m(\nabla^{\mathsf{M},s}\Phi^* f) + B_{s-1}^m \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{M}}(\nabla^{\mathsf{M},s}\Phi^* f) \\ &+ \sum_{j=1}^{m} \mathrm{Ins}_j(B_s^m(\nabla^{\mathsf{M},s}\Phi^* f), B_\Phi)\Bigg) + \nabla^{\mathsf{M}} B_0^m(\Phi^* f) + \sum_{j=1}^{m} \mathrm{Ins}_j(B_0^m(\Phi^* f), B_\Phi),\end{aligned}$$

and the lemma follows from this. ■

With this data, we have the following result.

5.40 Lemma: (Iterated symmetrised covariant differentials of pull-backs of functions I) *Let* $r \in \{\infty, \omega\}$ *and let* M *and* N *be* C^r*-manifolds with* C^r*-affine connections* ∇^{M} *and* ∇^{N}*, respectively. For* $m \in \mathbb{Z}_{\geq 0}$*, there exist* C^r*-vector bundle mappings*

$$(\widehat{A}_s^m, \mathrm{id}_{\mathsf{M}}) \in \mathrm{VB}^r(\mathrm{S}^s(\Phi^*\mathsf{T}^*\mathsf{N}); \mathrm{S}^m(\mathsf{T}^*\mathsf{M})), \qquad s \in \{0, 1, \dots, m\},$$

such that

$$\mathrm{Sym}_m \circ \nabla^{\mathsf{M},m}\Phi^* f = \sum_{s=0}^{m} \widehat{A}^m_s(\mathrm{Sym}_s \circ \Phi^*\nabla^{\mathsf{N},s} f)$$

for all $f \in \mathrm{C}^m(\mathsf{N})$*.*

Proof: This follows from Lemma 5.38 in the same way as Lemma 5.3 follows from Lemma 5.1. ■

Next we consider the "inverse" of the preceding lemma.

5.41 Lemma: (Iterated symmetrised covariant differentials of horizontal lifts of functions II) *Let* $r \in \{\infty, \omega\}$ *and let* M *and* N *be* C^r*-manifolds with* C^r*-affine connections* ∇^{M} *and* ∇^{N}*, respectively. For* $m \in \mathbb{Z}_{\geq 0}$*, there exist* C^r*-vector bundle mappings*

$$(\widehat{B}^m_s, \mathrm{id}_{\mathsf{M}}) \in \mathrm{VB}^r(\mathrm{S}^s(\mathsf{T}^*\mathsf{M}); \mathrm{S}^m(\Phi^*\mathsf{T}^*\mathsf{N})), \qquad s \in \{0, 1, \dots, m\},$$

such that

$$\mathrm{Sym}_m \circ \Phi^*\nabla^{\mathsf{N},m} f = \sum_{s=0}^{m} \widehat{B}^m_s(\mathrm{Sym}_s \circ \nabla^{\mathsf{M},s}\Phi^* f)$$

for all $f \in \mathrm{C}^m(\mathsf{N})$*.*

Proof: This follows from Lemma 5.39 in the same way as Lemma 5.4 follows from Lemma 5.2. ■

The following lemma provides two decompositions of $\mathsf{T}^{*m}_{\Phi}\mathsf{M}$, one "in the domain" and one "in the codomain," and the relationship between them. The assertion simply results from an examination of the preceding four lemmata.

5.42 Lemma: (Decomposition of jets of pull-backs of functions) *Let* $r \in \{\infty, \omega\}$ *and let* M *and* N *be* C^r*-manifolds with* C^r*-affine connections* ∇^{M} *and* ∇^{N}*, respectively. Then there exist* C^r*-vector bundle mappings*

$$A^m_{\nabla^{\mathsf{M}},\nabla^{\mathsf{N}}} \in \mathrm{VB}^r(\mathsf{T}^{*m}_{\Phi}\mathsf{M}; \mathrm{S}^{\leq m}(\Phi^*\mathsf{T}^*\mathsf{N})), \quad B^m_{\nabla^{\mathsf{M}},\nabla^{\mathsf{N}}} \in \mathrm{VB}^r(\mathsf{T}^{*m}_{\Phi}\mathsf{M}; \mathrm{S}^{\leq m}(\mathsf{T}^*\mathsf{M})),$$

defined by

$$\begin{aligned} A^m_{\nabla^{\mathsf{M}},\nabla^{\mathsf{N}}}(j_m(\Phi^* f)(x)) &= \mathrm{Sym}_{\leq m}(\Phi^* f(x), \Phi^*\nabla^{\mathsf{N}} f(x), \dots, \Phi^*\nabla^{\mathsf{N},m} f(x)), \\ B^m_{\nabla^{\mathsf{M}},\nabla^{\mathsf{N}}}(j_m(\Phi^* f)(x)) &= \mathrm{Sym}_{\leq m}(\Phi^* f(x), \nabla^{\mathsf{M}}\Phi^* f(x), \dots, \nabla^{\mathsf{M},m}\Phi^* f(x)). \end{aligned}$$

Moreover, $A^m_{\nabla^{\mathsf{M}},\nabla^{\mathsf{N}}}$ *is an isomorphism,* $B^m_{\nabla^{\mathsf{M}},\nabla^{\mathsf{N}}}$ *is injective, and*

$$\begin{aligned} B^m_{\nabla^{\mathsf{M}},\nabla^{\mathsf{N}}} &\circ (A^m_{\nabla^{\mathsf{M}},\nabla^{\mathsf{N}}})^{-1} \circ (\mathrm{Sym}_{\leq m}(\Phi^* f(e), \Phi^*\nabla^{\mathsf{N}} f(x), \dots, \Phi^*\nabla^{\mathsf{N},m} f(x)) \\ &= \left(A^0_0(\Phi^* f(x)), \sum_{s=0}^{1} \widehat{A}^1_s(\mathrm{Sym}_s \circ \Phi^*\nabla^{\mathsf{N},s} f(x)), \dots, \sum_{s=0}^{m} \widehat{A}^m_s(\mathrm{Sym}_s \circ \Phi^*\nabla^{\mathsf{N},s} f(x))\right) \end{aligned}$$

and

$$A^m_{\nabla^{\mathsf{M}},\nabla^{\mathsf{N}}} \circ (B^m_{\nabla^{\mathsf{M}},\nabla^{\mathsf{N}}})^{-1} \circ \mathrm{Sym}_{\leq m}(\Phi^* f(x), \nabla^{\mathsf{M}}\Phi^* f(x), \dots, \nabla^{\mathsf{M},m}\Phi^* f(x))$$
$$= \left(B^0_0(\Phi^* f(x)), \sum_{s=0}^{1} \widehat{B}^1_s(\mathrm{Sym}_s \circ \nabla^{\mathsf{M},s}\Phi^* f(x)), \dots, \sum_{s=0}^{m} \widehat{B}^m_s(\mathrm{Sym}_s \circ \nabla^{\mathsf{M},s}\Phi^* f(x)) \right),$$

where the vector bundle mappings $\widehat{A}^m_s$ and $\widehat{B}^m_s$, $s \in \{0, 1, \dots, m\}$, are as in Lemmata 5.40 and 5.41.

6. Fibre norms for some useful jet bundles

In Section 5 we saw how to make decompositions for jets of sections of vector bundles and jets of various lifts to the total space of a vector bundle $\pi_{\mathsf{E}}\colon \mathsf{E} \to \mathsf{M}$, using the Levi-Civita affine connection induced by a natural Riemannian metric on E. In this section we consider fibre norms for these jet bundles. The fibre norm for the space of jets of sections of a vector bundle is deduced in a natural way from a Riemannian metric on M and a fibre metric in $\pi_{\mathsf{E}}\colon \mathsf{E} \to \mathsf{M}$. For fibre norms of lifted objects, the story is more complicated. Since the objects are lifted from M, there are two natural fibre norms in each case, one coming from the Riemannian metric on E, and the other coming from the Riemannian metric on M and the fibre metric on the vector bundle.

The setup is the following. We let $r \in \{\infty, \omega\}$ and let $\pi_{\mathsf{E}}\colon \mathsf{E} \to \mathsf{M}$ be a C^r-vector bundle. We consider a Riemannian metric $\mathbb{G}_{\mathsf{M}}$ on M, a fibre metric $\mathbb{G}_{\pi_{\mathsf{E}}}$ on E, the Levi-Civita connection ∇^{M} on M, and a vector bundle connection $\nabla^{\pi_{\mathsf{E}}}$ in E, all being of class C^r. This gives the Riemannian metric $\mathbb{G}_{\mathsf{E}}$ of (4.1) and the associated Levi-Civita connection ∇^{E}. This data gives the fibre metrics for all sorts of tensors defined on the total space E. We, however, are interested only in the lifted tensors such as are described in Section 3.

The reader will definitely observe a certain repetitiveness to our constructions in this section, rather similar to that seen in Section 5. However, the ideas here are important and the notation is confusing, so we do not skip anything.

We treat the smooth and real analytic cases simultaneously in this section. In the smooth case, the formulae we give are useful for applying the methods of the paper to the setting of the paper in smooth category.

6.1. Fibre norms for horizontal lifts of functions. We let $r \in \{\infty, \omega\}$ and let $\pi_{\mathsf{E}}\colon \mathsf{E} \to \mathsf{M}$ be a C^r-vector bundle. For $f \in \mathrm{C}^m(\mathsf{M})$, we have $\pi_{\mathsf{E}}^* f \in \mathrm{C}^m(\mathsf{E})$. We can, therefore, think of the m-jet of $\pi_{\mathsf{E}}^* f$ as being characterised by $j_m f$, as well as by $j_m \pi_{\mathsf{E}}^* f$, and of comparing these two characterisations. Thus we have the two fibre norms

$$\|j_m f(x)\|^2_{\mathbb{G}_{\mathsf{M},m}} = \sum_{j=0}^{m} \frac{1}{(j!)^2} \|\nabla^{\mathsf{M},j} f(x)\|^2_{\mathbb{G}_{\mathsf{M}}}$$

and

$$\|j_m \pi_{\mathsf{E}}^* f(e)\|^2_{\mathbb{G}_{\mathsf{E},m}} = \sum_{j=0}^{m} \frac{1}{(j!)^2} \|\nabla^{\mathsf{E},j} \pi_{\mathsf{E}}^* f(e)\|^2_{\mathbb{G}_{\mathsf{E}}}. \tag{6.1}$$

These fibre norms can be related by virtue of Lemma 5.5. To do so, we make use of the following lemma.

6.1 Lemma: (Fibre norms for horizontal lifts of functions)

$$\|\pi_{\mathsf{E}}^*\nabla^{\mathsf{M},m}f(e)\|_{\mathbb{G}_{\mathsf{E}}} = \|\nabla^{\mathsf{M},m}f(\pi_{\mathsf{E}}(e))\|_{\mathbb{G}_{\mathsf{M}}}.$$

Proof: We have the fibre metric $\mathbb{G}_{\mathsf{E}}^{-1}$ on $\mathsf{T}^*\mathsf{E}$ associated with the Riemannian metric $\mathbb{G}_{\mathsf{E}}$. The subbundles $\mathsf{H}^*\mathsf{E}$ and $\mathsf{V}^*\mathsf{E}$ are $\mathbb{G}_{\mathsf{E}}^{-1}$-orthogonal. We note that $T_e^*\pi_{\mathsf{E}}\colon \mathsf{T}^*_{\pi_{\mathsf{E}}(e)}\mathsf{M} \to \mathsf{H}_e^*\mathsf{E}$ is an isometry. Thus we have the formula

$$\|\pi_{\mathsf{E}}^*B\|_{\mathbb{G}_{\mathsf{E}}} = \|B\|_{\mathbb{G}_{\mathsf{M}}}, \qquad B \in \Gamma^0(\mathrm{T}^m(\mathsf{T}^*\mathsf{M})),$$

and the assertion of the lemma is merely a special case of this formula. ■

We note that the fibre norm (6.1) makes use of the vector bundle mapping

$$B^m_{\nabla^{\mathsf{E}}} \in \mathrm{VB}^r(\mathsf{P}^{*m}\mathsf{E}; \mathrm{S}^{\leq m}(\mathsf{T}^*\mathsf{E}))$$

from Lemma 5.5. If instead we use the vector bundle mapping

$$A^m_{\nabla^{\mathsf{E}}} \in \mathrm{VB}^r(\mathsf{P}^{*m}\mathsf{E}; \mathrm{S}^{\leq m}(\pi_{\mathsf{E}}^*\mathsf{T}^*\mathsf{M}))$$

from Lemma 5.5, then we have the alternative fibre norm

$$\|j_m\pi_{\mathsf{E}}^*f(e)\|'^2_{\mathbb{G}_{\mathsf{E},m}} = \sum_{j=0}^m \frac{1}{(j!)^2}\|\pi_{\mathsf{E}}^*\nabla^{\mathsf{M},j}f(e)\|^2_{\mathbb{G}_{\mathsf{E}}} = \sum_{j=0}^m \frac{1}{(j!)^2}\|\nabla^{\mathsf{M},j}f(\pi_{\mathsf{E}}(e))\|^2_{\mathbb{G}_{\mathsf{M}}}.$$

The relationship between the fibre norms $\|\cdot\|_{\mathbb{G}_{\mathsf{E},m}}$ and $\|\cdot\|'_{\mathbb{G}_{\mathsf{E},m}}$ can be phrased as, "What is the relationship between the jet of the lift and the lift of the jet?" This is a question we will phrase below for other sorts of lifts, and will address comprehensively when we prove the continuity of the various lifting operations in Section 9.3.

6.2. Fibre norms for vertical lifts of sections. We let $r \in \{\infty, \omega\}$ and let $\pi_{\mathsf{E}}\colon \mathsf{E} \to \mathsf{M}$ be a C^r-vector bundle. For $\xi \in \Gamma^m(\mathsf{E})$, we have $\xi^{\mathrm{v}} \in \Gamma^m(\mathsf{TE})$. We can, therefore, think of the m-jet of ξ^{v} as being characterised by $j_m\xi$, as well as by $j_m\xi^{\mathrm{v}}$, and of comparing these two characterisations. Thus we have the two fibre norms

$$\|j_m\xi(x)\|^2_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}},m}} = \sum_{j=0}^m \frac{1}{(j!)^2}\|\nabla^{\mathsf{M},\pi_{\mathsf{E}},j}\xi(x)\|^2_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}}}$$

and

$$\|j_m\xi^{\mathrm{v}}(e)\|^2_{\mathbb{G}_{\mathsf{E},m}} = \sum_{j=0}^m \frac{1}{(j!)^2}\|\nabla^{\mathsf{E},j}\xi^{\mathrm{v}}(e)\|^2_{\mathbb{G}_{\mathsf{E}}}. \tag{6.2}$$

These fibre norms can be related by virtue of Lemma 5.10. To do so, we make use of the following lemma.

6.2 Lemma: (Fibre norms for vertical lifts of sections)

$$\|(\nabla^{\mathsf{M},\pi_{\mathsf{E}},m}\xi)^{\mathrm{v}}(e)\|_{\mathbb{G}_{\mathsf{E}}} = \|\nabla^{\mathsf{M},\pi_{\mathsf{E}},m}\xi(\pi_{\mathsf{E}}(e))\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}}}.$$

Proof: The subbundles HE and VE are $\mathbb{G}_{\mathsf{E}}$-orthogonal and the subbundles $\mathsf{H}^*\mathsf{E}$ and $\mathsf{V}^*\mathsf{E}$ are $\mathbb{G}_{\mathsf{E}}^{-1}$-orthogonal. We note that the identification $\mathsf{V}_e\mathsf{E} \simeq \mathsf{E}_{\pi_{\mathsf{E}}(e)}$ is an isometry and that $T_e^*\pi_{\mathsf{E}}\colon \mathsf{T}^*_{\pi_{\mathsf{E}}(e)}\mathsf{M} \to \mathsf{H}_e^*\mathsf{E}$ is an isometry. Thus we have the formula

$$\|B^{\mathrm{v}}\|_{\mathbb{G}_{\mathsf{E}}} = \|B\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}}}, \qquad B \in \Gamma^0(\mathrm{T}^m(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{E}),$$

and the assertion of the lemma is merely a special case of this formula. ∎

We note that the fibre norm (6.2) makes use of the vector bundle mapping

$$B^m_{\nabla^{\mathsf{E}}} \in \mathrm{VB}^r(\mathsf{P}^{*m}\mathsf{E} \otimes \mathsf{VE}; \mathrm{S}^{\leq m}(\mathsf{T}^*\mathsf{E}) \otimes \mathsf{VE})$$

from Lemma 5.10. If instead we use the vector bundle mapping

$$A^m_{\nabla^{\mathsf{E}}} \in \mathrm{VB}^r(\mathsf{P}^{*m}\mathsf{E} \otimes \mathsf{VE}; \mathrm{S}^{\leq m}(\pi_{\mathsf{E}}^*\mathsf{T}^*\mathsf{M}) \otimes \mathsf{VE})$$

from Lemma 5.10, then we have the alternative fibre norm

$$\|j_m\xi^{\mathrm{v}}(e)\|'^2_{\mathbb{G}_{\mathsf{E},m}} = \sum_{j=0}^m \frac{1}{(j!)^2}\|(\nabla^{\mathsf{M},\pi_{\mathsf{E}},j}\xi)^{\mathrm{v}}(e)\|^2_{\mathbb{G}_{\mathsf{E}}} = \sum_{j=0}^m \frac{1}{(j!)^2}\|\nabla^{\mathsf{M},\pi_{\mathsf{E}},j}\xi(\pi_{\mathsf{E}}(e))\|^2_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}}}.$$

Again, this points out the matter of the relationship between the jet of a lift versus the lift of the jet, and this matter will be considered in detail in the continuity results of Section 9.3.

6.3. Fibre norms for horizontal lifts of vector fields. We let $r \in \{\infty, \omega\}$ and let $\pi_{\mathsf{E}}\colon \mathsf{E} \to \mathsf{M}$ be a C^r-vector bundle. For $X \in \Gamma^m(\mathsf{TM})$, we have $X^{\mathrm{h}} \in \Gamma^m(\mathsf{TE})$. We can, therefore, think of the m-jet of X^{h} as being characterised by j_mX, as well as by j_mX^{h}, and of comparing these two characterisations. Thus we have the two fibre norms

$$\|j_mX(x)\|^2_{\mathbb{G}_{\mathsf{M},m}} = \sum_{j=0}^m \frac{1}{(j!)^2}\|\nabla^{\mathsf{M},j}X(x)\|^2_{\mathbb{G}_{\mathsf{M}}}$$

and

$$\|j_mX^{\mathrm{h}}(e)\|^2_{\mathbb{G}_{\mathsf{E},m}} = \sum_{j=0}^m \frac{1}{(j!)^2}\|\nabla^{\mathsf{E},j}X^{\mathrm{h}}(e)\|^2_{\mathbb{G}_{\mathsf{E}}}. \tag{6.3}$$

These fibre norms can be related by virtue of Lemma 5.15. To do so, we make use of the following lemma.

6.3 Lemma: (Fibre norms for horizontal lifts of vector fields)

$$\|(\nabla^{\mathsf{M},m}X)^{\mathrm{h}}(e)\|_{\mathbb{G}_{\mathsf{E}}} = \|\nabla^{\mathsf{M},m}X(\pi_{\mathsf{E}}(e))\|_{\mathbb{G}_{\mathsf{M}}}.$$

Proof: The subbundles HE and VE are $\mathbb{G}_{\mathsf{E}}$-orthogonal. We note that the identification $\mathsf{H}_e\mathsf{E} \simeq \mathsf{T}_{\pi_{\mathsf{E}}(e)}\mathsf{M}$ is an isometry and that $T_e^*\pi_{\mathsf{E}} \colon \mathsf{T}^*_{\pi_{\mathsf{E}}(e)}\mathsf{M} \to \mathsf{H}_e^*\mathsf{E}$ is an isometry. Thus we have the formula

$$\|B^{\mathrm{h}}\|_{\mathbb{G}_{\mathsf{E}}} = \|B\|_{\mathbb{G}_{\mathsf{M}}}, \qquad B \in \Gamma^0(\mathrm{T}^m(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{TM}),$$

and the assertion of the lemma is merely a special case of this formula. ■

We note that the fibre norm (6.3) makes use of the vector bundle mapping

$$B^m_{\nabla^{\mathsf{E}}} \in \mathrm{VB}^r(\mathsf{P}^{*m}\mathsf{E} \otimes \mathsf{HE}; \mathrm{S}^{\leq m}(\mathsf{T}^*\mathsf{E}) \otimes \mathsf{HE})$$

from Lemma 5.15. If instead we use the vector bundle mapping

$$A^m_{\nabla^{\mathsf{E}}} \in \mathrm{VB}^r(\mathsf{P}^{*m}\mathsf{E} \otimes \mathsf{HE}; \mathrm{S}^{\leq m}(\pi_{\mathsf{E}}^*\mathsf{T}^*\mathsf{M}) \otimes \mathsf{HE})$$

from Lemma 5.15, then we have the alternative fibre norm

$$\|j_mX^{\mathrm{h}}(e)\|'^2_{\mathbb{G}_{\mathsf{E}},m} = \sum_{j=0}^m \frac{1}{(j!)^2}\|(\nabla^{\mathsf{M},j}X)^{\mathrm{h}}(e)\|^2_{\mathbb{G}_{\mathsf{E}}} = \sum_{j=0}^m \frac{1}{(j!)^2}\|\nabla^{\mathsf{M},j}X(\pi_{\mathsf{E}}(e))\|^2_{\mathbb{G}_{\mathsf{M}}}.$$

Again, this points out the matter of the relationship between the jet of a lift versus the lift of the jet, and this matter will be considered in detail in the continuity results of Section 9.3.

6.4. Fibre norms for vertical lifts of dual sections. We let $r \in \{\infty, \omega\}$ and let $\pi_{\mathsf{E}} \colon \mathsf{E} \to \mathsf{M}$ be a C^r-vector bundle. For $\lambda \in \Gamma^m(\mathsf{E}^*)$, we have $\lambda^{\mathrm{v}} \in \Gamma^m(\mathsf{T}^*\mathsf{E})$. We can, therefore, think of the m-jet of λ^{v} as being characterised by $j_m\lambda$, as well as by $j_m\lambda^{\mathrm{v}}$, and of comparing these two characterisations. Thus we have fibre norms

$$\|j_m\lambda(x)\|^2_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}},m}} = \sum_{j=0}^m \frac{1}{(j!)^2}\|\nabla^{\mathsf{M},\pi_{\mathsf{E}},j}\lambda(x)\|^2_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}}}$$

and

$$\|j_m\lambda^{\mathrm{v}}(e)\|^2_{\mathbb{G}_{\mathsf{E},m}} = \sum_{j=0}^m \frac{1}{(j!)^2}\|\nabla^{\mathsf{E},j}\lambda^{\mathrm{v}}(e)\|^2_{\mathbb{G}_{\mathsf{E}}}. \tag{6.4}$$

These fibre norms can be related by virtue of Lemma 5.20. To do so, we make use of the following lemma.

6.4 Lemma: (Fibre norms for vertical lifts of dual sections)

$$\|(\nabla^{\mathsf{M},\pi_{\mathsf{E}},m}\lambda)^{\mathrm{v}}(e)\|_{\mathbb{G}_{\mathsf{E}}} = \|\nabla^{\mathsf{M},\pi_{\mathsf{E}},m}\lambda(\pi_{\mathsf{E}}(e))\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}}}.$$

Proof: The subbundles $\mathsf{H}^*\mathsf{E}$ and $\mathsf{V}^*\mathsf{E}$ are $\mathbb{G}_{\mathsf{E}}^{-1}$-orthogonal. We note that the identification $\mathsf{V}_e^*\mathsf{E} \simeq \mathsf{E}^*_{\pi_{\mathsf{E}}(e)}$ is an isometry and that $T_e^*\pi_{\mathsf{E}} \colon \mathsf{T}^*_{\pi_{\mathsf{E}}(e)}\mathsf{M} \to \mathsf{H}_e^*\mathsf{E}$ is an isometry. Thus we have the formula

$$\|B^{\mathrm{v}}\|_{\mathbb{G}_{\mathsf{E}}} = \|B\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}}}, \qquad B \in \Gamma^0(\mathrm{T}^m(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{E}^*),$$

and the assertion of the lemma is merely a special case of this formula. ■

We note that the fibre norm (6.4) makes use of the vector bundle mapping

$$B^m_{\nabla^{\mathsf{E}}} \in \mathrm{VB}^r(\mathsf{P}^{*m}\mathsf{E} \otimes \mathsf{V}^*\mathsf{E}; \mathrm{S}^{\leq m}(\mathsf{T}^*\mathsf{E}) \otimes \mathsf{V}^*\mathsf{E})$$

from Lemma 5.20. If instead we use the vector bundle mapping

$$A^m_{\nabla^{\mathsf{E}}} \in \mathrm{VB}^r(\mathsf{P}^{*m}\mathsf{E} \otimes \mathsf{V}^*\mathsf{E}; \mathrm{S}^{\leq m}(\pi^*_{\mathsf{E}}\mathsf{T}^*\mathsf{M}) \otimes \mathsf{V}^*\mathsf{E})$$

from Lemma 5.20, then we have the alternative fibre norm

$$\|j_m\lambda^{\mathrm{v}}(e)\|'^2_{\mathbb{G}_{\mathsf{E},m}} = \sum_{j=0}^m \frac{1}{(j!)^2}\|(\nabla^{\mathsf{M},\pi_{\mathsf{E}},j}\lambda)^{\mathrm{v}}(e)\|^2_{\mathbb{G}_{\mathsf{E}}} = \sum_{j=0}^m \frac{1}{(j!)^2}\|\nabla^{\mathsf{M},\pi_{\mathsf{E}},j}\lambda(\pi_{\mathsf{E}}(e))\|^2_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}}}.$$

Again, this points out the matter of the relationship between the jet of a lift versus the lift of the jet, and this matter will be considered in detail in the continuity results of Section 9.3.

6.5. Fibre norms for vertical lifts of endomorphisms. We let $r \in \{\infty, \omega\}$ and let $\pi_{\mathsf{E}} \colon \mathsf{E} \to \mathsf{M}$ be a C^r-vector bundle. For $L \in \Gamma^m(\mathrm{T}^1_1(\mathsf{E}))$, we have $L^{\mathrm{v}} \in \Gamma^m(\mathrm{T}^1_1(\mathsf{E}))$. We can, therefore, think of the m-jet of L^{v} as being characterised by j_mL, as well as by j_mL^{v}, and of comparing these two characterisations. Thus we have the two fibre norms

$$\|j_mL(x)\|^2_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}},m}} = \sum_{j=0}^m \frac{1}{(j!)^2}\|\nabla^{\mathsf{M},\pi_{\mathsf{E}},j}L(x)\|^2_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}}}$$

and

$$\|j_mL^{\mathrm{v}}(e)\|^2_{\mathbb{G}_{\mathsf{E},m}} = \sum_{j=0}^m \frac{1}{(j!)^2}\|\nabla^{\mathsf{E},j}L^{\mathrm{v}}(e)\|^2_{\mathbb{G}_{\mathsf{E}}}. \tag{6.5}$$

These fibre norms can be related by virtue of Lemma 5.25. To do so, we make use of the following lemma.

6.5 Lemma: (Fibre norms for vertical lifts of endomorphisms)

$$\|(\nabla^{\mathsf{M},\pi_{\mathsf{E}},m}L)^{\mathrm{v}}(e)\|_{\mathbb{G}_{\mathsf{E}}} = \|\nabla^{\mathsf{M},\pi_{\mathsf{E}},m}L(\pi_{\mathsf{E}}(e))\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}}}.$$

Proof: The subbundles $\mathsf{H}^*\mathsf{E}$ and $\mathsf{V}^*\mathsf{E}$ are $\mathbb{G}^{-1}_{\mathsf{E}}$-orthogonal. We note that the identifications $\mathsf{V}_e\mathsf{E} \simeq \mathsf{E}_{\pi_{\mathsf{E}}(e)}$ and $\mathsf{V}^*_e\mathsf{E} \simeq \mathsf{E}^*_{\pi_{\mathsf{E}}(e)}$ are isometries, and that $T^*_e\pi_{\mathsf{E}} \colon \mathsf{T}^*_{\pi_{\mathsf{E}}(e)}\mathsf{M} \to \mathsf{H}^*_e\mathsf{E}$ is an isometry. Thus we have the formula

$$\|B^{\mathrm{v}}\|_{\mathbb{G}_{\mathsf{E}}} = \|B\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}}}, \qquad B \in \Gamma^0(\mathrm{T}^m(\mathsf{T}^*\mathsf{M}) \otimes \mathrm{T}^1_1(\mathsf{E})),$$

and the assertion of the lemma is merely a special case of this formula. ■

We note that the fibre norm (6.5) makes use of the vector bundle mapping

$$B^m_{\nabla^{\mathsf{E}}} \in \mathrm{VB}^r(\mathsf{P}^{*m}\mathsf{E} \otimes \mathrm{T}^1_1(\mathsf{VE}); \mathrm{S}^{\leq m}(\mathsf{T}^*\mathsf{E}) \otimes \mathrm{T}^1_1(\mathsf{VE}))$$

from Lemma 5.25. If instead we use the vector bundle mapping

$$A^m_{\nabla^{\mathsf{E}}} \in \mathrm{VB}^r(\mathsf{P}^{*m}\mathsf{E} \otimes \mathrm{T}^1_1(\mathsf{VE}); \mathrm{S}^{\leq m}(\pi^*_{\mathsf{E}}\mathsf{T}^*\mathsf{M}) \otimes \mathrm{T}^1_1(\mathsf{VE}))$$

from Lemma 5.25, then we have the alternative fibre norm

$$\|j_mL^{\mathrm{v}}(e)\|'^2_{\mathbb{G}_{\mathsf{E},m}} = \sum_{j=0}^m \frac{1}{(j!)^2}\|(\nabla^{\mathsf{M},\pi_{\mathsf{E}},j}L)^{\mathrm{v}}(e)\|^2_{\mathbb{G}_{\mathsf{E}}} = \sum_{j=0}^m \frac{1}{(j!)^2}\|\nabla^{\mathsf{M},\pi_{\mathsf{E}},j}L(\pi_{\mathsf{E}}(e))\|^2_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}}}.$$

Again, this points out the matter of the relationship between the jet of a lift versus the lift of the jet, and this matter will be considered in detail in the continuity results of Section 9.3.

6.6. Fibre norms for vertical evaluations of dual sections. We let $r \in \{\infty, \omega\}$ and let $\pi_{\mathsf{E}} \colon \mathsf{E} \to \mathsf{M}$ be a C^r-vector bundle. For $\lambda \in \Gamma^m(\mathsf{E}^*)$, we have $\lambda^{\mathrm{e}} \in \mathrm{C}^m(\mathsf{E})$. We can, therefore, think of the m-jet of λ^{e} as being characterised by $j_m\lambda$, as well as by $j_m\lambda^{\mathrm{e}}$, and of comparing these two characterisations.

Thus we have the two fibre norms

$$\|j_m\lambda(x)\|^2_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}},m}} = \sum_{j=0}^m \frac{1}{(j!)^2}\|\nabla^{\mathsf{M},\pi_{\mathsf{E}},j}\lambda(x)\|^2_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}}}$$

and

$$\|j_m\lambda^{\mathrm{e}}(e)\|^2_{\mathbb{G}_{\mathsf{E},m}} = \sum_{j=0}^m \frac{1}{(j!)^2}\|\nabla^{\mathsf{E},j}\lambda^{\mathrm{e}}(e)\|^2_{\mathbb{G}_{\mathsf{E}}}. \tag{6.6}$$

These fibre norms can be related by virtue of Lemma 5.30. To do so, we make use of the following lemma.

6.6 Lemma: (Fibre norms for vertical evaluations of dual sections)

$$\|(\nabla^{\mathsf{M},\pi_{\mathsf{E}},m}\lambda)^{\mathrm{e}}(e)\|_{\mathbb{G}_{\mathsf{E}}} = \|\nabla^{\mathsf{M},\pi_{\mathsf{E}},m}\lambda(\pi_{\mathsf{E}}(e))(e)\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}}}.$$

Proof: The subbundles $\mathsf{H}^*\mathsf{E}$ and $\mathsf{V}^*\mathsf{E}$ are $\mathbb{G}_{\mathsf{E}}^{-1}$-orthogonal. We note that the identification $\mathsf{V}^*_e\mathsf{E} \simeq \mathsf{E}^*_{\pi_{\mathsf{E}}(e)}$ is an isometry, and that $T^*_e\pi_{\mathsf{E}} \colon \mathsf{T}^*_{\pi_{\mathsf{E}}(e)}\mathsf{M} \to \mathsf{H}^*_e\mathsf{E}$ is an isometry. Thus we have the formula

$$\|B^{\mathrm{e}}(e)\|_{\mathbb{G}_{\mathsf{E}}} = \|B(\pi_{\mathsf{E}}(e))(e)\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}}}, \qquad B \in \Gamma^0(\mathrm{T}^m(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{E}^*),$$

and the assertion of the lemma is merely a special case of this formula. ■

We note that the fibre norm (6.6) makes use of the vector bundle mapping

$$B^m_{\nabla^{\mathsf{E}}} \in \mathrm{VB}^r(\mathsf{P}^{*m}\mathsf{E}; \mathrm{S}^{\leq m}(\mathsf{T}^*\mathsf{E}))$$

from Lemma 5.30. If instead we use the vector bundle mapping

$$A^m_{\nabla^{\mathsf{E}}} \in \mathrm{VB}^r(\mathsf{P}^{*m}\mathsf{E}; \mathrm{S}^{\leq m}(\pi^*_{\mathsf{E}}\mathsf{T}^*\mathsf{M}))$$

from Lemma 5.30, then we have the alternative fibre norm

$$\|j_m\lambda^{\mathrm{e}}(e)\|'^2_{\mathbb{G}_{\mathsf{E},m}} = \sum_{j=0}^m \frac{1}{(j!)^2}\|(\nabla^{\mathsf{M},\pi_{\mathsf{E}},j}\lambda)^{\mathrm{e}}(e)\|^2_{\mathbb{G}_{\mathsf{E}}} = \sum_{j=0}^m \frac{1}{(j!)^2}\|\nabla^{\mathsf{M},\pi_{\mathsf{E}},j}\lambda(\pi_{\mathsf{E}}(e))(e)\|^2_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}}}.$$

Again, this points out the matter of the relationship between the jet of a lift versus the lift of the jet, and this matter will be considered in detail in the continuity results of Section 9.3.

