Rigid body mechanics in Galilean spacetimes*

Ajit Bhand†

2002/09/10

"There is no subject so old that something new cannot be said about it."

–Fyodor Mikhailovich Dostoyevsky.

Abstract

An observer–independent formulation of rigid body dynamics is provided in the general setting of an abstract spacetime. In particular, definitions of (body and spatial) angular and linear velocities and momenta are provided independent of an observer. Rigid motions are defined and their properties are studied in detail. A rigid body is defined in this general setting and its various properties are explored. The equations governing the motion of a rigid body undergoing a rigid motion in a Galilean spacetime are derived using the conservation of spatial linear and spatial angular momenta. The canonical Galilean group is shown to decompose into rotations, space translations, velocity boosts and time translations and its properties are discussed. The abstract Galilean group is studied in detail and is found to be isomorphic to the canonical Galilean group in the presence of an observer. Various subgroups of the abstract Galilean group are characterized. Total velocities corresponding to a rigid motion are defined. The formulation of rigid body dynamics is then studied in the presence of an observer. It is shown that in this case, the velocities and momenta defined previously project to the well known quantities. Finally, it is seen that the equations of motion derived earlier describe the usual motion given by the solutions of the Euler equations for a rigid body undergoing rigid motion.

Contents

*Report for project in fulfilment of requirements for MSc

†Assistant Professor, Mathematics Department, Indian Institute of Science Education and Research, Bhopal, India

Email: abhand@iiserbhopal.ac.in, URL: http://home.iiserbhopal.ac.in/~abhand/

Work performed while a graduate student at Queen's University.

Research supported in part by a grant from the Natural Sciences and Engineering Research Council of Canada.

1. Introduction

The main aim of this project is to understand the dynamics of a rigid body in the general framework of an abstract Galilean spacetime. Underlying the spacetime is an affine space which is thought of as a "translation" of a vector space. We wish to study how the motion of a rigid body is related to rigid motions– the set of mappings belonging to the group of Galilean transformations from an affine space to itself, called the Galilean group. Our

objective is also to formulate rigid body dynamics in an observer–independent manner. We note that the angular and linear momenta associated with a rigid body undergoing motion in a Galilean spacetime are thought of as observer dependent quantities in the literature (c.f. [Artz 1981]). In this project we take the view that momentum is an inherent property of a rigid body in motion and that it is possible to define it without using any external structure. The observer merely affects the way the momentum is *measured*.

The role of Galilean structure of the Newtonian spacetime has been understood in the case of the dynamics of a particle [Arnol′d 1978, Artz 1981] and of course, the dynamics of a rigid body in a fixed Galilean frame has been investigated quite throughly [Arnol′d 1978]. However, to our knowledge, the role played by the Galilean structure has not been explained for rigid body mechanics. In particular, it is not entirely clear what quantities in the description of the dynamics of a rigid body are dependent on the choice of a Galilean observer, and in what manner the dependence arises.

An observer in a Galilean spacetime, apart from providing a reference frame for observing Newton's laws, also plays its role by providing an isomorphism from the abstract Galilean group to the "standard" or canonical Galilean group which consists of rotations, spatial translations, uniform velocity boosts and time translations. This isomorphism is the key to understanding the structure of the Galilean group.

Our treatment is more general also because we allow arbitrary velocity boosts in our analysis. This facilitates a better understanding of the "generalized" Euler equations for a rigid body. In the presence of an observer, the solutions to these equations are found to agree with the motion of a rigid body in the classical setup.

This report is organized as follows. In Chapter 2, we present the mathematical preliminaries relevant to our investigation. Several important terms like affine spaces and subspaces, observers, Galilean spacetimes and the Galilean group are introduced and their various properties are described. With this background, we first look at rigid motions in Chapter 3. Various quantities associated with a rigid motion, such as the body and spatial linear and angular velocities are defined. Throughout this chapter, the treatment is observer–independent. The notion of a rigid body along with its attendant features is introduced. In particular, the inertia tensor of a rigid body is defined and and its properties are thoroughly explained. The discussion then focuses on angular and spatial momenta and finally the equations of motion for a rigid body are derived. In Chapter 4, the structure of the canonical as well as the abstract Galilean group is investigated in detail. It is shown that an observer induces an isomorphism between the Galilean group to the canonical group. Various subgroups of the abstract Galilean group are characterized Next, "total velocities" are defined and the Lie algebra of the canonical Galilean group is also described. In Chapter 5, the formulation presented in chapter 3 is studied in the *presence* of an observer. It is shown that in such a case, we recover the familiar quantities associated with the classical treatment of rigid body mechanics. The Euler equations are also studied and, as mentioned earlier, are found to have solutions consistent with the motion of a rigid body in the classical framework. Finally, in Chapter 6, we summarize our results and discuss some directions for further investigations.

2. Preliminaries

In this chapter, we present the mathematical background and introduce the notation to be used in the following chapters. In section 2.1, we define affine spaces and subspaces– the principle objects of interest in this project. In Section 2.2, we introduce the idea of a Galilean spacetime and describe the affine spaces naturally associated with it. We also introduce the set of Galilean velocities. Next, we define observers in a Galilean spacetime and discuss their properties. Finally, in Section 2.4, we define the Galilean group and introduce the fundamental maps associated with a Galilean mapping.

2.1. Affine spaces. In this section, we define affine spaces and subspaces and record some of their properties. We refer to [Berger 1987] for more details. We start by defining group actions.

2.1 Notation: Given a set X, we represent by $\mathrm{inv}(X)$, the set of invertible maps from X to itself. Given vector spaces V and U, we denote by $L(V;U)$, the set of linear maps from V to U.

2.2 Definition: Let G be a group and X a set. A G–action on X is a homomorphism $\Theta : G \to \mathrm{inv}(X)$.

We shall identify the map Θ with the associated map from $G \times X \to X$.

Given $x \in X$, the *orbit* of x under the group action is given by

$$\mathrm{orb}_G(x) = \{\Theta(g,x) \mid \ g \in G\}.$$

The *isotropy group* of x is given by

$$\mathrm{iso}_G(x) = \{g \in G \mid \ \Theta(g,x) = x\}$$

2.3 Definition: Let Θ be a G–action on a set X.

1. Θ is *transitive* if there is only one orbit.
2. Θ is *faithful* if $\Theta(g,x) = x$ for every $x \in X$ implies $g = e$
3. Θ is *free* if $\Theta(g,x) = x$ for any $x \in X$ implies $g = e$. In this case, $\mathrm{iso}_G(x) = e$
4. Θ is *simply–transitive* if it is free and transitive.

Next we introduce the concept of an affine space. It may be thought of as a vector space without an origin. Thus, it makes sense only to consider the "difference" of two elements of an affine space as being a vector. The elements themselves are not to be regarded as vectors.

2.4 Definition: Let $\mathscr{A}$ be a set and V a real vector space. An *affine space* modeled on V is a triple $(\mathscr{A}, V, \Theta)$ where Θ is a faithful, transitive V–action on $\mathscr{A}$.

We shall now cease to use the map Θ and instead use the more suggestive notation $\Theta(v,x) = v + x$. By definition, if $x, y \in \mathscr{A}$, there exists $v \in V$ such that $y = x + v$.

2.5 Notation: In this case we shall denote $v = y - x$.

2.6 Remark: The minus sign here is simply notation; we have not really defined "subtraction" in $\mathscr{A}$. The idea is that to any two points in $\mathscr{A}$, we may assign a unique vector in V and we notationally write this as the difference between the two elements.

All this leads to the following result which is easily proved once all the symbols are understood. By a slight abuse of notation, we shall denote an affine space $(\mathscr{A}, V, \Theta)$ simply by $\mathscr{A}$.

2.7 Proposition: *Let $\mathscr{A}$ be a $\mathbb{R}$- affine space modeled on V. For fixed $x \in \mathscr{A}$ define vector addition on $\mathscr{A}$ by*

$$y_1 + y_2 = ((y_1 - x) + (y_2 - x)) + x, \qquad \forall y_1, y_2 \in \mathscr{A}$$

and scalar multiplication on $\mathscr{A}$ by

$$ay = (a(y - x)) + x, \qquad \forall y \in \mathscr{A}.$$

These operations make $\mathscr{A}$ a $\mathbb{R}$–vector space and $y \mapsto y - x$ is an isomorphism of this $\mathbb{R}$–vector space with V.