6.7. Fibre norms for vertical evaluations of endomorphisms. We let $r \in \{\infty, \omega\}$ and let $\pi_{\mathsf{E}} \colon \mathsf{E} \to \mathsf{M}$ be a C^r-vector bundle. For $L \in \Gamma^m(\mathrm{T}^1_1(\mathsf{E}))$, we have $L^{\mathrm{e}} \in \Gamma^m(\mathsf{TE})$. We can, therefore, think of the m-jet of L^{e} as being characterised by j_mL, as well as by j_mL^{e}, and of comparing these two characterisations. Thus we have the two fibre norms

$$\|j_mL(x)\|^2_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}},m}} = \sum_{j=0}^m \frac{1}{(j!)^2}\|\nabla^{\mathsf{M},\pi_{\mathsf{E}},j}L(x)\|^2_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}}}$$

and

$$\|j_m L^{\mathrm{e}}(e)\|^2_{\mathbb{G}_{\mathsf{E},m}} = \sum_{j=0}^{m} \frac{1}{(j!)^2} \|\nabla^{\mathsf{E},j} L^{\mathrm{e}}(e)\|^2_{\mathbb{G}_{\mathsf{E}}}. \tag{6.7}$$

These fibre norms can be related by virtue of Lemma 5.35. To do so, we make use of the following lemma.

6.7 Lemma: (Fibre norms for vertical evaluations of endomorphisms)

$$\|(\nabla^{\mathsf{M},\pi_{\mathsf{E}},m} L)^{\mathrm{e}}(e)\|_{\mathbb{G}_{\mathsf{E}}} = \|\nabla^{\mathsf{M},\pi_{\mathsf{E}},m} L(\pi_{\mathsf{E}}(e))(e)\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}}}.$$

Proof: The subbundles $\mathsf{H}^*\mathsf{E}$ and $\mathsf{V}^*\mathsf{E}$ are $\mathbb{G}_{\mathsf{E}}^{-1}$-orthogonal. We note that the identification $\mathsf{V}^*_e\mathsf{E} \simeq \mathsf{E}^*_{\pi_{\mathsf{E}}(e)}$ is an isometry and that $T^*_e\pi_{\mathsf{E}}\colon \mathsf{T}^*_{\pi_{\mathsf{E}}(e)}\mathsf{M} \to \mathsf{H}^*_e\mathsf{E}$ is an isometry. Thus we have the formula

$$\|B^{\mathrm{e}}(e)\|_{\mathbb{G}_{\mathsf{E}}} = \|B(\pi_{\mathsf{E}}(e))(e)\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}}}, \qquad B \in \Gamma^0(\mathrm{T}^m(\mathsf{T}^*\mathsf{M}) \otimes \mathrm{T}^1_1(\mathsf{E})),$$

and the assertion of the lemma is merely a special case of this formula. ■

We note that the fibre norm (6.7) makes use of the vector bundle mapping

$$B^m_{\nabla^{\mathsf{E}}} \in \mathrm{VB}^r(\mathsf{P}^{*m}\mathsf{E} \otimes \mathsf{VE}; \mathrm{S}^{\leq m}(\mathsf{T}^*\mathsf{E}) \otimes \mathsf{VE})$$

from Lemma 5.35. If instead we use the vector bundle mapping

$$A^m_{\nabla^{\mathsf{E}}} \in \mathrm{VB}^r(\mathsf{P}^{*m}\mathsf{E} \otimes \mathsf{VE}; \mathrm{S}^{\leq m}(\pi^*_{\mathsf{E}}\mathsf{T}^*\mathsf{M}) \otimes \mathsf{VE})$$

from Lemma 5.35, then we have the alternative fibre norm

$$\|j_m L^{\mathrm{e}}(e)\|'^2_{\mathbb{G}_{\mathsf{E},m}} = \sum_{j=0}^{m} \frac{1}{(j!)^2} \|(\nabla^{\mathsf{M},\pi_{\mathsf{E}},j} L)^{\mathrm{e}}(e)\|^2_{\mathbb{G}_{\mathsf{E}}} = \sum_{j=0}^{m} \frac{1}{(j!)^2} \|\nabla^{\mathsf{M},\pi_{\mathsf{E}},j} L(e)\|^2_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}}}.$$

Again, this points out the matter of the relationship between the jet of a lift versus the lift of the jet, and this matter will be considered in detail in the continuity results of Section 9.3.

6.8. Fibre norms for pull-backs of functions. We let $r \in \{\infty, \omega\}$ and let M and N be C^r-manifolds, and let $\Phi \in \mathrm{C}^r(\mathsf{M}; \mathsf{N})$. For $f \in \mathrm{C}^m(\mathsf{N})$, we have $\Phi^* f \in \mathrm{C}^m(\mathsf{M})$. We can, therefore, think of the m-jet of $\Phi^* f$ as being characterised by $j_m f$, as well as by $j_m \Phi^* f$, and of comparing these two characterisations. Thus we have the two fibre norms

$$\|j_m f(x)\|^2_{\mathbb{G}_{\mathsf{N},m}} = \sum_{j=0}^{m} \frac{1}{(j!)^2} \|\nabla^{\mathsf{N},j} f(x)\|^2_{\mathbb{G}_{\mathsf{N}}}$$

and

$$\|j_m \Phi^* f(e)\|^2_{\mathbb{G}_{\mathsf{M},m}} = \sum_{j=0}^{m} \frac{1}{(j!)^2} \|\nabla^{\mathsf{M},j} \Phi^* f(e)\|^2_{\mathbb{G}_{\mathsf{M}}}. \tag{6.8}$$

These fibre norms can be related by virtue of Lemma 5.42. To make use of this relationship, we shall also need to relate the norms of the terms in these expressions. In the preceding sections, this was easy to do since the Riemannian metric on E was related in a specific way to the Riemannian metric on M and the fibre metric in E. Here, this is not so simple since, if we choose a Riemannian metric $\mathbb{G}_{\mathsf{M}}$ on M and a Riemannian metric $\mathbb{G}_{\mathsf{N}}$ on N, these will be have no useful relationship. So, rather than getting an equality between certain norms, the best we can achieve (and all that we need) is a useful bound, and this is the content of the next lemma.

6.8 Lemma: (Fibre norms for pull-backs of functions) *For a compact set $\mathcal{K} \subseteq \mathsf{M}$:*

(i) there exists $C \in \mathbb{R}_{>0}$ such that

$$\|\Phi^*\nabla^{\mathsf{N},m} f(x)\|_{\mathbb{G}_\mathsf{M}} \leq C^m \|\nabla^{\mathsf{N},m} f(\Phi(x))\|_{\mathbb{G}_\mathsf{N}}, \qquad x \in \mathcal{K},\ m \in \mathbb{Z}_{\geq 0};$$

(ii) if Φ is a submersion or an injective immersion, then C from part (i) can be chosen so that it holds that

$$\|\nabla^{\mathsf{N},m} f(\Phi(x))\|_{\mathbb{G}_\mathsf{N}} \leq C^m \|\Phi^*\nabla^{\mathsf{N},m} f(x)\|_{\mathbb{G}_\mathsf{M}}, \qquad x \in \mathcal{K},\ m \in \mathbb{Z}_{\geq 0}.$$

Proof: The essential part of the proof is the following linear algebraic sublemma.

1 Sublemma: *Let $(\mathsf{U}, \mathbb{G}_\mathsf{U})$ and $(\mathsf{V}, \mathbb{G}_\mathsf{V})$ be finite-dimensional $\mathbb{R}$-inner product spaces and let $\Phi \in \mathrm{Hom}_\mathbb{R}(\mathsf{U}; \mathsf{V})$. Then there exists $C \in \mathbb{R}_{>0}$ such that*

$$\|\Phi^* A\|_{\mathbb{G}_\mathsf{U}} \leq C^k \|A\|_{\mathbb{G}_\mathsf{V}}$$

for every $A \in \mathrm{T}^k(\mathsf{V}^*)$, $k \in \mathbb{Z}_{\geq 0}$. If, additionally, Φ is a surjective submersion or an injective immersion, then C can be chosen so that, additionally, it holds that

$$\|A\|_{\mathbb{G}_\mathsf{V}} \leq C^k \|\Phi^* A\|_{\mathbb{G}_\mathsf{U}}$$

for every $A \in \mathrm{T}^k(\mathsf{V}^*)$, $k \in \mathbb{Z}_{\geq 0}$.

Proof: Let $(f_1, \dots, f_m)$ and $(e_1, \dots, e_n)$ be orthonormal bases for U and V with dual bases $(f^1, \dots, f^m)$ and $(e^1, \dots, e^n)$. Write

$$A = \sum_{j_1,\dots,j_k=1}^n A_{j_1\cdots j_k} e^{j_1} \otimes \cdots \otimes e^{j_k}$$

and

$$\Phi = \sum_{j=1}^n \sum_{a=1}^m \Phi^j_a e_j \otimes f^a.$$

Then

$$\Phi^* A = \sum_{j_1,\dots,j_k=1}^n \sum_{a_1,\dots,a_k=1}^m \Phi^{j_1}_{a_1} \cdots \Phi^{j_k}_{a_k} A_{j_1\cdots j_k} f^{a_1} \otimes \cdots \otimes f^{a_k}.$$

Denote

$$\|\Phi\|_\infty = \max\left\{\left|\Phi^j_a\right| \ \middle|\ a \in \{1, \dots, m\}, j \in \{1, \dots, n\}\right\}.$$

We have

$$\begin{aligned}
\|\Phi^* A\|^2_{\mathbb{G}_\mathsf{U}} &= \sum_{a_1,\dots,a_k=1}^m \left(\sum_{j_1,\dots,j_k=1}^n \Phi^{j_1}_{a_1} \cdots \Phi^{j_k}_{a_k} A_{j_1\cdots j_k}\right)^2 \\
&\leq \sum_{a_1,\dots,a_k=1}^m \left(\sum_{j_1,\dots,j_k=1}^n \left|\Phi^{j_1}_{a_1} \cdots \Phi^{j_k}_{a_k} A_{j_1\cdots j_k}\right|\right)^2 \\
&\leq \sum_{a_1,\dots,a_k=1}^m \left(\sum_{j_1,\dots,j_k=1}^n \left|\Phi^{j_1}_{a_1} \cdots \Phi^{j_k}_{a_k}\right|^2\right) \left(\sum_{j_1,\dots,j_k=1}^n \left|A_{j_1\cdots j_k}\right|^2\right) \\
&\leq (nm\|\Phi\|^2_\infty)^k \|A\|^2_{\mathbb{G}_\mathsf{V}}.
\end{aligned}$$

The first part of the result follows by taking $C = \sqrt{nm}\|\Phi\|_\infty$.

If Φ is surjective, let $\Psi \in \mathrm{Hom}_{\mathbb{R}}(\mathsf{V};\mathsf{U})$ be a right-inverse for Φ. Then, by the first part of the result, there exists $C \in \mathbb{R}_{>0}$ such that

$$\|A\|_{\mathbb{G}_\mathsf{V}} = \|(\Phi\circ\Psi)^*A\|_{\mathbb{G}_\mathsf{V}} \leq \|\Psi^*\Phi^*A\|_{\mathbb{G}_\mathsf{V}} \leq C^k\|\Phi^*A\|_{\mathbb{G}_\mathsf{U}}$$

for every $A \in \mathrm{T}^k(\mathsf{V}^*)$, $k \in \mathbb{Z}_{\geq 0}$.

If Φ is injective, we choose the orthonormal basis $(e_1,\dots,e_n)$ so that $(e_1,\dots,e_m)$ is a basis for image(Φ). In this case we have

$$\Phi = \sum_{a,b=1}^{m} \Phi_a^b e_b \otimes f^a,$$

where the $m \times m$ matrix with components Φ_a^b, $a,b \in \{1,\dots,m\}$, is invertible, and

$$\Phi^*A = \sum_{b_1,\dots,b_k=1}^{m}\sum_{a_1,\dots,a_k=1}^{m} \Phi_{a_1}^{b_1}\cdots\Phi_{a_k}^{b_k} A_{b_1\cdots b_k} f^{a_1}\otimes\cdots\otimes f^{a_k}.$$

Letting Ψ_a^b, $a,b \in \{1,\dots,m\}$, be defined by

$$\Psi_a^c\Phi_c^b = \begin{cases} 1, & a = b,\\ 0, & a \neq b,\end{cases}$$

we have

$$A = \sum_{b_1,\dots,b_k=1}^{m}\sum_{a_1,\dots,a_k=1}^{m} \Psi_{a_1}^{b_1}\cdots\Psi_{a_k}^{b_k} (\Phi^*A)_{b_1\cdots b_k} e^{a_1}\otimes\cdots\otimes e^{a_k},$$

and the conclusion in this case follows just as in the first part of the proof since, locally, a surjective submersion and an injective immersion can be made linear in an appropriate set of coordinates [Abraham, Marsden, and Ratiu 1988, Theorems 3.5.2 and 3.5.7]. ▼

To prove the first part of the lemma, let $x \in \mathcal{K}$ and take $C_x \in \mathbb{R}_{>0}$ as in the sublemma such that

$$\|\Phi^*\nabla^{\mathsf{N},m} f(x)\|_{\mathbb{G}_\mathsf{M}} \leq C_x^m \|\nabla^{\mathsf{N},m} f(\Phi(x))\|_{\mathbb{G}_\mathsf{N}}, \qquad m \in \mathbb{Z}_{\geq 0}.$$

By continuity, and noting the exact form of the constant C from the sublemma (i.e., depending on the size of the derivative of $T_x\Phi$), there exists a neighbourhood $\mathcal{U}_x$ of x such that

$$\|\Phi^*\nabla^{\mathsf{N},m} f(y)\|_{\mathbb{G}_\mathsf{M}} \leq (2C_x)^m \|\nabla^{\mathsf{N},m} f(\Phi(y))\|_{\mathbb{G}_\mathsf{N}}, \qquad y \in \mathcal{N}_x,\ m \in \mathbb{Z}_{>0}.$$

Then take $x_1,\dots,x_k \in \mathcal{K}$ such that $\mathcal{K} \subseteq \cup_{j=1}^k \mathcal{U}_{x_j}$. The first part of the lemma then follows by taking

$$C = \max\{2C_{x_1},\dots,2C_{x_k}\}.$$

The second part of the lemma follows, *mutatis mutandis*, from the second part of the sublemma. ■

We note that the fibre norm (6.8) makes use of the vector bundle mapping

$$B^m_{\nabla^{\mathsf{E}}} \in \mathrm{VB}^r(\mathsf{T}^{*m}_{\Phi}\mathsf{M}; \mathrm{S}^{\leq m}(\mathsf{T}^*\mathsf{M}))$$

from Lemma 5.42. If instead we use the vector bundle mapping

$$A^m_{\nabla^{\mathsf{E}}} \in \mathrm{VB}^r(\mathsf{T}^{*m}_{\Phi}\mathsf{M}; \mathrm{S}^{\leq m}(\Phi^*\mathsf{T}^*\mathsf{N}))$$

from Lemma 5.42, then we have the alternative fibre norm

$$\|j_m\Phi^* f(e)\|'^2_{\mathbb{G}_{\mathsf{M},m}} = \sum_{j=0}^{m} \frac{1}{(j!)^2}\|\Phi^*\nabla^{\mathsf{N},j} f(e)\|^2_{\mathbb{G}_{\mathsf{M}}}.$$

The relationship between the fibre norms $\|\cdot\|_{\mathbb{G}_{\mathsf{M},m}}$ and $\|\cdot\|'_{\mathbb{G}_{\mathsf{M},m}}$ can be phrased as, "What is the relationship between the jet of the pull-back and the pull-back of the jet?" This is a question we will phrase below for other sorts of lifts, and will address comprehensively in the proof of continuity of pull-back in Theorem 9.3.

7. Estimates related to jet bundle norms

In Section 5 we gave formulae relating derivatives of geometric objects to derivatives of their lifts, and vice versa. In Section 6 we defined fibre metrics associated with spaces of lifted objects. In each of the multitude of constructions, there arose certain vector bundle mappings that satisfied recursion relations. In order to establish some important comparison results for different characterisations of topologies, we will need some rather detailed technical estimates concerning the growth of these recursively defined vector bundle mappings in the real analytic case, and we develop these here. As a part of this, we establish a number of fairly simple, linear algebraic estimates. It is not the existence of these estimates that are of interest, but the form they take. As we shall see, for the real analytic topology, the dimensions of various tensor spaces show up in ways that need to be bookkept.

The results in this section are important, but somewhat elaborate. Moreover, they apply specifically to the real analytic setting. The algebraic computations and estimates of Section 7.1, when applied in the smooth setting, do not require the very particular forms we give here.

7.1. Algebraic estimates. To work with the topologies we present in Section 2.4, we will have to compute and estimate high-order derivatives of various sorts of tensors. In this section we collect the fairly elementary formulae we shall need. All norms on tensor products are those induced by an inner product as in Lemma 2.2. For simplicity, therefore, we shall often omit any particular symbols attached to "$\|\cdot\|$" to connote which norm we are talking about; all vector spaces have a unique norm (given the data) that we shall use.

We start by giving the norm of the identity mapping on tensors.

7.1 Lemma: (The norm of the identity map) *If* V *is a finite-dimensional* $\mathbb{R}$*-vector space with inner product* $\mathbb{G}$*, then* $\|\mathrm{id}_{\mathsf{V}}\| = \sqrt{\dim_{\mathbb{R}}(\mathsf{V})}$.

Proof: Let $(e_1, \dots, e_n)$ be an orthonormal basis for V with dual basis $(e^1, \dots, e^n)$ the dual basis. Write

$$\mathrm{id}_{\mathsf{V}} = \sum_{j=1}^{n} \sum_{k=1}^{n} \delta_j^k e_k \otimes e^j.$$

We have

$$\|A\|^2 = \sum_{j=1}^{n} \sum_{k=1}^{n} (\delta_j^k)^2 = n,$$

as claimed. ■

Next we consider the norm of the tensor product of linear maps.

7.2 Lemma: (Norms of tensor products) *Let U, V, W, and X be finite-dimensional $\mathbb{R}$-vector spaces with inner products. Then, for $A \in \mathrm{Hom}_{\mathbb{R}}(\mathsf{U}; \mathsf{V})$ and $B \in \mathrm{Hom}_{\mathbb{R}}(\mathsf{W}; \mathsf{X})$,*

$$\|A \otimes B\| = \|A\| \|B\|.$$

Proof: Let $(e_1, \dots, e_n)$, $(f_1, \dots, f_m)$, $(g_1, \dots, g_k)$, and $(h_1, \dots, h_l)$ be orthonormal bases for U, V, W, and X, respectively. Let $(e^1, \dots, e^n)$, $(f^1, \dots, f^m)$, $(g^1, \dots, g^k)$, and $(h^1, \dots, h^l)$ be the dual bases. Write

$$A = \sum_{j=1}^{n} \sum_{a=1}^{m} A_j^a f_a \otimes e^j, \quad B = \sum_{i=1}^{k} \sum_{b=1}^{l} B_i^b h_b \otimes g^i$$

so that

$$A \otimes B = \sum_{j=1}^{n} \sum_{i=1}^{k} \sum_{a=1}^{m} \sum_{b=1}^{l} A_j^a B_i^b (f_a \otimes h_b) \otimes (e^j \otimes g^i).$$

Then

$$\begin{aligned} \|A \otimes B\|^2 &= \sum_{j=1}^{n} \sum_{i=1}^{k} \sum_{a=1}^{m} \sum_{b=1}^{l} (A_j^a B_i^b)^2 \\ &\le \left(\sum_{j=1}^{n} \sum_{a=1}^{m} (A_j^a)^s \right) \left(\sum_{i=1}^{k} \sum_{b=1}^{l} (B_i^b)^2 \right) = \|A\|^2 \|B\|^2, \end{aligned}$$

as claimed. ■

Our next estimate concerns the relationship between norms of tensors evaluated on arguments.

7.3 Lemma: (Norm of tensor evaluation) *Let U and V be finite-dimensional $\mathbb{R}$-vector spaces with inner products $\mathbb{G}$ and $\mathbb{H}$, respectively. Then*

$$\|L(u)\| \le \|L\| \, \|u\|$$

for all linear mappings $L \in \mathrm{Hom}_{\mathbb{R}}(\mathsf{U}; \mathsf{V})$ and for all $u \in \mathsf{U}$.

Proof: Let $(f_1, \dots, f_m)$ and $(e_1, \dots, e_n)$ be an orthonormal basis for U and V. For $L \in \mathrm{Hom}_{\mathbb{R}}(\mathsf{U}; \mathsf{V})$, write

$$L = \sum_{a=1}^{m} \sum_{j=1}^{n} L_a^j e_j \otimes f^a.$$

Then we compute, using Cauchy–Schwarz,

$$\begin{aligned}\|L(u)\|^2 = \sum_{j=1}^{n} \left(\sum_{a=1}^{m} L_a^j u^a \right)^2 &\le \sum_{j=1}^{n} \left(\sum_{a=1}^{m} \left| L_a^j u^a \right| \right)^2 \\ &\le \sum_{j=1}^{n} \left(\sum_{a=1}^{m} \left| L_a^j \right|^2 \right) \left(\sum_{a=1}^{m} |u^a|^2 \right) = \|L\|^2 \|u\|^2,\end{aligned}$$

giving the lemma. ■

We shall also make use of a sort of "reverse inequality" related to the above.

7.4 Lemma: (Upper bound for norm of linear map) *Let U and V be finite-dimensional $\mathbb{R}$-vector spaces with inner products $\mathbb{G}_{\mathsf{U}}$ and $\mathbb{G}_{\mathsf{V}}$. For $L \in \mathrm{Hom}_{\mathbb{R}}(\mathsf{U}; \mathsf{V})$,*

$$\|L\| \le \sqrt{\dim_{\mathbb{R}}(\mathsf{U})} \sup\{\|L(u)\| \mid \|u\| = 1\}.$$

Proof: The result is true with equality and without the constant if one uses the induced norm for $\mathrm{Hom}_{\mathbb{R}}(\mathsf{U}; \mathsf{V})$, rather than the tensor norm as we do here. So the statement of the lemma is really about relating the induced norm with the tensor norm.

The tensor norm, in the case of linear mappings as we have here, is really the Frobenius norm, and as such it is computed as the ℓ^2-norm of the vector of the set of $\dim_{\mathbb{R}}(\mathsf{U})$ eigenvalues of $\sqrt{L^T \circ L}$. On the other hand, the induced norm is the ℓ^∞ norm of this same vector of eigenvalues of $\sqrt{L^T \circ L}$. These interpretations can be found in [Bhatia 1997, page 7]. For this reason, an application of (1.3) gives the result. ■

Another tensor estimate we shall find useful concerns symmetrisation.

7.5 Lemma: (Norms of symmetrised tensors) *Let V be a finite-dimensional $\mathbb{R}$-vector space and let $\mathbb{G}$ be an inner product on V. Then*

$$\|\mathrm{Sym}_k(A)\| \le \|A\|$$

for every $A \in \mathrm{T}^k(\mathsf{V}^*)$ and $k \in \mathbb{Z}_{>0}$.

Proof: The result follows from the following sublemma.

1 Sublemma: *The map $\mathrm{Sym}_k\colon \mathrm{T}^k(\mathsf{V}^*) \to \mathrm{S}^k(\mathsf{V}^*)$ is the orthogonal projection.*

Proof: Let us simply denote by $\mathbb{G}$ the inner product on $\mathrm{T}^k(\mathsf{V}^*)$, defined as in Lemma 2.2. It suffices to show that $\mathbb{G}(A, S) = \mathbb{G}(\mathrm{Sym}_k(A), S)$ for every $A \in \mathrm{T}^k(\mathsf{V}^*)$ and $S \in \mathrm{S}^k(\mathsf{V}^*)$. It suffices to show that this is true as A runs over a set of generators for $\mathrm{T}^k(\mathsf{V}^*)$ and S runs over a set of generators for $\mathrm{S}^k(\mathsf{V}^*)$.

Thus we let $(e_1, \dots, e_n)$ be an orthonormal basis for V with dual basis $(e^1, \dots, e^n)$. Then we have generators

$$e^{a_1} \otimes \cdots \otimes e^{a_k}, \qquad a_1, \dots, a_k \in \{1, \dots, n\},$$

for $\mathrm{T}^k(\mathsf{V}^*)$ and

$$\mathrm{Sym}_k(e^{b_1} \otimes \cdots \otimes e^{b_k}), \qquad b_1, \dots, b_k \in \{1, \dots, n\},$$

for $\mathrm{S}^k(\mathsf{V}^*)$. For $a_1, \dots, a_k, b_1, \dots, b_k \in \{1, \dots, n\}$, we wish to show that the inner product

$$\begin{aligned}\mathbb{G}(e^{a_1} \otimes \cdots \otimes e^{a_k}, \mathrm{Sym}_k(e^{b_1} \otimes \cdots \otimes e^{b_k})) \\ = \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} \mathbb{G}(e^{a_1} \otimes \cdots \otimes e^{a_k}, e^{b_{\sigma(1)}} \otimes \cdots \otimes e^{b_{\sigma(k)}}) \\ = \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} \mathbb{G}(e^{a_1}, e^{b_{\sigma(1)}}) \cdots \mathbb{G}(e^{a_k}, e^{b_{\sigma(k)}})\end{aligned}$$

is equal to

$$\mathbb{G}(\mathrm{Sym}_k(e^{a_1} \otimes \cdots \otimes e^{a_k}), \mathrm{Sym}_k(e^{b_1} \otimes \cdots \otimes e^{b_k})).$$

Unless $\{a_1, \dots, a_k\}$ and $\{b_1, \dots, b_k\}$ agree as multisets, we have

$$0 = \mathbb{G}(e^{a_1} \otimes \cdots \otimes e^{a_k}, \mathrm{Sym}_k(e^{b_1} \otimes \cdots \otimes e^{b_k})) = \mathbb{G}(\mathrm{Sym}_k(e^{a_1} \otimes \cdots \otimes e^{a_k}), \mathrm{Sym}_k(e^{b_1} \otimes \cdots \otimes e^{b_k})).$$

Thus we can suppose that $\{a_1, \dots, a_k\}$ and $\{b_1, \dots, b_k\}$ agree as multisets.

In this case, since

$$\mathrm{Sym}_k(e^{a_1} \otimes \cdots \otimes e^{a_k}) = \mathrm{Sym}_k(e^{b_1} \otimes \cdots \otimes e^{b_k}),$$

we can assume, without loss of generality, that $a_j = b_j$, $j \in \{1, \dots, k\}$. For $l \in \{1, \dots, n\}$, let $k_l^{\boldsymbol{a}} \in \mathbb{Z}_{\geq 0}$ be the number of occurrences of l in the list $(a_1, \dots, a_k)$. Let $\mathfrak{S}_k^{\boldsymbol{a}} \subseteq \mathfrak{S}_k$ be those permutations σ for which $a_j = a_{\sigma(j)}$, $j \in \{1, \dots, k\}$. Note that $\mathrm{card}(\mathfrak{S}_k^{\boldsymbol{a}}) = k_1^{\boldsymbol{a}}! \cdots k_n^{\boldsymbol{a}}!$ since $\mathfrak{S}_k^{\boldsymbol{a}}$ consists of compositions of permutations that permute all the 1's, all the 2's, etc., in the list $(a_1, \dots, a_k)$. With these bits of notation, we have

$$e^{a_1} \otimes \cdots \otimes e^{a_k} = e^{a_{\sigma(1)}} \otimes \cdots \otimes e^{a_{\sigma(k)}} \quad \Longleftrightarrow \quad \sigma \in \mathfrak{S}_k^{\boldsymbol{a}}.$$

Therefore,

$$\mathbb{G}(e^{a_1} \otimes \cdots \otimes e^{a_k}, e^{a_{\sigma(1)}} \otimes \cdots \otimes e^{a_{\sigma(k)}}) = \begin{cases} 1, & \sigma \in \mathfrak{S}_k^{\boldsymbol{a}}, \\ 0, & \text{otherwise.} \end{cases}$$

We then have

$$\begin{aligned}\mathbb{G}(e^{a_1} \otimes \cdots \otimes e^{a_k}, \mathrm{Sym}_k(e^{a_1} \otimes \cdots \otimes e^{a_k})) \\ = \frac{k_1^{\boldsymbol{a}}! \cdots k_n^{\boldsymbol{a}}!}{k!} \mathbb{G}(e^{a_1} \otimes \cdots \otimes e^{a_k}, e^{a_1} \otimes \cdots \otimes e^{a_k}) = \frac{k_1^{\boldsymbol{a}}! \cdots k_n^{\boldsymbol{a}}!}{k!}.\end{aligned}$$

Next we calculate

$$\mathbb{G}(\mathrm{Sym}_k(e^{a_1} \otimes \cdots \otimes e^{a_k}), \mathrm{Sym}_k(e^{a_1} \otimes \cdots \otimes e^{a_k})).$$

Let $\sigma \in \mathfrak{S}_k$ and, for $l \in \{1, \dots, n\}$, let $k_l^{\sigma(\boldsymbol{a})} \in \mathbb{Z}_{\geq 0}$ be the number of occurrences of l in the list $(a_{\sigma(1)}, \dots, a_{\sigma(k)})$. Let $\mathfrak{S}_k^{\sigma(\boldsymbol{a})} \subseteq \mathfrak{S}_k$ be those permutations σ' for which $a_{\sigma(j)} = a_{\sigma'(j)}$, $j \in \{1, \dots, k\}$. As above, $\mathrm{card}(\mathfrak{S}_k^{\sigma(\boldsymbol{a})}) = k_1^{\sigma(\boldsymbol{a})}! \cdots k_n^{\sigma(\boldsymbol{a})}!$. Also as above, we then have

$$\mathbb{G}(e^{\sigma(1)} \otimes \cdots \otimes e^{\sigma(k)}, \mathrm{Sym}_k(e^{a_1} \otimes \cdots \otimes e^{a_k})) = \frac{k_1^{\sigma(\boldsymbol{a})}! \cdots k_n^{\sigma(\boldsymbol{a})}!}{k!} = \frac{k_1^{\boldsymbol{a}}! \cdots k_n^{\boldsymbol{a}}!}{k!},$$

if $k_1^{\boldsymbol{a}},\dots,k_n^{\boldsymbol{a}}$ are as in the preceding paragraph. Therefore,

$$\begin{aligned}\mathbb{G}(\mathrm{Sym}_k(e^{a_1}\otimes\cdots\otimes e^{a_k}),\mathrm{Sym}_k(e^{a_1}\otimes\cdots\otimes e^{a_k}))\\ &=\frac{1}{k!}\sum_{\sigma\in\mathfrak{S}_k}\mathbb{G}(e^{\sigma(1)}\otimes\cdots\otimes e^{\sigma(k)},\mathrm{Sym}_k(e^{a_1}\otimes\cdots\otimes e^{a_k}))\\ &=\frac{1}{k!}\sum_{\sigma\in\mathfrak{S}_k}\frac{k_1^{\boldsymbol{a}}!\cdots k_n^{\boldsymbol{a}}!}{k!}=\frac{k_1^{\boldsymbol{a}}!\cdots k_n^{\boldsymbol{a}}!}{k!},\end{aligned}$$

and so we have

$$\begin{aligned}\mathbb{G}(e^{a_1}\otimes\cdots\otimes e^{a_k},\mathrm{Sym}_k(e^{b_1}\otimes\cdots\otimes e^{b_k}))\\ =\mathbb{G}(\mathrm{Sym}_k(e^{a_1}\otimes\cdots\otimes e^{a_k}),\mathrm{Sym}_k(e^{a_1}\otimes\cdots\otimes e^{a_k})),\end{aligned}$$

and the sublemma follows. ▼

Now, given $A\in\mathrm{T}^k(\mathsf{V}^*)$, we write $A=\mathrm{Sym}_k(A)+A_1$ where A_1 is orthogonal to $\mathrm{S}^k(\mathsf{V}^*)$. We then have $\|A\|^2=\|\mathrm{Sym}_k(A)\|^2+\|A_1\|^2$, from which the lemma follows. ■

The sublemma from the preceding lemma is proved, differently, by Neuberger [1968, page 124].

Let us also determine the norm of various insertion operators that we shall use. We shall use notation that is specific to the manner in which we shall use these estimates, and this will seem unmotivated out of context. Let U, V, and W be finite-dimensional $\mathbb{R}$-vector spaces, let $m,s,r\in\mathbb{Z}_{>0}$ and $a\in\{0,1,\dots,r\}$, let $S\in\mathrm{T}^1_{r-a+2}(\mathsf{U})$, and let

$$A\in\mathrm{T}^{s+r-a+1}_{m+a+1}(\mathsf{U})\otimes\mathsf{W}\otimes\mathsf{V}^*.$$

We then have the mapping

$$I^1_{A,S,j}\colon\mathrm{T}^s(\mathsf{U}^*)\otimes\mathsf{V}\to\mathrm{T}^{m+r+1}(\mathsf{U}^*)\otimes\mathsf{W}$$

defined by

$$I^1_{A,S,j}(\beta)=A(\mathrm{Ins}_j(\beta,S)).$$

Here we implicitly use the isomorphism

$$\kappa\colon\mathrm{T}^s_m(\mathsf{U})\to\mathrm{Hom}_{\mathbb{R}}(\mathrm{T}^s(\mathsf{U}^*);\mathrm{T}^m(\mathsf{U}^*)),$$

for a finite-dimensional $\mathbb{R}$-vector space U and for $m,s\in\mathbb{Z}_{\geq0}$, via

$$\kappa(v_1\otimes\cdots\otimes v_s\otimes\alpha^1\otimes\cdots\otimes\alpha^m)(\beta^1\otimes\cdots\otimes\beta^s)=\langle\beta^1;v_1\rangle\cdots\langle\beta^s;v_s\rangle\alpha^1\otimes\cdots\otimes\alpha^m,$$

for $v_a\in\mathsf{U}$, $a\in\{1,\dots,s\}$, and $\alpha^j,\beta^b\in\mathsf{U}^*$, $b\in\{1,\dots,s\}$, $j\in\{1,\dots,m\}$. Thus, for additional finite-dimensional $\mathbb{R}$-vector spaces V and W, we have the identification

$$\mathrm{T}^m_s(\mathsf{U})\otimes\mathsf{W}\otimes\mathsf{V}^*\simeq\mathrm{Hom}_{\mathbb{R}}(\mathrm{T}^s(\mathsf{U}^*)\otimes\mathsf{V};\mathrm{T}^m(\mathsf{U}^*)\otimes\mathsf{W}).$$

We now have the following result.

7.6 Lemma: (Norm of composition with tensor insertion I) *With the preceding notation,*

$$\|I^1_{A,S,j}\| \le \|A\|\|S\|.$$

Proof: Let $(f_1,\dots,f_m)$ be an orthonormal basis for U with dual basis $(f^1,\dots,f^m)$. Let $(e_1,\dots,e_n)$ be an orthonormal basis for V with $(e^1,\dots,e^n)$ the dual basis. Let $(g_1,\dots,g_k)$ be an orthonormal basis for W with $(g^1,\dots,g^k)$ the dual basis. Let us write

$$S=\sum_{a=1}^m\sum_{a_1,\dots,a_{r-a+2}}^m S^a_{a_1\cdots a_{r-a+2}} f_a\otimes f^{a_1}\otimes\dots\otimes f^{a_{r-a+2}}$$

and

$$\begin{aligned}A=\sum_{a_1,\dots,a_{s+r-a+1}=1}^m\ \sum_{b_1,\dots,b_{m+a+1}=1}^m\ \sum_{\alpha=1}^k\sum_{l=1}^n A^{a_a\cdots a_{s+r-a+1}\alpha}_{b_1\cdots b_{m+a+1}l}&\\ \times f^{b_1}\otimes\dots\otimes f^{b_{m+a+1}}\otimes f_{a_1}\otimes\dots\otimes f_{a_{s+1}}\otimes g_\alpha\otimes e^l&.\end{aligned}$$

We then have, for $a_1,\dots,a_s\in\{1,\dots,m\}$, $\alpha\in\{1,\dots,k\}$, and $l\in\{1,\dots,n\}$,

$$\begin{aligned}&\mathrm{Ins}_{S,j}(f^{a_1}\otimes\dots\otimes f^{a_s}\otimes g_\alpha\otimes e^l)\\ &\quad=\mathrm{Ins}_j(f^{a_1}\otimes\dots\otimes f^{a_j}\otimes\dots\otimes f^{a_s}\otimes g_\alpha\otimes e^l,S)\\ &\quad=\sum_{b_1,\dots,b_{r-a+2}=1}^m S^{a_j}_{b_1\cdots b_{r-a+2}}\\ &\qquad\times f^{a_1}\otimes\dots\otimes f^{a_{j-1}}\otimes f^{b_1}\otimes\dots\otimes f^{b_{r-a+2}}\otimes f^{a_{j+1}}\otimes\dots\otimes f^{a_s}\otimes g_\alpha\otimes e^l\\ &\quad=\sum_{c_1,\dots,c_{j-1}=1}^m\ \sum_{c_{j+1},\dots,c_s=1}^m\ \sum_{b_1,\dots,b_{r-a+2}=1}^n\ \sum_{\beta=1}^k\sum_{p=1}^n S^{a_j}_{b_1\cdots b_{r-a+2}}\delta^{a_1}_{c_1}\cdots\delta^{a_{j-1}}_{c_{j-1}}\delta^{a_{j+1}}_{c_{j+1}}\cdots\delta^{a_s}_{c_s}\delta^\beta_\alpha\delta^l_p\\ &\qquad\times f^{c_1}\otimes\dots\otimes f^{c_{j-1}}\otimes f^{b_1}\otimes\dots\otimes f^{b_{r-a+2}}\otimes f^{c_{j+1}}\otimes\dots\otimes f^{c_s}\otimes g_\beta\otimes e^p.\end{aligned}$$

Thus

$$\begin{aligned}&I^1_{A,S,j}(f^{a_1}\otimes\dots\otimes f^{a_s}\otimes g_\alpha\otimes e^l)\\ &\quad=\sum_{b_1,\dots,b_{r-a+2}=1}^m\ \sum_{d_1,\dots,d_{m+a+1}=1}^m\ \sum_{\alpha=1}^k\sum_{l=1}^n A^{a_1\cdots a_{j-1}b_1\cdots b_{r-a+2}a_{j+1}\cdots a_s\alpha}_{d_1\cdots d_{m+a+1}l}S^{a_j}_{b_1\cdots b_{r-a+2}}\\ &\qquad\qquad\times f^{d_1}\otimes\dots\otimes f^{d_{m+a+1}}\otimes g_\alpha\otimes e^l.\end{aligned}$$

Then we calculate, using Cauchy–Schwarz,

$$\begin{aligned}
\|I^1_{A,S,j}\|^2 &= \sum_{a_1,\dots,a_s=1}^m \sum_{d_1,\dots,d_{m+a+1}=1}^m \sum_{\alpha=1}^k \sum_{l=1}^n \left(\sum_{b_1,\dots,b_{r-a+2}=1}^m A^{a_1\cdots a_{j-1}b_1\cdots b_{r-a+2}a_{j+1}\cdots a_s\alpha}_{d_1\cdots d_{m+a+1}l} S^{a_j}_{b_1\cdots b_{r-a+2}} \right)^2 \\
&\le \sum_{a_1,\dots,a_s=1}^m \sum_{d_1,\dots,d_{m+a+1}=1}^m \sum_{\alpha=1}^k \sum_{l=1}^n \left(\sum_{b_1,\dots,b_{r-a+2}=1}^m \left| A^{a_1\cdots a_{j-1}b_1\cdots b_{r-a+2}a_{j+1}\cdots a_s\alpha}_{d_1\cdots d_{m+a+1}l} S^{a_j}_{b_1\cdots b_{r-a+2}} \right| \right)^2 \\
&\le \sum_{a_1,\dots,a_s=1}^m \sum_{d_1,\dots,d_{m+a+1}=1}^m \sum_{\alpha=1}^k \sum_{l=1}^n \left(\sum_{b_1,\dots,b_{r-a+2}=1}^m \left| A^{a_1\cdots a_{j-1}b_1\cdots b_{r-a+2}a_{j+1}\cdots a_s}_{d_1\cdots d_{m+a+1}l} \right|^2 \right) \\
&\quad \times \left(\sum_{b_1,\dots,b_{r-a+2}=1}^m \left| S^{a_j}_{b_1\cdots b_{r-a+2}} \right|^2 \right) \\
&\le \|A\|^2\|S\|^2,
\end{aligned}$$

as claimed. ■

Now we perform the same sort of estimate for a similar construction. We take U, V, and W as above, and m, s, r, and a as above. We also still take $S \in \mathrm{T}^1_{r-a+2}(\mathsf{U})$, but here we take

$$B \in \mathrm{T}^s_{m+a}(\mathsf{U}) \otimes \mathsf{W} \otimes \mathsf{V}^*.$$

We then have the mapping

$$I^2_{B,S,j} \colon \mathrm{T}^s(\mathsf{U}^*) \otimes \mathsf{V} \to \mathrm{T}^{m+r+1}(\mathsf{U}^*) \otimes \mathsf{W}$$

defined by

$$I^2_{B,S,j}(\beta) = \mathrm{Ins}_j(B(\beta), S)$$

We now have the following result, whose proof follows from direct computation, just as does Lemma 7.6.

7.7 Lemma: (Norm of composition with tensor insertion II) *With the preceding notation,*

$$\|I^2_{B,S,j}\| \le \|B\|\|S\|.$$

7.2. Tensor field estimates. We next turn to providing estimates for the tensors A^m_s, B^m_s, C^m_s, and D^m_s, $m \in \mathbb{Z}_{\ge 0}$, $s \in \{0, 1, \dots, m\}$, that appear in the lemmata from Section 5. In this section is where all of our seemingly pointless computations from Sections 3 and 4, and our only slightly less seemingly pointless constructions from Sections 5 and 6, bear fruit. We first develop a general estimate, and then show how this estimate can be made to apply to all of the required tensors from Section 5.

We work with real analytic vector bundles $\pi_{\mathsf{E}} \colon \mathsf{E} \to \mathsf{M}$ and $\pi_{\mathsf{F}} \colon \mathsf{F} \to \mathsf{M}$. The rôle of $\pi_{\mathsf{E}} \colon \mathsf{E} \to \mathsf{M}$ in this discussion and that in Section 5 is different. One should think of E in Section 5 as being played by M here. This is because the tensors in Section 5 are defined as having E as their base space. So here we rename this base space as M. As a consequence of this, one should think of (1) the rôle of M in the lemma below as being played by E in the lemmata of Section 5, (2) the rôle of ∇^{M} in the lemma below as being played by ∇^{E}

in the lemmata of Section 5, and (2) the rôle of $\nabla^{\pi_{\mathsf{E}}}$ in the lemma below as being played by the induced connection in an appropriate tensor bundle in the lemmata of Section 5. In our development here, we use the symbol $\nabla^{\mathsf{M},\pi_{\mathsf{E}}}$ to denote the connection induced in any of the myriad bundles formed by taking tensor products of TM, $\mathsf{T}^*\mathsf{M}$, E, and E^*, cf. the constructions at the beginning of Section 2.2.

With this as backdrop, the main technical result we have is the following.

7.8 Lemma: (Bound for families of real analytic tensors defined by recursion) *Let $\pi_{\mathsf{E}}\colon \mathsf{E}\to\mathsf{M}$ and $\pi_{\mathsf{F}}\colon\mathsf{F}\to\mathsf{M}$ be real analytic vector bundles, let ∇^{M} be a real analytic affine connection on M, let $\nabla^{\pi_{\mathsf{E}}}$ and $\nabla^{\pi_{\mathsf{F}}}$ be real analytic vector bundle connections in E and F, respectively. Let $\mathbb{G}_{\mathsf{M}}$ be a real analytic Riemannian metric on M, and let $\mathbb{G}_{\pi_{\mathsf{E}}}$ and $\mathbb{G}_{\pi_{\mathsf{F}}}$ be real analytic fibre metrics for E and F, respectively. Suppose that we are given the following data:*

(i) $\phi_m\in\Gamma^\omega(\mathrm{T}^m_m(\mathsf{TM})\otimes\mathsf{F}\otimes\mathsf{E}^*)$, $m\in\mathbb{Z}_{\geq 0}$;

(ii) $\Phi^s_m\in\Gamma^\omega(\mathrm{End}(\mathrm{T}^s_{m+1}(\mathsf{TM})\otimes\mathsf{F}\otimes\mathsf{F}^*))$, $m\in\mathbb{Z}_{\geq 0}$, $s\in\{0,1,\dots,m\}$;

(iii) $\Psi^s_{jm}\in\Gamma^\omega(\mathrm{Hom}(\mathrm{T}^s_m(\mathsf{TM})\otimes\mathsf{F}\otimes\mathsf{E}^*;\mathrm{T}^s_{m+1}(\mathsf{TM})\otimes\mathsf{F}\otimes\mathsf{E}^*))$, $m\in\mathbb{Z}_{\geq 0}$, $s\in\{0,1,\dots,m\}$, $j\in\{0,1,\dots,m\}$;

(iv) $\Lambda^s_m\in\Gamma^\omega(\mathrm{Hom}(\mathrm{T}^{s-1}_m(\mathsf{TM})\otimes\mathsf{F}\otimes\mathsf{E}^*;\mathrm{T}^s_{m+1}(\mathsf{TM})\otimes\mathsf{F}\otimes\mathsf{E}^*))$, $m\in\mathbb{Z}_{\geq 0}$, $s\in\{1,\dots,m\}$;

(v) $A^m_s\in\Gamma^\omega(\mathrm{T}^s_m(\mathsf{TM})\otimes\mathsf{F}\otimes\mathsf{E}^*)$, $m\in\mathbb{Z}_{\geq 0}$, $s\in\{0,1,\dots,m\}$,

and that the data satisfies the recursion relations prescribed by $A^0_0=\phi_0$ and

$$\begin{aligned}
A^{m+1}_{m+1}&=\Phi^{m+1}_m\circ\phi_{m+1}, && m\in\mathbb{Z}_{\geq 0}\\
A^{m+1}_s&=\Phi^s_m\circ\nabla^{\mathsf{M},\pi_{\mathsf{E}}\otimes\pi_{\mathsf{F}}}A^m_s+\sum_{j=0}^m\Psi^s_{jm}\circ A^m_s+\Lambda^s_m\circ A^m_{s-1}, && m\in\mathbb{Z}_{>0},\ s\in\{1,\dots,m\},\\
A^{m+1}_0&=\Phi^0_m\circ\nabla^{\mathsf{M},\pi_{\mathsf{E}}\otimes\pi_{\mathsf{F}}}A^m_0+\sum_{j=0}^m\Psi^0_{jm}\circ A^m_0, && m\in\mathbb{Z}_{\geq 0}.
\end{aligned}$$

Suppose that the data are such that, for each compact $\mathcal{K}\subseteq\mathsf{M}$, there exist $C_1,\sigma_1\in\mathbb{R}_{>0}$ satisfying

(i) $\|D^r_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}\otimes\pi_{\mathsf{F}}}}\phi_m(x)\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}\otimes\pi_{\mathsf{F}}}}\leq C_1\sigma_1^{-r}r!$, $m,r\in\mathbb{Z}_{\geq 0}$;

(ii) $\|D^r_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{F}}}}\Phi^s_m(x)\circ A\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}\otimes\pi_{\mathsf{F}}}}\leq C_1\sigma_1^{-r}r!\|A\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}\otimes\pi_{\mathsf{F}}}}$, $A\in\mathrm{T}^s_{m+a}(\mathsf{T}_x\mathsf{M}\otimes\mathsf{F}_x\otimes\mathsf{E}^*_x)$, $m,r,a\in\mathbb{Z}_{\geq 0}$, $s\in\{0,1,\dots,m+1\}$;

(iii) $\|D^r_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}\otimes\pi_{\mathsf{F}}}}\Psi^s_{jm}(x)\circ A\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}\otimes\pi_{\mathsf{F}}}}\leq C_1\sigma_1^{-r}r!\|A\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}\otimes\pi_{\mathsf{F}}}}$, $A\in\mathrm{T}^s_{m+a}(\mathsf{T}_x\mathsf{M}\otimes\mathsf{F}_x\otimes\mathsf{E}^*_x)$, $m,r,a\in\mathbb{Z}_{\geq 0}$, $s\in\{0,1,\dots,m\}$, $j\in\{0,1,\dots,m\}$;

(iv) $\|D^r_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}\otimes\pi_{\mathsf{F}}}}\Lambda^s_m(x)\circ A\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}\otimes\pi_{\mathsf{F}}}}\leq C_1\sigma_1^{-r}r!\|A\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}\otimes\pi_{\mathsf{F}}}}$, $A\in\mathrm{T}^{s-1}_{m+a}(\mathsf{T}_x\mathsf{M}\otimes\mathsf{F}_x\otimes\mathsf{E}^*_x)$, $m,r,a\in\mathbb{Z}_{\geq 0}$, $s\in\{0,1,\dots,m\}$.

for $x\in\mathcal{K}$.

Then, for $\mathcal{K}\subseteq\mathsf{M}$ compact, there exist $C,\sigma,\rho\in\mathbb{R}_{>0}$ such that

$$\|D^r_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}\otimes\pi_{\mathsf{F}}}}A^m_s(x)\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}\otimes\pi_{\mathsf{F}}}}\leq C\sigma^{-m}\rho^{-(m+r-s)}(m+r-s)!$$

for $m,r\in\mathbb{Z}_{\geq 0}$, $s\in\{0,1,\dots,m\}$, and $x\in\mathcal{K}$.

Proof: We prove the lemma with a sort of meandering induction, covering various special cases of m and s before giving a proof for the general case.

Before we embark on the proof, we organise some data that will arise in the estimate that we prove.

1. We take $\mathcal{K} \subseteq \mathsf{M}$ compact and define $C_1, \sigma_1 \in \mathbb{R}_{>0}$ as in the statement of the lemma. We shall assume, without loss of generality, that $C_1 > 1$ and $\sigma_1 < 1$.
2. Choose $\beta \in \mathbb{R}_{>0}$ sufficiently large that
$$\sum_{k=0}^{\infty} \beta^{-k} < \infty,$$
and let $\alpha = \frac{\beta}{\beta-1} > 1$ denote the value of this sum. Let $\gamma = 4\alpha$.
3. We note that, for any $a, b, c \in \mathbb{Z}_{>0}$ with $b < c$, we have
$$\frac{(a+b)!}{b!} < \frac{(a+c)!}{c!}.$$
This is a direct computation:
$$\frac{(a+b)!}{b!} = (1+b)\cdots(a+b) < (1+c)\cdots(a+c) = \frac{(a+c)!}{c!}.$$
4. For $m \in \mathbb{Z}_{\geq 0}$ and $s \in \{0, 1, \dots, m\}$, we denote
$$C_{m,s} = \begin{cases} 1, & m = 0 \text{ or } s = 0, \\ \binom{m-1}{s-1}, & \text{otherwise.} \end{cases}$$
We note that
   (a) $C_{m,m} = 1$, that
   (b) $C_{m,s} \leq C_{m+1,s}$, that
   (c) $C_{m,s} \leq C_{m+1,s+1}$, and that
   (d) $mC_{m,s} \leq (m+1-s)C_{m+1,s}$.

   The first and second of these assertions is obvious. For the third, for $m, s \in \mathbb{Z}_{>0}$ with $s \leq m$, we compute
$$C_{m,s} = \frac{(m-1)!}{(s-1)!(m-s)!} \leq \frac{m}{s}\frac{(m-1)!}{(s-1)!(m-s)!} = C_{m+1,s+1}.$$
For the fourth, for $m \in \mathbb{Z}_{>0}$ and $s \in \mathbb{Z}_{>0}$ satisfying $s \leq m$, we compute
$$mC_{m,s} = m\frac{(m-1)!}{(s-1)!(m-s)!} = (m-s+1)\frac{m!}{(s-1)!(m+1-s)!} = (m-s+1)C_{m+1,s}.$$
5. We shall have occasion below, and also subsequently, to use a standard multinomial estimate. First let $\alpha_1, \dots, \alpha_n \in \mathbb{R}_{>0}$ and note that
$$(\alpha_1 + \dots + \alpha_n)^m = \sum_{m_1+\dots+m_n=m} \frac{m!}{m_1!\cdots m_n!}\alpha_1^{m_1}\cdots\alpha_n^{m_n}.$$

Taking $\alpha_1 = \cdots = \alpha_n = 1$, we see that

$$\frac{m!}{m_1!\cdots m_n!} \le n^m \tag{7.1}$$

whenever $m_1, \dots, m_n \in \mathbb{Z}_{\ge 0}$ sum to m.
Given all of this, we shall prove that

$$\|D^r_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}\otimes\pi_{\mathsf{F}}}} A^m_s(x)\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}\otimes\pi_{\mathsf{F}}}} \le C_1(C_1\sigma_1^{-1}\gamma)^m C_{m,s}\left(\frac{\beta}{\sigma_1}\right)^{m+r-s}(m+r-s)! \tag{7.2}$$

for $m, r \in \mathbb{Z}_{\ge 0}$, $s \in \{0, 1, \dots, m\}$, and $x \in \mathcal{K}$.

Case $m = s = 0$:

Directly using the hypotheses, we have

$$\begin{aligned}\|D^r_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}\otimes\pi_{\mathsf{F}}}} A^0_0(x)\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}\otimes\pi_{\mathsf{F}}}} &= \|D^r_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}\otimes\pi_{\mathsf{F}}}} \phi_0(x)\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}\otimes\pi_{\mathsf{F}}}}\\ &\le C_1\sigma_1^{-r} r! \le C_1(C_1\sigma_1^{-1}\gamma)^0 C_{0,0}\left(\frac{\beta}{\sigma_1}\right)^{0+r-0}(0+r-0)!\end{aligned}$$

for $r \in \mathbb{Z}_{\ge 0}$ and $x \in \mathcal{K}$. This gives (7.2) in this case.

Case $m \in \mathbb{Z}_{>0}$ and $s = m$:

By Lemma 4.4, we have

$$D^r_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}\otimes\pi_{\mathsf{F}}}} A^m_m = D^r_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}\otimes\pi_{\mathsf{F}}}}(\Phi^m_{m-1}\circ\phi_m) = \sum_{a=0}^{r}\binom{r}{a} D^a_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}}\Phi^m_{m-1}(D^{r-a}_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}\otimes\pi_{\mathsf{F}}}}\phi_m)$$

for $m, r \in \mathbb{Z}_{\ge 0}$. Therefore, by Lemma 7.3, using the hypotheses, and by the preliminary observation 3 above,

$$\begin{aligned}\|D^r_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}\otimes\pi_{\mathsf{F}}}} A^m_m\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}\otimes\pi_{\mathsf{F}}}} &\le \sum_{a=0}^{r}\frac{r!}{a!(r-a)!}(C_1\sigma_1^{-a}a!)(C_1\sigma_1^{-(m+r-a)}(r-a)!)\\ &\le C_1C_1\sigma_1^{-m} r! \sum_{a=0}^{r}\sigma_1^{-a}\left(\frac{\beta}{\sigma_1}\right)^{r-a} \le C_1C_1\sigma_1^{-m}\left(\frac{\beta}{\sigma_1}\right)^{r} r!\sum_{a=0}^{r}\beta^{-a}\\ &\le C_1(C_1\sigma_1^{-1}\gamma)^m C_{m,m}\left(\frac{\beta}{\sigma_1}\right)^{m+r-m}(m+r-m)!.\end{aligned}$$

As this holds for every $m \in \mathbb{Z}_{>0}$, $r \in \mathbb{Z}_{\ge 0}$, and $x \in \mathcal{K}$, this gives (7.2) in this case.

Case $m = 1$ and $s = 0$:

By Lemma 4.4 we have

$$D^r_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}\otimes\pi_{\mathsf{F}}}} A^1_0 = \underbrace{\sum_{a=0}^{r} \binom{r}{a} D^a_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}} \Phi^0_0(D^{r-a}_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}\otimes\pi_{\mathsf{F}}}} \nabla^{\mathsf{M},\pi_{\mathsf{E}}\otimes\pi_{\mathsf{F}}} A^0_0)}_{\text{term 1}}$$

$$+ \underbrace{\sum_{a=0}^{r} \binom{r}{a} D^a_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}\otimes\pi_{\mathsf{F}}}} \Psi^0_{00}(D^{r-a}_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}\otimes\pi_{\mathsf{F}}}} A^0_0)}_{\text{term 2(a)}} + \underbrace{\sum_{a=0}^{r} \binom{r}{a} D^a_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}\otimes\pi_{\mathsf{F}}}} \Psi^0_{10}(D^{r-a}_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}\otimes\pi_{\mathsf{F}}}} A^0_0)}_{\text{term 2(b)}}.$$

As we showed in the proof of Lemma 4.8, we have

$$D^{r-a}_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}\otimes\pi_{\mathsf{F}}}}(\nabla^{\mathsf{M},\pi_{\mathsf{E}}\otimes\pi_{\mathsf{F}}} A^0_0) = D^{r-a+1}_{\nabla^{\mathsf{M},\pi_{\mathsf{E}}\otimes\pi_{\mathsf{F}}}} A^0_0.$$

Therefore, by Lemma 7.3 and using the hypotheses,

$$\begin{aligned}\|\text{term 1}(x)\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}\otimes\pi_{\mathsf{F}}}} &\le \sum_{a=0}^{r} \frac{r!}{a!(r-a)!}(C_1\sigma_1^{-a}a!)(C_1\sigma_1^{-(r-a+1)}(r-a+1)!) \\ &\le C_1C_1(r+1)!\sum_{a=0}^{r}\sigma_1^{-a}\left(\frac{\beta}{\sigma_1}\right)^{r-a+1} \le C_1C_1\left(\frac{\beta}{\sigma_1}\right)^{r+1}(r+1)!\sum_{a=0}^{r}\beta^{-a} \\ &\le C_1(C_1\alpha)\left(\frac{\beta}{\sigma_1}\right)^{r+1}(r+1)!.\end{aligned}$$

In a similar manner,

$$\|\text{term 2(a)}(x)\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}\otimes\pi_{\mathsf{F}}}}, \|\text{term 2(b)}(x)\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}\otimes\pi_{\mathsf{F}}}} \le C_1(C_1\alpha)\left(\frac{\beta}{\sigma_1}\right)^{r} r!.$$

Therefore, for $r \in \mathbb{Z}_{\ge 0}$ and $x \in \mathcal{K}$,

$$\begin{aligned}\|D^r_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}\otimes\pi_{\mathsf{F}}}} A^1_0(x)\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}\otimes\pi_{\mathsf{F}}}} &\le \frac{1}{4}C_1(C_1\gamma)\left(\frac{\beta}{\sigma_1}\right)^{r+1}(r+1)! + \frac{1}{2}C_1(C_1\gamma)\left(\frac{\beta}{\sigma_1}\right)^{r} r! \\ &\le C_1(C_1\sigma_1^{-1}\gamma)^1 C_{1,0}\left(\frac{\beta}{\sigma_1}\right)^{1+r-0}(1+r-0)!\end{aligned}$$

and this gives (7.2) in this case.

Case $m \in \mathbb{Z}_{>0}$ and $s = 0$:

We use induction on m, the desired estimate having been shown to be true for $m = 1$. By Lemma 4.4 we have

$$D^r_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}\otimes\pi_{\mathsf{F}}}} A_0^{m+1} = \underbrace{\sum_{a=0}^{r} \binom{r}{a} D^a_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}} \Phi^0_m (D^{r-a}_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}\otimes\pi_{\mathsf{F}}}} \nabla^{\mathsf{M},\pi_{\mathsf{E}}\otimes\pi_{\mathsf{F}}} A_0^m)}_{\text{term 1}}$$
$$+ \underbrace{\sum_{a=0}^{r} \binom{r}{a} D^a_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}\otimes\pi_{\mathsf{F}}}} \Psi^0_{0m} (D^{r-a}_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}\otimes\pi_{\mathsf{F}}}} A_0^m)}_{\text{term 2(a)}}$$
$$+ \underbrace{\sum_{j=1}^{m}\sum_{a=0}^{r} \binom{r}{a} D^a_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}\otimes\pi_{\mathsf{F}}}} \Psi^0_{jm} (D^{r-a}_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}\otimes\pi_{\mathsf{F}}}} A_0^m)}_{\text{term 2(b)}}.$$

For term 1, as above for the case $m = 1$ and $s = 0$, we have

$$\|D^{r-a}_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}\otimes\pi_{\mathsf{F}}}} (\nabla^{\mathsf{M},\pi_{\mathsf{E}}\otimes\pi_{\mathsf{F}}} A_0^m)\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}\otimes\pi_{\mathsf{F}}}} \leq \|D^{r-a+1}_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}\otimes\pi_{\mathsf{F}}}} A_0^m\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}\otimes\pi_{\mathsf{F}}}}.$$

We now use Lemma 7.3, the hypotheses, the induction hypotheses, and the preliminary observation 3 above to determine that

$$\begin{aligned}
\|\text{term 1}(x)\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}\otimes\pi_{\mathsf{F}}}} \leq & \sum_{a=0}^{r} \frac{r!}{a!(r-a)!} (C_1 \sigma_1^{-a} a!) \\
& \times \left(C_1 (C_1 \sigma_1^{-1} \gamma)^m C_{m,0} \left(\frac{\beta}{\sigma_1}\right)^{m+r-a+1} (m+r-a+1)! \right) \\
\leq & \, C_1 (C_1 \sigma_1^{-1} \gamma)^m C_{m,0} \left(\frac{\beta}{\sigma_1}\right)^{m+r+1} (m+r+1)! \sum_{a=0}^{r} \beta^{-a} \\
\leq & \, C_1 (C_1 \sigma_1^{-1} \gamma)^m C_{m,0} \alpha \left(\frac{\beta}{\sigma_1}\right)^{m+r+1} (m+r+1)!
\end{aligned}$$

By a similar computation, we have

$$\|\text{term 2(a)}(x)\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}\otimes\pi_{\mathsf{F}}}} \leq C_1 (C_1 \sigma_1^{-1} \gamma)^m C_{m,0} \alpha \left(\frac{\beta}{\sigma_1}\right)^{m+r} (m+r)!.$$

We also have, making use of our observation 4 from above,

$$\begin{aligned}
\|\text{term 2(b)}(x)\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}\otimes\pi_{\mathsf{F}}}} & \leq C_1 (C_1 \sigma_1^{-1} \gamma)^m m C_{m,0} \alpha \left(\frac{\beta}{\sigma_1}\right)^{m+r} (m+r)! \\
& \leq C_1 (C_1 \sigma_1^{-1} \gamma)^m (m+1) C_{m+1,0} \alpha \left(\frac{\beta}{\sigma_1}\right)^{m+r} (m+r)! \\
& \leq C_1 (C_1 \sigma_1^{-1} \gamma)^m C_{m+1,0} \alpha \left(\frac{\beta}{\sigma_1}\right)^{m+r} (m+r+1)!.
\end{aligned}$$

Thus, for $x \in \mathcal{K}$,

$$\begin{aligned}\|D^r_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}\otimes\pi_{\mathsf{F}}}} A^{m+1}_0(x)\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}\otimes\pi_{\mathsf{F}}}} &\le \frac{1}{4}C_1(C_1\sigma_1^{-1}\gamma)^m C_{m,0}\gamma\left(\frac{\beta}{\sigma_1}\right)^{m+r+1}(m+r+1)! \\ &\quad + \frac{1}{4}C_1(C_1\sigma_1^{-1}\gamma)^m C_{m,0}\gamma\left(\frac{\beta}{\sigma_1}\right)^{m+r}(m+r)! \\ &\quad + \frac{1}{4}C_1(C_1\sigma_1^{-1}\gamma)^m C_{m+1,0}\gamma\left(\frac{\beta}{\sigma_1}\right)^{m+r}(m+r+1)! \\ &\le C_1(C_1\sigma_1^{-1}\gamma)^{m+1} C_{m+1,0}\left(\frac{\beta}{\sigma_1}\right)^{m+1+r-0}(m+1+r-0)!.\end{aligned}$$

This proves (7.2) by induction in this case.

Case $m \in \mathbb{Z}_{>0}$ and $s \in \{1,\dots,m-1\}$:

We use induction first on m (the result having been proved for the case $m = 0$) and, for fixed m, by induction on s (the result having been proved for the case $s = 0$). By Lemma 4.4 we have

$$\begin{aligned}D^r_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}\otimes\pi_{\mathsf{F}}}} A^{m+1}_s &= \underbrace{\sum_{a=0}^{r}\binom{r}{a} D^a_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}}\Phi^s_m(D^{r-a}_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}\otimes\pi_{\mathsf{F}}}}\nabla^{\mathsf{M},\pi_{\mathsf{E}}\otimes\pi_{\mathsf{F}}} A^m_s)}_{\text{term 1}} \\ &\quad + \underbrace{\sum_{a=0}^{r}\binom{r}{a} D^a_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}\otimes\pi_{\mathsf{F}}}}\Psi^s_{0m}(D^{r-a}_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}\otimes\pi_{\mathsf{F}}}} A^m_s)}_{\text{term 2(a)}} \\ &\quad + \underbrace{\sum_{j=1}^{m}\sum_{a=0}^{r}\binom{r}{a} D^a_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}\otimes\pi_{\mathsf{F}}}}\Psi^s_{jm}(D^{r-a}_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}\otimes\pi_{\mathsf{F}}}} A^m_s)}_{\text{term 2(b)}} \\ &\quad + \underbrace{\sum_{a=0}^{r}\binom{r}{a} D^a_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}\otimes\pi_{\mathsf{F}}}}\Lambda^s_m(D^{r-a}_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}\otimes\pi_{\mathsf{F}}}} A^m_{s-1})}_{\text{term 3}}.\end{aligned}$$

We can argue just as in the preceding paragraph that, for $x \in \mathcal{K}$,

$$\begin{aligned}\|\text{term 1}(x)\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}\otimes\pi_{\mathsf{F}}}} &\le C_1(C_1\sigma_1^{-1}\gamma)^m C_{m,s}\alpha\left(\frac{\beta}{\sigma_1}\right)^{m+r+1-s}(m+r+1-s)! \\ \|\text{term 2(a)}(x)\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}\otimes\pi_{\mathsf{F}}}} &\le C_1(C_1\sigma_1^{-1}\gamma)^m C_{m,s}\alpha\left(\frac{\beta}{\sigma_1}\right)^{m+r-s}(m+r-s)! \\ \|\text{term 2(b)}(x)\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}\otimes\pi_{\mathsf{F}}}} &\le C_1(C_1\sigma_1^{-1}\gamma)^m C_{m+1,s}\alpha\left(\frac{\beta}{\sigma_1}\right)^{m+r-s}(m+r+1-s)! \\ \|\text{term 3}(x)\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}\otimes\pi_{\mathsf{F}}}} &\le C_1(C_1\sigma_1^{-1}\gamma)^m C_{m,s-1}\alpha\left(\frac{\beta}{\sigma_1}\right)^{m+r-s+1}(m+r-s+1)!.\end{aligned}$$

Adding these as in the previous case and using our observation 4 above, we have

$$\|D^r_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}\otimes\pi_{\mathsf{F}}}} A^{m+1}_s(x)\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}\otimes\pi_{\mathsf{F}}}} \leq C_1(C_1\sigma_1^{-1}\gamma)^{m+1}C_{m+1,s}\left(\frac{\beta}{\sigma_1}\right)^{m+1+r-s}(m+1+r-s)!,$$

proving (7.2) by induction in this case.

We now note that a standard binomial estimate via (7.1) gives $C_{m,s} \leq 2^m$. The lemma now follows from (7.2) by taking

$$C = C_1, \;\; \sigma = 2C_1\sigma_1^{-1}\gamma, \;\; \rho = \frac{\beta}{\sigma_1}. \qquad \blacksquare$$

We now apply the lemma to the recursion relations that we proved in Lemmata 5.1, 5.2, 5.6, 5.7, 5.11, 5.12, 5.16, 5.17, 5.21, 5.22, 5.26, 5.27, 5.31, 5.32, 5.38, and 5.39. We first provide the correspondence between the data from the preceding lemmata with the data of Lemma 7.8.