2.8 Notation: We shall denote the vector space introduced in the above result by $\mathscr{A}_x$.

Let $\mathscr{A}$ and $\mathscr{B}$ be affine spaces modeled on vector spaces V and U respectively and let $f : \mathscr{A} \to \mathscr{B}$ be a map. Given $x_0 \in \mathscr{A}$, let $\mathscr{A}_{x_0}$ and $\mathscr{B}_{f(x_0)}$ be the corresponding vector spaces defined in Proposition 2.7. $\mathscr{A}_{x_0}$ is isomorphic to V with the isomorphism $x \mapsto x - x_0$ and $\mathscr{B}_{f(x_0)}$ is isomorphic to U with the isomorphisms $y \mapsto y - f(x_0)$. We denote these isomorphism by $\Phi_{x_0} : \mathscr{A}_{x_0} \to V$ and $\Phi_{f(x_0)} : \mathscr{B}_{f(x_0)} \to U$, respectively. The next proposition provides a definition of an affine map between two affine spaces.

2.9 Proposition: *The following conditions are equivalent:*

(i) $f \in L(\mathscr{A}_{x_0}; \mathscr{B}_{f(x_0)})$ *for some* $x_0 \in \mathscr{A}$*;*

(ii) $f \in L(\mathscr{A}_{x_0}; \mathscr{B}_{f(x_0)})$ *for all* $x_0 \in \mathscr{A}$*;*

(iii) $\Phi_{f(x_0)} \circ f \circ \Phi_{x_0}^{-1} \in L(V; U)$ *for some* $x_0 \in \mathscr{A}$*;*

(iv) $\Phi_{f(x_0)} \circ f \circ \Phi_{x_0}^{-1} \in L(V; U)$ *for all* $x_0 \in \mathscr{A}$

Moreover, if these conditions are satisfied, $\Phi_{f(x_0)} \circ f \circ \Phi_{x_0}^{-1} \in L(V; U)$ depends only on f, and will be denoted by f_V. A map satisfying the properties above is called an (affine) morphism, or an **affine map***, between V and U. We say that f is an (affine) isomorphism if f is a morphism and is bijective; f is an automorphism if f is an isomorphism from $\mathscr{A}$ to itself.*

Next, we define affine subspaces.

2.10 Definition: Given an affine space $(\mathscr{A}, V, \Theta)$ and U a subspace of V, a subset $\mathscr{B} \subset \mathscr{A}$ is called an *affine subspace* if the triple $(\mathscr{B}, U, \Theta|_U)$ is an affine space modeled on U.

The following result characterizes affine subspaces.

2.11 Proposition: *Let $\mathscr{A}$ be a $\mathbb{R}$–affine space modeled on V and let $\mathscr{B} \subset \mathscr{A}$. The following are equivalent:*

(i) $\mathscr{B}$ *is an affine subspace of* $\mathscr{A}$*;*

(ii) there exists a subspace U of V so that for some fixed $x \in \mathscr{B}$,

$$\mathscr{B} = \{u + x |\ u \in U\}$$

(iii) if $x \in \mathscr{B}$ then $\{y - x |\ y \in \mathscr{B}\} \subset V$ is a subspace.

2.2. Time and distance. We begin by giving the basic definition of a Galilean spacetime and by providing meaning to the intuitive notions of time and distance.

2.12 Definition: A *Galilean spacetime* is a quadruple $\mathscr{G} = (\mathscr{E}, V, g, \tau)$ where

GSp1. V is a 4-dimensional $\mathbb{R}$–vector space.

GSp2. $\tau : V \to \mathbb{R}$ is a surjective linear map called the *time map*.

GSp3. g is an inner product on $\ker(\tau)$, and

GSp4. $\mathscr{E}$ is an affine space modeled on V.

Points in $\mathscr{E}$ are called *events*–thus $\mathscr{E}$ is a model for the spatio-temporal world of Newtonian mechanics. With the time map, we may measure the time between two events $x_1, x_2 \in \mathscr{E}$ as $\tau(x_2 - x_1)$. Note, however, that it does not make sense to talk about the "time" of a particular event $x \in \mathscr{E}$. Events $x_1, x_2 \in \mathscr{E}$ are called *simultaneous* if $\tau(x_2 - x_1) = 0$; that is, if $x_2 - x_1 \in \ker(\tau)$).

Using the definition, we may define the *distance* between simultaneous events $x_1, x_2 \in \mathscr{E}$ to be $\sqrt{g(x_2 - x_1, x_2 - x_1)}$. Note that this method for defining distance does not allow us to measure distances between events that are *not* simultaneous. In particular, it doesn't make sense to talk about two non-simultaneous events as occurring in the same place.

Simultaneity is an equivalence relation on $\mathscr{E}$ and the quotient we denote by $I_{\mathscr{G}} = \mathscr{E}/\sim$, with $\sim$ denoting the relation of simultaneity. $I_{\mathscr{G}}$ is simply the collection of equivalence classes of simultaneous events. We call it the *set of instants*. We denote, by $\pi_{\mathscr{G}} : \mathscr{E} \to I_{\mathscr{G}}$ the canonical projection.

For $s \in I_{\mathscr{G}}$, we denote by $\mathscr{E}(s)$ the collection of events $x \in \mathscr{E}$ with the property that $\pi_{\mathscr{G}}(x) = s$. Thus events in $\mathscr{E}(s)$ are simultaneous. The following result shows that $\mathscr{E}(s)$ is an affine space modeled on $\ker(\tau)$.

2.13 Lemma: *For each $s \in I_{\mathscr{G}}$, $\mathscr{E}(s)$ is a 3-dimensional affine space modeled on* $\ker(\tau)$.

Proof: The affine action of $\ker(\tau)$ on $\mathscr{E}(s)$ is that obtained by restricting the affine action of V on $\mathscr{E}$. We must show that this restriction is well defined. That is, given $v \in \ker(\tau)$ we need to show that $v+x \in \mathscr{E}(s)$ for every $x \in \mathscr{E}(s)$. If $x \in \mathscr{E}(s)$ then $\tau((v+x)-x) = \tau(v) = 0$ which means that $v + x \in \mathscr{E}(s)$ as claimed. Also, for $x_1, x_2 \in \mathscr{E}(s)$, by definition, there exists $w \in \ker(\tau)$ such that $x_2 = w + x_1$. This shows that the action is transitive which therefore means that for a fixed $x \in \mathscr{E}(s)$, we have

$$\mathscr{E}(s) = \{u + x |\ u \in \ker(\tau)\}$$

It now follows from Proposition 2.11 that $\mathscr{E}(s)$ is an affine subspace of $\mathscr{E}$ modeled on $\ker(\tau)$. ■

Just as a single set of simultaneous events is an affine space, so too is the set of instants.

2.14 Lemma: *$I_{\mathscr{G}}$ is a 1-dimensional affine space modeled on $\mathbb{R}$.*

Proof: The affine action of $\mathbb{R}$ on $I_{\mathscr{G}}$ is defined as follows. For $t \in \mathbb{R}$ and $s_1 \in I_{\mathscr{G}}$, we define $t + s_1$ to be $s_2 = \pi_{I_{\mathscr{G}}}(x_2)$ where $\tau(x_2 - x_1) = t$ for some $x_1 \in \mathscr{E}(s_1)$ and $x_2 \in \mathscr{E}(s_2)$. We need o show that this definition does not depend on the choice of x_1 and x_2. Let $\bar{x}_1 \in \mathscr{E}(s_1)$ and $\bar{x}_2 \in \mathscr{E}(s_2)$. Since $\bar{x}_1 \in \mathscr{E}(s_1)$, we have $\bar{x}_1 - x_1 = v_1 \in \ker(\tau)$ and similarly $\bar{x}_2 - x_2 = v_2 \in \ker(\tau)$. Therefore we have

$$\tau(\bar{x}_2 - \bar{x}_1) = \tau((v_2 + x_2) - (v_1 + x_1)) = \tau((v_2 - v_1) + (x_2 - x_1)) \qquad = \tau(x_2 - x_1),$$

where we have used the associativity of affine addition. Therefore the condition that $\tau(x_2 - x_1) = t$ does not depend on the choice of x_1 and x_2. ■

We next denote by $V_{\mathscr{G}}$ the vectors $v \in V$ for which $\tau(v) = 1$. We call vectors in $V_{\mathscr{G}}$ *Galilean velocities*. Elementary linear algebra shows that $V_{\mathscr{G}}$ is an affine space modeled over $\ker(\tau)$ with the affine action given by vector addition in V.

2.3. Observers. An observer is to be thought of intuitively as someone who is present at each instant. Such an observer should be moving at a uniform velocity. Note that in a Galilean spacetime, the notion of "stationarity" makes no sense. We now provide our definition of an observer.

2.15 Definition: An *inertial observer* in a Galilean spacetime $\mathscr{G} = (\mathscr{E}, V, g, \tau)$. is a 1-dimensional affine subspace $\mathscr{O}$ of $\mathscr{E}$ with the property that $\pi_{\mathscr{G}}(\mathscr{O})$ consists of more than one point.

The definition thus requires that $\mathscr{O}$ not be comprised entirely of simultaneous events. It is easy to see that the definition implies that $\pi_{\mathscr{G}}(\mathscr{O}) = I_{\mathscr{G}}$. As a consequence of the definition, we have the following result.

2.16 Proposition: *If $\mathscr{O}$ is an observer in a Galilean spacetime $\mathscr{G} = (\mathscr{E}, V, g, \tau)$ then for each $s_0 \in I_{\mathscr{G}}$ there exists a unique point $x \in \mathscr{O} \cap \mathscr{E}(s_0)$.*

Proof: Since $\mathscr{O}$ is a 1-dimensional affine subspace of $\mathscr{E}$ whose projection consists of more than one point, given $s_0 \in I_{\mathscr{G}}$ there exists $x_0 \in \mathscr{O}$ such that $\pi_{\mathscr{G}}(x_0) = s_0$. By Proposition 2.11, there exists a 1-dimensional subspace $W \subset V$ such that

$$\mathscr{O} = \{w + x_0 |\ w \in W\}$$

Clearly, $x_0 \in \mathscr{E}(s_0) \cap \mathscr{O}$. This means that for each $s_0 \in I_{\mathscr{G}}$, the intersection $\mathscr{E}(s_0) \cap \mathscr{O}$ is non-empty. Next, assume that there exists $y_0 \in \mathscr{E}(s_0)$ with $y_0 \neq x_0$ such that $y_0 \in \mathscr{E}(s_0) \cap \mathscr{O}$. Now, $y_0 \in \mathscr{E}(s_0)$ implies that $y_0 - x_0 \in \ker(\tau)$. On the other hand, $y_0 \in \mathscr{O}$ implies that $y_0 - x_0 \in W$. This means that $y_0 = x_0$ which proves the uniqueness of x_0. ■

This means that an observer does exactly what it should: it resides in exactly one place at each instant. By requiring that $\mathscr{O}$ be an affine subspace, we ensure that it has a uniform velocity, and so is an appropriate reference for observing Newton's laws. We shall often refer to an inertial observer as merely an "observer" when there's no danger in doing so. We shall denote by $\mathscr{O}_s$ the unique point in the intersection $\mathscr{O} \cap \mathscr{E}(s)$.

Since an observer $\mathscr{O}$ is a 1-dimensional affine subspace, there is a unique 1-dimensional subspace U of V upon which $\mathscr{O}$ is modeled. Therefore, there exists a unique vector $v_{\mathscr{O}} \in V_{\mathscr{G}}$ with the property that $\tau(v_{\mathscr{O}}) = 1$. Conversely, given $v \in V_{\mathscr{G}}$ and $x \in \mathscr{E}$, there exists a unique observer $\mathscr{O}$ with the properties (1) $x \in \mathscr{O}$ (2) $v = v_{\mathscr{O}}$. We call $v_{\mathscr{O}}$ the *Galilean velocity* of the observer $\mathscr{O}$. It provides a reference velocity with which we can measure other velocities. Indeed, given an observer $\mathscr{O}$ we may define an associated map $P_{\mathscr{O}} : V \to \ker(\tau)$ by

$$P_{\mathscr{O}}(v) = v - (\tau(v))v_{\mathscr{O}}$$

In particular, if $v \in V_{\mathscr{G}}$, we note that $v = v_{\mathscr{O}} + P_{\mathscr{O}}(v)$. Thus $P_{\mathscr{O}}$ can be thought of as giving the velocity of v relative to the observer's Galilean velocity $v_{\mathscr{O}}$. Note that such velocities always live in the 3-dimensional vector space $\ker(\tau)$ that is to be thought of as the space of velocities that are familiar in mechanics. Such velocities are, however, only defined in the presence of an observer.

World lines. Intuitively, a world line is to be thought of as being the spatio-temporal history of something experiencing the spacetime. We make the following definition.

2.17 Definition: Let $\mathscr{G} = (\mathscr{E}, V, g, \tau)$ be a Galilean spacetime. A *world line* in $\mathscr{G}$ is a section of $\pi_{\mathscr{G}} : \mathscr{E} \to I_{\mathscr{G}}$.

A world line $c : I_{\mathscr{G}} \to \mathscr{E}$ is *differentiable* at $s_0 \in I_{\mathscr{G}}$ if the limit

$$c'(s_0) := \lim_{t \to 0} \frac{c(t + s_0) - c(s_0)}{t}$$

exists. Since c is a section of $\pi_{\mathscr{G}}$, $\tau(c(t + s_0) - c(s_0)) = t$ and so $c'(s_0) \in V_{\mathscr{G}}$, provided it exists. Similarly, for a differentiable world line, if the limit

$$\lim_{t \to 0} \frac{c'(t + s_0) - c'(s_0))}{t}$$

exists, we denote it by $c''(s_0)$, the acceleration of the world line at the instant s_0. Since

$$\tau(c''(s_0)) = \lim_{t \to 0} \frac{\tau(c'(t + s_0) - c'(s_0))}{t} = \lim_{t \to 0} \frac{1 - 1}{t} = 0,$$

we have $c''(s_0) \in \ker(\tau)$.

2.4. Galilean mappings. If $\mathscr{G}_i = (\mathscr{E}_i, V_i, g_i, \tau_i)$, $i = 1, 2$, are two Galilean spacetimes, a *Galilean mapping* from $\mathscr{G}_1$ to $\mathscr{G}_2$ is a map $\psi : \mathscr{E}_1 \to \mathscr{E}_2$ with the following properties:

GM1. ψ is an affine map.

GM2. $\tau_2(\psi(x_1) - \psi(x_2)) = \tau_1(x_1 - x_2)$ for $x_1, x_2 \in \mathscr{E}_1$;

GM3. $g_2(\psi(x_1) - \psi(x_2), \psi(x_1) - \psi(x_2)) = g_1(x_1 - x_2, x_1 - x_2)$ for simultaneous events $x_1, x_2 \in \mathscr{E}_1$.

A Galilean mapping $\phi : \mathscr{G} \to \mathbb{R}^3 \times \mathbb{R}$ into the canonical Galilean spacetime is a *coordinate system* for $\mathscr{G}$. A Galilean mapping from $\mathscr{G}$ to itself is a *Galilean transformation*, and these form a Lie group under composition. We shall denote this group by $\mathrm{Gal}(\mathscr{G})$ and its Lie algebra by $\mathfrak{gal}(\mathscr{G})$. If $\psi \in \mathrm{Gal}(\mathscr{G})$ then there are induced natural mappings of $V, I_{\mathscr{G}}$, and $\mathbb{R}$ as follows.

2.18 Lemma: *Let $\mathscr{G} = (\mathscr{E}, V, g, \tau)$ be a Galilean spacetime with $\psi \in \mathrm{Gal}(\mathscr{G})$. The following mappings are well defined:*

(i) the mapping $\psi_V : V \to V$ defined by $\psi_V(v) = \psi(v + x_0) - \psi(x_0)$ where $x_0 \in \mathscr{E}$;

(ii) the mapping $\psi_{I_{\mathscr{G}}} : I_{\mathscr{G}} \to I_{\mathscr{G}}$ defined by $\psi_{I_{\mathscr{G}}}(s) = \pi_{\mathscr{G}}(\psi(x))$ where $x \in \mathscr{E}(s)$;

(iii) the mapping $\psi_\tau : \mathbb{R} \to \mathbb{R}$ defined by $\psi_\tau(t) = t + \psi_{I_{\mathscr{G}}}(s) - s$ for $s \in I_{\mathscr{G}}$.

Furthermore,

(iv) $\psi_V(v) = \psi(x_1) - \psi(x_2)$ where $x_2 - x_2 = v$, and

(v) there exists $t_\psi \in \mathbb{R}$ so that $\psi_{I_{\mathscr{G}}}(s) = s + t_\psi$ and $\psi_\tau(t) = t + t_\psi$.

Proof: Let $x_0, \bar{x}_0 \in \mathscr{E}$. We have

$$\begin{aligned}
\psi_V(v) &= \psi(v + x_0) - \psi(x_0) \\
&= \psi((v + (x_0 - \bar{x}_0)) + \bar{x}_0) - \psi((x_0 - \bar{x}_0) + \bar{x}_0) \\
&= \psi((x_0 - \bar{x}_0) + \bar{x}_0) + \psi(v + \bar{x}_0) - \psi(\bar{x}_0) - \psi((x_0 - \bar{x}_0) + \bar{x}_0) \\
&= \psi(v + \bar{x}_0) - \psi(\bar{x}_0).
\end{aligned}$$

where we have used the property

$$\psi((v_1 + v_2) + x) = (\psi(v_1 + x) + \psi(v_2 + x)) - \psi(x)$$

since ψ is an affine map. This property is readily verified using the definition of an affine map.