1. Lemma 5.1: We have
   (a) $\mathsf{M} = \mathsf{E}$, $\mathsf{E} = \mathsf{F} = \mathbb{R}_{\mathsf{E}}$,
   (b) $\phi_m(\beta_m) = \beta_m$, $\beta_m \in \mathrm{T}^m(\mathsf{T}^*\mathsf{M})$, $m \in \mathbb{Z}_{\geq 0}$,
   (c) $\Phi^s_m(\alpha^{m+1}_s) = \alpha^{m+1}_s$, $\alpha^{m+1}_s \in \mathrm{T}^s_{m+1}(\mathsf{T}^*\mathsf{M}\otimes\mathsf{F}\otimes\mathsf{E}^*))$, $m \in \mathbb{Z}_{\geq 0}$, $s \in \{0,1,\dots,m+1\}$,
   (d) $\Psi^s_{jm}(\alpha^m_s)(\beta_s) = -\alpha^m_s \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{M}}(\mathrm{Ins}_j(\beta_s, B_{\mathsf{E}}))$, $\alpha^m_s \in \mathrm{Hom}(\mathrm{T}^s(\mathsf{T}^*\mathsf{M})\otimes\mathsf{E}; \mathrm{T}^m(\mathsf{T}^*\mathsf{M})\otimes\mathsf{F})$, $\beta_s \in \mathrm{T}^s(\mathsf{T}^*\mathsf{M})\otimes\mathsf{E}$, $m \in \mathbb{Z}_{>0}$, $s \in \{1,\dots,m\}$, $j \in \{1,\dots,s\}$,
   (e) $\Lambda^s_m(\alpha^m_{s-1}) = \alpha^m_{s-1}\otimes\mathrm{id}_{\mathsf{T}^*\mathsf{M}}$, $\alpha^m_{s-1} \in \mathrm{Hom}(\mathrm{T}^{s-1}(\mathsf{T}^*\mathsf{M})\otimes\mathsf{E}; \mathrm{T}^m(\mathsf{T}^*\mathsf{M})\otimes\mathsf{F})$, $m \in \mathbb{Z}_{>0}$, $s \in \{1,\dots,m\}$,
   (f) $\Psi^0_{jm} = 0$, $m \in \mathbb{Z}_{\geq 0}$, and
   (g) $\Lambda^0_m = 0$, $m \in \mathbb{Z}_{\geq 0}$.
2. Lemma 5.2: We have
   (a) $\mathsf{M} = \mathsf{E}$, $\mathsf{E} = \mathsf{F} = \mathbb{R}_{\mathsf{E}}$,
   (b) $\phi_m(\beta_m) = \beta_m$, $\beta_m \in \mathrm{T}^m(\mathsf{T}^*\mathsf{M})$, $m \in \mathbb{Z}_{\geq 0}$,
   (c) $\Phi^s_m(\alpha^{m+1}_s) = \alpha^{m+1}_s$, $\alpha^{m+1}_s \in \mathrm{T}^s_{m+1}(\mathsf{T}^*\mathsf{M}\otimes\mathsf{F}\otimes\mathsf{E}^*)$, $m \in \mathbb{Z}_{\geq 0}$, $s \in \{0,1,\dots,m+1\}$,
   (d) $\Psi^s_{jm}(\alpha^m_s)(\beta_s) = \mathrm{Ins}_j(\alpha^m_s(\beta_s), B_{\mathsf{E}})$, $\alpha^m_s \in \mathrm{Hom}(\mathrm{T}^s(\mathsf{T}^*\mathsf{M})\otimes\mathsf{E}; \mathrm{T}^m(\mathsf{T}^*\mathsf{M})\otimes\mathsf{F})$, $\beta_s \in \mathrm{T}^s(\mathsf{T}^*\mathsf{M})\otimes\mathsf{E}$, $m \in \mathbb{Z}_{>0}$, $s \in \{1,\dots,m\}$, $j \in \{1,\dots,m\}$, and
   (e) $\Lambda^s_m(\alpha^m_{s-1}) = \alpha^m_{s-1}\otimes\mathrm{id}_{\mathsf{T}^*\mathsf{M}}$, $\alpha^m_{s-1} \in \mathrm{Hom}(\mathrm{T}^{s-1}(\mathsf{T}^*\mathsf{M})\otimes\mathsf{E}; \mathrm{T}^m(\mathsf{T}^*\mathsf{M})\otimes\mathsf{F})$, $m \in \mathbb{Z}_{>0}$, $s \in \{0,\dots,m\}$.
3. Lemma 5.6: We have
   $$\mathsf{M} = \mathsf{E}, \;\; \mathsf{E} = \mathsf{F} = \mathsf{VE},$$
   and all other data derived from Lemma 5.6, similarly to the case of Lemma 5.1.
4. Lemma 5.7: We have
   $$\mathsf{M} = \mathsf{E}, \;\; \mathsf{E} = \mathsf{F} = \mathsf{VE},$$
   and all other data derived from Lemma 5.7, similarly to the case of Lemma 5.2.
5. Lemma 5.11: We have
   $$\mathsf{M} = \mathsf{E}, \;\; \mathsf{E} = \mathsf{F} = \mathsf{HE},$$
   and all other data derived from Lemma 5.11, similarly to the case of Lemma 5.1.

6. Lemma 5.12: We have
$$\mathsf{M} = \mathsf{E}, \;\; \mathsf{E} = \mathsf{F} = \mathsf{HE},$$
and all other data derived from Lemma 5.12, similarly to the case of Lemma 5.2.
7. Lemma 5.16: We have
$$\mathsf{M} = \mathsf{E}, \;\; \mathsf{E} = \mathsf{F} = \mathsf{V}^*\mathsf{E},$$
and all other data derived from Lemma 5.16, similarly to the case of Lemma 5.1.
8. Lemma 5.17: We have
$$\mathsf{M} = \mathsf{E}, \;\; \mathsf{E} = \mathsf{F} = \mathsf{V}^*\mathsf{E},$$
and all other data derived from Lemma 5.16, similarly to the case of Lemma 5.2.
9. Lemma 5.21: We have
$$\mathsf{M} = \mathsf{E}, \;\; \mathsf{E} = \mathsf{F} = \mathrm{T}^1_1(\mathsf{VE}),$$
and all other data derived from Lemma 5.21, similarly to the case of Lemma 5.1.
10. Lemma 5.22: We have
$$\mathsf{M} = \mathsf{E}, \;\; \mathsf{E} = \mathsf{F} = \mathrm{T}^1_1(\mathsf{VE}),$$
and all other data derived from Lemma 5.22, similarly to the case of Lemma 5.2.
11. Lemma 5.26: We have
    (a) $\mathsf{M} = \mathsf{E}$, $\mathsf{E} = \mathbb{R}_{\mathsf{E}} \oplus \mathsf{V}^*\mathsf{E}$, $\mathsf{F} = \mathbb{R}_{\mathsf{E}} \oplus \mathbb{R}_{\mathsf{E}}$,
    (b) $\phi_m(\beta_m, \delta_m) = \beta_m$, $(\beta_m, \delta_m) \in \mathrm{T}^m(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{E}$, $m \in \mathbb{Z}_{\geq 0}$,
    (c) $\Phi^s_m(\alpha^{m+1}_s, \gamma^{m+1}_s) = (\alpha^{m+1}_m, \gamma^{m+1}_s)$, $(\alpha^{m+1}_s, \gamma^{m+1}_s) \in \mathrm{T}^s_{m+1}(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{F} \otimes \mathsf{E}^*$, $m \in \mathbb{Z}_{\geq 0}$, $s \in \{0, 1, \dots, m-1\}$,
    (d) $\Phi^m_m(\alpha^{m+1}_m, \gamma^{m+1}_m) = (0, 0)$, $(\alpha^{m+1}_m, \gamma^{m+1}_m) \in \mathrm{T}^m_{m+1}(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{F} \otimes \mathsf{E}^*$, $m \in \mathbb{Z}_{\geq 0}$,
    (e) $\Psi^s_{jm}(\alpha^m_s, \gamma^m_s)(\beta_s, \delta_s) = (-\alpha^m_s \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{M}}(\mathrm{Ins}_j(\beta_s, B_{\mathsf{E}})),$ $-\sum_{j=1}^{s+1} \gamma^m_s \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{M}}(\mathrm{Ins}_j(\delta_s, B_{\mathsf{E}})))$, $(\alpha^m_s, \gamma^m_s) \in \mathrm{T}^s_m(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{F} \otimes \mathsf{E}^*$, $(\beta_s, \delta_s) \in \mathrm{T}^s(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{E}$, $m \geq 2$, $s \in \{1, \dots, m-1\}$, $j \in \{1, \dots, s\}$,
    (f) $\Psi^m_{jm}(\alpha^m_m, \gamma^m_m)(\beta_m, \delta_m) = (-\mathrm{Ins}_j(\beta_m, B_{\mathsf{E}}), \delta_m)$, $(\alpha^m_m, \gamma^m_m) \in \mathrm{T}^m_m(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{F} \otimes \mathsf{E}^*$, $(\beta_m, \delta_m) \in \mathrm{T}^m(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{E}$, $m \geq 2$, $j \in \{1, \dots, m\}$,
    (g) $\Psi^m_{00}(\alpha^m_0, \gamma^m_0)(\beta_0, \delta_0) = (0, -\gamma^m_0 \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{M}}(\mathrm{Ins}_1(\delta_0, B_{\mathsf{E}})))$, $(\alpha^m_0, \gamma^m_0) \in \mathrm{T}^m_0(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{F} \otimes \mathsf{E}^*$, $(\beta_0, \delta_0) \in \mathsf{E}$, $m \geq 2$,
    (h) $\Lambda^s_m(\alpha^m_s, \gamma^m_s) = (\alpha^m_{s-1} \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{M}}, \gamma^m_{s-1} \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{M}})$, $m \geq 2$, $s \in \{1, \dots, m\}$.
12. Lemma 5.27: We have
    (a) $\mathsf{M} = \mathsf{E}$, $\mathsf{E} = \mathbb{R}_{\mathsf{E}} \oplus \mathsf{V}^*\mathsf{E}$, $\mathsf{F} = \mathbb{R}_{\mathsf{E}} \oplus \mathbb{R}_{\mathsf{E}}$,
    (b) $\phi_m(\beta_m, \delta_m) = \beta_m$, $(\beta_m, \delta_m) \in \mathrm{T}^m(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{E}$, $m \in \mathbb{Z}_{\geq 0}$,
    (c) $\Phi^s_m(\alpha^{m+1}_s, \gamma^{m+1}_s) = (\alpha^{m+1}_m, \gamma^{m+1}_s)$, $(\alpha^{m+1}_s, \gamma^{m+1}_s) \in \mathrm{T}^s_{m+1}(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{F} \otimes \mathsf{E}^*$, $m \in \mathbb{Z}_{\geq 0}$, $s \in \{0, 1, \dots, m-1\}$,
    (d) $\Phi^m_m(\alpha^{m+1}_m, \gamma^{m+1}_m) = (0, 0)$, $(\alpha^{m+1}_m, \gamma^{m+1}_m) \in \mathrm{T}^m_{m+1}(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{F} \otimes \mathsf{E}^*$, $m \in \mathbb{Z}_{\geq 0}$,
    (e) $\Psi^s_{jm}(\alpha^m_s, \gamma^m_s)(\beta_s, \delta_s) = (\mathrm{Ins}_j(\alpha^m_s(\beta_s), B_{\mathsf{E}}) - \mathrm{Ins}_{m+1}(\alpha^m_s(\beta_s), B^*_{\mathsf{E}}), -\overline{B}^m_s)$, $(\alpha^m_s, \gamma^m_s) \in \mathrm{T}^s_m(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{F} \otimes \mathsf{E}^*$, $(\beta_s, \delta_s) \in \mathrm{T}^s(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{E}$, $m \geq 2$, $s \in \{1, \dots, m-1\}$, $j \in \{1, \dots, m\}$,

(f) $\Psi^m_{jm}(\alpha^m_m,\gamma^m_m)(\beta_m,\delta_m)=(\mathrm{Ins}_j(\beta_m,B_{\mathsf{E}})-\mathrm{Ins}_{m+1}(\beta_m,B^*_{\mathsf{E}}),-\delta_m)$, $(\alpha^m_m,\gamma^m_m)\in\mathrm{T}^m_m(\mathsf{T}^*\mathsf{M})\otimes\mathsf{F}\otimes\mathsf{E}^*$, $(\beta_m,\delta_m)\in\mathrm{T}^m(\mathsf{T}^*\mathsf{M})\otimes\mathsf{E}$, $m\geq 2$, $j\in\{1,\dots,m\}$,

(g) $\Psi^m_{00}(\alpha^m_0,\gamma^m_0)(\beta_0,\delta_0)=(0,-\gamma^m_0\otimes\mathrm{id}_{\mathsf{T}^*\mathsf{M}}(\mathrm{Ins}_1(\delta_0,B_{\mathsf{E}})))$, $(\alpha^m_0,\gamma^m_0)\in\mathrm{T}^m_0(\mathsf{T}^*\mathsf{M})\otimes\mathsf{F}\otimes\mathsf{E}^*$, $(\beta_0,\delta_0)\in\mathsf{E}$, $m\geq 2$,

(h) $\Lambda^s_m(\alpha^m_s,\gamma^m_s)=(\alpha^m_{s-1}\otimes\mathrm{id}_{\mathsf{T}^*\mathsf{M}},\gamma^m_{s-1}\otimes\mathrm{id}_{\mathsf{T}^*\mathsf{M}})$, $m\geq 2$, $s\in\{1,\dots,m\}$.

13. Lemma 5.31: We have

$$\mathsf{M}=\mathsf{E},\ \ \mathsf{E}=\mathsf{VE}\oplus\mathrm{T}^1_1(\mathsf{VE}),\ \ \mathsf{F}=\mathsf{VE}\oplus\mathsf{VE},$$

and all other data derived from Lemma 5.31, similarly to the case of Lemma 5.26.

14. Lemma 5.32: We have

$$\mathsf{M}=\mathsf{E},\ \ \mathsf{E}=\mathsf{VE}\oplus\mathrm{T}^1_1(\mathsf{VE}),\ \ \mathsf{F}=\mathsf{VE}\oplus\mathsf{VE},$$

and all other data derived from Lemma 5.32, similarly to the case of Lemma 5.27.

15. Lemma 5.38: We have

$$\mathsf{M}=\mathsf{M},\ \ \mathsf{E}=\Phi^*\mathsf{T}^*\mathsf{N},\ \ \mathsf{F}=\mathsf{T}^*\mathsf{M},$$

and all other data derived from Lemma 5.38, similarly to the case of Lemma 5.1.

16. Lemma 5.39: We have

$$\mathsf{M}=\mathsf{M},\ \ \mathsf{E}=\mathsf{T}^*\mathsf{M},\ \ \mathsf{F}=\Phi^*\mathsf{T}^*\mathsf{N},$$

and all other data derived from Lemma 5.39, similarly to the case of Lemma 5.2.

Having now translated the lemmata of Section 5 to the general Lemma 7.8, we now need to show that the data of the lemmata of Section 5 satisfy the hypotheses of Lemma 7.8. As is easily seen, there are a few sorts of expressions that appear repeatedly, and we shall simply give estimates for these terms and leave to the reader the putting together of the pieces.

The following lemma gives the required bounds.

7.9 Lemma: (Specific bounds for terms coming from recursion) *Let $\pi_{\mathsf{E}}\colon\mathsf{E}\to\mathsf{M}$ be a real analytic vector bundle, let ∇^{M} be a real analytic affine connection on M, and let $\nabla^{\pi_{\mathsf{E}}}$ be a real analytic vector bundle connection in E. Let $\mathbb{G}_{\mathsf{M}}$ be a real analytic Riemannian metric on M and let $\mathbb{G}_{\mathsf{E}}$ be a real analytic fibre metrics for E. Let $S\in\Gamma^\omega(\mathrm{T}^1_2(\mathsf{TM}))$. Let $\mathcal{K}\subseteq\mathsf{M}$ be compact and let n be the larger of the dimension of M and the fibre dimension of E and let $\sigma_0=n^{-1}$. Let $m,r,a\in\mathbb{Z}_{\geq 0}$ and $s\in\{0,1,\dots,m\}$. Then we have the following bounds for $x\in\mathcal{K}$:*

(i) $\|D^r_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}}\mathrm{id}_{\mathrm{T}^m(\mathsf{T}^*\mathsf{M})\otimes\mathsf{E}}(x)\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}}}\leq\sigma_0^{m+r+1}$*;*

(ii) $\|D^r_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}}\mathrm{id}_{\mathrm{T}^m_s(\mathsf{T}^*\mathsf{M})\otimes\mathsf{E}}(x)\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}}}\leq\sigma_0^{2m+r+1}$*;*

(iii) if $\Phi^s_m(\alpha^{m+1}_s)=\alpha^{m+1}_s$, $\alpha^{m+1}_s\in\mathrm{T}^s_{m+1}(\mathsf{T}^*\mathsf{M})\otimes\mathsf{E}$*, then there exist* $C_1,\sigma_1\in\mathbb{R}_{>0}$ *such that*

$$\|D^r_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}}\Phi^s_m\circ D^a_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}}A^{m+1}_s(x)\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}}}\leq\|D^a_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}}A^{m+1}_s(x)\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}}};$$

(iv) *if*

$$\Psi^s_{jm}(\alpha^m_s)(\beta_s) = (\alpha^m_s \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{M}})(\mathrm{Ins}_j(\beta_s, S)),$$
$$\alpha^m_s \in \mathrm{Hom}(\mathrm{T}^s(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{E}; \mathrm{T}^m(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{E}),\ \ \beta_s \in \mathrm{T}^s(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{E},$$

then there exist $C_1, \sigma_1 \in \mathbb{R}_{>0}$ *such that*

$$\|D^r_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}}\Psi^s_{jm} \circ D^a_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}} A^m_s(x)\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}}} \le C_1\sigma_1^{-r} r! \|D^a_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}} A^m_s(x)\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}}};$$

(v) *if*

$$\Psi^s_{jm}(\alpha^m_s)(\beta_s) = \mathrm{Ins}_j(\alpha^m_s(\beta_s), S),$$
$$\alpha^m_s \in \mathrm{Hom}(\mathrm{T}^s(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{E}; \mathrm{T}^m(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{E}),\ \ \beta_s \in \mathrm{T}^s(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{E},$$

then there exist $C_1, \sigma_1 \in \mathbb{R}_{>0}$ *such that*

$$\|D^r_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}}\Psi^s_{jm} \circ D^a_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}} A^m_s(x)\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}}} \le C_1\sigma_1^{-r} r! \|D^a_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}} A^m_s(x)\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}}};$$

(vi) *if*

$$\Lambda^s_m(\alpha^m_{s-1}) = \alpha^m_{s-1} \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{M}}, \qquad \alpha^m_s \in \mathrm{Hom}(\mathrm{T}^{s-1}(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{E}; \mathrm{T}^m(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{E}),$$

then there exist $C_1, \sigma_1 \in \mathbb{R}_{>0}$ *such that*

$$\|D^r_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}}\Lambda^s_m \circ D^a_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}} A^m_{s-1}(x)\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}}} \le \|D^a_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}} A^m_{s-1}(x)\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}}}.$$

Proof: Parts (i) and (ii) follow from Lemma 7.1 along with the fact that the covariant derivative of the identity tensor is zero. Part (iii) is a tautology, but one that arises in the lemmata of Section 5.

For the next two parts of the proof, let $C_1, \sigma_1 \in \mathbb{R}_{>0}$ be such that

$$\|D^r_{\nabla^{\mathsf{M}}} S(x)\|_{\mathbb{G}_{\mathsf{M}}} \le C_1\sigma_1^{-r} r!, \qquad x \in \mathcal{K}, \tag{7.3}$$

this being possible by Lemma 2.3, and recalling the rôle of the factorials in the definition (2.6) of the fibre norms.

(iv) Let us define

$$\hat{\Psi}^s_{jm}(\beta^m_{s+1})(\alpha_s) = (\beta^m_{s+1})(\mathrm{Ins}_j(\alpha_s, S)),$$
$$\beta^m_{s+1} \in \mathrm{Hom}(\mathrm{T}^{s+1}(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{E}; \mathrm{T}^m(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{E}),\ \ \alpha_s \in \mathrm{T}^s(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{E}$$

and

$$\tau^s_m(\alpha^m_s) = \alpha^m_s \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{M}}, \qquad \alpha^m_s \in \mathrm{Hom}(\mathrm{T}^s(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{E}; \mathrm{T}^m(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{E})$$

so that $\Psi^s_{jm} = \hat{\Psi}^s_m \circ \tau^s_m$. Note that $\hat{\Psi}^s_{jm} = \mathrm{Ins}_{S,j}$ so that, by Lemma 4.3,

$$D^r_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}}\hat{\Psi}^s_{jm}(D^a_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}} B^m_{s+1}) = \mathrm{Ins}_{D^r_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}} S,j}(D^a_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}} B^m_{s+1}).$$

Since the covariant derivative of the identity tensor is zero,

$$D^a_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}}(A^m_s \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{M}}) = (D^a_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}} A^m_s) \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{M}}),$$

from which we deduce that $D^a_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}}\tau^s_m = \tau^s_{m+a}$. Thus

$$\begin{aligned} D^r_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}}\Psi^s_{jm} \circ D^a_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}}A^m_s &= D^r_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}}(\hat{\Psi}^s_{jm} \circ \tau^s_m) \circ D^a_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}}A^m_s \\ &= \mathrm{Ins}_{D^r_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}}S,j}(D^a_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}}A^m_s \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{M}}). \end{aligned}$$

By Lemmata 7.6 and 7.2, this part of the lemma follows immediately.

(v) Here we have $\Psi^s_m(\alpha^m_s) = \mathrm{Ins}_{S,j} \circ \alpha^m_s$ and, following the arguments from the preceding part of the proof,

$$D^r_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}}\Psi^s_m \circ D^a_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}}A^m_s = \mathrm{Ins}_{D^r_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}}S,j}(D^a_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}}A^m_s),$$

and so this part of the lemma follows from Lemma 7.7.

(vi) This follows from Lemma 7.2 and the fact that the covariant derivative of the identity tensor is zero. ■

8. Independence of topologies on connections and metrics

The seminorms introduced in Section 2.4 for defining topologies for the space of real analytic sections of a vector bundle $\pi_{\mathsf{E}}\colon \mathsf{E} \to \mathsf{M}$ are made upon a choice of various objects, namely (1) an affine connection ∇^{M} on M, (2) a vector bundle connection $\nabla^{\pi_{\mathsf{E}}}$ in E, (3) a Riemannian metric $\mathbb{G}_{\mathsf{M}}$ on M, and (4) a fibre metric $\mathbb{G}_{\pi_{\mathsf{E}}}$ for E. In order for these topologies to be useful, they should be independent of all of these choices. This is made more urgent by our very specific choice in Section 4.1 of a Riemannian metric $\mathbb{G}_{\mathsf{E}}$ on the total space E and its Levi-Civita connection. These choices were made because they made available to us the geometric constructions of Section 4, constructions of which substantial use was made in Sections 5 and 7, and of which will be made in Section 9, as well as in the present section.

That the topologies are independent of choices of geometric objects is more or less clear in the smooth case, but we will rather precisely point out why this is so in our developments below. In the real analytic case, one must make use of all of the technical developments of Sections 3–7.

8.1. Comparison of iterated covariant derivatives for different connections.

Our constructions start by comparing how covariant derivatives of high-order differ when one changes connection. The reader will see substantial similarity between the results in this section and those in Sections 4.3 and 5.

We let $r \in \{\infty, \omega\}$ and let $\pi_{\mathsf{E}}\colon \mathsf{E} \to \mathsf{M}$ be a C^r-vector bundle. We consider C^r-affine connections ∇^{M} and $\overline{\nabla}^{\mathsf{M}}$ on M, and vector bundle connections $\nabla^{\pi_{\mathsf{E}}}$ and $\overline{\nabla}^{\pi_{\mathsf{E}}}$ in E. It then holds that

$$\overline{\nabla}^{\mathsf{M}}_X Y = \nabla^{\mathsf{M}}_X Y + S_{\mathsf{M}}(Y, X), \quad \overline{\nabla}^{\pi_{\mathsf{E}}}_X \xi = \nabla^{\pi_{\mathsf{E}}}_X \xi + S_{\pi_{\mathsf{E}}}(\xi, X)$$

for $S_{\mathsf{M}} \in \Gamma^r(\mathrm{T}^1_2(\mathsf{TM}))$ and $S_{\pi_{\mathsf{E}}} \in \Gamma^r(\mathsf{E}^* \otimes \mathsf{T}^*\mathsf{M} \otimes \mathsf{E})$.

First we relate covariant derivatives of higher-order tensors.

8.1 Lemma: (Covariant derivatives of higher-order tensors with respect to different connections) *Let $r \in \{\infty, \omega\}$ and let $\pi_{\mathsf{E}} \colon \mathsf{E} \to \mathsf{M}$ be a C^r-vector bundle. Consider C^r-affine connections ∇^{M} and $\overline{\nabla}^{\mathsf{M}}$ on M, and C^r-vector bundle connections $\nabla^{\pi_{\mathsf{E}}}$ and $\overline{\nabla}^{\pi_{\mathsf{E}}}$ in E. If $k \in \mathbb{Z}_{>0}$ and if $B \in \Gamma^1(\mathrm{T}^k(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{E})$, then*

$$\overline{\nabla}^{\mathsf{M},\pi_{\mathsf{E}}} B = \nabla^{\mathsf{M},\pi_{\mathsf{E}}} B - \sum_{j=1}^{k} \mathrm{Ins}_j(B, S_{\mathsf{M}}) - \mathrm{Ins}_{k+1}(B, S_{\pi_{\mathsf{E}}}).$$

Proof: We have

$$\begin{aligned}
\mathscr{L}_{X_{k+1}}(B(X_1,\dots,X_k,\alpha)) &= (\overline{\nabla}^{\mathsf{M},\pi_{\mathsf{E}}}_{X_{k+1}} B)(X_1,\dots,X_k,\alpha) \\
&\quad + \sum_{j=1}^{k} B(X_1,\dots,\overline{\nabla}^{\mathsf{M}}_{X_{k+1}} X_j,\dots,X_k,\alpha) + B(X_1,\dots,X_k,\overline{\nabla}^{\pi_{\mathsf{E}}}_{X_{k+1}}\alpha) \\
&= (\overline{\nabla}^{\mathsf{M},\pi_{\mathsf{E}}}_{X_{k+1}} B)(X_1,\dots,X_k,\alpha) + \sum_{j=1}^{k} B(X_1,\dots,\nabla^{\mathsf{M}}_{X_{k+1}} X_j,\dots,X_k,\alpha) \\
&\quad + \sum_{j=1}^{k} B(X_1,\dots,S_{\mathsf{M}}(X_j,X_{k+1}),\dots,X_k,\alpha) + B(X_1,\dots,X_k,\nabla^{\pi_{\mathsf{E}}}_{X_{k+1}}\alpha) \\
&\quad + B(X_1,\dots,X_k,S_{\pi_{\mathsf{E}}}(\alpha,X_{k+1})).
\end{aligned}$$

This gives

$$\overline{\nabla}^{\mathsf{M},\pi_{\mathsf{E}}} B = \nabla^{\mathsf{M},\pi_{\mathsf{E}}} B - \sum_{j=1}^{k} \mathrm{Ins}_j(B, S_{\mathsf{M}}) - \mathrm{Ins}_{k+1}(B, S_{\pi_{\mathsf{E}}}),$$

as claimed. ■

With this lemma, we can provide the following characterisation of iterated covariant differentials of sections of E with respect to different connections.

8.2 Lemma: (Iterated covariant differentials of sections with respect to different connections I) *Let $r \in \{\infty, \omega\}$ and let $\pi_{\mathsf{E}} \colon \mathsf{E} \to \mathsf{M}$ be a C^r-vector bundle. Consider C^r-affine connections ∇^{M} and $\overline{\nabla}^{\mathsf{M}}$ on M, and C^r-vector bundle connections $\nabla^{\pi_{\mathsf{E}}}$ and $\overline{\nabla}^{\pi_{\mathsf{E}}}$ in E. For $m \in \mathbb{Z}_{\geq 0}$, there exist C^r-vector bundle mappings*

$$(A^m_s, \mathrm{id}_{\mathsf{E}}) \in \mathrm{VB}^r(\mathrm{T}^s(\mathsf{T}^*\mathsf{M} \otimes \mathsf{E}); \mathrm{T}^m(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{E}), \qquad s \in \{0,1,\dots,m\},$$

such that

$$\overline{\nabla}^{\mathsf{M},\pi_{\mathsf{E}},m}\xi = \sum_{s=0}^{m} A^m_s(\nabla^{\mathsf{M},\pi_{\mathsf{E}},s}\xi)$$

for all $\xi \in \Gamma^m(\mathsf{E})$. Moreover, the vector bundle mappings $A^m_0, A^m_1, \dots, A^m_m$ satisfy the *recursion relations prescribed by* $A_0^0(\beta_0)=\beta_0$ *and*

$$\begin{aligned}
A_{m+1}^{m+1}&(\beta_{m+1})=\beta_{m+1},\\
A_s^{m+1}(\beta_s)&=(\overline{\nabla}^{\mathsf{M},\pi_{\mathsf{E}}}A_s^m)(\beta_s)+A_{s-1}^m\otimes\mathrm{id}_{\mathsf{T}^*\mathsf{M}}(\beta_s)-\sum_{j=1}^s A_s^m\otimes\mathrm{id}_{\mathsf{T}^*\mathsf{M}}(\mathrm{Ins}_j(\beta_s,S_{\mathsf{M}}))\\
&\qquad -A_s^m\otimes\mathrm{id}_{\mathsf{T}^*\mathsf{M}}(\mathrm{Ins}_{s+1}(\beta_s,S_{\pi_{\mathsf{E}}})),\quad s\in\{1,\dots,m\},\\
A_0^{m+1}(\beta_0)&=(\overline{\nabla}^{\mathsf{M},\pi_{\mathsf{E}}}A_0^m)(\beta_0)-A_0^m\otimes\mathrm{id}_{\mathsf{T}^*\mathsf{M}}(\mathrm{Ins}_1(\beta_0,S_{\pi_{\mathsf{E}}})),
\end{aligned}$$

where $\beta_s\in\mathrm{T}^s(\mathsf{T}^*\mathsf{M})\otimes\mathsf{E}$*,* $s\in\{0,1,\dots,m\}$*.*

Proof: The assertion clearly holds for $m=0$, so suppose it true for $m\in\mathbb{Z}_{>0}$. Thus

$$\overline{\nabla}^{\mathsf{M},\pi_{\mathsf{E}},m}\xi=\sum_{s=0}^m A_s^m(\nabla^{\mathsf{M},\pi_{\mathsf{E}},s}\xi),$$

where the vector bundle mappings A_s^a, $a\in\{0,1,\dots,m\}$, $s\in\{0,1,\dots,a\}$, satisfy the recursion relations from the statement of the lemma. Then

$$\begin{aligned}
\overline{\nabla}^{\mathsf{M},\pi_{\mathsf{E}},m+1}\xi&=\sum_{s=0}^m(\overline{\nabla}^{\mathsf{M},\pi_{\mathsf{E}}}A_s^m)(\nabla^{\mathsf{M},\pi_{\mathsf{E}},s}\xi)+\sum_{s=0}^m A_s^m\otimes\mathrm{id}_{\mathsf{T}^*\mathsf{M}}(\overline{\nabla}^{\mathsf{M},\pi_{\mathsf{E}}}\nabla^{\mathsf{M},\pi_{\mathsf{E}},s}\xi)\\
&=\sum_{s=0}^m(\overline{\nabla}^{\mathsf{M},\pi_{\mathsf{E}}}A_s^m)(\nabla^{\mathsf{M},\pi_{\mathsf{E}},s}\xi)+\sum_{s=0}^m A_s^m\otimes\mathrm{id}_{\mathsf{T}^*\mathsf{M}}(\nabla^{\mathsf{M},\pi_{\mathsf{E}},s+1}\xi)\\
&\quad-\sum_{s=1}^m\sum_{j=1}^s A_s^m\otimes\mathrm{id}_{\mathsf{T}^*\mathsf{M}}(\mathrm{Ins}_j(\nabla^{\mathsf{M},\pi_{\mathsf{E}},s}\xi,S_{\mathsf{M}}))\\
&\quad-\sum_{s=1}^m A_s^m\otimes\mathrm{id}_{\mathsf{T}^*\mathsf{M}}(\mathrm{Ins}_{s+1}(\nabla^{\mathsf{M},\pi_{\mathsf{E}},s}\xi,S_{\pi_{\mathsf{E}}}))-A_0^m\otimes\mathrm{id}_{\mathsf{T}^*\mathsf{M}}(\mathrm{Ins}_1(\xi,S_{\pi_{\mathsf{E}}}))\\
&=\nabla^{\mathsf{M},\pi_{\mathsf{E}},m+1}\xi+\sum_{s=1}^m\Bigg((\overline{\nabla}^{\mathsf{M},\pi_{\mathsf{E}}}A_s^m)(\nabla^{\mathsf{M},\pi_{\mathsf{E}},s}\xi)+A_{s-1}^m\otimes\mathrm{id}_{\mathsf{T}^*\mathsf{M}}(\nabla^{\mathsf{M},\pi_{\mathsf{E}},s}\xi)\\
&\quad-\sum_{j=1}^s A_s^m\otimes\mathrm{id}_{\mathsf{T}^*\mathsf{M}}(\mathrm{Ins}_j(\nabla^{\mathsf{M},\pi_{\mathsf{E}},s}\xi,S_{\mathsf{M}}))-A_s^m\otimes\mathrm{id}_{\mathsf{T}^*\mathsf{M}}(\mathrm{Ins}_{s+1}(\nabla^{\mathsf{M},\pi_{\mathsf{E}},s}\xi,S_{\pi_{\mathsf{E}}}))\Bigg)\\
&\quad-(\overline{\nabla}^{\mathsf{M},\pi_{\mathsf{E}}}A_0^m)(\xi)-A_0^m\otimes\mathrm{id}_{\mathsf{T}^*\mathsf{M}}(\mathrm{Ins}_1(\xi,S_{\pi_{\mathsf{E}}}))
\end{aligned}$$

by Lemma 8.1. From this, the lemma follows. ■

The lemma has an "inverse" which we state next.