We will now show that $\ker(\tau)$ is an invariant submanifold for ψ_V. We let $x, \tilde{x} \in \mathscr{E}$ have the property that $x - \tilde{x} = u \in \ker(\tau)$. Then

$$\tau(\psi_V(u)) = \tau(\psi(x) - \psi(\tilde{x})) = \tau(u) = 0,$$

where we have used the property GM2.

(ii) Let $x, \tilde{x} \in \mathscr{E}(s)$. There exists $u \in \ker(\tau)$ so that $\tilde{x} = u + x$. Now we compute

$$\begin{aligned}
\pi_{\mathscr{G}}(\psi(\tilde{x})) - \pi_{\mathscr{G}}(\psi(x)) &= \tau(\psi(\tilde{x}) - \psi(x)) \\
&= \tau(\psi_V(\tilde{x} - x)) \\
&= \tau(\psi_V(u)) = 0,
\end{aligned}$$

using the fact that $\ker(\tau)$ is an invariant submanifold for ψ_V.

(iii) We wish to show that the definition is independent of the choice of $s \in I_{\mathscr{G}}$. For $\tilde{s} \in I_{\mathscr{G}}$, we compute

$$\begin{aligned}
t + \psi_{I_{\mathscr{G}}}(\tilde{s}) - \tilde{s} &= t + \psi_{I_{\mathscr{G}}}(s + (\tilde{s} - s)) - s + (s - \tilde{s}) \\
&= t + \psi_{I_{\mathscr{G}}}(s + (\tilde{s} - s)) - \psi_{I_{\mathscr{G}}}(s) + \psi_{I_{\mathscr{G}}}(s) - s + (s - \tilde{s}) \\
&= t + \psi_{I_{\mathscr{G}}}(s) - s
\end{aligned}$$

(iv) We have

$$\begin{aligned}\psi(x_1) - \psi(x_2) &= \psi((x_1 - x_2) + x_2) - \psi(x_2)\\ &= \psi_V(x_1 - x_2),\end{aligned}$$

as desired.

(v) Let $x_0 \in \mathscr{E}$ and let $t_\psi = \tau(\psi(x_0) - x_0)$. For $s \in I_{\mathscr{G}}$, let $x \in \mathscr{E}(s)$. We then have

$$\begin{aligned}\psi_{I_{\mathscr{G}}}(s) - s &= \psi_{I_{\mathscr{G}}}(s) - \pi_{\mathscr{G}}(x)\\ &= \pi_{\mathscr{G}}(\psi(x)) - \pi_{\mathscr{G}}(x)\\ &= \tau(\psi(x) - \psi(x_0)) + \tau(\psi(x_0) - x_0) + \tau(x_0 - x)\\ &= \tau(\psi(x_0) - x_0) = t_\psi,\end{aligned}$$

where we have used the property GM2 of Galilean mappings. This shows that the definition of t_ψ is independent of x_0 and that $\psi_{I_{\mathscr{G}}}(s) = t_\psi + s$, as desired. From (iii) it also follows that $\psi_\tau(t) = t_\psi + t$. ■

2.19 Remarks: (i) The above lemma shows that given $\psi \in \mathrm{Gal}(\mathscr{G})$, ψ_V leaves $\ker(\tau)$ invariant. We shall see in the next chapter that $\psi_V|_{\ker(\tau)}$ has a special meaning, at least in the context of rigid motions.

(ii) Using the definition of a Galilean mapping, it is easy to see that $V_{\mathscr{G}}$ is also invariant under ψ_V.

3. Observer-independent formulation of rigid body dynamics

In this chapter, we provide an observer-independent formulation of rigid body dynamics. To our knowledge, such a formulation is yet to be done in the general setting of a Galilean spacetime although dynamics of a particle are well understood when an observer is present [Artz 1981]. In Section 3.1, we define rigid motions in a Galilean spacetime and list their fundamental properties. In the next section we define the body and spatial velocities associated with a rigid motion. Here again, the treatment is completely observer- independent. In Section 3.3 we provide our definition of a rigid body and discuss in it's implications in detail. Finally, in section 3.4 we we define momenta and derive the equations of motion for a rigid body undergoing a rigid motion in a Galilean spacetime.

3.1. Rigid motions. A *rigid motion* in a Galilean spacetime $\mathscr{G} = (\mathscr{E}, V, g, \tau)$ is a smooth mapping $\Psi : \mathbb{R} \to \mathrm{Gal}(\mathscr{G})$ with the property that $\Psi_t(x) \in \mathscr{E}(t + \pi_{\mathscr{G}}(x))$ for each $x \in \mathscr{E}(s)$. Thus a rigid motion maps points in $\mathscr{E}(s)$ to $\mathscr{E}(s)$ at $t = 0$ and for $t \neq 0$ the points get shifted by the affine action of $\mathbb{R}$ on $I_{\mathscr{G}}$. A rigid motion is *proper* when $\Psi_0 : \mathscr{E}(s) \to \mathscr{E}(s)$ preserves orientation for a given (and hence for all) $s \in I_{\mathscr{G}}$. Note that we do not demand that $\Psi_0 = \mathrm{id}_{\mathscr{E}}$ although it makes sense that this will often be the case.

Let us give some of the basic properties of rigid motions. To state these properties, let $O(\ker(\tau))$ denote the g-orthogonal linear mappings of $\ker(\tau)$, with $SO(\ker(\tau)) \subset O(\ker(\tau))$ those mappings with determinant 1. The Lie algebra of $O(\ker(\tau))$ we denote by $\mathfrak{o}(\ker(\tau))$, recalling that it is the collection of g-skew symmetric linear mappings of $\ker(\tau)$. The following result is immediate.

3.1 Lemma: *Given a rigid motion Ψ, for each $x \in \mathscr{E}$ the map $I_{\mathscr{G}} \ni s \mapsto \Psi_{s-\pi_{\mathscr{G}}(x)}(x) \in \mathscr{E}$ is a section of $\pi_{\mathscr{G}} : \mathscr{E} \to I_{\mathscr{G}}$.*

Proof: We need only show that $\Psi_{s-\pi_{\mathscr{G}}(x)}(x) \in \mathscr{E}(s)$. However, this follows directly from the definition of a rigid motion. ■

In view of Definition 2.17, we may think of the map $s \mapsto \Psi_{s-\pi_{\mathscr{G}}(x)}(x)$ as a world line of a point $x \in \mathscr{E}$ in the Galilean spacetime.

3.2 Proposition: *Let Ψ be a rigid motion in a Galilean spacetime $\mathscr{G} = (\mathscr{E}, V, g, \tau)$. Then, for each $t \in \mathbb{R}$, $\Psi_{t,V}|_{\ker(\tau)} \in O(\ker(\tau))$.*

Proof: From Lemma 2.18, it is clear that $\Psi_{t,V}$ maps $\ker(\tau)$ to itself. Next for simultaneous events x_1 and x_2, we compute

$$\begin{aligned} g(\Psi_{t,V}|_{\ker(\tau)}(x_1 - x_2), \Psi_{t,V}|_{\ker(\tau)}(x_1 - x_2)) &= g(\Psi_{t,V}(x_1 - x_2), \Psi_{t,V}(x_1 - x_2)) \\ &= g(\Psi_t(x_1) - \Psi_t(x_2), \Psi_t(x_1) - \Psi_t(x_2)) \\ &= g(x_1 - x_2, x_1 - x_2), \end{aligned}$$

where we have used the properties of the rigid motion and Lemma 2.18. This shows that $\Psi_{t,V}|_{\ker(\tau)} \in O(\ker(\tau))$ as desired. ■

This proposition shows that given a rigid motion in a Galilean spacetime, we can associate to this rigid motion, a unique map from $\mathbb{R} \to O(\ker(\tau))$. We denote this map by R_Ψ.

The next proposition gives a geometrical interpretation of the map $\Psi_{t,V}$.