8.3 Lemma: (Iterated covariant differentials of sections with respect to different connections II) *Let* $r\in\{\infty,\omega\}$ *and let* $\pi_{\mathsf{E}}\colon\mathsf{E}\to\mathsf{M}$ *be a* C^r*-vector bundle. Consider* C^r*-affine connections* ∇^{M} *and* $\overline{\nabla}^{\mathsf{M}}$ *on* M*, and* C^r*-vector bundle connections* $\nabla^{\pi_{\mathsf{E}}}$ *and* $\overline{\nabla}^{\pi_{\mathsf{E}}}$ *in* E*. For* $m\in\mathbb{Z}_{\geq 0}$*, there exist* C^r*-vector bundle mappings*

$$(B_s^m,\mathrm{id}_{\mathsf{E}})\in\mathrm{VB}^r(\mathrm{T}^s(\mathsf{T}^*\mathsf{M}\otimes\mathsf{E});\mathrm{T}^m(\mathsf{T}^*\mathsf{M})\otimes\mathsf{E}),\qquad s\in\{0,1,\dots,m\},$$

such that

$$\nabla^{\mathsf{M},\pi_{\mathsf{E}},m}\xi = \sum_{s=0}^{m} B^m_s(\overline{\nabla}^{\mathsf{M},\pi_{\mathsf{E}},s}\xi)$$

for all $\xi \in \Gamma^m(\mathsf{E})$*. Moreover, the vector bundle mappings* $B^m_0, B^m_1, \dots, B^m_m$ *satisfy the recursion relations prescribed by* $B^0_0(\alpha_0) = \beta_0$ *and*

$$\begin{aligned}
B^{m+1}_{m+1}(\alpha_{m+1}) &= \alpha_{m+1},\\
B^{m+1}_s(\alpha_s) &= (\overline{\nabla}^{\mathsf{M},\pi_{\mathsf{E}}} B^m_s)(\alpha_s) + B^m_{s-1}\otimes \mathrm{id}_{\mathsf{T}^*\mathsf{M}}(\alpha_s) + \sum_{j=1}^{m} \mathrm{Ins}_j(B^m_s(\alpha_s), S_{\mathsf{M}})\\
&\quad + \mathrm{Ins}_{m+1}(B^m_s(\alpha_s\xi), S_{\pi_{\mathsf{E}}}), \quad s\in\{1,\dots,m\},\\
B^{m+1}_0(\alpha_0) &= (\overline{\nabla}^{\mathsf{M},\pi_{\mathsf{E}}} B^m_0)(\alpha_0) + \sum_{j=1}^{m}\mathrm{Ins}_j(B^m_0(\alpha_0), S_{\mathsf{M}}) + \mathrm{Ins}_{m+1}(B^m_0(\alpha_0), S_{\pi_{\mathsf{E}}}),
\end{aligned}$$

where $\alpha_s \in \mathrm{T}^s(\mathsf{T}^*\mathsf{M})\otimes \mathsf{E}$*,* $s \in \{0,1,\dots,m\}$*.*

Proof: The lemma is clearly true for $m = 0$, so suppose it true for $m \in \mathbb{Z}_{>0}$. Thus

$$\nabla^{\mathsf{M},\pi_{\mathsf{E}},m}\xi = \sum_{s=0}^{m} B^m_s(\overline{\nabla}^{\mathsf{M},\pi_{\mathsf{E}},s}\xi), \tag{8.1}$$

where the vector bundle mappings B^a_s, $a \in \{0,1,\dots,m\}$, $s \in \{0,1,\dots,a\}$, satisfy the recursion relations given in the lemma. Then, working with the left-hand side of this relation,

$$\begin{aligned}
\overline{\nabla}^{\mathsf{M},\pi_{\mathsf{E}}}\nabla^{\mathsf{M},\pi_{\mathsf{E}},m}\xi &= \nabla^{\mathsf{M},\pi_{\mathsf{E}},m+1}\xi - \sum_{j=1}^{m}\mathrm{Ins}_j(\nabla^{\mathsf{M},\pi_{\mathsf{E}},m}\xi, S_{\mathsf{M}}) - \mathrm{Ins}_{m+1}(\nabla^{\mathsf{M},\pi_{\mathsf{E}},m}\xi, S_{\pi_{\mathsf{E}}})\\
&= \nabla^{\mathsf{M},\pi_{\mathsf{E}},m+1}\xi - \sum_{s=0}^{m}\sum_{j=1}^{m}\mathrm{Ins}_j(B^m_s(\overline{\nabla}^{\mathsf{M},\pi_{\mathsf{E}},s}\xi), S_{\mathsf{M}})\\
&\quad - \sum_{s=0}^{m}\mathrm{Ins}_{m+1}(B^m_s(\overline{\nabla}^{\mathsf{M},\pi_{\mathsf{E}},s}\xi), S_{\pi_{\mathsf{E}}}),
\end{aligned}$$

by Lemma 8.1. Now, working with the right-hand side of (8.1),

$$\overline{\nabla}^{\mathsf{M},\pi_{\mathsf{E}}}\nabla^{\mathsf{M},\pi_{\mathsf{E}},m}\xi = \sum_{s=0}^{m}(\overline{\nabla}^{\mathsf{M},\pi_{\mathsf{E}}}B^m_s)(\overline{\nabla}^{\mathsf{M},\pi_{\mathsf{E}},m}\xi) + \sum_{s=0}^{m} B^m_s\otimes\mathrm{id}_{\mathsf{T}^*\mathsf{M}}(\overline{\nabla}^{\mathsf{M},\pi_{\mathsf{E}},m+1}\xi).$$

Combining the preceding two computations,

$$\begin{aligned}
\nabla^{\mathsf{M},\pi_{\mathsf{E}},m+1}\xi = \overline{\nabla}^{\mathsf{M},\pi_{\mathsf{E}},m+1}\xi + \sum_{s=1}^{m}\Bigg(&(\overline{\nabla}^{\mathsf{M},\pi_{\mathsf{E}}}B^m_s)(\overline{\nabla}^{\mathsf{M},\pi_{\mathsf{E}},s}\xi) + B^m_{s-1}\otimes\mathrm{id}_{\mathsf{T}^*\mathsf{M}}(\overline{\nabla}^{\mathsf{M},\pi_{\mathsf{E}},s}\xi)\\
&+ \sum_{j=1}^{m}\mathrm{Ins}_j(B^m_s(\overline{\nabla}^{\mathsf{M},\pi_{\mathsf{E}},s}\xi), S_{\mathsf{M}}) + \mathrm{Ins}_{m+1}(B^m_s(\overline{\nabla}^{\mathsf{M},\pi_{\mathsf{E}},s}\xi), S_{\pi_{\mathsf{E}}})\Bigg)\\
&+ (\overline{\nabla}^{\mathsf{M},\pi_{\mathsf{E}}}B^m_0)(\xi) + \sum_{j=1}^{m}\mathrm{Ins}_j(B^m_0(\xi), S_{\mathsf{M}}) + \mathrm{Ins}_{m+1}(B^m_0(\xi), S_{\pi_{\mathsf{E}}}),
\end{aligned}$$

and from this the lemma follows. ■

Now we give symmetrised versions of the preceding lemmata, since it is these that are required for computations with jets.

8.4 Lemma: (Iterated symmetrised covariant differentials of sections with respect to different connections I) *Let $r \in \{\infty, \omega\}$ and let $\pi_{\mathsf{E}}\colon \mathsf{E} \to \mathsf{M}$ be a C^r-vector bundle. Consider C^r-affine connections ∇^{M} and $\overline{\nabla}^{\mathsf{M}}$ on M, and C^r-vector bundle connections $\nabla^{\pi_{\mathsf{E}}}$ and $\overline{\nabla}^{\pi_{\mathsf{E}}}$ in E. For $m \in \mathbb{Z}_{\geq 0}$, there exist C^r-vector bundle mappings*

$$(\widehat{A}^m_s, \mathrm{id}_{\mathsf{E}}) \in \mathrm{VB}^r(\mathrm{T}^s(\mathsf{T}^*\mathsf{M} \otimes \mathsf{E}); \mathrm{T}^m(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{E}), \qquad s \in \{0, 1, \dots, m\},$$

such that

$$(\mathrm{Sym}_m \otimes \mathrm{id}_{\mathsf{E}}) \circ \overline{\nabla}^{\mathsf{M},\pi_{\mathsf{E}},m}\xi = \sum_{s=0}^{m} \widehat{A}^m_s((\mathrm{Sym}_s \otimes \mathrm{id}_{\mathsf{E}}) \circ \nabla^{\mathsf{M},\pi_{\mathsf{E}},s}\xi)$$

for all $\xi \in \Gamma^m(\mathsf{E})$.

Proof: We define $A^m\colon \mathrm{T}^{\leq m}(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{E} \to \mathrm{T}^{\leq m}(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{E}$ by

$$A^m(\xi, \nabla^{\pi_{\mathsf{E}}}\xi, \dots, \nabla^{\mathsf{M},\pi_{\mathsf{E}},m}\xi) = \left(A^0_0(\xi), \sum_{s=0}^{1} A^1_s(\nabla^{\mathsf{M},\pi_{\mathsf{E}},s}\xi), \dots, \sum_{s=0}^{m} A^m_s(\nabla^{\mathsf{M},\pi_{\mathsf{E}},s}\xi)\right).$$

Let us organise the mappings we require into the following diagram:

$$\begin{array}{ccccc} \mathrm{T}^{\leq m}(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{E} & \xrightarrow{\mathrm{Sym}_{\leq m} \otimes \mathrm{id}_{\mathsf{E}}} & \mathrm{S}^{\leq m}(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{E} & \xrightarrow{S^m_{\overline{\nabla}^{\mathsf{M}},\overline{\nabla}^{\pi_{\mathsf{E}}}}} & \mathsf{J}^m\mathsf{E} \\ {\scriptstyle A^m}\downarrow & & \downarrow{\scriptstyle \widehat{A}^m} & & \Vert \\ \mathrm{T}^{\leq m}(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{E} & \xrightarrow{\mathrm{Sym}_{\leq m} \otimes \mathrm{id}_{\mathsf{E}}} & \mathrm{S}^{\leq m}(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{E} & \xrightarrow{S^m_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}}} & \mathsf{J}^m\mathsf{E} \end{array} \tag{8.2}$$

Here $\widehat{A}^m$ is defined so that the right square commutes. We shall show that the left square also commutes. Indeed,

$$\begin{aligned} \widehat{A}^m \circ \mathrm{Sym}_{\leq m} \otimes \mathrm{id}_{\mathsf{E}}&(\xi, \nabla^{\pi_{\mathsf{E}}}\xi, \dots, \nabla^{\mathsf{M},\pi_{\mathsf{E}},m}\xi) \\ &= (S^m_{\overline{\nabla}^{\mathsf{M}},\overline{\nabla}^{\pi_{\mathsf{E}}}})^{-1} \circ S^m_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}} \circ (\mathrm{Sym}_{\leq m} \otimes \mathrm{id}_{\mathsf{E}})(\xi, \overline{\nabla}^{\pi_{\mathsf{E}}}\xi, \dots, \nabla^{\mathsf{M},\pi_{\mathsf{E}},m}\xi) \\ &= \mathrm{Sym}_{\leq m} \otimes \mathrm{id}_{\mathsf{VE}}(\xi, \overline{\nabla}^{\pi_{\mathsf{E}}}\xi, \dots, \overline{\nabla}^{\mathsf{M},\pi_{\mathsf{E}},m}\xi) \\ &= (\mathrm{Sym}_{\leq m} \otimes \mathrm{id}_{\mathsf{E}}) \circ A^m(\xi, \nabla^{\pi_{\mathsf{E}}}\xi, \dots, \nabla^{\mathsf{M},\pi_{\mathsf{E}},m}\xi). \end{aligned}$$

Thus the diagram (8.2) commutes. Thus, if we define

$$\widehat{A}^m_s((\mathrm{Sym}_s \otimes \mathrm{id}_{\mathsf{E}}) \circ \nabla^{\mathsf{M},\pi_{\mathsf{E}},s}\xi) = (\mathrm{Sym}_m \otimes \mathrm{id}_{\mathsf{E}}) \circ A^m_s(\nabla^{\mathsf{M},\pi_{\mathsf{E}},s}\xi), \tag{8.3}$$

then we have

$$(\mathrm{Sym}_m \otimes \mathrm{id}_{\mathsf{E}}) \circ \overline{\nabla}^{\mathsf{M},\pi_{\mathsf{E}},m}\xi = \sum_{s=0}^{m} \widehat{A}^m_s((\mathrm{Sym}_s \otimes \mathrm{id}_{\mathsf{E}}) \circ \nabla^{\mathsf{M},\pi_{\mathsf{E}},s}\xi),$$

as desired. ■

The previous lemma has an "inverse" which we state next.

8.5 Lemma: (Iterated symmetrised covariant differentials of sections with respect to different connections II) *Let $r \in \{\infty, \omega\}$ and let $\pi_{\mathsf{E}} \colon \mathsf{E} \to \mathsf{M}$ be a C^r-vector bundle. Consider C^r-affine connections ∇^{M} and $\overline{\nabla}^{\mathsf{M}}$ on M, and C^r-vector bundle connections $\nabla^{\pi_{\mathsf{E}}}$ and $\overline{\nabla}^{\pi_{\mathsf{E}}}$ in E. For $m \in \mathbb{Z}_{\geq 0}$, there exist C^r-vector bundle mappings*

$$(\widehat{B}^m_s, \mathrm{id}_{\mathsf{E}}) \in \mathrm{VB}^r(\mathrm{T}^s(\mathsf{T}^*\mathsf{M} \otimes \mathsf{E}); \mathrm{T}^m(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{E}), \qquad s \in \{0, 1, \dots, m\},$$

such that

$$(\mathrm{Sym}_m \otimes \mathrm{id}_{\mathsf{E}}) \circ \nabla^{\mathsf{M},\pi_{\mathsf{E}},m}\xi = \sum_{s=0}^m \widehat{B}^m_s((\mathrm{Sym}_s \otimes \mathrm{id}_{\mathsf{E}}) \circ \overline{\nabla}^{\mathsf{M},\pi_{\mathsf{E}},s}\xi)$$

for all $\xi \in \Gamma^m(\mathsf{E})$.

Proof: The proof here is identical with the proof of Lemma 8.4, making the obvious notational transpositions. ■

The preceding four lemmata combine to give the following result.

8.6 Lemma: (Decompositions of jets of sections with respect to different connections) *Let $r \in \{\infty, \omega\}$ and let $\pi_{\mathsf{E}} \colon \mathsf{E} \to \mathsf{M}$ be a C^r-vector bundle. Consider C^r-affine connections ∇^{M} and $\overline{\nabla}^{\mathsf{M}}$ on M, and C^r-vector bundle connections $\nabla^{\pi_{\mathsf{E}}}$ and $\overline{\nabla}^{\pi_{\mathsf{E}}}$ in E. For $m \in \mathbb{Z}_{\geq 0}$, there exist C^r-vector bundle mappings*

$$A^m \in \mathrm{VB}^r(\mathsf{J}^m\mathsf{E}; \mathrm{S}^{\leq m}(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{E}), \quad B^m \in \mathrm{VB}^r(\mathsf{J}^m\mathsf{E}; \mathrm{S}^{\leq m}(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{E}),$$

defined by

$$\begin{aligned} A^m(j_m\xi(x)) &= \mathrm{Sym}_{\leq m} \otimes \mathrm{id}_{\mathsf{E}}(\xi(x), \nabla^{\pi_{\mathsf{E}}}\xi(x), \dots, \nabla^{\mathsf{M},\pi_{\mathsf{E}},m}\xi(x)), \\ B^m(j_m\xi(x)) &= \mathrm{Sym}_{\leq m} \otimes \mathrm{id}_{\mathsf{VE}}(\xi(x), \overline{\nabla}^{\pi_{\mathsf{E}}}\xi(x), \dots, \overline{\nabla}^{\mathsf{M},\pi_{\mathsf{E}},m}\xi(x)). \end{aligned}$$

Moreover, A^m and B^m are isomorphisms, and

$$\begin{aligned} &B^m \circ (A^m)^{-1} \circ (\mathrm{Sym}_{\leq m} \otimes \mathrm{id}_{\mathsf{E}})(\xi(x), \nabla^{\pi_{\mathsf{E}}}\xi(x), \dots, \nabla^{\mathsf{M},\pi_{\mathsf{E}},m}\xi(x)) \\ &= \left(\xi(x), \sum_{s=0}^1 \widehat{A}^1_s((\mathrm{Sym}_s \otimes \mathrm{id}_{\mathsf{E}}) \circ \nabla^{\mathsf{M},\pi_{\mathsf{E}},s}\xi(x)), \dots, \sum_{s=0}^m \widehat{A}^m_s((\mathrm{Sym}_s \otimes \mathrm{id}_{\mathsf{E}}) \circ \nabla^{\mathsf{M},\pi_{\mathsf{E}},s}\xi(x))\right) \end{aligned}$$

and

$$\begin{aligned} &A^m \circ (B^m)^{-1} \circ (\mathrm{Sym}_{\leq m} \otimes \mathrm{id}_{\mathsf{E}})(\xi(x), \overline{\nabla}^{\pi_{\mathsf{E}}}\xi(x), \dots, \overline{\nabla}^{\mathsf{M},\pi_{\mathsf{E}},m}\xi(x)) \\ &= \left(\xi(x), \sum_{s=0}^1 \widehat{B}^1_s((\mathrm{Sym}_s \otimes \mathrm{id}_{\mathsf{E}}) \circ \overline{\nabla}^{\mathsf{M},\pi_{\mathsf{E}},s}\xi(e)), \dots, \sum_{s=0}^m \widehat{B}^m_s((\mathrm{Sym}_s \otimes \mathrm{id}_{\mathsf{E}}) \circ \nabla^{\mathsf{M},\pi_{\mathsf{E}},s}\xi(x))\right), \end{aligned}$$

where the vector bundle mappings $\widehat{A}^m_s$ and $\widehat{B}^m_s$, $s \in \{0, 1, \dots, m\}$, are as in Lemmata 8.4 and 8.5.

8.2. Comparison of metric-related notions for different connections and metrics. We next consider how various constructions involving Riemannian metrics and fibre metrics vary when one varies the fibre metrics. The first result concerns fibre norms for tensor products induced by a fibre metric.

8.7 Lemma: (Comparison of fibre norms for different fibre metrics) *Let $\pi_{\mathsf{E}}\colon \mathsf{E}\to\mathsf{M}$ be a smooth vector bundle and let $\mathbb{G}_1$ and $\mathbb{G}_2$ be smooth fibre metrics on E. Let $\mathcal{K}\subseteq\mathsf{M}$ be compact. Then there exist $C,\sigma\in\mathbb{R}_{>0}$ such that*

$$\frac{\sigma^{r+s}}{C}\|A(x)\|_{\mathbb{G}_2}\le\|A(x)\|_{\mathbb{G}_1}\le\frac{C}{\sigma^{r+s}}\|A(x)\|_{\mathbb{G}_2}$$

for all $A\in\Gamma^0(\mathrm{T}^r_s(\mathsf{E}))$, $r,s\in\mathbb{Z}_{\ge0}$, and $x\in\mathcal{K}$.

Proof: We begin by proving a linear algebra result.

1 Sublemma: *If $\mathbb{G}_1$ and $\mathbb{G}_2$ are inner products on a finite-dimensional $\mathbb{R}$-vector space V, then there exists $C\in\mathbb{R}_{>0}$ such that*

$$C^{-1}\mathbb{G}_1(v,v)\le\mathbb{G}_2(v,v)\le C\mathbb{G}_1(v,v)$$

for all $v\in\mathsf{V}$.

Proof: Let $\mathbb{G}_j^\flat\in\mathrm{Hom}_{\mathbb{R}}(\mathsf{V};\mathsf{V}^*)$ and $\mathbb{G}_j^\sharp\in\mathrm{Hom}_{\mathbb{R}}(\mathsf{V}^*;\mathsf{V})$, $j\in\{1,2\}$, be the induced linear maps. Note that

$$\mathbb{G}_1(\mathbb{G}_1^\sharp\circ\mathbb{G}_2^\flat(v_1),v_2)=\mathbb{G}_2(v_1,v_2)=\mathbb{G}_2(v_2,v_1)=\mathbb{G}_1(\mathbb{G}_1^\sharp\circ\mathbb{G}_2^\flat(v_2),v_1),$$

showing that $\mathbb{G}_1^\sharp\circ\mathbb{G}_2^\flat$ is $\mathbb{G}_1$-symmetric. Let $(e_1,\dots,e_n)$ be a $\mathbb{G}_1$-orthonormal basis for V that is also a basis of eigenvectors for $\mathbb{G}_1^\sharp\circ\mathbb{G}_2^\flat$. The matrix representatives of $\mathbb{G}_1$ and $\mathbb{G}_2$ are then

$$[\mathbb{G}_1]=\begin{bmatrix}1&0&\cdots&0\\0&1&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&1\end{bmatrix},\qquad[\mathbb{G}_2]=\begin{bmatrix}\lambda_1&0&\cdots&0\\0&\lambda_2&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&\lambda_n\end{bmatrix},$$

where $\lambda_1,\dots,\lambda_n\in\mathbb{R}_{>0}$. Let us assume without loss of generality that

$$\lambda_1\le\cdots\le\lambda_n.$$

Then taking $C=\max\{\lambda_n,\lambda_1^{-1}\}$ gives the result, as one can verify directly. ▼

Next we use the preceding sublemma to give the linear algebraic version of the lemma.

2 Sublemma: *Let V be a finite-dimensional $\mathbb{R}$-vector space and let $\mathbb{G}_1$ and $\mathbb{G}_2$ be inner products on V. Then there exist $C,\sigma\in\mathbb{R}_{>0}$ such that*

$$\frac{\sigma^{r+s}}{C}\|A\|_{\mathbb{G}_2}\le\|A\|_{\mathbb{G}_1}\le\frac{C}{\sigma^{r+s}}\|A\|_{\mathbb{G}_2}$$

for all $A\in\mathrm{T}^r_s(\mathsf{V})$, $r,s\in\mathbb{Z}_{\ge0}$.

Proof: As in the proof of Sublemma 1, let $(e_1,\dots,e_n)$ be a $\mathbb{G}_1$-orthonormal basis for V consisting of eigenvectors for $\mathbb{G}_1^\sharp\circ\mathbb{G}_2^\flat$. Let $\lambda_1,\dots,\lambda_n\in\mathbb{R}_{>0}$ be the corresponding eigenvalues, supposing that

$$\lambda_1\le\cdots\le\lambda_n.$$

Note that $\mathbb{G}_2(e_j, e_k) = \delta_{jk}\lambda_j$, $j \in \{1, \dots, n\}$, (δ_{jk} being the Kronecker delta symbol) so that $(\widehat{e}_1 \triangleq \lambda_1^{-1} e_1, \dots, \widehat{e}_n \triangleq \lambda_n^{-1} e_n)$ is a $\mathbb{G}_2$-orthonormal basis. Denote by $(e^1, \dots, e^n)$ and $(\widehat{e}^1, \dots, \widehat{e}^n)$ be the dual bases. Note that $\widehat{e}^j = \lambda_j e^j$, $j \in \{1, \dots, n\}$.

Now let $A \in \mathrm{T}^r_s(\mathsf{V})$ and write

$$A = \sum_{j_1,\dots,j_r=1}^{n} \sum_{k_1,\dots,k_s=1}^{n} A^{j_1\cdots j_r}_{k_1\cdots k_s} e_{j_1} \otimes \cdots \otimes e_{j_r} \otimes e^{k_1} \otimes \cdots \otimes e^{k_s}$$

and

$$A = \sum_{j_1,\dots,j_r=1}^{n} \sum_{k_1,\dots,k_s=1}^{n} \widehat{A}^{j_1\cdots j_r}_{k_1\cdots k_s} \widehat{e}_{j_1} \otimes \cdots \otimes \widehat{e}_{j_r} \otimes \widehat{e}^{k_1} \otimes \cdots \otimes \widehat{e}^{k_s}.$$

We necessarily have

$$\widehat{A}^{j_1\cdots j_r}_{k_1\cdots k_s} = \lambda_{j_1} \cdots \lambda_{j_r} \lambda_{k_1}^{-1} \cdots \lambda_{k_s}^{-1} A^{j_1\cdots j_r}_{k_1\cdots k_s}, \qquad j_1, \dots, j_r, k_1, \dots, k_s \in \{1, \dots, n\}.$$

We have

$$\|A\|_{\mathbb{G}_1} = \left(\sum_{j_1,\dots,j_r=1}^{n} \sum_{k_1,\dots,k_s=1}^{n} \left| A^{j_1\cdots j_r}_{k_1\cdots k_s} \right|^2 \right)^{1/2}, \quad \|A\|_{\mathbb{G}_2} = \left(\sum_{j_1,\dots,j_r=1}^{n} \sum_{k_1,\dots,k_s=1}^{n} \left| \widehat{A}^{j_1\cdots j_r}_{k_1\cdots k_s} \right|^2 \right)^{1/2}.$$

Therefore, if we let $\sigma = \min\{\lambda_1, \lambda_n^{-1}\}$, we have

$$\|A\|_{\mathbb{G}_2} \leq \sigma^{-(r+s)} \|A\|_{\mathbb{G}_1}.$$

This gives one half of the estimate in the sublemma, and the other is established analogously. ▼

The lemma follows from the preceding sublemma since C and σ depend only on $\mathbb{G}_1$ and $\mathbb{G}_2$ through the largest and smallest eigenvalues of $\mathbb{G}_1^\sharp \circ \mathbb{G}_2^\flat$, which are uniformly bounded above and below on $\mathcal{K}$. ■

Now we can compare fibre norms for jet bundles associated with different metrics and connections.

8.8 Lemma: (Comparison of fibre norms for jet bundles for different metrics and connections) *Let $r \in \{\infty, \omega\}$ and let $\pi_{\mathsf{E}} \colon \mathsf{E} \to \mathsf{M}$ be a C^r-vector bundle. Consider C^r-affine connections ∇^{M} and $\overline{\nabla}^{\mathsf{M}}$ on M, and C^r-vector bundle connections $\nabla^{\pi_{\mathsf{E}}}$ and $\overline{\nabla}^{\pi_{\mathsf{E}}}$ in E. Consider C^r-Riemannian metrics $\mathbb{G}_{\mathsf{M}}$ and $\overline{\mathbb{G}}_{\mathsf{M}}$ for M, and C^r-fibre metrics $\mathbb{G}_{\pi_{\mathsf{E}}}$ and $\overline{\mathbb{G}}_{\pi_{\mathsf{E}}}$ for E. Let $\mathcal{K} \subseteq \mathsf{M}$ be compact. Then there exist $C, \sigma \in \mathbb{R}_{>0}$ such that*

$$\frac{\sigma^m}{C} \|j_m\xi(x)\|_{\overline{\mathbb{G}}_{\mathsf{M},\pi_{\mathsf{E}},m}} \leq \|j_m\xi(x)\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}},m}} \leq \frac{C}{\sigma^m} \|j_m\xi(x)\|_{\overline{\mathbb{G}}_{\mathsf{M},\pi_{\mathsf{E}},m}}$$

for all $\xi \in \Gamma^m(\mathsf{E})$, $m \in \mathbb{Z}_{\geq 0}$, and $x \in \mathcal{K}$.

Proof: We first make some preliminary constructions that will be useful.

By Lemma 8.4, we have

$$\begin{aligned} \xi(x) &= \widehat{A}^0_0\xi(x), \\ (\mathrm{Sym}_1\otimes\mathrm{id}_{\mathsf{E}})\overline{\nabla}^{\pi_{\mathsf{E}}}\xi(x) &= \widehat{A}^1_1(\nabla^{\pi_{\mathsf{E}}}\xi(x)) + \widehat{A}^1_0(\xi(x)), \\ &\vdots \\ (\mathrm{Sym}_m\otimes\mathrm{id}_{\mathsf{E}})\circ\overline{\nabla}^{\mathsf{M},\pi_{\mathsf{E}},m}\xi(x) &= \sum_{s=0}^m \widehat{A}^m_s((\mathrm{Sym}_s\otimes\mathrm{id}_{\mathsf{E}})\circ\nabla^{\mathsf{M},\pi_{\mathsf{E}},s}\xi(x)). \end{aligned} \tag{8.4}$$

In like manner, by Lemma 8.5, we have

$$\begin{aligned} \xi(x) &= \widehat{B}^0_0\xi(x), \\ (\mathrm{Sym}_1\otimes\mathrm{id}_{\mathsf{E}})\nabla^{\pi_{\mathsf{E}}}\xi(x) &= \widehat{B}^1_1(\overline{\nabla}^{\pi_{\mathsf{E}}}\xi(x)) + \widehat{B}^1_0(\xi(x)), \\ &\vdots \\ (\mathrm{Sym}_m\otimes\mathrm{id}_{\mathsf{E}})\circ\nabla^{\mathsf{M},\pi_{\mathsf{E}},m}\xi(x) &= \sum_{s=0}^m \widehat{B}^m_s((\mathrm{Sym}_s\otimes\mathrm{id}_{\mathsf{E}})\circ\overline{\nabla}^{\mathsf{M},\pi_{\mathsf{E}},s}\xi(x)). \end{aligned} \tag{8.5}$$

By Lemma 7.3, we have

$$\|A^m_s(\beta_s)\|_{\overline{\mathbb{G}}_{\mathsf{M},\pi_{\mathsf{E}}}} \le \|A^m_s\|_{\overline{\mathbb{G}}_{\mathsf{M},\pi_{\mathsf{E}}}}\|\beta_s\|_{\overline{\mathbb{G}}_{\mathsf{M},\pi_{\mathsf{E}}}}$$

for $\beta_s \in \mathrm{T}^s(\mathsf{T}^*\mathsf{M})\otimes\mathsf{E}$, $m\in\mathbb{Z}_{>0}$, $s\in\{0,1,\dots,m\}$. By Lemma 7.5,

$$\|\mathrm{Sym}_s(A)\|_{\overline{\mathbb{G}}_{\mathsf{M},\pi_{\mathsf{E}}}} \le \|A\|_{\overline{\mathbb{G}}_{\mathsf{M},\pi_{\mathsf{E}}}}$$

for $A\in\mathrm{T}^s(\mathsf{T}^*\mathsf{E})$ and $s\in\mathbb{Z}_{>0}$. Thus, recalling (8.3),

$$\|\widehat{A}^m_s(\mathrm{Sym}_s(\beta_s))\|_{\overline{\mathbb{G}}_{\mathsf{M},\pi_{\mathsf{E}}}} = \|\mathrm{Sym}_m\circ A^m_s(\beta_s)\|_{\overline{\mathbb{G}}_{\mathsf{M},\pi_{\mathsf{E}}}} \le \|A^m_s\|_{\overline{\mathbb{G}}_{\mathsf{E}}}\|\beta_s\|_{\overline{\mathbb{G}}_{\mathsf{M},\pi_{\mathsf{E}}}},$$

for $\beta_s\in\mathrm{T}^s(\pi^*_{\mathsf{E}}\mathsf{T}^*\mathsf{M})\otimes\mathsf{E}$, $m\in\mathbb{Z}_{>0}$, $s\in\{1,\dots,m\}$.

By Lemmata 7.8 and 7.9 with $r=0$, there exist $\sigma_1,\rho_1\in\mathbb{R}_{>0}$ such that

$$\|A^k_s(x)\|_{\overline{\mathbb{G}}_{\mathsf{M},\pi_{\mathsf{E}}}} \le \sigma_1^{-k}\rho_1^{-(k-s)}(k-s)!, \qquad k\in\mathbb{Z}_{\ge0},\ s\in\{0,1,\dots,k\},\ x\in\mathcal{K}.$$

Without loss of generality, we assume that $\sigma_1,\rho_1\le 1$. Thus, abbreviating $\sigma_2=\sigma_1\rho_1$, we have

$$\|\widehat{A}^k_s((\mathrm{Sym}_s\otimes\mathrm{id}_{\mathsf{E}})\circ\overline{\nabla}^{\mathsf{M},\pi_{\mathsf{E}},s}\xi(x)\|_{\overline{\mathbb{G}}_{\mathsf{M},\pi_{\mathsf{E}}}} \le C_1\sigma_2^{-k}(k-s)!\|(\mathrm{Sym}_s\otimes\mathrm{id}_{\mathsf{E}})\circ\nabla^{\mathsf{M},\pi_{\mathsf{E}},s}\xi(x)\|_{\overline{\mathbb{G}}_{\mathsf{M},\pi_{\mathsf{E}}}}$$

for $m\in\mathbb{Z}_{\ge0}$, $k\in\{0,1,\dots,m\}$, $s\in\{0,1,\dots,k\}$, $x\in\mathcal{K}$. Thus, by (1.3) and (8.5),

$$\begin{aligned} \|j_m\xi(x)\|_{\overline{\mathbb{G}}_{\mathsf{M},\pi_{\mathsf{E}},m}} &\le \sum_{k=0}^m \frac{1}{k!}\|(\mathrm{Sym}_k\otimes\mathrm{id}_{\mathsf{E}})\circ\nabla^{\mathsf{M},\pi_{\mathsf{E}},k}\xi(x)\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}}} \\ &= \sum_{k=0}^m \frac{1}{k!}\left\|\sum_{s=0}^k \widehat{A}^k_s((\mathrm{Sym}_s\otimes\mathrm{id}_{\mathsf{E}})\circ\nabla^{\mathsf{M},\pi_{\mathsf{E}},s}\xi(x))\right\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}}} \\ &\le \sum_{k=0}^m\sum_{s=0}^k C_1\sigma_2^{-k}\frac{s!(k-s)!}{k!}\frac{1}{s!}\|(\mathrm{Sym}_s\otimes\mathrm{id}_{\mathsf{E}})\circ\nabla^{\mathsf{M},\pi_{\mathsf{E}},s}\xi(x)\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}}} \end{aligned}$$

for $x \in \mathcal{K}$ and $m \in \mathbb{Z}_{\geq 0}$. Now note that

$$\frac{s!(k-s)!}{k!} \leq 1, \quad C_1\sigma_2^{-k} \leq C_1\sigma_2^{-m},$$

for $s \in \{0,1,\dots,m\}$, $k \in \{0,1,\dots,s\}$, since $\sigma_2 \leq 1$. Then

$$\begin{aligned}\|j_m\xi(x)\|_{\overline{\mathbb{G}}_{\mathsf{M},\pi_{\mathsf{E}},m}} &\leq C_1\sigma_2^{-m}\sum_{k=0}^{m}\sum_{s=0}^{k}\frac{1}{s!}\|(\mathrm{Sym}_s\otimes\mathrm{id}_{\mathsf{E}})\circ\nabla^{\mathsf{M},\pi_{\mathsf{E}},s}\xi(x)\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}}}\\ &\leq C_1\sigma_2^{-m}\sum_{k=0}^{m}\sum_{s=0}^{m}\frac{1}{s!}\|(\mathrm{Sym}_s\otimes\mathrm{id}_{\mathsf{E}})\circ\nabla^{\mathsf{M},s}\xi(x)\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}}}\\ &= (m+1)C_1\sigma_2^{-m}\sum_{s=0}^{m}\frac{1}{s!}\|(\mathrm{Sym}_s\otimes\mathrm{id}_{\mathsf{E}})\circ\nabla^{\mathsf{M},\pi_{\mathsf{E}},s}\xi(x)\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}}}.\end{aligned}$$

Now let $\sigma < \sigma_2$ and note that

$$\lim_{m\to\infty}(m+1)\frac{\sigma_2^{-m}}{\sigma^{-m}} = 0.$$

Thus there exists $N \in \mathbb{Z}_{>0}$ such that

$$(m+1)C_1\sigma_2^{-m} \leq C_1\sigma^{-m}, \qquad m \geq N.$$

Let

$$C = \max\left\{C_1, 2C_1\frac{\sigma}{\sigma_2}, 3C_1\left(\frac{\sigma}{\sigma_2}\right)^2,\dots,(N+1)C_1\left(\frac{\sigma}{\sigma_2}\right)^N\right\}.$$

We then immediately have $(m+1)C_1\sigma_2^{-m} \leq C\sigma^{-m}$ for all $m \in \mathbb{Z}_{\geq 0}$. We then have, using (1.3),

$$\begin{aligned}\|j_m\xi(x)\|_{\overline{\mathbb{G}}_{\mathsf{M},\pi_{\mathsf{E}},m}} &\leq C\sigma^{-m}\sum_{s=0}^{m}\frac{1}{s!}\|(\mathrm{Sym}_s\otimes\mathrm{id}_{\mathsf{E}})\circ\nabla^{\mathsf{M},\pi_{\mathsf{E}},s}\xi(x)\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}}}\\ &= C\sqrt{m}\sigma^{-m}\|j_m\xi(x)\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}},m}}.\end{aligned}$$

After modifying C and σ in the manner of the computations just preceding, we have

$$\|j_m\xi(x)\|_{\overline{\mathbb{G}}_{\mathsf{M},\pi_{\mathsf{E}},m}} \leq C\sigma^{-m}\|j_m\xi(x)\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}},m}}.$$

This gives one half of the desired pair of estimates.

For the other half of the estimate, we use (8.5), and Lemmata 7.8 and 7.9 in the computations above to arrive at the estimate

$$\|j_m\xi(x)\|_{\overline{\mathbb{G}}_{\mathsf{M},\pi_{\mathsf{E}},m}} \leq C\sigma^{-m}\|j_m\xi(x)\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}},m}},$$

which gives the result. ■

8.9 Remark: (Adaptation to the smooth case) The preceding result holds in the smooth case, and with a much easier proof. In the result, one can replace "$C\sigma^{-m}$" with a fixed constant "C" for each m. For this reason, the proof is also far simpler, as one need not keep track of all the factorial terms that give rise to the exponential component in the estimates. •

8.3. Local descriptions of the real analytic topology. We endeavour to make our presentation as unencumbered of coordinates as possible. While the intrinsic jet bundle characterisations of the seminorms are useful for abstract definitions and proofs, concrete proofs often require local descriptions of the topologies. In this section we provide these local descriptions of the topologies. By proving that these local descriptions are equivalent to the intrinsic descriptions above, we also prove that these intrinsic descriptions of topologies do not depend on the choice of metrics or connections.

Let us develop the notation for working with local descriptions of topologies. Let $\mathcal{U} \subseteq \mathbb{R}^n$ be an open set and let $\boldsymbol{\Phi} \in \mathrm{C}^\omega(\mathcal{U};\mathbb{R}^k)$. We define local seminorms as follows. For $\mathcal{K} \subseteq \mathcal{U}$ compact and for $\boldsymbol{a} \in \mathrm{c}_0(\mathbb{Z}_{\geq 0};\mathbb{R}_{>0})$, denote

$$p'^{\omega}_{\mathcal{K},\boldsymbol{a}}(\boldsymbol{\Phi}) = \sup\Big\{\frac{a_0a_1\cdots a_m}{I!}|\boldsymbol{D}^I\Phi^a(\boldsymbol{x})|\ \Big|\ \boldsymbol{x}\in\mathcal{K},\ a\in\{1,\dots,k\},\ I\in\mathbb{Z}^n_{\geq 0},\ |I|\leq m,\ m\in\mathbb{Z}_{\geq 0}\Big\}.$$

These seminorms, defined for all compact $\mathcal{K} \subseteq \mathcal{U}$ and $\boldsymbol{a} \in \mathrm{c}_0(\mathbb{Z}_{\geq 0};\mathbb{R}_{>0})$, define the ***local*** $\mathbf{C}^{\boldsymbol{\omega}}$***-topology*** for $\mathrm{C}^\omega(\mathcal{U};\mathbb{R}^k)$.

There are many possible variations of the seminorms that one can use, and these variations are equivalent to the seminorms above. For example, rather than using the ∞-vector norm, one might use the 2-vector norm. In doing so, one uses (1.3) to give

$$\begin{aligned}\sup\{|\boldsymbol{D}^I\Phi^a(\boldsymbol{x})|\ |\ I\in\mathbb{Z}^n_{\geq 0},\ |I|=m,\ a\in\{1,\dots,k\}\}&\leq\|\boldsymbol{D}^m\boldsymbol{\Phi}(\boldsymbol{x})\|\\&\leq\sqrt{kn^m}\sup\{|\boldsymbol{D}^I\Phi^a(\boldsymbol{x})|\ |\ I\in\mathbb{Z}^n_{\geq 0},\ |I|=m,\ a\in\{1,\dots,k\}\}.\end{aligned}$$

If we define

$$b_0 = 2\sqrt{k}a_0,\ b_j = 2\sqrt{n}a_j,\qquad j\in\mathbb{Z}_{>0},$$

then, noting that $n^j \leq n^m$ for $j \in \{0,1,\dots,m\}$ and that $m+1 \leq 2^m$ for $m \in \mathbb{Z}_{\geq 0}$, we have

$$p'^{\omega}_{\mathcal{K},\boldsymbol{a}}(\boldsymbol{\Phi}) \leq \sup\Big\{\frac{a_0a_1\cdots a_m}{I!}\|\boldsymbol{D}^m\boldsymbol{\Phi}(\boldsymbol{x})\|\ \Big|\ \boldsymbol{x}\in\mathcal{K},\ I\in\mathbb{Z}^n_{\geq 0},\ |I|\leq m,\ m\in\mathbb{Z}_{\geq 0}\Big\} \leq p'^{\omega}_{\mathcal{K},\boldsymbol{b}}(\boldsymbol{\Phi}),$$

and this gives equivalence of the topologies using the ∞- and 2-norms. Another variation in the seminorms is that one might scale the derivatives by $\frac{1}{|I|!}$ rather than $\frac{1}{I!}$. In this case, we use the standard multinomial estimate (7.1) to give

$$\frac{|I|!}{I!} \leq n^m.$$

Thus, if we take

$$b_0 = a_0,\ b_j = na_j,\qquad j\in\mathbb{Z}_{>0},$$

we have

$$p'^{\omega}_{\mathcal{K},\boldsymbol{b}}(\boldsymbol{\Phi}) \leq \sup\left\{\frac{a_0 a_1 \cdots a_m}{|I|!}|\boldsymbol{D}^I\Phi^a(\boldsymbol{x})| \;\middle|\; \boldsymbol{x}\in\mathcal{K},\ a\in\{1,\dots,k\},\ I\in\mathbb{Z}^n_{\geq 0},\ |I|\leq m,\ m\in\mathbb{Z}_{\geq 0}\right\} \leq p'^{\omega}_{\mathcal{K},\boldsymbol{a}}(\boldsymbol{\Phi}).$$

This gives the equivalence of the topologies defined using the scaling factor $\frac{1}{|I|!}$ for derivatives in place of $\frac{1}{I!}$. One can also combine the previous modifications. Indeed, if we use the 2-norm and the scaling factor $\frac{1}{|I!|}$, then one readily sees that we recover the intrinsic seminorms on the trivial vector bundle $\mathbb{R}^k_{\mathcal{U}}$ of Section 2.4 using (1) the Euclidean inner product for the Riemannian metric on $\mathcal{U}$ and for the fibre metric on $\mathbb{R}^k$ and (2) standard differentiation as covariant differentiation. We shall use this observation in the proof of Theorem 8.10 below.

We wish to show that these local topologies can be used to define a topology for $\Gamma^\omega(\mathsf{E})$ that is equivalent to the intrinsic topologies defined in Section 2.4 using jet bundles, connections, and metrics. To state the result, let us indicate some notation. Let $(\mathcal{V},\psi)$ be a vector bundle chart for $\pi_{\mathsf{E}}\colon \mathsf{E}\to\mathsf{M}$ with $(\mathcal{U},\phi)$ the induced chart for M. Suppose that $\psi(\mathcal{V}) = \phi(\mathcal{U})\times\mathbb{R}^k$. Given a section ξ, we define $\psi_*(\xi)\colon \phi(\mathcal{U})\to\mathbb{R}^k$ by requiring that

$$\psi\circ\xi\circ\phi^{-1}(\boldsymbol{x}) = (\boldsymbol{x},\psi_*(\xi)(\boldsymbol{x})).$$

With this notation, we have the following result.

8.10 Theorem: (Agreement of intrinsic and local topologies) *Let $\pi_{\mathsf{E}}\colon\mathsf{E}\to\mathsf{M}$ be a C^ω-vector bundle. Let $\mathbb{G}_{\mathsf{M}}$ be a Riemannian metric on M, let $\mathbb{G}_{\pi_{\mathsf{E}}}$ be a vector bundle metric on E, let ∇^{M} be an affine connection on M, and let $\nabla^{\pi_{\mathsf{E}}}$ be a vector bundle connection on E, with all of these being of class C^ω. Then the following two collections of seminorms for $\Gamma^\omega(\mathsf{E})$ define the same topology:*

(i) $p^{\omega}_{\mathcal{K},\boldsymbol{a}}$, $\boldsymbol{a}\in\mathrm{c}_0(\mathbb{Z}_{\geq0};\mathbb{R}_{>0})$, $\mathcal{K}\subseteq\mathsf{M}$ *compact;*

(ii) $p'^{\omega}_{\mathcal{K},\boldsymbol{a}}\circ\psi_*$, $\boldsymbol{a}\in\mathrm{c}_0(\mathbb{Z}_{\geq0};\mathbb{R}_{>0})$, $\mathcal{K}\subseteq\phi(\mathcal{U})$ *compact,* $(\mathcal{V},\psi)$ *is a vector bundle chart for* E *with* $(\mathcal{U},\phi)$ *the induced chart for* M.

Proof: As alluded to in the discussion above, it suffices to use the norm

$$\|\boldsymbol{D}^m\boldsymbol{\Phi}(\boldsymbol{x})\|_2 = \left(\sum_{\substack{I\in\mathbb{Z}^n_{\geq0}\\|I|=m}}\sum_{a=1}^k|\boldsymbol{D}^I\Phi^a(\boldsymbol{x})|^2\right)^{1/2}$$

for derivatives of $\mathbb{R}^k$-valued functions on $\mathcal{U}\subseteq\mathbb{R}^n$. If we denote

$$j_m\boldsymbol{\Phi}(\boldsymbol{x}) = (\boldsymbol{\Phi}(\boldsymbol{x}),\boldsymbol{D}\boldsymbol{\Phi}(\boldsymbol{x}),\dots,\boldsymbol{D}^m\boldsymbol{\Phi}(\boldsymbol{x})),$$

then we define

$$\|j_m\boldsymbol{\Phi}(\boldsymbol{x})\|^2_{2,m} = \sum_{j=0}^m\frac{1}{(j!)^2}\|\boldsymbol{D}^j\boldsymbol{\Phi}(\boldsymbol{x})\|^2_2,$$

this norm agreeing with the fibre norms used in Section 2.4 with the flat connections and with the Euclidean inner products. We use these norms to define seminorms that we denote by q' in place of the local seminorms p' as above.

We might like to use Lemma 8.8 in this proof. However, we cannot do so. The reason for this is that the proof of Lemma 8.8 makes reference to Lemma 7.8. The proof of this lemma relies on the bound (7.3), which is deduced from Lemma 2.3. The proof of Lemma 2.3 in [Jafarpour and Lewis 2014], we note, relies on exactly what we are now proving. To intrude on the potential circular logic, we must give a proof of this part of the theorem that does not rely on Lemma 8.8. In fact, the only part of the chain of results that we need to prove independently is the bound (7.3). In particular, if we can show that Lemma 8.8 holds in the current situation where

1. $\mathsf{M} = \mathcal{U} \subseteq \mathbb{R}^n$ and $\mathsf{E} = \mathbb{R}^k_{\mathcal{U}}$,
2. $\overline{\mathbb{G}}_{\mathcal{U}}$ and $\overline{\mathbb{G}}_{\pi_{\mathsf{E}}}$ are the Euclidean inner products, and
3. $\overline{\nabla}^{\mathsf{M}}$ and $\overline{\nabla}^{\pi_{\mathsf{E}}}$ are the flat connections,

this will be enough to make use of this result.

Let $(\mathcal{V}, \psi)$ be a vector bundle chart for E with $(\mathcal{U}, \phi)$ the chart for M. Standard estimates for real analytic functions [e.g., Krantz and Parks 2002, Proposition 2.2.10] give $C_1, \sigma_1 \in \mathbb{R}_{>0}$ such that

$$\|D^r_{\overline{\nabla}_{\mathcal{U}}, \overline{\nabla}^{\pi_{\mathsf{E}}}} S_{\mathcal{U}}(\boldsymbol{x})\|_2, \|D^r_{\overline{\nabla}_{\mathcal{U}}, \overline{\nabla}^{\pi_{\mathsf{E}}}} S_{\pi_{\mathsf{E}}}(\boldsymbol{x})\|_2 \leq C_1 \sigma_1^{-r} r!, \qquad \boldsymbol{x} \in \mathcal{K}.$$

This gives the bound (7.3) in this case, and so we can use Lemma 7.9, and then Lemma 7.8, and then the computation of Lemma 8.8 to give

$$\frac{\sigma^m}{C} \|j_m \xi\|_{\mathsf{G}_{\mathsf{M},\pi_{\mathsf{E}},m}} \leq \|j_m(\psi_*(\xi))(\phi(x))\|_{2,m} \leq \frac{C}{\sigma^m} \|j_m \xi\|_{\mathsf{G}_{\mathsf{M},\pi_{\mathsf{E}},m}}.$$

Now, having established Lemma 8.8 in the case of interest, we proceed with the proof, making use of this fact.

Let $\mathcal{K} \subseteq \phi(\mathcal{U})$ be compact and let $\boldsymbol{a} \in \mathrm{c}_0(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0})$. As per our appropriate version of Lemma 8.8, there exist $C, \sigma \in \mathbb{R}_{>0}$ such that

$$\|j_m(\psi_*(\xi))(\phi(x))\|_{2,m} \leq \frac{C}{\sigma^m} \|j_m \xi(x)\|_{\mathsf{G}_{\mathsf{M},\pi_{\mathsf{E}},m}}$$

for every $\xi \in \Gamma^\omega(\mathsf{E})$, $x \in \phi^{-1}(\mathcal{K})$, and $m \in \mathbb{Z}_{\geq 0}$. Then

$$a_0 a_1 \cdots a_m \|j_m(\psi_*(\xi))(\phi(x))\|_{2,m} \leq \frac{C a_0 a_1 \cdots a_m}{\sigma^m} \|j_m \xi(x)\|_{\mathsf{G}_{\mathsf{M},\pi_{\mathsf{E}},m}}$$

for every $\xi \in \Gamma^\omega(\mathsf{E})$, $x \in \phi^{-1}(\mathcal{K})$, and $m \in \mathbb{Z}_{\geq 0}$. Define $\boldsymbol{b} \in \mathrm{c}_0(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0})$ by

$$b_0 = C a_0, \ b_j = \frac{a_j}{\sigma}, \qquad j \in \mathbb{Z}_{>0}.$$

Then, taking supremums of the preceding inequality gives

$$q'^{\omega}_{\mathcal{K},\boldsymbol{a}} \circ \psi_*(\xi) \leq p^{\omega}_{\phi^{-1}(\mathcal{K}),\boldsymbol{b}}(\xi)$$

for $\xi \in \Gamma^\omega(\mathsf{E})$.

Now let $\mathcal{K} \subseteq \mathsf{M}$ be compact and let $\boldsymbol{a} \in c_0(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0})$. Let $x \in \mathcal{K}$ and let $(\mathcal{V}_x, \psi_x)$ be a vector bundle chart for E with $(\mathcal{U}_x, \phi_x)$ the chart for M with $x \in \mathcal{U}_x$. We suppose that $\mathcal{U}_x$ is relatively compact and that, by our appropriate version of Lemma 8.8, there exist $C_x, \sigma_x \in \mathbb{R}_{>0}$ such that

$$\|j_m\xi(y)\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}},m}} \leq \frac{C_x}{\sigma_x^m}\|j_m(\psi_*\xi)(y)\|_{m,2}$$

for $\xi \in \Gamma^\omega(\mathsf{E})$, $y \in \mathrm{cl}(\mathcal{U}_x)$, $m \in \mathbb{Z}_{\geq 0}$. Therefore,

$$a_0 a_1 \cdots a_m \|j_m\xi(y)\|_{\mathbb{G}_{\pi_{\mathsf{E}},m}} \leq \frac{C_x a_0 a_1 \cdots a_m}{\sigma_x^m}\|j_m(\psi_*\xi)(y)\|_{m,2}$$

for $\xi \in \Gamma^\omega(\mathsf{E})$, $y \in \mathrm{cl}(\mathcal{U}_x)$, $m \in \mathbb{Z}_{\geq 0}$. Compactness of $\mathcal{K}$ gives $x_1, \dots, x_s \in \mathcal{K}$ such that $\mathcal{K} \subseteq \cup_{j=1}^s \mathcal{U}_{x_j}$ and we then take

$$C = \max\{C_{x_1}, \dots, C_{x_s}\}, \quad \sigma = \min\{\sigma_{x_1}, \dots, \sigma_{x_s}\}.$$

We define $\boldsymbol{b} \in c_0(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0})$ by

$$b_0 = Ca_0, \; b_j = \frac{a_j}{\sigma}, \qquad j \in \mathbb{Z}_{>0}.$$

We then arrive at the inequality

$$p^\omega_{\mathcal{K},\boldsymbol{a}}(\xi) \leq q'^\omega_{\mathrm{cl}(\mathcal{U}_{x_1}),\boldsymbol{b}} \circ \psi_{x_1*}(\xi) + \cdots + q'^\omega_{\mathrm{cl}(\mathcal{U}_{x_s}),\boldsymbol{b}} \circ \psi_{x_s*}(\xi)$$

which is valid for $\xi \in \Gamma^\omega(\mathsf{E})$. ■

8.11 Remark: (Adaptation to the smooth case) The preceding theorem holds in the smooth case. The proof is slightly simpler in the smooth case, unlike in the proof of Lemma 8.8 where the smooth case is significantly simpler than the real analytic case. Note also that, in the smooth case, one does not need the local estimates for derivatives of real analytic functions, so this also significantly simplifies the logic. •

An immediate consequence of the theorem is that the topologies defined by the seminorms of Section 2.4 are independent of the choice of connections ∇^{M} and $\nabla^{\pi_{\mathsf{E}}}$, Riemannian metric $\mathbb{G}_{\mathsf{M}}$, and vector bundle metric $\mathbb{G}_{\pi_{\mathsf{E}}}$, since the preceding result shows that all such topologies are the same as the one defined by local seminorms.

9. Continuity of standard geometric operations

In this section we put to use the somewhat complicated results of the preceding sections to prove the continuity of standard algebraic and differential operations on real analytic manifolds. The reader will notice as they go through the proofs that there are definite themes that emerge from the various proofs of continuity. Moreover, we take full advantage of the results from Section 7.1 that were nominally developed to prove the bounds of Lemma 7.8, so illustrating their general utility. We hope that a demonstration of the collection of results—some easy, other less easy—will prove useful.

As a general comment on the results in this section, we shall prove in many cases that certain linear mappings between spaces of sections of real analytic vector bundles are continuous and open onto their image, i.e., homeomorphisms onto their image. One might hope to do this with a general Open Mapping Theorem. Indeed, since the space of real analytic sections of a vector bundle is both webbed and ultrabornological, one is in a perhaps in a position to use the Open Mapping Theorem of De Wilde [1967] (see also [Meise and Vogt 1997, Theorem 24.30]). However, since the images of our mappings are not necessarily ultrabornological (even closed subspaces of ultrabornological spaces may not be ultrabornological), we typically prove the openness by a direct argument, by virtue of our having given in Section 5 relations between iterated covariant derivatives going "both ways." Moreover, the use of seminorms to prove these results is in keeping with the general tenor of this work.

As we have indicated as we have been going along, the results in this section are applicable to the smooth case. We shall indicate the modifications required in sample cases, with the general situation following easily from these.

9.1. Continuity of algebraic operations. We begin with a consideration of continuity of standard algebraic operations with vector bundles.

9.1 Theorem: (Continuity of algebraic operations) *Let* $\pi_{\mathsf{E}}\colon \mathsf{E}\to\mathsf{M}$ *and* $\pi_{\mathsf{F}}\colon \mathsf{F}\to\mathsf{M}$ *be* C^ω*-vector bundles. Then the following mappings are continuous:*

(i) $\Gamma^\omega(\mathsf{E})\oplus\Gamma^\omega(\mathsf{E})\ni(\xi,\eta)\mapsto\xi+\eta\in\Gamma^\omega(\mathsf{E})$*;*

(ii) $\Gamma^\omega(\mathsf{F}\otimes\mathsf{E}^*)\times\Gamma^\omega(\mathsf{E})\ni(L,\xi)\mapsto L\circ\xi\in\Gamma^\omega(\mathsf{F})$*.*

Also, fixing an injective vector bundle mapping $L\in\Gamma^\omega(\mathsf{F}\otimes\mathsf{E}^*)$*, the following mapping is open onto its image:*

(iii) $\Gamma^\omega(\mathsf{E})\ni\xi\mapsto L\circ\xi\in\Gamma^\omega(\mathsf{F})$*.*

Proof: We suppose that we have a real analytic affine connection ∇^{M} on M, and real analytic vector bundle connections $\nabla^{\pi_{\mathsf{E}}}$ and $\nabla^{\pi_{\mathsf{F}}}$ in E and F, respectively. We suppose that we have a real analytic Riemannian metric $\mathbb{G}_{\mathsf{M}}$ on M, and real analytic fibre metrics $\mathbb{G}_{\pi_{\mathsf{E}}}$ and $\mathbb{G}_{\pi_{\mathsf{F}}}$ on E and F, respectively. This gives the seminorms $p^\omega_{\mathcal{K},\boldsymbol{a}}$ and $q^\omega_{\mathcal{K},\boldsymbol{a}}$, $\mathcal{K}\subseteq\mathsf{M}$ compact, $\boldsymbol{a}\in\mathrm{c}_0(\mathbb{Z}_{\ge0};\mathbb{R}_{>0})$, for $\Gamma^\omega(\mathsf{E})$ and $\Gamma^\omega(\mathsf{F})$, respectively. We denote the induced seminorms for $\Gamma^\omega(\mathsf{F}\otimes\mathsf{E}^*)$ by $q^\omega_{\mathcal{K},\boldsymbol{a}}\otimes p^\omega_{\mathcal{K},\boldsymbol{a}}$, $\mathcal{K}\subseteq\mathsf{M}$ compact, $\boldsymbol{a}\in\mathrm{c}_0(\mathbb{Z}_{\ge0};\mathbb{R}_{>0})$.

(i) The fibre norms from Section 2.3 satisfy the triangle inequality, and this readily gives

$$p^\omega_{\mathcal{K},\boldsymbol{a}}(\xi+\eta)\le p^\omega_{\mathcal{K},\boldsymbol{a}}(\xi)+p^\omega_{\mathcal{K},\boldsymbol{a}}(\eta),$$

which immediately gives this part of the result.

(ii) Let us make some preliminary computations from which this part of the theorem will follow easily.

First, by Lemma 7.3, we have

$$\|L\circ\xi(x)\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{F}}}}\le\|L(x)\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{F}}\otimes\pi_{\mathsf{E}}}}\|\xi(x)\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}}}.\tag{9.1}$$

Next, by Lemmata 4.4, 7.3, and 7.5, we have

$$\|D^k_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{F}}}}(L\circ\xi(x))\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{F}}}}\le\sum_{j=0}^k\binom{k}{j}\|D^j_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{F}}\otimes\pi_{\mathsf{E}}}}L(x)\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{F}}\otimes\pi_{\mathsf{E}}}}\|D^{k-j}_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}}\xi(x)\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}}}$$

for $k \in \mathbb{Z}_{>0}$. By (1.3) (twice) we have

$$\begin{aligned}\|j_m(L\circ\xi)(x)\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{F}},m}} &\le \sum_{k=0}^m \frac{1}{k!}\|D^k_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{F}}}}(L\circ\xi(x))\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{F}}}}\\ &\le \sum_{k=0}^m \frac{1}{k!}\sum_{j=0}^k \binom{k}{j}\|D^j_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{F}}\otimes\pi_{\mathsf{E}}}}L(x)\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{F}}\otimes\pi_{\mathsf{E}}}}\|D^{k-j}_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}}\xi(x)\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}}}\\ &= \sum_{k=0}^m\sum_{j=0}^k \frac{\|D^j_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{F}}\otimes\pi_{\mathsf{E}}}}L(x)\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{F}}\otimes\pi_{\mathsf{E}}}}}{j!}\frac{\|D^{k-j}_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}}\xi(x)\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}}}}{(k-j)!}\\ &\le (m+1)^2\sup\left\{\frac{\|D^j_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{F}}\otimes\pi_{\mathsf{E}}}}L(x)\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{F}}\otimes\pi_{\mathsf{E}}}}}{j!}\;\middle|\; j\le m\right\}\\ &\qquad\times\sup\left\{\frac{\|D^{k-j}_{\nabla^{\mathsf{M}},\nabla^{\pi_{\mathsf{E}}}}\xi(x)\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}}}}{(k-j)!}\;\middle|\; j\le m\right\}\\ &\le (m+1)^{5/2}\|j_mL(x)\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{F}}\otimes\pi_{\mathsf{E}},m}}\|j_m\xi(x)\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}},m}}.\end{aligned}$$

Noting that $(m+1)^{5/2} \le 3^{m+1}$, $m \in \mathbb{Z}_{>0}$, we finally get

$$\|j_m(L\circ\xi)(x)\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{F}},m}} \le 3^{m+1}\|j_mL(x)\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{F}}\otimes\pi_{\mathsf{E}},m}}\|j_m\xi(x)\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}},m}}. \tag{9.2}$$

Let $\mathcal{K} \subseteq \mathsf{M}$ be compact and let $\boldsymbol{a} \in c_0(\mathbb{Z}_{\ge 0};\mathbb{R}_{>0})$. Define define $\boldsymbol{a}' \in c_0(\mathbb{Z}_{\ge 0};\mathbb{R}_{>0})$ by $a'_j = \sqrt{3}a_j$, $j \in \mathbb{Z}_{\ge 0}$. We then have

$$q^\omega_{\mathcal{K},\boldsymbol{a}}(L\circ\xi) \le Cq^\omega_{\mathcal{K},\boldsymbol{a}'}\otimes p^\omega_{\mathcal{K},\boldsymbol{a}'}(L)p^\omega_{\mathcal{K},\boldsymbol{a}'}(\xi) = C(q^\omega_{\mathcal{K},\boldsymbol{a}'}\otimes p^\omega_{\mathcal{K},\boldsymbol{a}'})\otimes p^\omega_{\mathcal{K},\boldsymbol{a}'}(L\otimes\xi).$$

By [Jarchow 1981, Theorem 15.1.2], this gives continuity of the bilinear map $(L,\xi)\mapsto L\circ\xi$.

(iii) We first prove a couple of technical lemmata.

1 Lemma: *Let U and V be locally convex topological vector spaces, and let $L \in \mathsf{L}(\mathsf{U};\mathsf{V})$. If, for every continuous seminorm q for U, there exists a continuous seminorm p for V such that*

$$q(u) \le p(L(u)), \qquad u \in \mathsf{U},$$

then L is an open mapping onto its image.

Proof: First we prove that there are 0-bases $\mathscr{B}_{\mathsf{U}}$ for $\mathcal{U}$ and $\mathscr{B}_{\mathsf{V}}$ for V such that, for each $\mathcal{B} \in \mathscr{B}_{\mathsf{U}}$, there exists $\mathcal{C} \in \mathscr{B}_{\mathsf{V}}$ such that

$$\mathcal{C}\cap\operatorname{image}(L) \subseteq L(\mathcal{B}).$$

To see this, first let q be a continuous seminorm for U and let p be a continuous seminorm for V satisfying

$$q(u) \le p(L(u)), \qquad u \in \mathsf{U}.$$

Then

$$p(L(u)) < 1 \implies q(u) < 1 \implies L(u) \in L(q^{-1}([0,1)).$$

Thus

$$p^{-1}([0,1)) \cap \operatorname{image}(L) \subseteq L(q^{-1}([0,1))).$$

Now let $\mathscr{B}_{\mathsf{U}}$ be the collection of all 0-neighbourhoods of the form

$$\mathcal{B} = \cap_{j=1}^{k} q_j^{-1}([0,1)), \qquad k \in \mathbb{Z}_{>0},\ q_j \text{ a continuous seminorm}, j \in \{1,\dots,k\}.$$

This is a 0-base for U. For each such $\mathcal{B}$, we let p_j be continuous seminorm for V corresponding to q_j by

$$q_j(u) \le p_j(L(u)), \qquad u \in \mathsf{U},\ j \in \{1,\dots,k\}.$$

Then, by our above computations,

$$\left(\cap_{j=1}^{k} p_j^{-1}([0,1))\right) \cap \operatorname{image}(L) \subseteq L\left(\cap_{j=1}^{k} q_j^{-1}([0,1))\right).$$

Thus, the 0-base

$$\cap_{j=1}^{k} p_j^{-1}([0,1)), \qquad k \in \mathbb{Z}_{>0}, p_j \text{ a continuous seminorm}, j \in \{1,\dots,k\},$$

for V has the desired property.

Now let $\mathcal{O} \subseteq \mathsf{V}$ be open and let $u \in \mathcal{O}$. Let $\mathcal{B} \in \mathscr{B}_{\mathsf{U}}$ be such that $u + \mathcal{B} \subseteq \mathcal{O}$ and let $\mathcal{C} \in \mathscr{B}_{\mathsf{V}}$ be such that $\mathcal{C} \cap \operatorname{image}(L) \subseteq L(\mathcal{B})$. Then

$$L(u) + \mathcal{C} \cap \operatorname{image}(L) \subseteq L(u) + L(\mathcal{B}) = L(u + \mathcal{B}) \subseteq L(\mathcal{O}).$$

Thus $L(u) + \mathcal{C} \cap \operatorname{image}(L)$ is a neighbourhood of $L(u)$ in $L(\mathcal{O})$ which shows that $L(\mathcal{O})$ is open in $\operatorname{image}(L)$. ▼

2 Lemma: *If L is injective, then there exists a left-inverse $L' \in \Gamma^{\omega}(\mathsf{E} \otimes \mathsf{F}^*)$.*

Proof: First we note that $\operatorname{image}(L)$ is a C^{ω}-subbundle of F and that L is a C^{ω}-vector bundle isomorphism onto $\operatorname{image}(L)$. Let $\mathsf{G} \subseteq \mathsf{F}$ be the $\mathbb{G}_{\pi_{\mathsf{E}}}$-orthogonal complement to $\operatorname{image}(L)$ which is then itself a C^{ω}-subbundle of F. Clearly, $\mathsf{F} = \operatorname{image}(L) \oplus \mathsf{G}$. Let

$$\begin{aligned} L' \colon \operatorname{image}(L) \oplus \mathsf{G} &\to \mathsf{E} \\ (L(e), g) &\mapsto e, \end{aligned}$$

and note that L' is obviously a left-inverse of L. It is also of class C^{ω} since the projection from F to the summand $\operatorname{image}(L)$ is of class C^{ω}. ▼

By Lemma 2 we suppose that there is a C^{ω}-vector bundle mapping L' that is a left-inverse for L. Then, from the first part of the proof, for a compact $\mathcal{K} \subseteq \mathsf{M}$ and for $\boldsymbol{a} \in \mathrm{c}_0(\mathbb{Z}_{\ge 0}; \mathbb{R}_{>0})$, let $C \in \mathbb{R}_{>0}$ be such that

$$p^{\omega}_{\mathcal{K},\boldsymbol{a}}(L' \circ \eta) \le C q^{\omega}_{\mathcal{K},\boldsymbol{a}}(\eta), \qquad \eta \in \Gamma^{\omega}(\mathsf{F}).$$

We then have, for $\xi \in \Gamma^{\omega}(\mathsf{E})$,

$$p^{\omega}_{\mathcal{K},\boldsymbol{a}}(\xi) = p^{\omega}_{\mathcal{K},\boldsymbol{a}}(L' \circ L \circ \xi) \le C q^{\omega}_{\mathcal{K},\boldsymbol{a}}(L \circ \xi).$$

By Lemma 1, this suffices to establish that L is open onto its image. ■

9.2 Remark: (Adaptation to the smooth case) The preceding proof works equally well in the smooth case. Indeed, the proof is a little easier since one does not need to carefully keep track of the growth in m of the coefficient of the norm of the m-jet. •

The following result is an important one, and is very much nontrivial in the real analytic case. It is established during the course of the proof of their Lemma 2.5 by Jafarpour and Lewis [2014] using a local description of the real analytic topology. Here we use an intrinsic proof.

9.3 Theorem: (Composition induces a continuous map between function spaces) *Let* M *and* N *be* C^ω*-manifolds. If* $\Phi \in \mathrm{C}^\omega(\mathsf{M};\mathsf{N})$*, then the mapping*

$$\begin{aligned}\Phi^*\colon \mathrm{C}^\omega(\mathsf{N}) &\to \mathrm{C}^\omega(\mathsf{M})\\ f &\mapsto f\circ\Phi\end{aligned}$$

is continuous. Moreover, if Φ *is a proper surjective submersion or a proper embedding, then* Φ^* *is open onto its image. In case* Φ *is a proper embedding, for any compact* $\mathcal{K}\subseteq\mathsf{M}$ *and any* $\boldsymbol{a}\in \mathrm{c}_0(\mathbb{Z}_{\ge0};\mathbb{R}_{>0})$*, there exists* $\boldsymbol{a}'\in \mathrm{c}_0(\mathbb{Z}_{\ge0};\mathbb{R}_{>0})$ *such that*

$$q^\omega_{\Phi(\mathcal{K}),\boldsymbol{a}}(f)\le p^\omega_{\mathcal{K},\boldsymbol{a}'}(\Phi^*f),\qquad f\in \mathrm{C}^\omega(\mathsf{N}).$$

Proof: We let ∇^{M} and ∇^{N} be C^ω-affine connections on M and N, respectively, and let $\mathbb{G}_{\mathsf{M}}$ and $\mathbb{G}_{\mathsf{N}}$ be C^ω-Riemannian metrics on M and N, respectively. For $\mathcal{K}\subseteq\mathsf{M}$ and $\mathcal{L}\subseteq\mathsf{N}$ compact, and for $\boldsymbol{a}\in \mathrm{c}_0(\mathbb{Z}_{\ge0};\mathbb{R}_{>0})$, we denote by $p^\omega_{\mathcal{K},\boldsymbol{a}}$ and $q^\omega_{\mathcal{L},\boldsymbol{a}}$ the associated seminorms for $\mathrm{C}^\omega(\mathsf{M})$ and $\mathrm{C}^\omega(\mathsf{N})$, respectively.