3.3 Proposition: *Let Ψ be a rigid motion in a Galilean spacetime $\mathscr{G} = (\mathscr{E}, V, g, \tau)$. For $x \in \mathscr{E}$, the* **tangent map at** *x, denoted $T_x\Psi_t : \mathscr{E}_x \to \mathscr{E}_{\Psi_t(x)}$ is given by*

$$T_x\Psi_t(v) = \Psi_{t,V}(v)$$

Proof: Let us choose a curve $c(\tilde{t}) = v\tilde{t} + x$ where $v \in V$ such that $c'(\tilde{t})\big|_{\tilde{t}=0} = v$. The definition of the tangent map then gives

$$\begin{aligned} T_x\Psi_t(v) &= \frac{d}{d\tilde{t}}\bigg|_{\tilde{t}=0} \Psi_t(v\tilde{t} + x) \\ &= \lim_{\tilde{t}\to 0} \frac{\Psi_t(v\tilde{t} + x) - \Psi_t(x)}{\tilde{t}} \\ &= \lim_{\tilde{t}\to 0} \frac{\Psi_{t,V}(\tilde{t}v)}{\tilde{t}} \\ &= \Psi_{t,V}(v) \end{aligned}$$

as desired. ■

3.4 Remark: Although we have proved it for a rigid motion Ψ, this result is true for a general Galilean mapping. This is reflected in the fact that we do not use any properties of the rigid motion in the proof.

3.2. Body and spatial velocities. In this section we provide our observer–independent definitions for the body and spatial velocities corresponding to a rigid motion in a Galilean spacetime.

Angular velocities. To define the body and spatial angular velocities, we require the notation of the map from $\ker(\tau)$ to $\mathfrak{o}(\ker(\tau))$ given by $\omega \mapsto \hat{\omega}$. This is a generalization of of the map from $\mathbb{R}^3$ to $\mathfrak{o}(3)$ defined by

$$\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \mapsto \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

and may be explicitly defined by choosing an orthonormal basis for $\ker(\tau)$ and then applying this transformation to the components in this basis. Since the vector product in $\mathbb{R}^3$ commutes with orthogonal transformations, this definition is independent of the choice of orthonormal basis. In like manner one can define $u_1 \times u_2$ for any $u_1, u_2 \in \ker(\tau)$ as the generalization of the $\mathbb{R}^3$ vector product.

We may now define the body and spatial angular velocities associated with a given rigid motion.

3.5 Definition: Let Ψ be a rigid motion in a Galilean spacetime $\mathscr{G} = (\mathscr{E}, V, g, \tau)$ with R_Ψ as defined in Proposition 3.2.

(i) The **body angular velocity** is the map $\Omega_\Psi : \mathbb{R} \to \ker(\tau)$ satisfying

$$\hat{\Omega}_\Psi(t) = R_\Psi^{-1}(t) \circ \dot{R}_\Psi(t),$$

(ii) The **spatial angular velocity** is the map $\omega_\Psi : \mathbb{R} \to \ker(\tau)$ satisfying

$$\hat{\omega}_\Psi(t) = \dot{R}_\Psi(t) \circ R_\Psi^{-1}(t).$$

Linear velocities. It is possible to define linear velocities associated with a rigid motion once a point $x \in \mathscr{E}$ is chosen. This establishes a reference point with respect to which the "translations" can be measured. We now define body linear and spatial linear velocities in this setting.

3.6 Definition: Let Ψ be a rigid motion in a Galilean spacetime $\mathscr{G} = (\mathscr{E}, V, g, \tau)$.

(i) The *body linear velocity*, is the map $V_\Psi^b : \mathscr{E} \times \mathbb{R} \to V_{\mathscr{G}}$ satisfying

$$V_\Psi^b(x,t) = \Psi_{t,V}^{-1}(\frac{d}{dt}\Psi_t(x)).$$

(ii) The *spatial linear velocity*, is the map $v_\Psi^s : \mathscr{E} \times \mathbb{R} \to V_{\mathscr{G}}$ satisfying

$$v_\Psi^s(x,t) = -\Psi_{t,V}(\frac{d}{dt}(\Psi_t^{-1}(x))).$$

3.7 Remarks: (i) Notice that, by the definition of a rigid motion Ψ in a Galilean spacetime, the quantity $\frac{d}{dt}(\Psi_t(x)) \in V_{\mathscr{G}}$. Also, $\Psi_{t,V}$ leaves $V_{\mathscr{G}}$ invariant.

(ii) The body linear velocity is obtained by first computing the velocity of the point undergoing the motion Ψ at time t and then pulling this quantity back through $\Psi_{t,V}^{-1}$.

(iii) The interpretation of spatial linear velocity is less intuitive. It will become clearer in chapter 5 when we deal with this situation in the presence of an observer. In such a case, we shall recover the familiar expressions for the body and spatial linear velocities.

3.3. Rigid bodies. In order to talk about momenta, we need the notion a of a rigid body. In this section we provide our definition for a rigid body and provide the implications of this definition. We begin by proving in our Galilean setting some of the basic properties of the inertia tensor of a rigid body.

Definitions. Let $\mathscr{G} = (\mathscr{E}, V, g, \tau)$ be a Galilean spacetime. A *rigid body* is a pair $(\mathcal{B}, \mu)$ where $\mathcal{B} \subset \mathscr{E}(s_0)$ is a compact subset of simultaneous events, and μ is a mass-distribution on $\mathscr{E}(s_0)$ with support equal to $\mathcal{B}$. Our definition thus allows such degenerate rigid bodies as point masses, and bodies whose mass distribution is contained in a line in $\mathscr{E}(s_0)$. We denote

$$\mu(\mathcal{B}) = \int d\mu$$

as the *mass* of the body.

The *center of mass* of the body $(\mathcal{B}, \mu)$ is the point

$$x_c = \frac{1}{\mu(\mathcal{B})}(\int (x - x_0) d\mu) + x_0.$$

Note that the integrand is in $\ker(\tau)$ and so too will be the integral. The following lemma gives some of the basic properties of this definition. If $\mathscr{S} \subset \mathscr{A}$ is a subset of an affine space $\mathscr{A}$. we let $\mathrm{conv}(\mathscr{S})$ denote the convex hull of $\mathscr{S}$ and $\mathrm{aff}(\mathscr{S})$ denote the affine hull of $\mathscr{S}$. If X is a topological space with subsets $T \subset S \subset X$, $\mathrm{int}_S(T)$ denotes the interior of T relative to the induced topology on S.

3.8 Lemma: *Let $(\mathcal{B}, \mu)$ be a rigid body in a Galilean spacetime with $\mathcal{B} \subset \mathscr{E}(s_0)$. The following statements hold:*

(i) the expression

$$x_c = \frac{1}{\mu(\mathcal{B})}(\int (x - x_0) d\mu) + x_0$$

is independent of the choice of $x_0 \in \mathscr{E}(s_0)$;

(ii) x_c is the unique point in $\mathscr{E}(s_0)$ with the property that

$$\int (x - x_c) d\mu = 0$$

(iii) $x_c \in \mathrm{int}_{\mathrm{aff}(\mathcal{B})}(\mathrm{conv}(\mathcal{B}))$.

Proof: (i) To check that the definition of x_c is independent of $x_0 \in \mathscr{E}(s_0)$, we let $\tilde{x}_0 \in \mathscr{E}(s_0)$ and compute

$$\begin{aligned}\frac{1}{\mu(\mathcal{B})}(\int (x-\tilde{x}_0)d\mu)+\tilde{x}_0 &= \frac{1}{\mu(\mathcal{B})}(\int (x-x_0)d\mu)+\frac{1}{\mu(\mathcal{B})}(\int (x_0-\tilde{x}_0)d\mu)\\ &\quad +(\tilde{x}_0-x_0)+x_0\\ &= \frac{1}{\mu(\mathcal{B})}(\int (x-x_0)d\mu)+x_0\end{aligned}$$

(ii) By definition of x_c and by part (i) we have

$$x_c = \frac{1}{\mu(\mathcal{B})}(\int (x-x_c)d\mu)+x_c.$$

from which it follows that

$$\int (x-x_c)d\mu = \mu(\mathcal{B})(x_c-x_c)=0$$

Now suppose that $\tilde{x}_c \in \mathscr{E}(s_0)$ is an arbitrary point with the property that

$$\int (x-\tilde{x}_c)d\mu = 0$$

Then, by (i),

$$x_c = \frac{1}{\mu(\mathcal{B})}(\int (x-\tilde{x}_c)d\mu)+\tilde{x}_c.$$

from which we conclude that $\tilde{x}_c = x_c$.

(iii) If x_c is on the relative boundary of $\mathrm{conv}(\mathcal{B})$ or not in $\mathcal{B}$ at all, then there exists a hyperplane P in $\mathscr{E}(s_0)$ passing through x_c so that there are points in $\mathcal{B}$ which lie on one side of P, but there are no points in $\mathcal{B}$ on the opposite side. In other words, there exists $\lambda \in \ker(\tau)^\star$ so that the set

$$\{x \in \mathcal{B} |\ \lambda(x-x_c) > 0\}$$

is non-empty, but the set

$$\{x \in \mathcal{B} |\ \lambda(x-x_c) < 0\}$$

is empty. But this would imply that

$$\int \lambda(x-x_c)d\mu > 0$$

contradicting (ii). ■

The inertia tensor. The properties of a rigid body are characterized by three things: (1)its mass, (2) its center of mass and (3) its inertia tensor. We now define the latter. Let $x_0 \in \mathscr{E}(s_0)$. We define the inertia tensor about x_0 of a rigid body $(\mathcal{B},\mu)$ to be the linear map $\mathbb{I}_{x_0} : \ker(\tau) \to \ker(\tau)$ defined by

$$\mathbb{I}_{x_0}(u) = \int (x-x_0)\times(u\times(x-x_0))d\mu.$$

We denote the inertia tensor about the center of mass of $(\mathcal{B},\mu)$ by $\mathbb{I}_c$. Next, we record some basic properties of the inertia tensor.

3.9 Proposition: *The inertia tensor $\mathbb{I}_{x_0}$ of a rigid body $(\mathcal{B}, \mu)$ is symmetric with respect to the inner product g.*

Proof: Using the vector identity

$$g(u, v \times w) = g(w, u \times v)$$

we compute

$$\begin{aligned} g(\mathbb{I}_{x_0}(u_1), u_2) &= \int g((x - x_0) \times (u_1 \times (x - x_0)), u_2) d\mu \\ &= \int g(u_1 \times (x - x_0), (u_2 \times (x - x_0)) d\mu \\ &= \int g(u_1, (x - x_0) \times (u_2 \times (x - x_0))) d\mu \\ &= g(u_1, \mathbb{I}_{x_0}(u_2)), \end{aligned}$$

which is what we wished to show. ■

It is often useful to be able to compute the inertia tensor about a general point by first computing it about the center of mass. The following result shows how this is done.

3.10 Proposition: $\mathbb{I}_{x_0}(u) = \mathbb{I}_c(u) + \mu(\mathcal{B})(x_c - x_0) \times (u \times (x_c - x_0))$.

Proof: We compute

$$\begin{aligned} \mathbb{I}_{x_0} &= \int (x - x_0) \times (u \times (x - x_0)) d\mu \\ &= \int (x - x_c) + (x_c - x_0)) \times (u \times ((x - x_c) + (x_c - x_0))) d\mu \\ &= \int (x - x_c) \times (u \times (x - x_c)) d\mu + \int (x_c - x_0) \times (u \times (x_c - x_0)) d\mu \\ &\quad + \int (x - x_c) \times (u \times (x_c - x_0)) d\mu + \int (x_c - x_0) \times (u \times (x - x_c)) d\mu \end{aligned}$$

It then follows from Lemma 3.8 that the last two terms vanish, and from this the result follows. ■

Eigenvalues of the inertia tensor. Since $\mathbb{I}_{x_0}$ is symmetric, its eigenvalues are real. Furthermore, they are non-negative. The following result demonstrates this, as well as other eigenvalue related assertions.

3.11 Proposition: *Let $(\mathcal{B}, \mu)$ be a rigid body with $\mathcal{B} \in \mathscr{E}(s_0)$ and let $x_0 \in \mathscr{E}(s_0)$. Let $\mathbb{I}_{x_0}$ denote the inertia tensor of $(\mathcal{B}, \mu)$ about x_0. The following statements hold:*

(i) the eigenvalues of the inertia tensor $\mathbb{I}_{x_0}$ of a rigid body are never negative;

(ii) if $\mathbb{I}_{x_0}$ has a zero eigenvalue, then the other two eigenvalues are equal.

(iii) if $\mathbb{I}_{x_0}$ has two zero eigenvalues, then $\mathbb{I}_{x_0} = 0$.

Proof: (i) Since $\mathbb{I}_{x_0}$ is symmetric, it's eigenvalues will be non-negative if and only if the quadratic form $u \mapsto g(\mathbb{I}_{x_0}(u), u)$ is positive semi-definite. For $u \in \ker(\tau)$ we compute

$$\begin{aligned} g(\mathbb{I}_{x_0}(u), u) &= \int g(u, (x - x_0) \times (u \times (x - x_0)))d\mu \\ &= \int g(u \times (x - x_0), u \times (x - x_0))d\mu. \end{aligned}$$

Since the integrand is non-negative, so too will be the integral.

(ii) Let I_1 be the zero eigenvalue with v_1 a unit eigenvector. We claim that the support of the mass distribution μ must be contained in the line

$$\ell_{v_1} = \{sv_1 + x_0 |\ s \in \mathbb{R}\}.$$

To see that this must be so, suppose that the support of μ is not contained in ℓ_{v_1}. Then there exists a Borel set $S \subset \mathscr{E}(s_0) \setminus \ell_{v_1}$ so that $\mu(S) > 0$. This would imply that

$$\begin{aligned} g(\mathbb{I}_{x_0}(v_1), v_1) &= \int g(v_1 \times (x - x_0), v_1 \times (x - x_0))d\mu \\ &\geq \int_S g(v_1 \times (x - x_0), v_1 \times (x - x_0))d\mu. \end{aligned}$$

Since $S \cap \ell_{v_1} = \varnothing$ it follows that for all points $x \in S$, the vector $x - x_0$ is not collinear with v_1. Therefore

$$g(v_1 \times (x - x_0), v_1 \times (x - x_0)) > 0$$

for all $x \in S$, and this would imply that $\mathbb{I}_{x_0}(v_1, v_1) > 0$. But this contradicts v_1 being an eigenvector with zero eigenvalue, and so the support of $\mathcal{B}$ must be contained in the line ℓ_{v_1}.

To see that this implies the remaining two eigenvectors are equal, we shall show that any vector that is g-orthogonal to v_1 is an eigenvector for $\mathbb{I}_{x_0}$. First write

$$x - x_0 = f^1(x)v_1 + f^2(x)v_2 + f^3(x)v_3$$

for functions $f^i : \mathscr{E}(s_0) \to \mathbb{R},\ i = 1, 2, 3$. Since the support of μ is contained in the line ℓ_{v_1} we have

$$\int (x - x_0) \times (u \times (x - x_0))d\mu = v_1 \times (u \times v_1) \int (f^1(x))^2 d\mu$$

for all $u \in \ker(\tau)$. Now recall the property of the cross product that $v_1 \times (u \times v_1) = u$ provided u is orthogonal to v_1 and that v_1 has unit length. Therefore we see that for any u that is orthogonal to v_1 we have

$$\mathbb{I}_{x_0}(u) = \left(\int (f^1(x))^2 d\mu \right) u,$$

meaning that all such vectors u are eigenvectors with the same eigenvalue, which is what we wished to show.

(iii) It follows from the above arguments that if two eigenvalues I_1 and I_2 are zero, then the support μ must lie in the intersection of the lines ℓ_{v_1} and ℓ_{v_2} (here v_i is an eigenvector for $I_i,\ i = 1, 2$), and this intersection is a single point, that must therefore be x_0. From this and the definition of $\mathbb{I}_{x_0}$, it follows that $\mathbb{I}_{x_0} = 0$. ■

Note that in proving the result we have proved the following corollary.

3.12 corollary: *Let $(\mathcal{B}, \mu)$ be a rigid body with inertia tensor $\mathbb{I}_{x_0}$. The following statements are true:*

(i) $\mathbb{I}_{x_0}$ has a zero eigenvalue if and only if $\mathcal{B}$ is contained in a line through x_0;

(ii) if $\mathbb{I}_{x_0}$ has two zero eigenvalues then $\mathcal{B} = \{x_0\}$, i.e., $\mathcal{B}$ is a particle located at x_0;

(iii) if there is no line through x_0 that contains the support of μ, then the inertia tensor is an isomorphism.

In coming to an understanding of the "appearance" of a rigid body, it is most convenient to refer to its inertia tensor $\mathbb{I}_c$ about it's center of mass. Let $\{I_1, I_2, I_3\}$ be the eigenvalues of $\mathbb{I}_c$ that we call the *principal inertias* of $(\mathcal{B}, \mu)$. If $\{v_1, v_2, v_3\}$ are orthonormal eigenvectors associated with these eigenvalues, we call these the *principal axes* of $(\mathcal{B}, \mu)$. Related to these is the the *inertial ellipsoid* which is the ellipsoid in $\ker(\tau)$ given by

$$E(\mathcal{B}) = \left\{x^1 v_1 + x^2 v_2 + x^3 v_3 \in \ker(\tau) |\ I_1 (x^1)^2 + I_2 (x^2)^2 + I_3 (x^3)^2 = 1\right\},$$

provided that none of the eigenvalues of $\mathbb{I}_{x_0}$ are zero. If one of the eigenvalue *does* vanish, then by Proposition 3.11, the other two eigenvalues are equal. If we suppose that $I_1 = 0$ and $I_2 = I_3 = I$ then in the case of a single zero eigenvalue, the inertial ellipsoid is

$$E(\mathcal{B}) = \left\{x^1 v_1 + x^2 v_2 + x^3 v_3 \in \ker(\tau) |\ x^2 = x^3 = 0,\ x^1 \in \left\{-\frac{1}{\sqrt{I}}, \frac{1}{\sqrt{I}}\right\}\right\}.$$

in the most degenerate case, when all eigenvalues are zero, we define $E(\mathcal{B}) = \{0\}$. These latter two inertia ellipsoids, correspond to cases (i) and (ii) in Corollary 3.12.