From Lemma 5.40 we have the formula

$$\mathrm{Sym}_m\circ\nabla^{\mathsf{M},m}\Phi^*f=\sum_{s=0}^m \widehat{A}^m_s(\mathrm{Sym}_s\circ\Phi^*\nabla^{\mathsf{N},s}f). \tag{9.3}$$

By Lemma 7.3, we have

$$\|A^m_s(\beta_s)\|_{\mathbb{G}_{\mathsf{M}}}\le\|A^m_s\|_{\mathbb{G}_{\mathsf{M}},\mathbb{G}_{\mathsf{N}}}\|\beta_s\|_{\mathbb{G}_{\mathsf{N}}}$$

for $\beta_s\in\mathrm{T}^s(\mathsf{T}^*_x\mathsf{M})$, $m\in\mathbb{Z}_{>0}$, and $s\in\{0,1,\dots,m\}$. By Lemma 7.5,

$$\|\mathrm{Sym}_s(A)\|_{\mathbb{G}_{\mathsf{M}},\mathbb{G}_{\mathsf{N}}}\le\|A\|_{\mathbb{G}_{\mathsf{M}},\mathbb{G}_{\mathsf{N}}}$$

for $A\in\mathrm{T}^s(\mathsf{T}^*\mathsf{N})$ and $s\in\mathbb{Z}_{>0}$. Thus, recalling (5.3) (and its analogue that would arise in a spelled out proof of Lemma 5.40),

$$\|\widehat{A}^m_s(\mathrm{Sym}_s(\beta_s))\|_{\mathbb{G}_{\mathsf{M}}}=\|\mathrm{Sym}_m\circ A^m_s(\beta_s)\|_{\mathbb{G}_{\mathsf{M}}}\le\|A^m_s\|_{\mathbb{G}_{\mathsf{M}},\mathbb{G}_{\mathsf{N}}}\|\beta_s\|_{\mathbb{G}_{\mathsf{M}}},$$

for $\beta_s\in\mathrm{T}^s(\Phi^*\mathsf{T}^*\mathsf{N})$, $m\in\mathbb{Z}_{\ge0}$, $s\in\{1,\dots,m\}$.

Let $\mathcal{K}\subseteq\mathsf{M}$ be compact. By Lemmata 7.8 and 7.9 with $r=0$, there exist $C_1,\sigma_1,\rho_1\in\mathbb{R}_{>0}$ such that

$$\|A^k_s(x)\|_{\mathbb{G}_{\mathsf{M}},\mathbb{G}_{\mathsf{N}}}\le C_1\sigma_1^{-k}\rho_1^{-(k-s)}(k-s)!,\qquad k\in\mathbb{Z}_{\ge0},\ s\in\{0,1,\dots,k\},\ x\in\mathcal{K}.$$

By Lemma 6.8, let $C_2\in\mathbb{R}_{>0}$ be such that

$$\|\Phi^*\nabla^{\mathsf{N},m}f(x)\|_{\mathbb{G}_{\mathsf{M}}}\le C_2^m\|\nabla^{\mathsf{N},m}f(\Phi(x))\|_{\mathbb{G}_{\mathsf{N}}},\qquad x\in\mathcal{K},\ m\in\mathbb{Z}_{\ge0}.$$

Without loss of generality, we assume that $C_1, C_2 \geq 1$ and $\sigma_1, \rho_1 \leq 1$. Thus, abbreviating $\sigma_2 = \sigma_1\rho_1$, we have

$$\|\widehat{A}^k_s(\Phi^* \operatorname{Sym}_s \circ \nabla^{\mathsf{N},s} f(x))\|_{\mathbb{G}_\mathsf{M}} \leq C_1 C_2^s \sigma_2^{-k}(k-s)!\|\operatorname{Sym}_s \circ \nabla^{\mathsf{N},s} f(\Phi(x))\|_{\mathbb{G}_\mathsf{N}}$$

for $k \in \mathbb{Z}_{\geq 0}$, $s \in \{0, 1, \dots, k\}$, $x \in \mathcal{K}$. Thus, by (1.3) and (9.3),

$$\begin{aligned}\|j_m(\Phi^* f)(x)\|_{\mathbb{G}_{\mathsf{M},m}} &\leq \sum_{k=0}^{m} \frac{1}{k!}\|\operatorname{Sym}_k \circ \nabla^{\mathsf{M},k}\Phi^* f(x)\|_{\mathbb{G}_\mathsf{M}} \\ &= \sum_{k=0}^{m} \frac{1}{k!}\left\|\sum_{s=0}^{k} \widehat{A}^k_s(\Phi^* \operatorname{Sym}_s \circ \nabla^{\mathsf{N},s} f(\Phi(x)))\right\|_{\mathbb{G}_\mathsf{M}} \\ &\leq \sum_{k=0}^{m}\sum_{s=0}^{k} C_1\sigma_2^{-k}\frac{s!(k-s)!}{k!}\frac{C_2^s}{s!}\|\operatorname{Sym}_s \circ \nabla^{\mathsf{N},s} f(\Phi(x))\|_{\mathbb{G}_\mathsf{N}}\end{aligned}$$

for $x \in \mathcal{K}$ and $m \in \mathbb{Z}_{\geq 0}$. Now note that

$$\frac{s!(k-s)!}{k!} \leq 1, \quad C_1\sigma_2^{-k}C_2^s \leq C_1 C_2^m \sigma_2^{-m},$$

for $s \in \{0, 1, \dots, m\}$, $k \in \{0, 1, \dots, s\}$, since $\sigma_2 \leq 1$. Then

$$\begin{aligned}\|j_m(\Phi^* f)(x)\|_{\mathbb{G}_{\mathsf{M},m}} &\leq C_1 C_2^m \sigma_2^{-m}\sum_{k=0}^{m}\sum_{s=0}^{k}\frac{1}{s!}\|\operatorname{Sym}_s \circ \nabla^{\mathsf{N},s} f(\Phi(x))\|_{\mathbb{G}_\mathsf{N}} \\ &\leq C_1 C_2^m \sigma_2^{-m}\sum_{k=0}^{m}\sum_{s=0}^{m}\frac{1}{s!}\|\operatorname{Sym}_s \circ \nabla^{\mathsf{N},s} f(\Phi(x))\|_{\mathbb{G}_\mathsf{N}} \\ &= (m+1)C_1 C_2^m \sigma_2^{-m}\sum_{s=0}^{m}\frac{1}{s!}\|\operatorname{Sym}_s \circ \nabla^{\mathsf{N},s} f(\Phi(x))\|_{\mathbb{G}_\mathsf{N}}.\end{aligned}$$

Now let $\sigma < C_2^{-1}\sigma_2$ and note that

$$\lim_{m\to\infty}(m+1)\frac{C_2^m\sigma_2^{-m}}{\sigma^{-m}} = 0.$$

Thus there exists $N \in \mathbb{Z}_{>0}$ such that

$$(m+1)C_1C_2^m\sigma_2^{-m} \leq C_1\sigma^{-m}, \qquad m \geq N.$$

Let

$$C = \max\left\{C_1, 2C_1C_2\frac{\sigma}{\sigma_2}, 3C_1C_2^2\left(\frac{\sigma}{\sigma_2}\right)^2, \dots, (N+1)C_1C_2^N\left(\frac{\sigma}{\sigma_2}\right)^N\right\}.$$

We then immediately have $(m+1)C_1C_2^m\sigma_2^{-m} \leq C\sigma^{-m}$ for all $m \in \mathbb{Z}_{\geq 0}$. We then have, using (1.3),

$$\begin{aligned}\|j_m(\Phi^* f)(x)\|_{\mathbb{G}_{\mathsf{M},m}} &\leq C\sigma^{-m}\left(\sum_{s=0}^{m}\frac{1}{s!}\|\operatorname{Sym}_s \circ \nabla^{\mathsf{N},s} f(\Phi(x))\|_{\mathbb{G}_\mathsf{N}}\right) \\ &= C\sqrt{m+1}\sigma^{-m}\|j_m f(\Phi(x))\|_{\mathbb{G}_{\mathsf{N},m}}.\end{aligned}$$

By modifying C and σ guided by what we did just preceding, we get

$$\|j_m(\Phi^* f)(x)\|_{\mathbb{G}_{\mathsf{M},m}} \le C\sigma^{-m}\|j_m f(\Phi(x))\|_{\mathbb{G}_{\mathsf{N},m}}.$$

Now, for $\boldsymbol{a} \in c_0(\mathbb{Z}_{\ge 0}; \mathbb{R}_{>0})$, let $\boldsymbol{a}' \in c_0(\mathbb{Z}_{\ge 0}; \mathbb{R}_{>0})$ be defined by $a'_0 = Ca_0$ and $a'_j = a_j\sigma^{-1}$, $j \in \mathbb{Z}_{>0}$. Then we have

$$p^{\omega}_{\mathcal{K},\boldsymbol{a}}(\Phi^* f) \le q^{\omega}_{\Phi(\mathcal{K}),\boldsymbol{a}'}(f),$$

and this gives continuity of Φ^*.

Now we turn to the final assertion concerning the openness of Φ^* in particular cases. First we note that, by Lemma 5.41, we have

$$\mathrm{Sym}_m \circ \Phi^* \nabla^{\mathsf{N},m} f(x) = \sum_{s=0}^{m} \widehat{B}^m_s(\mathrm{Sym}_s \circ \nabla^{\mathsf{M},s} \Phi^* f(x)).$$

First consider the case where Φ is a proper surjective submersion. For $\mathcal{L} \subseteq \mathsf{N}$ compact and for $y \in \mathcal{L}$, since Φ is surjective, there exists $x \in \mathsf{M}$ such that $\Phi(x) = y$. Also, since Φ is proper, $\Phi^{-1}(\mathcal{L})$ is compact. We can now reproduce the steps from the proof above, now making use of the second part of Lemma 6.8, to prove that

$$q^{\omega}_{\mathcal{L},\boldsymbol{a}}(f) \le p^{\omega}_{\Phi^{-1}(\mathcal{L}),\boldsymbol{a}'}(\Phi^* f),$$

which suffices to prove the openness of Φ^* by Lemma 1 from the proof of Theorem 9.1.

Finally consider the case where Φ is a proper embedding. Here we make use of a lemma.

1 Lemma: *Let* M *be a* C^{ω}*-manifold and let* $\mathsf{S} \subseteq \mathsf{M}$ *be a* C^{ω}*-embedded submanifold with* $\iota_{\mathsf{S}} \colon \mathsf{S} \to \mathsf{M}$ *the inclusion. Then*

$$\iota_{\mathsf{S}}^* \colon \mathrm{C}^{\omega}(\mathsf{M}) \to \mathrm{C}^{\omega}(\mathsf{S})$$

is an epimorphism, i.e., continuous, surjective, and open.

Proof: First note that we can use Sublemma 2 from the proof of Lemma 4.5 to show that $\iota_{\mathsf{S}}^* \colon \mathrm{C}^{\omega}(\mathsf{M}) \to \mathrm{C}^{\omega}(\mathsf{S})$ is surjective. It, therefore, remains to show that ι_{S}^* is continuous and open. Continuity follows from Theorem 9.3. Since $\mathrm{C}^{\omega}(\mathsf{S})$ and $\mathrm{C}^{\omega}(\mathsf{M})$ are ultrabornological webbed spaces, the De Wilde Open Mapping Theorem [Meise and Vogt 1997, Theorem 24.30] implies that ι_{S}^* is open. ▼

The lemma immediately gives openness of Φ^* in the case that Φ is a proper embedding. For the final assertion, we can follow the same argument as was sketched for the openness of Φ^* when Φ is a proper surjective submersion to give

$$q^{\omega}_{\Phi(\mathcal{K}),\boldsymbol{a}}(f) \le p^{\omega}_{\mathcal{K},\boldsymbol{a}'}(\Phi^* f),$$

as desired. ■

The matter of determining general conditions under which Φ^* is an homeomorphism onto its image or has closed image are taken up by Domański and Langenbruch [2003, 2006]. The linear operator $f \mapsto \Phi^* f$ is often called a composition operator. Also of interest is the nonlinear operator $\Phi \mapsto f \circ \Phi$, which is variously called a "superposition operator," a "nonlinear composition operator," or the "Nemytskii operator." Both operators are of substantial interest in various areas of mathematics.

9.4 Remark: (Adaptation to the smooth case) The preceding proof can be adapted to the smooth case. Indeed, much of the elaborate work of the proof can be simplified by not having to pay attention to the exponential growth of m-jet norms as $m \to \infty$. In the smooth case, one works with fixed orders of derivatives. This comment applies to all of our subsequent proofs in this section. •

9.2. Continuity of operations involving differentiation. Next we consider a general version of the assertion that "differentiation is continuous."

9.5 Theorem: (Prolongation of sections is continuous map) *Let $\pi_{\mathsf{E}} \colon \mathsf{E} \to \mathsf{M}$ be a C^ω-vector bundle. If $k \in \mathbb{Z}_{\geq 0}$, then the map*

$$\begin{aligned} J_k^\omega \colon \Gamma^\omega(\mathsf{E}) &\to \Gamma^\omega(\mathsf{J}^k\mathsf{E}) \\ \xi &\mapsto j_k\xi \end{aligned}$$

is continuous.

Proof: We let ∇^{M} be a C^ω-affine connection on M, ∇^π be a C^ω-vector bundle connection in E, $\mathbb{G}_{\mathsf{M}}$ be a C^ω-Riemannian metric on M, and $\mathbb{G}_\pi$ be a C^ω-vector bundle connection in E. We denote the associated seminorms for $\Gamma^\omega(\mathsf{E})$ by $p^\omega_{\mathcal{K},\boldsymbol{a}}$ and for $\Gamma^\omega(\mathsf{J}^k\mathsf{E})$ by $p^{k,\omega}_{\mathcal{K},\boldsymbol{a}}$, for $\mathcal{K} \subseteq \mathsf{M}$ compact and $\boldsymbol{a} \in \mathrm{c}_0(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0})$.

We recall from Section 4.4 that we have the vector bundle mapping $\pi_{m,k} \colon \mathsf{J}^{k+m}\mathsf{E} \to \mathsf{J}^m\mathsf{J}^k\mathsf{E}$ defined by the requirement that $\pi_{m,k} \circ j_{k+m}\xi = j_m j_k \xi$. We begin the proof by doing some computations that give norm estimates for this vector bundle map. To do this, we use the representation $\widehat{\Delta}_{m,k}$ of $\pi_{m,k}$ relative to our decompositions of jet bundles, as in Lemma 4.8. We let $x \in \mathsf{M}$ and let $A_j \in \mathrm{S}^j(\mathsf{T}_x^*\mathsf{M}) \otimes \mathsf{E}_x$, $j \in \{0, 1, \dots, k+m\}$, and compute, using Lemmata 4.7 and 7.5, (4.15), and (1.3),

$$\begin{aligned} \|\widehat{\Delta}_{m,k}\pi_{\mathsf{E}}(A_0, A_1, \dots, A_{m+k})\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}}} &\leq \sum_{l=0}^m \frac{1}{l!} \sum_{j=0}^k \|\Delta_{j,l}(A_{j+l})\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}}} \\ &\leq \sum_{l=0}^m \frac{(k+l)!}{l!} \sum_{j=0}^k \frac{1}{(j+l)!} \|A_{j+l}\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}}} \\ &\leq \frac{(k+m)!}{m!} \sum_{l=0}^m \sum_{j=0}^k \frac{1}{(j+l)!} \|A_{j+l}\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}}} \\ &\leq (m+k)^k(m+1) \sum_{j=0}^{k+m} \frac{1}{j!} \|A_j\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}}}. \end{aligned}$$

For $\sigma \in (0, 1)$,

$$\lim_{m \to 0} \sigma^{-m}(m+k)^k(m+1) = 0.$$

Thus let $N \in \mathbb{Z}_{>0}$ be such that

$$\sigma^{-m}(m+k)^k(m+1) < 0, \qquad m \geq N.$$

Next let

$$C = \max\left\{k^k, \frac{2(1+k)^k}{\sigma}, \dots, \frac{(N+1)(N+k)^k}{\sigma^N}\right\}.$$

Then, for any $m \in \mathbb{Z}_{\geq 0}$,
$$(m+k)^k(m+1) \leq C\sigma^{-m},$$
and so, using (1.3),
$$\|\widehat{\Delta}_{m,k}\pi_{\mathsf{E}}(A_0, A_1, \dots, A_{m+k})\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}}} \leq \sqrt{k+m+1}C\sigma^{-m}\|(A_0, A_1, \dots, A_{m+k})\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}}}.$$
Modifying C and σ similarly to our constructions above shows that
$$\|j_m j_k\xi(x)\|_{\mathbb{G}_{\mathsf{M},\pi_k,m}} \leq C\sigma^{-m}\|j_{k+m}\xi(x)\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}},k+m}}, \qquad x \in \mathsf{M}.$$

Let $\mathcal{K} \subseteq \mathsf{M}$ be compact and let $\boldsymbol{a} \in \mathrm{c}_0(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0})$. Define $\boldsymbol{a}' \in \mathrm{c}_0(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0})$ by $a'_0 = a_0$, $a'_j = C$, $j \in \{1, \dots, k\}$, and $a'_j = \sigma^{-1}a_{j-k}$, $j \in \{k+1, k+2, \dots\}$. The computations from the beginning of the proof then give
$$\begin{aligned} a_0 a_1 \cdots a_m \|j_m j_k\xi(x)\| &\leq C\sigma^{-m} a_0 a_1 \cdots a_m \|j_{k+m}\xi(x)\| \\ &\leq a_0 C^k(\sigma^{-1}a_1)\cdots(\sigma^{-1}a_m)\|j_{k+m}\xi(x)\| \\ &= a'_0 a'_1 \cdots a'_{k+m}\|j_{k+m}\xi(x)\|, \end{aligned}$$
since $C \geq 1$. We then immediately have
$$p^{k,\omega}_{\mathcal{K},\boldsymbol{a}}(j_k\xi) \leq p^{\omega}_{\mathcal{K},\boldsymbol{a}'}(\xi),$$
which gives the theorem. ■

We can now prove a collection of results regarding standard operations of differentiation, derived from the preceding result about basic prolongation.

9.6 Corollary: (Continuity of differential) *Let* M *be a* C^ω*-manifold. Then the mapping*
$$\begin{aligned} \mathrm{d}\colon \mathrm{C}^\omega(\mathsf{M}) &\to \Gamma^\omega(\mathsf{T}^*\mathsf{M}) \\ f &\mapsto \mathrm{d}f \end{aligned}$$
is continuous.

Proof: Note that $\mathsf{J}^1(\mathsf{M}; \mathbb{R}) \simeq \mathbb{R}_{\mathsf{M}} \oplus \mathsf{T}^*\mathsf{M}$ and that, under this identification, $j_1 f = f \oplus \mathrm{d}f$. Thus $\mathrm{d}f = \mathrm{pr}_2 \circ j_1 f$, where $\mathrm{pr}_2\colon \mathsf{J}^1(\mathsf{M}; \mathbb{R}) \to \mathsf{T}^*\mathsf{M}$ is the C^ω-vector bundle mapping of projection onto the second factor. The result then immediately follows from Theorem 9.1(ii) and Theorem 9.5. ■

9.7 Corollary: (Continuity of Lie derivative) *Let* M *be a* C^ω*-manifold. Then the map*
$$\begin{aligned} \mathscr{L}\colon \Gamma^\omega(\mathsf{TM}) \times \mathrm{C}^\omega(\mathsf{M}) &\to \mathrm{C}^\omega(\mathsf{M}) \\ (X, f) &\mapsto \mathscr{L}_X f \end{aligned}$$
is continuous.

Proof: We think of X as being a C^ω-vector bundle mapping via
$$\begin{aligned} X\colon \mathsf{T}^*\mathsf{M} &\to \mathbb{R}_{\mathsf{M}} \\ \alpha_x &\mapsto \langle \alpha_x; X(x)\rangle. \end{aligned}$$
Then the bilinear mapping of the lemma is given by the composition
$$(X, f) \mapsto (X, \mathrm{d}f) \mapsto X(\mathrm{d}f).$$
The left mapping is continuous since it is the product of the continuous mappings id and d. The right mapping is continuous by Theorem 9.1(ii), and so the corollary follows. ■

9.8 Corollary: (Continuity of covariant derivative) *Let $\pi_{\mathsf{E}}\colon \mathsf{E} \to \mathsf{M}$ be a C^ω-vector bundle with a C^ω-vector bundle connection $\nabla^{\pi_{\mathsf{E}}}$. Then the map*

$$\begin{aligned}\nabla^{\pi_{\mathsf{E}}}\colon \Gamma^\omega(\mathsf{TM}) \times \Gamma^\omega(\mathsf{E}) &\to \Gamma^\omega(\mathsf{E})\\ (X,\xi) &\mapsto \nabla^{\pi_{\mathsf{E}}}_X \xi\end{aligned}$$

is continuous.

Proof: As in the proof of Lemma 2.1, we have a C^ω-vector bundle mapping $S_{\nabla^{\pi_{\mathsf{E}}}}\colon \mathsf{E} \to \mathsf{J}^1\mathsf{E}$ over id_{M} that determines the connection $\nabla^{\pi_{\mathsf{E}}}$ by

$$\nabla^{\pi_{\mathsf{E}}}\xi(x) = j_1\xi(x) - S_{\nabla^{\pi_{\mathsf{E}}}}(\xi(x)).$$

The mapping $\xi \mapsto \nabla^{\pi_{\mathsf{E}}}\xi$ is continuous by Theorems 9.5 and 9.1. We note that $\nabla^{\pi_{\mathsf{E}}}\xi$ is to be thought of as a C^ω-vector bundle mapping by

$$\begin{aligned}\nabla^{\pi_{\mathsf{E}}}\xi\colon \mathsf{TM} &\to \mathsf{E}\\ X &\mapsto \nabla^{\pi_{\mathsf{E}}}_X\xi.\end{aligned}$$

The bilinear mapping of the lemma is then given by the composition

$$(X,\xi) \mapsto (X, \nabla^{\pi_{\mathsf{E}}}\xi) \mapsto \nabla^{\pi_{\mathsf{E}}}\xi(X).$$

The left mapping is continuous since it is the product of the continuous mappings id and $\xi \mapsto \nabla^{\pi_{\mathsf{E}}}\xi$. The right mapping is continuous by Theorem 9.1(ii), and so the lemma follows. ■

9.9 Corollary: (Continuity of Lie bracket) *Let M be a C^ω-manifold. Then the map*

$$\begin{aligned}[\cdot,\cdot]\colon \Gamma^\omega(\mathsf{TM}) \times \Gamma^\omega(\mathsf{TM}) &\to \Gamma^\omega(\mathsf{TM})\\ (X,Y) &\mapsto [X,Y]\end{aligned}$$

is continuous.

Proof: Let $\mathbb{G}_{\mathsf{M}}$ be a real analytic Riemannian metric on M and let ∇^{M} be the associated Levi-Civita connection. Since

$$[X,Y] = \nabla^{\mathsf{M}}_X Y - \nabla^{\mathsf{M}}_X Y,$$

the result follows from Corollary 9.8. ■

9.10 Corollary: (Continuity of linear partial differential operators) *Let $\pi_{\mathsf{E}}\colon \mathsf{E} \to \mathsf{M}$ and $\pi_{\mathsf{F}}\colon \mathsf{F} \to \mathsf{M}$ be C^ω-vector bundles and let $\Phi \in \mathrm{VB}^\omega(\mathsf{J}^k\mathsf{E};\mathsf{F})$. Then the kth-order linear partial differential operator $D_\Phi\colon \Gamma^\omega(\mathsf{E}) \to \Gamma^\omega(\mathsf{F})$ defined by $D_\Phi(\xi)(x) = \Phi(j_k\xi(x))$, $x \in \mathsf{M}$, is continuous.*

Proof: The operator D_Φ is the composition of the continuous mappings $\xi \mapsto j_k\xi$ $\Gamma^\omega(\mathsf{E})$ to $\Gamma^\omega(\mathsf{J}^k\mathsf{E})$ and $\Xi \mapsto \Phi \circ \Xi$ from $\Gamma^\omega(\mathsf{J}^k\mathsf{E})$ to $\Gamma^\omega(\mathsf{F})$. ■

The reader can no doubt imagine many extensions of results such as the ones we give, and we leave these for the reader to figure out as they need them.

9.3. Continuity of lifting operations. In Sections 3.1–3.4 we introduced a variety of constructions for lifting objects from the base space of a vector bundle to the total space. In Section 5 we considered how to differentiate these constructions in multiple ways, and how relate these multiple differentiations. In Sections 6.1–6.7 we described fibre norms to give norms for these lifted objects. In this section, we put this all together to prove results that are the entire *raison d'être* for all of these constructions, some of them quite elaborate. That is, we show that these lift operations are homeomorphisms onto their images. Many of the proofs are similar to one another, so we only give representative proofs.

We begin by considering horizontal lifts of functions. We note that continuity of the mapping in the next theorem follows from Theorem 9.3, but openness does not since the vector bundle projection is not proper. In any case, we give an independent proof of continuity, as it is a model for the proof of subsequent statements for which we will not give detailed proofs.

9.11 Theorem: (Horizontal lift of functions is an homeomorphism onto its image) *Let $\pi_{\mathsf{E}}\colon \mathsf{E}\to\mathsf{M}$ be a C^ω-vector bundle. Then the mapping*

$$\mathrm{C}^\omega(\mathsf{M})\ni f\mapsto \pi^*f\in\mathrm{C}^\omega(\mathsf{E})$$

is an homeomorphism onto its image.

Proof: It is clear that the asserted map is injective, so we focus on its topological attributes.

We let $\mathbb{G}_{\mathsf{M}}$ be a C^ω-Riemannian metric on M, $\mathbb{G}_\pi$ be a C^ω-vector bundle connection in E, ∇^{M} be the Levi-Civita connection for $\mathbb{G}_{\mathsf{M}}$, and ∇^π be a C^ω-vector bundle connection in E. Corresponding to this, we have a Riemannian metric $\mathbb{G}_{\mathsf{E}}$ on E with its Levi-Civita connection ∇^{E}, as in Section 4.1. We denote the associated seminorms for $\mathrm{C}^\omega(\mathsf{M})$ and $\mathrm{C}^\omega(\mathsf{E})$ by $p^\omega_{\mathcal{K},\boldsymbol{a}}$ and $q^\omega_{\mathcal{L},\boldsymbol{a}}$ for $\mathcal{K}\subseteq\mathsf{M}$ and $\mathcal{L}\subseteq\mathsf{E}$ compact, and for $\boldsymbol{a}\in\mathrm{c}_0(\mathbb{Z}_{\ge0};\mathbb{R}_{>0})$.

Let us make some preliminary computations.

By Lemma 5.3, we have

$$\mathrm{Sym}_m\circ\nabla^{\mathsf{E},m}\pi^*_{\mathsf{E}}f(e)=\sum_{s=0}^m\widehat{A}^m_s(\mathrm{Sym}_{s+1}\circ\pi^*_{\mathsf{E}}\nabla^{\mathsf{M},s}f(e)).\tag{9.4}$$

By Lemma 7.3, we have

$$\|A^m_s(\beta_s)\|_{\mathbb{G}_{\mathsf{E}}}\le\|A^m_s\|_{\mathbb{G}_{\mathsf{E}}}\|\beta_s\|_{\mathbb{G}_{\mathsf{E}}}$$

for $\beta_s\in\mathrm{T}^s(\mathsf{T}^*_e\mathsf{E})$, $m\in\mathbb{Z}_{>0}$, and $s\in\{0,1,\dots,m\}$. By Lemma 7.5,

$$\|\mathrm{Sym}_s(A)\|_{\mathbb{G}_{\mathsf{E}}}\le\|A\|_{\mathbb{G}_{\mathsf{E}}}$$

for $A\in\mathrm{T}^s(\mathsf{T}^*\mathsf{N})$ and $s\in\mathbb{Z}_{>0}$. Thus, recalling (5.3),

$$\|\widehat{A}^m_s(\mathrm{Sym}_s(\beta_s))\|_{\mathbb{G}_{\mathsf{E}}}=\|\mathrm{Sym}_m\circ A^m_s(\beta_s)\|_{\mathbb{G}_{\mathsf{E}}}\le\|A^m_s\|_{\mathbb{G}_{\mathsf{E}}}\|\beta_s\|_{\mathbb{G}_{\mathsf{E}}},$$

for $\beta_s\in\mathrm{T}^s(\pi^*_{\mathsf{E}}\mathsf{T}^*\mathsf{M})$, $m\in\mathbb{Z}_{>0}$, $s\in\{1,\dots,m\}$.

Let $\mathcal{L}\subseteq\mathsf{E}$ be compact. By Lemmata 7.8 and 7.9 with $r=0$, there exist $C_1,\sigma_1,\rho_1\in\mathbb{R}_{>0}$ such that

$$\|A^k_s(e)\|_{\mathbb{G}_{\mathsf{E}}}\le C_1\sigma_1^{-k}\rho_1^{-(k-s)}(k-s)!,\qquad k\in\mathbb{Z}_{>0},\ s\in\{0,1,\dots,k-1\},\ e\in\mathcal{L}.$$

Without loss of generality, we assume that $C_1 \geq 1$ and $\sigma_1, \rho_1 \leq 1$. Thus, using Lemma 6.1 and abbreviating $\sigma_2 = \sigma_1\rho_1$, we have

$$\|\widehat{A}^k_s(\pi^*_{\mathsf{E}} \operatorname{Sym}_s \circ \nabla^{\mathsf{M},s} f(e))\|_{\mathbb{G}_{\mathsf{E}}} \leq C_1\sigma_2^{-k}(k-s)!\|\operatorname{Sym}_s \circ \nabla^{\mathsf{M},s} \mathrm{d}f(\pi_{\mathsf{E}}(e))\|_{\mathbb{G}_{\mathsf{M}}}$$

for $k \in \mathbb{Z}_{\geq 0}$, $s \in \{0, 1, \dots, k\}$, $e \in \mathcal{L}$. Thus, by (1.3), (9.4), and Lemma 6.8,

$$\begin{aligned}
\|j_m(\pi^*_{\mathsf{E}} f)(e)\|_{\mathbb{G}_{\mathsf{E},m}} &\leq \sum_{k=0}^m \frac{1}{k!}\|\operatorname{Sym}_k \circ \nabla^{\mathsf{E},k}\pi^*_{\mathsf{E}} f(e)\|_{\mathbb{G}_{\mathsf{E}}} \\
&= \sum_{k=0}^m \frac{1}{k!}\left\|\sum_{s=0}^k \widehat{A}^k_s(\pi^*_{\mathsf{E}} \operatorname{Sym}_s \circ \nabla^{\mathsf{M},s} f(e))\right\|_{\mathbb{G}_{\mathsf{E}}} \\
&\leq \sum_{k=0}^m \sum_{s=0}^k C_1\sigma_2^{-k}\frac{s!(k-s)!}{k!}\frac{1}{s!}\|\operatorname{Sym}_s \circ \nabla^{\mathsf{M},s} f(\pi_{\mathsf{E}}(e))\|_{\mathbb{G}_{\mathsf{M}}}
\end{aligned}$$

for $e \in \mathcal{L}$ and $m \in \mathbb{Z}_{\geq 0}$. Now note that

$$\frac{s!(k-s)!}{k!} \leq 1, \quad C_1\sigma_2^{-k} \leq C_1\sigma_2^{-m},$$

for $s \in \{0, 1, \dots, m\}$, $k \in \{0, 1, \dots, s\}$, since $\sigma_2 \leq 1$. Then

$$\begin{aligned}
\|j_m(\pi^*_{\mathsf{E}} f)(e)\|_{\mathbb{G}_{\mathsf{E},m}} &\leq C_1\sigma_2^{-m}\sum_{k=0}^m \sum_{s=0}^k \frac{1}{s!}\|\operatorname{Sym}_s \circ \nabla^{\mathsf{M},s} f(\pi_{\mathsf{E}}(e))\|_{\mathbb{G}_{\mathsf{M}}} \\
&\leq C_1\sigma_2^{-m}\sum_{k=0}^m \sum_{s=0}^m \frac{1}{s!}\|\operatorname{Sym}_s \circ \nabla^{\mathsf{M},s} f(\pi_{\mathsf{E}}(e))\|_{\mathbb{G}_{\mathsf{M}}} \\
&= (m+1)C_1\sigma_2^{-m}\sum_{s=0}^m \frac{1}{s!}\|\operatorname{Sym}_s \circ \nabla^{\mathsf{M},s} f(\pi_{\mathsf{E}}(e))\|_{\mathbb{G}_{\mathsf{M}}}.
\end{aligned}$$

Now let $\sigma < \sigma_2$ and note that

$$\lim_{m\to\infty}(m+1)\frac{\sigma_2^{-m}}{\sigma^{-m}} = 0.$$

Thus there exists $N \in \mathbb{Z}_{>0}$ such that

$$(m+1)C_1\sigma_2^{-m} \leq C_1\sigma^{-m}, \qquad m \geq N.$$

Let

$$C = \max\left\{C_1, C_1\frac{\sigma}{\sigma_2}, 2C_1\left(\frac{\sigma}{\sigma_2}\right)^2, \dots, NC_1\left(\frac{\sigma}{\sigma_2}\right)^N\right\}.$$

We then immediately have $(m+1)C_1\sigma_2^{-m} \leq C\sigma^{-m}$ for all $m \in \mathbb{Z}_{\geq 0}$. We then have, by (1.3),

$$\begin{aligned}
\|j_m(\pi^*_{\mathsf{E}} f)(e)\|_{\mathbb{G}_{\mathsf{E},m}} &\leq C\sigma^{-m}\sum_{s=0}^m \frac{1}{s!}\|\operatorname{Sym}_s \circ \nabla^{\mathsf{M},s} f(\pi_{\mathsf{E}}(e))\|_{\mathbb{G}_{\mathsf{M}}} \\
&\leq C\sqrt{m+1}\sigma^{-m}\|j_m f(\pi_{\mathsf{E}}(e))\|_{\mathbb{G}_{\mathsf{M},m}}.
\end{aligned}$$

By modifying C and σ guided by what we did just preceding, we get

$$\|j_m(\pi_{\mathsf{E}}^* f)(e)\|_{\mathbb{G}_{\mathsf{E},m}} \le C\sigma^{-m}\|j_m f(\pi_{\mathsf{E}}(e))\|_{\mathbb{G}_{\mathsf{M},m}}.$$

Now let $\boldsymbol{a} \in \mathrm{c}_0(\mathbb{Z}_{\ge 0};\mathbb{R}_{>0})$ and define $\boldsymbol{a}' \in \mathrm{c}_0(\mathbb{Z}_{\ge 0};\mathbb{R}_{>0})$ be defined by $a'_0 = Ca_0$ and $a'_j = a_j\sigma^{-1}$, $j \in \mathbb{Z}_{>0}$. Then we have

$$q^\omega_{\mathcal{L},\boldsymbol{a}}(\pi_{\mathsf{E}}^* f) \le p^\omega_{\pi_{\mathsf{E}}(\mathcal{L}),\boldsymbol{a}'}(f),$$

giving continuity in this case.