To relate these properties of the eigenvalues of $\mathbb{I}_c$ with the inertial ellipsoid $E(\mathcal{B}$. it is helpful to introduce the notion of an axis of symmetry for a rigid body. We let $\mathbb{I}_c$ be the inertia tensor about the center of mass, and denote by $\{I_1, I_2, I_3\}$ its eigenvalues and $\{v_1, v_2, v_3\}$ it's orthogonal eigenvectors. A vector $v \in \ker(\tau) \setminus \{0\}$ is an *axis of symmetry* for $(\mathcal{B}, \mu)$ if for every $R \in O(\ker(\tau))$ which fixes v we have $R(E(\mathcal{B})) = E(\mathcal{B})$. The following result gives the relationship between axes of symmetry and the eigenvalues of $\mathbb{I}_c$.

3.13 Proposition: *Let $(\mathcal{B}, \mu)$ be a rigid body with inertia tensor $\mathbb{I}_c$ about its center of mass. Let $\{I_1, I_2, I_3\}$ be the eigenvalues of $\mathbb{I}_c$ with orthonormal eigenvectors $\{v_1, v_2, v_3\}$. If $I_1 = I_2$ then v_3 is an axis of symmetry of $\mathcal{B}$.*

Conversely, if $v \in \ker(\tau)$ is an axis of symmetry, then v is an eigenvector of $\mathbb{I}_c$. If I is the eigenvalue for which v is an eigenvector, then the other two eigenvalues of $\mathbb{I}_c$ are equal.

Proof: Write $I_1 = I_2 = I_3$. We then see that any vector $v \in \mathrm{span}_{\mathbb{R}}\{v_1, v_2\}$ will have the property that

$$\mathbb{I}_c(v) = Iv$$

Now let $R \in O(\ker(\tau))$ fix the vector v_3. Because R is orthogonal, if we have $v \in \mathrm{span}_{\mathbb{R}}\{v_1, v_2\}$ then $R(v) \in \mathrm{span}_{\mathbb{R}}\{v_1, v_2\}$. Also, if

$$v = a^1 v_1 + a^2 v_2,$$

then,

$$R(v) = (\cos\theta a^1 + \sin\theta a^2) v_1 + (-\sin\theta a^1 + \cos\theta a^2) v_2 \tag{3.1}$$

for some $\theta \in \mathbb{R}$ since R is simply a rotation in the plane spanned by v_1, v_2. Now let $u \in E(\mathcal{B})$. We then write $u = x^1 v_1 + x^2 v_2 + x^3 v_3$ and note that

$$I(x^1)^2 + I(x^2)^2 + I(x_3)^3 = 1$$

It is now a straightforward but tedious calculation to verify that $R(u) \in E(\mathcal{B})$ using (3.1) and the fact that R fixes v_3. This shows that $R(E(\mathcal{B})) = E(\mathcal{B})$, and so v_3 is an axis of symmetry for $\mathcal{B}$.

For the second part of the proposition, note that $R \in O(\ker(\tau))$ has the property that $R(E(\mathcal{B})) = E(\mathcal{B})$ if and only if R maps the principal axes of the ellipsoid $E(\mathcal{B})$ to the principal axes. Since R is a rotation about *some* axis, this means that R fixes a principal axis of $E(\mathcal{B})$. Thus if $v \in \ker(\tau)$ is an axis of symmetry, then V must lie along a principal axis of the ellipsoid $E(\mathcal{B})$. By our definition of $E(\mathcal{B})$, this means that v is an eigenvector of $\mathbb{I}_c$. Let I be the associated eigenvalue and $\{I_1, I_2, I_3 = I\}$ be the collection of all eigenvalues of $\mathbb{I}_c$ with eigenvectors $\{v_1, v_2, v_3 = v\}$. Since v is an axis of symmetry, any rotation about v must map principal axes of $E(\mathcal{B})$ to principal axes. This means that for every $\theta \in \mathbb{R}$ the vectors

$$v_1' = \cos\theta v_1 - \sin\theta v_2, \qquad v_2' = \sin\theta v_1 + \cos\theta v_2$$

are eigenvectors for $\mathbb{I}_c$. This means that all non-zero vectors in $\mathrm{span}_{\mathbb{R}}\{v_1, v_2\}$ are eigenvectors for $\mathbb{I}_c$. This means that the restriction of $\mathbb{I}_c$ to $\mathrm{span}_{\mathbb{R}}\{v_1, v_2\}$s diagonal in every orthonormal basis for $\mathrm{span}_{\mathbb{R}}\{v_1, v_2\}$. Therefore, if $\{v_1, v_2\}$ are chosen to be orthonormal then $\{v_1', v_2'\}$ as defined previously are also orthonormal. Our conclusions assert the existence of $I_1', I_2' \in \mathbb{R}$ so that

$$\mathbb{I}_c(v_i') = I_i v_i', \qquad i = 1, 2.$$

but by the definition of v_1' and v_2' we also have

$$\begin{aligned}
\mathbb{I}_c(v_1') &= \cos\theta \mathbb{I}_c(v_1) - \sin\theta \mathbb{I}_c(v_2) \\
&= \cos\theta I_1 v_1 - \sin\theta v_2 \\
&= \cos\theta I_1(\cos\theta v_1' + \sin\theta v_2') - \sin\theta I_2(-\sin\theta v_1' + \cos\theta v_2').
\end{aligned}$$

Therefore,

$$I_1' v_1' = (\cos^2\theta I_1 + \sin^2\theta I_2) v_1' + \sin\theta\cos\theta (I_1 - I_2) v_2'$$

for every $\theta \in \mathbb{R}$. Since v_1', v_2' are orthogonal, this means that choosing θ so that $\sin\theta\cos\theta \neq 0$ implies that $I_1 - I_2 = 0$. This is what we wished to show. ■

3.4. Dynamics of rigid bodies. In this section, we derive the "equations of motion" for a rigid body undergoing a rigid motion in a Galilean spacetime. To do so, we need to define body and spatial momenta.

Spatial momenta. We let $\mathcal{G} = (\mathscr{E}, V, g, \tau)$ be a Galilean spacetime with Ψ a rigid motion and $(\mathcal{B}, \mu)$ a rigid body with $\mathcal{B} \subset \mathscr{E}(s_0)$. We wish to see how to define spatial linear momentum. To motivate our definition, we recall how it might be defined for a particle of mass m. If the particle is moving in $\mathbb{R}^3$ following the curve $t \mapsto x(t)$, its linear momentum is taken to be $m\dot{x}(t)$. The key point here is that one measures the linear momentum of the center of mass. Motivated by the classical definition, the angular momentum could be defined as the image of the angular velocity under the inertia tensor.

3.14 Definition: Let $(\mathcal{B}, \mu)$ be a rigid body in a Galilean spacetime $\mathscr{G} = (\mathscr{E}, V, g, \tau)$ and let Ψ be a rigid motion.

(i) The *spatial linear momentum* is the map $m^s_{\Psi,\mathcal{B}} : \mathbb{R} \to V_{\mathscr{G}}$ given by

$$m^s_{\Psi,\mathcal{B}}(t) = \mu(\mathcal{B})\frac{d}{dt}(\Psi_t(x_c)).$$

(ii) The *spatial angular momentum* is the map $\ell^s_{\Psi,\mathcal{B}} : \mathbb{R} \to \ker(\tau)$ given by

$$\ell^s_{\Psi,\mathcal{B}}(t) = \mathbb{I}_c(t)\omega_\Psi(t)$$

where

$$\mathbb{I}_c(t)(u(t)) = \int_{\mathcal{B}(t)} (\Psi_t(x) - \Psi_t(x_c)) \times (u(t) \times (\Psi_t(x) - \Psi_t(x_c))d\mu(t).$$

Notice that since the integrand is in $\ker(\tau)$, so too will be the integral. The following result shows what the spatial angular momentum looks like in terms of the inertia tensor of the body about its center of mass.