Now we show that π_{E}^* is open onto its image. The idea here is to make some preliminary observations to put ourselves in a position to be able to say, "Now proceed as above."

By Lemma 5.4, we have

$$\mathrm{Sym}_m \circ \pi_{\mathsf{E}}^* \nabla^{\mathsf{M},m} f(e) = \sum_{s=0}^m \widehat{B}^m_s(\mathrm{Sym}_s \circ \nabla^{\mathsf{E},s}\pi_{\mathsf{E}}^* f(e)). \tag{9.5}$$

For a compact $\mathcal{L} \subseteq \mathsf{E}$ we can proceed as above to give a bound

$$\|j_m f(\pi_{\mathsf{E}}(e))\|_{\mathbb{G}_{\mathsf{M},m}} \le C\sigma^{-m}\|j_m(\pi_{\mathsf{E}}^* f)(e)\|_{\mathbb{G}_{\mathsf{E},m}}, \qquad e \in \mathcal{L}.$$

We need to choose the compact set $\mathcal{L}$ in a specific way. We let $\mathcal{K} \subseteq \mathsf{M}$ be compact and choose a continuous section $\xi \in \Gamma^0(\mathsf{E})$, and then take $\mathcal{L} = \xi(\mathcal{K})$. Then we have the estimate

$$\|j_m f(x)\|_{\mathbb{G}_{\mathsf{M},m}} \le C\sigma^{-m}\|j_m(\pi_{\mathsf{E}}^* f)(\xi(x))\|_{\mathbb{G}_{\mathsf{E},m}}, \qquad x \in \mathcal{K}.$$

Now we can mirror the arguments above for continuity to give the bound

$$p^\omega_{\mathcal{K},\boldsymbol{a}}(f) \le q^\omega_{\xi(\mathcal{K}),\boldsymbol{a}'}(\pi_{\mathsf{E}}^* f),$$

and from this we conclude that $f \mapsto \pi_{\mathsf{E}}^* f$ is indeed open onto its image by Lemma 1 from the proof of Theorem 9.1. ■

Now we consider vertical lifts of sections.

9.12 Theorem: (Vertical lift of sections is an homeomorphism onto its image) *Let $\pi_{\mathsf{E}}\colon \mathsf{E} \to \mathsf{M}$ be a C^ω-vector bundle. Then the mapping*

$$\Gamma^\omega(\mathsf{E}) \ni \xi \mapsto \xi^{\mathrm{v}} \in \Gamma^\omega(\mathsf{TE})$$

is an homeomorphism onto its image.

Proof: This follows in the same manner as Theorem 9.11, using Lemmata 5.8, 5.9, and 6.2. ■

One has the similar result for vertical lifts of endomorphisms.

9.13 Theorem: (Vertical lift of endomorphisms is an homeomorphism onto its image) *Let $\pi_{\mathsf{E}}\colon \mathsf{E} \to \mathsf{M}$ be a C^ω-vector bundle. Then the mapping*

$$\Gamma^\omega(\mathrm{End}(\mathsf{E})) \ni L \mapsto L^{\mathrm{v}} \in \Gamma^\omega(\mathrm{End}(\mathsf{TE}))$$

is an homeomorphism onto its image.

Proof: This follows in the same manner as Theorem 9.11, using Lemmata 5.23, 5.24, and 6.5. ■

Now we consider horizontal lifts of vector fields.

9.14 Theorem: (Horizontal lift of vector fields is an homeomorphism onto its image) *Let $\pi_{\mathsf{E}}\colon \mathsf{E}\to\mathsf{M}$ be a C^ω-vector bundle. Then the mapping*

$$\Gamma^\omega(\mathsf{TM}) \ni X \mapsto X^{\mathrm{h}} \in \Gamma^\omega(\mathsf{TE})$$

is an homeomorphism onto its image.

Proof: This follows in the same manner as Theorem 9.11, using Lemmata 5.13, 5.14, and 6.3. ■

Now we consider vertical lifts of sections of the dual bundle.

9.15 Theorem: (Vertical lift of one-forms is an homeomorphism onto its image) *Let $\pi_{\mathsf{E}}\colon \mathsf{E}\to\mathsf{M}$ be a C^ω-vector bundle. Then the mapping*

$$\Gamma^\omega(\mathsf{E}^*) \ni \lambda \mapsto \lambda^{\mathrm{v}} \in \Gamma^\omega(\mathsf{T}^*\mathsf{E})$$

is an homeomorphism onto its image.

Proof: This follows in the same manner as Theorem 9.11, using Lemmata 5.18, 5.19, and 6.4. ■

Next we consider vertical evaluations of sections of the dual bundle.

9.16 Theorem: (Vertical evaluations of one-forms is an homeomorphism onto its image) *Let $\pi_{\mathsf{E}}\colon \mathsf{E}\to\mathsf{M}$ be a C^ω-vector bundle. Then the mapping*

$$\Gamma^\omega(\mathsf{E}^*) \ni \lambda \mapsto \lambda^{\mathrm{e}} \in \mathrm{C}^\omega(\mathsf{E})$$

is an homeomorphism onto its image.

Proof: Since the given map is clearly injective, we focus on its topological properties.

We let $\mathbb{G}_{\mathsf{M}}$ be a C^ω-Riemannian metric on M, $\mathbb{G}_\pi$ be a C^ω-vector bundle connection in E, ∇^{M} be the Levi-Civita connection for $\mathbb{G}_{\mathsf{M}}$, and ∇^π be a C^ω-vector bundle connection in E. Corresponding to this, we have a Riemannian metric $\mathbb{G}_{\mathsf{E}}$ on E with its Levi-Civita connection ∇^{E}, as in Section 4.1. We denote the associated seminorms for $\Gamma^\omega(\mathsf{E}^*)$ and $\mathrm{C}^\omega(\mathsf{E})$ by $p^\omega_{\mathcal{K},\boldsymbol{a}}$ and $q^\omega_{\mathcal{L},\boldsymbol{a}}$ for $\mathcal{K}\subseteq\mathsf{M}$ and $\mathcal{L}\subseteq\mathsf{E}$ compact, and for $\boldsymbol{a}\in \mathrm{c}_0(\mathbb{Z}_{\ge0};\mathbb{R}_{>0})$.

Let us make some preliminary computations.

By Lemma 5.28, we have

$$\begin{aligned}
\lambda^{\mathrm{e}}(e) &= \lambda^{\mathrm{e}}(e),\\
\nabla^{\mathsf{E}}\lambda^{\mathrm{e}}(e) &= \widehat{A}^1_1((\nabla^{\pi_{\mathsf{E}}}\lambda)^{\mathrm{e}}(e)) + \widehat{A}^1_0(\lambda^{\mathrm{e}}(e)) + \widehat{C}^1_0(\lambda^{\mathrm{v}}(e)),\\
&\ \ \vdots\\
(\mathrm{Sym}_m\otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}})\circ\nabla^{\mathsf{E},m}\lambda^{\mathrm{e}}(e) &= \sum_{s=0}^{m} \widehat{A}^m_s((\mathrm{Sym}_s\otimes\mathrm{id}_{\mathsf{T}^*\mathsf{E}})\circ(\nabla^{\mathsf{M},\pi_{\mathsf{E}},s}\lambda)^{\mathrm{e}}(e))\\
&\quad + \sum_{s=0}^{m-1} \widehat{C}^m_s((\mathrm{Sym}_s\otimes\mathrm{id}_{\mathsf{T}^*\mathsf{E}})\circ(\nabla^{\mathsf{M},\pi_{\mathsf{E}},s}\lambda)^{\mathrm{v}}(e)).
\end{aligned} \tag{9.6}$$

Just as in the proof of Theorem 9.11, by Lemmata 7.3 and 7.5, and the appropriate analogue of equation (5.6) that would appear in a fully fleshed out proof of Lemma 5.26, we have bounds

$$\|\widehat{A}^m_s(\mathrm{Sym}_s(\beta_s))\|_{\mathbb{G}_{\mathsf{E}}} = \|\mathrm{Sym}_m \circ A^m_s(\beta_s)\|_{\mathbb{G}_{\mathsf{E}}} \leq \|A^m_s\|_{\mathbb{G}_{\mathsf{E}}}\|\beta_s\|_{\mathbb{G}_{\mathsf{E}}}$$

and

$$\|\widehat{C}^m_s(\mathrm{Sym}_s(\gamma_s))\|_{\mathbb{G}_{\mathsf{E}}} = \|\mathrm{Sym}_m \circ C^m_s(\gamma_s)\|_{\mathbb{G}_{\mathsf{E}}} \leq \|C^m_s\|_{\mathbb{G}_{\mathsf{E}}}\|\gamma_s\|_{\mathbb{G}_{\mathsf{E}}}.$$

Let $\mathcal{L} \subseteq \mathsf{E}$ be compact. By Lemmata 7.8 and 7.9 with $r = 0$, there exist $C_1, \sigma_1, \rho_1 \in \mathbb{R}_{>0}$ such that

$$\|A^k_s(e)\|_{\mathbb{G}_{\mathsf{E}}} \leq C_1\sigma_1^{-k}\rho_1^{-(k-s)}(k-s)!, \qquad k \in \mathbb{Z}_{\geq 0},\ s \in \{0,1,\dots,k\},\ e \in \mathcal{L},$$

and

$$\|C^k_s(e)\|_{\mathbb{G}_{\mathsf{E}}} \leq C_1\sigma_1^{-k}\rho_1^{-(k-s)}(k-s)!, \qquad k \in \mathbb{Z}_{\geq 0},\ s \in \{0,1,\dots,k-1\},\ e \in \mathcal{L},$$

Without loss of generality, we assume that $C_1 \geq 1$ and $\sigma_1, \rho_1 \leq 1$. Thus, using Lemma 6.6 and abbreviating $\sigma_2 = \sigma_1\rho_1$, we have

$$\begin{aligned}\|\widehat{A}^k_s((\mathrm{Sym}_s \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}) \circ (\nabla^{\mathsf{M},\pi_{\mathsf{E}},s}\lambda)^{\mathrm{e}}(e))\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}}} \\ \leq C_1\sigma_2^{-k}(k-s)!\|(\mathrm{Sym}_s \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}) \circ (\nabla^{\mathsf{M},\pi_{\mathsf{E}},s}\lambda)^{\mathrm{e}}(e)\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}}}\end{aligned}$$

for $k \in \mathbb{Z}_{\geq 0}$, $s \in \{0,1,\dots,k\}$, $e \in \mathcal{L}$, and

$$\begin{aligned}\|\widehat{C}^k_s((\mathrm{Sym}_s \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}) \circ (\nabla^{\mathsf{M},\pi_{\mathsf{E}},s}\lambda)^{\mathrm{e}}(e))\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}}} \\ \leq C_1\sigma_2^{-k}(k-s)!\|(\mathrm{Sym}_s \otimes \mathrm{id}_{\mathsf{T}^*\mathsf{E}}) \circ (\nabla^{\mathsf{M},\pi_{\mathsf{E}},s}\lambda)^{\mathrm{e}}(e)\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}}}\end{aligned}$$

for $k \in \mathbb{Z}_{\geq 0}$, $s \in \{0,1,\dots,k-1\}$, $e \in \mathcal{L}$. Thus, by (1.3) and (9.6),

$$\begin{aligned}\|j_m\lambda^{\mathrm{e}}(e)\|_{\mathbb{G}_{\mathsf{E},m}} &\leq \sum_{k=0}^m \frac{1}{k!}\|\mathrm{Sym}_k \circ \nabla^{\mathsf{E},k}\lambda^{\mathrm{e}}(e)\|_{\mathbb{G}_{\mathsf{E}}} \\ &\leq \sum_{k=0}^m \frac{1}{k!}\left\|\sum_{s=0}^k \widehat{A}^k_s((\mathrm{Sym}_s \circ \mathrm{id}_{\mathsf{T}^*\mathsf{E}}) \circ (\nabla^{\mathsf{M},\pi_{\mathsf{E}},s}\lambda)^{\mathrm{e}}(e))\right\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}}} \\ &\quad + \sum_{k=0}^{m-1} \frac{1}{k!}\left\|\sum_{s=0}^k \widehat{C}^k_s((\mathrm{Sym}_s \circ \mathrm{id}_{\mathsf{T}^*\mathsf{E}}) \circ (\nabla^{\mathsf{M},\pi_{\mathsf{E}},s}\lambda)^{\mathrm{v}}(e))\right\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}}} \\ &\leq \sum_{k=0}^m C_1\sigma_2^{-k}\frac{s!(k-s)!}{k!}\frac{1}{s!}\left\|\sum_{s=0}^k (\mathrm{Sym}_s \circ \mathrm{id}_{\mathsf{T}^*\mathsf{E}}) \circ (\nabla^{\mathsf{M},\pi_{\mathsf{E}},s}\lambda)^{\mathrm{e}}(e)\right\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}}} \\ &\quad + \sum_{k=0}^{m-1} C_1\sigma_2^{-k}\frac{s!(k-s)!}{k!}\frac{1}{s!}\left\|\sum_{s=0}^k (\mathrm{Sym}_s \circ \mathrm{id}_{\mathsf{T}^*\mathsf{E}}) \circ (\nabla^{\mathsf{M},\pi_{\mathsf{E}},s}\lambda)^{\mathrm{v}}(e)\right\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}}}\end{aligned}$$

for $e \in \mathcal{L}$ and $m \in \mathbb{Z}_{\geq 0}$. Now note that

$$\frac{s!(k-s)!}{k!} \leq 1, \quad C_1\sigma_2^{-k} \leq C_1\sigma_2^{-m},$$

for $s \in \{0, 1, \dots, m-1\}$, $k \in \{0, 1, \dots, s\}$, since $\sigma_2 \leq 1$. Then

$$\begin{aligned}\|j_m\lambda^{\mathrm{e}}(e)\|_{\mathbb{G}_{\mathsf{E},m}} &\leq C_1\sigma_2^{-m}\sum_{k=0}^{m}\sum_{s=0}^{m}\frac{1}{s!}\left\|(\mathrm{Sym}_s\circ\mathrm{id}_{\mathsf{T}^*\mathsf{E}})\circ(\nabla^{\mathsf{M},\pi_{\mathsf{E}},s}\lambda)^{\mathrm{e}}(e)\right\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}}} \\ &\quad + C_1\sigma_2^{-m}\sum_{k=0}^{m-1}\sum_{s=0}^{m-1}\frac{1}{s!}\left\|(\mathrm{Sym}_s\circ\mathrm{id}_{\mathsf{T}^*\mathsf{E}})\circ(\nabla^{\mathsf{M},\pi_{\mathsf{E}},s}\lambda)^{\mathrm{v}}(e)\right\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}}} \\ &= (m+1)C_1\sigma_2^{-m}\sum_{s=0}^{m}\frac{1}{s!}\left\|(\mathrm{Sym}_s\circ\mathrm{id}_{\mathsf{T}^*\mathsf{E}})\circ(\nabla^{\mathsf{M},\pi_{\mathsf{E}},s}\lambda)^{\mathrm{e}}(e)\right\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}}} \\ &\quad + (m+1)C_1\sigma_2^{-m}\sum_{s=0}^{m-1}\frac{1}{s!}\left\|(\mathrm{Sym}_s\circ\mathrm{id}_{\mathsf{T}^*\mathsf{E}})\circ(\nabla^{\mathsf{M},\pi_{\mathsf{E}},s}\lambda)^{\mathrm{v}}(e)\right\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}}}.\end{aligned}$$

Now let $\sigma < \sigma_2$ and note that

$$\lim_{m\to\infty}(m+1)\frac{\sigma_2^{-m}}{\sigma^{-m}} = 0.$$

Thus there exists $N \in \mathbb{Z}_{>0}$ such that

$$(m+1)C_1\sigma_2^{-m} \leq C_1\sigma^{-m}, \qquad m \geq N.$$

Let

$$C = \max\left\{C_1, 2C_1\frac{\sigma}{\sigma_2}, 3C_1\left(\frac{\sigma}{\sigma_2}\right)^2, \dots, (N+1)C_1\left(\frac{\sigma}{\sigma_2}\right)^N\right\}.$$

We then immediately have $(m+1)C_1\sigma_2^{-m} \leq C\sigma^{-m}$ for all $m \in \mathbb{Z}_{\geq 0}$. We then have, using (1.3),

$$\begin{aligned}\|j_m\lambda^{\mathrm{e}}(e)\|_{\mathbb{G}_{\mathsf{E},m}} &\leq C\sigma^{-m}\left(\sum_{s=0}^{m}\frac{1}{s!}\left\|(\mathrm{Sym}_s\circ\mathrm{id}_{\mathsf{T}^*\mathsf{E}})\circ(\nabla^{\mathsf{M},\pi_{\mathsf{E}},s}\lambda)^{\mathrm{e}}(e)\right\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}}}\right. \\ &\qquad \left. + \sum_{s=0}^{m-1}\frac{1}{s!}\left\|(\mathrm{Sym}_s\circ\mathrm{id}_{\mathsf{T}^*\mathsf{E}})\circ(\nabla^{\mathsf{M},\pi_{\mathsf{E}},s}\lambda)^{\mathrm{v}}(e)\right\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}}}}\right) \\ &\leq C\sqrt{m+1}\,\sigma^{-m}(\|j_m\lambda(\pi_{\mathsf{E}}(e))(e)\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}},m}} + \|j_{m-1}\lambda(\pi_{\mathsf{E}}(e))\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}},m-1}}).\end{aligned}$$

By modifying C and σ just as we did in the preceding, we get

$$\|j_m\lambda^{\mathrm{e}}(e)\|_{\mathbb{G}_{\mathsf{E},m}} \leq C\sigma^{-m}(\|j_m\lambda(\pi_{\mathsf{E}}(e))(e)\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}},m}} + \|j_{m-1}\lambda(\pi_{\mathsf{E}}(e))\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}},m-1}}).$$

We take

$$\alpha = \max\{1, \sup\{\|e\|_{\mathbb{G}_{\pi_{\mathsf{E}}}} \mid e \in \mathcal{L}\}$$

and then use Lemma 7.3 to arrive at

$$\|j_m\lambda^{\mathrm{e}}(e)\|_{\mathbb{G}_{\mathsf{E},m}} \leq 2\alpha C\sigma^{-m}\|j_m\lambda(\pi_{\mathsf{E}}(e))(e)\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}},m}}$$

Now, given $\boldsymbol{a} \in \mathrm{c}_0(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0})$, we define $\boldsymbol{a}' \in \mathrm{c}_0(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0})$ by $a'_0 = 2\alpha C a_0$ and $a'_j = a_j\sigma^{-1}$, $j \in \mathbb{Z}_{>0}$, we then have

$$q^{\omega}_{\mathcal{L},\boldsymbol{a}}(\lambda^{\mathrm{e}}) \leq p^{\omega}_{\pi_{\mathsf{E}}(\mathcal{L}),\boldsymbol{a}'}(\lambda),$$

and this gives this part of the result.

Now we turn to showing that the mapping of the lemma is open onto its image. By Lemma 5.29, we have

$$\begin{aligned}\lambda^{\mathrm{e}}(e) &= \lambda^{\mathrm{e}}(e),\\ (\nabla^{\pi_{\mathsf{E}}}\lambda)^{\mathrm{e}}(e) &= \widehat{B}^1_1(\nabla^{\mathsf{E}}\lambda^{\mathrm{e}}(e)) + \widehat{B}^1_0(\lambda^{\mathrm{e}}(e)) + \widehat{D}^1_0(\lambda^{\mathrm{v}}(e)),\\ (\mathrm{Sym}_2\otimes\mathrm{id}_{\mathsf{TE}})\circ(\nabla^{\mathsf{M},\pi_{\mathsf{E}},2}\lambda)^{\mathrm{e}}(e) &= \widehat{B}^2_2(\nabla^{\mathsf{E},2}\lambda^{\mathrm{e}}(e)) + \widehat{B}^2_1(\nabla^{\mathsf{E}}\lambda^{\mathrm{e}}(e)) + \widehat{B}^2_0(\lambda^{\mathrm{e}}(e))\\ &\quad + \widehat{D}^2_1((\nabla^{\mathsf{M},\pi_{\mathsf{E}}}\lambda)^{\mathrm{v}}(e)) + \widehat{D}^1_0(\lambda^{\mathrm{v}}(e)),\\ &\vdots\\ (\mathrm{Sym}_m\otimes\mathrm{id}_{\mathsf{T}^*\mathsf{E}})\circ(\nabla^{\mathsf{M},\pi_{\mathsf{E}},m}\lambda)^{\mathrm{e}}(e) &= \sum_{s=0}^m \widehat{B}^m_s((\mathrm{Sym}_s\otimes\mathrm{id}_{\mathsf{T}^*\mathsf{E}})\circ\nabla^{\mathsf{E},s}\lambda^{\mathrm{e}}(e))\\ &\quad + \sum_{s=0}^{m-1}\widehat{D}^m_s((\mathrm{Sym}_s\otimes\mathrm{id}_{\mathsf{T}^*\mathsf{E}})\circ\nabla^{\mathsf{E},s}\lambda^{\mathrm{v}}(e)).\end{aligned}\tag{9.7}$$

Just as in the proof of Theorem 9.11, by Lemmata 7.3 and 7.5, and the appropriate analogue of equation (5.6) that would appear in a fully fleshed out proof of Lemma 5.27, we have bounds

$$\begin{aligned}\|\widehat{B}^m_s(\mathrm{Sym}_s(\beta_s))\|_{\mathbb{G}_{\mathsf{E}}} &= \|\mathrm{Sym}_m\circ B^m_s(\beta_s)\|_{\mathbb{G}_{\mathsf{E}}} \le \|B^m_s\|_{\mathbb{G}_{\mathsf{E}}}\|\beta_s\|_{\mathbb{G}_{\mathsf{E}}},\\ \|\widehat{D}^m_s(\mathrm{Sym}_s(\gamma_s))\|_{\mathbb{G}_{\mathsf{E}}} &= \|\mathrm{Sym}_m\circ D^m_s(\gamma_s)\|_{\mathbb{G}_{\mathsf{E}}} \le \|D^m_s\|_{\mathbb{G}_{\mathsf{E}}}\|\gamma_s\|_{\mathbb{G}_{\mathsf{E}}}.\end{aligned}$$

Proceeding analogously to the continuity proof above and using Lemma 6.6, we deduce that there exist $C_1,\sigma_1\in\mathbb{R}_{>0}$ such that

$$\|j_m\lambda(\pi_{\mathsf{E}}(e))(e)\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}},m}} \le C_1\sigma_1^{-m}(\|j_m\lambda^{\mathrm{e}}(e)\|_{\mathbb{G}_{\mathsf{E},m}} + \|j_{m-1}\lambda^{\mathrm{v}}(e)\|_{\mathbb{G}_{\mathsf{E},m-1}}),\qquad e\in\mathcal{L}.\tag{9.8}$$

Now let $\mathcal{K}\subseteq\mathsf{M}$ be compact and let $\boldsymbol{a}\in\mathrm{c}_0(\mathbb{Z}_{\ge0},\mathbb{R}_{>0})$. Define

$$\mathcal{L} = \pi_{\mathsf{E}}^{-1}(\mathcal{K})\cap\{e\in\mathsf{E}\mid\ \|e\|_{\mathbb{G}_{\pi_{\mathsf{E}}}} = 1\},$$

noting that $\mathcal{L}$ is compact. Let $n=\dim(\mathsf{M})$ and let k be the fibre dimension of E. By Lemma 7.4, and equations (1.2) and (7.1), we have

$$\begin{aligned}\|j_m\lambda(x)\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}},m}} &\le \sum_{j=0}^m\sqrt{k\binom{n+j-1}{j}}\sup\{\|j_m\lambda(\pi_{\mathsf{E}}(e))(e)\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}},m}}\mid\ e\in\mathcal{L}\}\\ &\le \sum_{j=0}^m k\binom{n+j-1}{j}\sup\{\|j_m\lambda(\pi_{\mathsf{E}}(e))(e)\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}},m}}\mid\ e\in\mathcal{L}\}\\ &\le m^2 2^{n+m}\sup\{\|j_m\lambda(\pi_{\mathsf{E}}(e))(e)\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}},m}}\mid\ e\in\mathcal{L}\}\end{aligned}$$

for $x\in\mathcal{K}$. For $\sigma_2<\frac{1}{2}$,

$$\lim_{m\to\infty} m^2\frac{2^m}{\sigma_2^{-m}} = 0.$$

By by now familiar arguments, one of which the reader can find in the first part of the proof, we can combine this with (9.8) to arrive at $C, \sigma \in \mathbb{R}_{>0}$ for which

$$\|j_m\lambda(x)\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}},m}} \leq C\sigma^{-m}(\sup\{\|j_m\lambda^{\mathrm{e}}(e)\|_{\mathbb{G}_{\mathsf{M},\pi_{\mathsf{E}},m}} \mid e \in \mathcal{L}\} + \sup\{\|j_m\lambda^{\mathrm{v}}(e)\|_{\mathbb{G}_{\mathsf{E},m}} \mid e \in \mathcal{L}\})$$

for $x \in \mathcal{K}$. Taking $\boldsymbol{a}' \in \mathrm{c}_0(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0})$ to be defined by $a'_0 = Ca_0$, $a'_j = \sigma^{-1}a_j$, $j \in \mathbb{Z}_{>0}$, we have

$$q^{\omega}_{\mathcal{K},\boldsymbol{a}}(\lambda) \leq p^{\omega}_{\mathcal{L},\boldsymbol{a}'}(\lambda^{\mathrm{e}}) + p^{\omega}_{\mathcal{L},\boldsymbol{a}'}(\lambda^{\mathrm{v}}).$$

By Lemma 1 from the proof of Theorem 9.1, this shows that the mapping

$$\Gamma^{\omega}(\mathsf{E}^*) \ni \lambda \mapsto (\lambda^{\mathrm{e}}, \lambda^{\mathrm{v}}) \in \mathrm{C}^{\omega}(\mathsf{E}) \oplus \Gamma^{\omega}(\mathsf{TE})$$

is open onto its image. This part of the lemma now follows from the following simple fact.

1 Lemma: *Let $\mathcal{S}$, $\mathcal{T}_1$, and $\mathcal{T}_2$ be topological spaces and let $\Phi\colon \mathcal{S} \to \mathcal{T}_1 \times \mathcal{T}_2$ be an open mapping onto its image. Then the mappings $\mathrm{pr}_1 \circ \Phi$ and $\mathrm{pr}_2 \circ \Phi$ are open onto their images.*

Proof: Let $\mathcal{O} \subseteq \mathcal{S}$ be open so that $\Phi(\mathcal{O})$ is open in image(Φ). Then, for each $(y_1, y_2) \in \mathcal{O}$, there exists a neighbourhood $\mathcal{N}_1 \subseteq \mathrm{image}(\mathrm{pr}_1 \circ \Phi)$ of y_1 and a neighbourhood $\mathcal{N}_2 \subseteq \mathrm{image}(\mathrm{pr}_2 \circ \Phi)$ of x_2 such that $\mathcal{N}_1 \times \mathcal{N}_2 \subseteq \Phi(\mathcal{O})$. This immediately gives the lemma. ▼

Thus we arrive at the conclusion that the mapping

$$\Gamma^{\omega}(\mathsf{E}^*) \ni \lambda \mapsto \lambda^{\mathrm{e}} \in \mathrm{C}^{\omega}(\mathsf{E})$$

is open onto its image, as desired. ■

Finally, we consider vertical evaluations of sections of the endomorphism bundle.

9.17 Theorem: (Vertical evaluation of endomorphisms is an homeomorphism onto its image) *Let $\pi_{\mathsf{E}}\colon \mathsf{E} \to \mathsf{M}$ be a C^{ω}-vector bundle. Then the mapping*

$$\Gamma^{\omega}(\mathrm{End}(\mathsf{E})) \ni L \mapsto L^{\mathrm{e}} \in \Gamma^{\omega}(\mathsf{TE})$$

is an homeomorphism onto its image.

Proof: This follows in the same manner as Theorem 9.16, using Lemmata 5.33, 5.34, and 6.7. ■

As an illustration of how continuity of these lifts can be helpful, let us consider the continuity of the map that assigns to a vector field on a manifold the tangent lift of that vector field. Precisely, let M be a real analytic manifold and let $X \in \Gamma^{\omega}(\mathsf{TM})$ be a real analytic vector field. The ***tangent lift*** of X is the vector field $X^T \in \Gamma^{\omega}(\mathsf{TTM})$ on TM whose flow is the derivative of the flow for X:

$$\Phi^{X^T}_t(v_x) = T_x\Phi^X_t(v_x) \quad \Longrightarrow \quad X^T = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} T_x\Phi^X_t(v_x). \tag{9.9}$$

Let us give a formula for the tangent lift that reduces the continuity of the mapping $X \mapsto X^T$ to continuity of familiar operations.

9.18 Lemma: (Decomposition of the tangent lift via an affine connection) *Let $r \in \{\infty, \omega\}$ and let M be a C^r-manifold with a C^r-affine connection ∇^{M}. Then*

$$X^T(v_x) = \mathrm{hlft}(v_x, X(x)) + \mathrm{vlft}(v_x, \nabla^{\mathsf{M}}_{v_x} X + T^{\mathsf{M}}(X(x), v_x)),$$

where T^{M} is the torsion of ∇^{M}.

Proof: Let $v_x \in \mathsf{TM}$ and let $Y \in \Gamma^r(\mathsf{TM})$ be such that $Y(x) = v_x$. Note that

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \Phi^X_t \circ \Phi^Y_s(x) = T_x\Phi^X_t(Y(x)).$$

Also compute

$$\begin{aligned}\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \Phi^X_t \circ \Phi^Y_s &= \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \Phi^Y_s \circ \Phi^X_t \circ \Phi^X_{-t}\Phi^Y_{-s} \circ \Phi^X_t \circ \Phi^Y_s(x) \\ &= Y(\Phi^X_t(x)) + T_x\Phi^X_t \left(\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \Phi^X_{-t} \circ \Phi^Y_{-s} \circ \Phi^X_t \circ \Phi^Y_s(\Phi^X_t(x)) \right).\end{aligned}$$

Note that, for $f \in \mathrm{C}^r(\mathsf{M})$,

$$f \circ \Phi^X_{-t} \circ \Phi^Y_{-s} \circ \Phi^X_t \circ \Phi^Y_s(x) = f(x) + st\mathscr{L}_{[Y,X]}f(x) + o(|st|),$$

by [Abraham, Marsden, and Ratiu 1988, Proposition 4.2.34]. Therefore,

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \Phi^X_{-t} \circ \Phi^Y_{-s} \circ \Phi^X_t \circ \Phi^Y_s(\Phi^X_t(x)) = t[Y,X](\Phi^X_t(x)).$$

Putting the above calculations together gives

$$T_x\Phi^X_t(Y(x)) = Y(\Phi^X_t(x)) - t[X,Y](\Phi^X_t(x)).$$

Thus, making use of (9.9),

$$\Phi^{X^{\mathrm{h}}}_t \circ \Phi^{X^T}_t(Y(x)) = \tau^{(t,0)}_{\gamma_-}(Y(\Phi^X_t(x)) - t[X,Y](\Phi^X_t(x))),$$

where γ_- is the integral curve of $-X$ through $\Phi^X_t(x)$ and τ_{γ_-} is parallel translation along γ_-. If γ is the integral curve of X through x note that $\tau^{(t,0)}_{\gamma_-} = \tau^{(0,t)}_{\gamma}$. Now we compute

$$\begin{aligned}\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Phi^{-X^{\mathrm{h}}}_t \circ \Phi^{X^T}_t(Y(x)) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \tau^{(0,t)}_{\gamma}(Y(\Phi^X_t(x)) - t[X,Y](\Phi^X_t(x))) \\ &= \nabla_X Y(x) - [X,Y](x) = \nabla_Y X(x) + T(X(x),Y(x)).\end{aligned}$$

Note that, since X^T and X^{h} are both vector fields over X, it follows that

$$t \mapsto \tau^{(0,t)}_{\gamma}(Y(\Phi^X_t(x)))$$

is a curve in $\mathsf{T}_x\mathsf{M}$. Thus the derivative of this curve at $t = 0$ is in $\mathsf{V}_{Y(x)}\mathsf{TM}$. Thus we have shown that

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Phi^{-X^{\mathrm{h}}}_t \circ \Phi^{X^T}_t(v_x) = \mathrm{vlft}(v_x, \nabla_{v_x} X(x) + T(X(x), v_x)).$$

Finally, for $f \in \mathrm{C}^r(\mathsf{M})$, by the BCH formula, we have

$$f \circ \Phi^{-X^{\mathrm{h}}}_t \circ \Phi^{X^T}_t(v_x) = f \circ \Phi^{X^T - X^{\mathrm{h}}}_t + o(|t|^2).$$

Differentiating with respect to t and evaluating at $t = 0$ gives the result. ■

Now we can combine Theorems 9.1(i), 9.13, and 9.14, and Corollary 9.8 to give the following result.

9.19 Corollary: (Continuity of tangent lift) *If* M *is a* C^ω*-manifold, then the mapping*

$$\Gamma^\omega(\mathsf{TM}) \ni X \mapsto X^T \in \Gamma^\omega(\mathsf{TTM})$$

is continuous.

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