3.15 Lemma: $\ell^s_{\Psi,\mathcal{B}}(t) = \mathbb{I}_c(R^{-1}_\Psi(t)\omega_\Psi(t))$.

Proof: We represent, by $\mathcal{B}(t)$ the rigid body after it has undergone the transformation Ψ_t and the corresponding mass distribution by $d\mu(t)$. We compute

$$\begin{aligned}
\ell^s_{\Psi,\mathcal{B}}(t) = \mathbb{I}_c(t)(\omega_\Psi(t)) &= \int_{\mathcal{B}(t)} (\Psi_t(x) - \Psi_t(x_c)) \times (\omega_\Psi(t) \times (\Psi_t(x) - \Psi_t(x_c)))\, d\mu(t) \\
&= \int_{\mathcal{B}(t)} (\Psi_{t,V}(x - x_c)) \times (\omega_\Psi(t) \times (\Psi_{t,V}(x - x_c)))\, d\mu(t) \\
&= \int_{\mathcal{B}(t)} (R_\Psi(t)(x - x_c)) \times (\omega_\Psi(t) \times (R_\Psi(t)(x - x_c)))\, d\mu(t) \\
&= \int_{\mathcal{B}(t)} (x - x_c) \times \left(R^{-1}_\Psi(t)\omega_\Psi(t) \times (x - x_c)\right) d\mu(t) \\
&= \int (x - x_c) \times \left(R^{-1}_\Psi(t)\omega_\Psi(t) \times (x - x_c)\right) d\mu \\
&= \mathbb{I}_c(R^{-1}_\Psi(t)\omega_\Psi(t))
\end{aligned}$$

where we have used the fact that $x - x_c \in \ker(\tau)$ and therefore $\Psi_{t,V}(x - x_c) = R_\Psi(t)(x - x_c)$. ■

Body momenta. We now proceed to define the body momenta.

3.16 Definition: Let $(\mathcal{B}, \mu)$ be a rigid body in a Galilean spacetime $\mathscr{G} = (\mathscr{E}, V, g, \tau)$ and let Ψ be a rigid motion.

(i) The *body linear momentum* is the map $M_{\Psi,\mathcal{B}} : \mathbb{R} \to V_{\mathscr{G}}$ satisfying

$$M_{\Psi,\mathcal{B}}(t) = \Psi^{-1}_{t,V}(m^s_{\Psi,\mathcal{B}}(t))$$

(ii) The *body angular momentum* is the map $L_{\Psi,\mathcal{B}} : \mathbb{R} \to \ker(\tau)$ satisfying

$$L_{\Psi,\mathcal{B}}(t) = R^{-1}_\Psi(t)(\ell^s_{\Psi,\mathcal{B}}(t))$$

3.17 Remarks: (i) Notice that $M_{\Psi,\mathcal{B}}(t) = \mu(\mathcal{B})V^b_\Psi(t)$.

(ii) The idea behind the definitions of body momenta is intuitively clear–they are the momenta "seen" by the body in it's reference frame.

Galilean–Euler equations. In this subsection, we derive the so-called "Euler equations" for the rigid body. These are the equations governing the motion of a rigid body undergoing a rigid motion in a Galilean spacetime obtained by using the principle of conservation of momenta. As we have remarked earlier, these equations are considerably general than those for the classical case of a rigid body in $\mathbb{R}^3 \times \mathbb{R}$ with rigid motions coming from the special Euclidean group $SE(3)$. We have the following result.

3.18 proposition: *Let $\mathcal{G} = (\mathcal{E}, V, g, \tau)$ be a Galilean spacetime with $(\mathcal{B}, \mu)$ a rigid body with $\mathcal{B} \in \mathcal{E}(s_0)$ and let Ψ be a rigid motion. The following statements are equivalent:*

1. *The spatial linear and spatial angular momenta are conserved.*
2. *The motion of the rigid body satisfies the following differential equations.*

$$\begin{aligned}\dot{R}_\Psi(t) &= R_\Psi(t) \circ \hat{\Omega}_\Psi(t)\\ \ddot{x}_c(t) &= 0\\ \dot{L}_{\Psi,\mathcal{B}}(t) &= L_{\Psi,\mathcal{B}}(t) \times \Omega_\Psi(t)\end{aligned}$$

where $\ddot{x}_c(t) = \frac{d^2}{dt^2}(\Psi_t(x_c))$.

Proof: The first equation says that $\hat{\Omega}(t) = R^{-1}_\Psi(t) \circ \dot{R}_\Psi(t)$ which is just the definition of Ω. Next, conservation of spatial linear momentum implies that

$$\frac{d}{dt} m_{\Psi,\mathcal{B}}(t) = \frac{d}{dt}(\mu(\mathcal{B})\dot{x}_c(t)) = 0$$

This gives us the second equation. To see the body angular momentum equation, we write

$$\ell_{\Psi,\mathcal{B}}(t) = R_\Psi(t)(L_{\Psi,\mathcal{B}}(t))$$

Differentiating this with respect to t and setting it equal to zero (since the spatial angular momentum is conserved), we get

$$\begin{aligned}0 &= \dot{R}_\Psi(t)L_{\Psi,\mathcal{B}}(t) + R_\Psi(t)\dot{L}_{\Psi,\mathcal{B}}(t)\\ &= R_\Psi(t)\hat{\Omega}_\Psi(t)L_{\Psi,\mathcal{B}}(t) + R_\Psi(t)\dot{L}_{\Psi,\mathcal{B}}(t)\end{aligned}$$

This gives us

$$\dot{L}_{\Psi,\mathcal{B}}(t) = L_{\Psi,\mathcal{B}}(t) \times \Omega_\Psi(t)$$

which is what we wanted to show. ■

Discussion. We call the equations given in part 2 of Proposition 3.18 the *Galilean–Euler* equations. These equations are quite formidable because they have been derived in the general setting of abstract Galilean spacetimes without requiring an observer. However, the generality of the treatment makes certain things less obvious. In particular, we are not able to recover an equation governing the evolution of the body angular momentum with time. It is implicit in these Galilean–Euler equations. We shall see that we come to the same conclusions in Chapter 5 when we look at these equations in the presence of an observer.

4. The structure of the Galilean group

As defined previously, the Galilean group of a Galilean spacetime $\mathscr{G} = (\mathscr{E}, V, g, \tau)$ is the set of affine maps from $\mathscr{E}$ to itself that preserve simultaneity of events and the distance between simultaneous events. In this chapter, we shall examine the Galilean group closely and describe its properties. We shall also look at its various subgroups. In Section 4.1 we study the canonical Galilean group and show that it consists of rotations, translations, velocity boosts and temporal origin shifts. We also look at it's subgroups and describe the various fundamental objects associated with it. In section 4.2, we study the abstract Galilean group $\mathrm{Gal}(\mathscr{G})$. We show that that in the presence of an observer, the Galilean group is isomorphic to the canonical Galilean group. We characterize the various subgroups of the abstract Galilean group and mention how these subgroups are related to the subgroups of the canonical group. Finally, in section 4.3, we introduce total velocities and describe their images under the isomorphism of the Lie algebras induced by the isomorphism constructed previously.

4.1. The canonical Galilean group. In this section, we study the Galilean group of a canonical Galilean spacetime which is a generalization of the "standard" Galilean spacetime $\mathbb{R}^3 \times \mathbb{R}$. To be precise, given a Galilean spacetime $\mathscr{G} = (\mathscr{E}, V, g, \tau)$, the *canonical* spacetime of $\mathscr{G}$ is the Galilean spacetime $\mathscr{G}_{\mathrm{can}} := (\mathscr{E}_{\mathrm{can}} := \ker(\tau) \oplus \mathbb{R}, V, g_{\mathrm{can}}, \tau)$. We wish to investigate the structure of the canonical Galilean group $\mathrm{Gal}(\mathscr{G}_{\mathrm{can}})$.

The structure of the canonical Galilean group. The next proposition shows that $\mathrm{Gal}(\mathscr{G}_{\mathrm{can}})$ decomposes into rotations, spatial translations, Galilean velocity boosts and temporal translations. We need some terminology and notation first. Recall that a subgroup H of a group G is *normal* if $gHg^{-1} \subset H$ for every $g \in G$. Next, a group G is a *semi-direct product* of H by N if

(i) N is a normal subgroup of G

(ii) $\{h \cdot n : h \in H, n \in N\} = G$

(iii) $H \cap N = e$.

4.1 Notation: We denote the semi-direct product of H by N as $H \ltimes N$.

4.2 proposition: *The Galilean group* $\mathrm{Gal}(\mathscr{G}_{\mathrm{can}})$ *of the canonical spacetime* $\mathscr{G}_{\mathrm{can}}$ *admits the decomposition*

$$\mathrm{Gal}(\mathscr{G}_{\mathrm{can}}) = (O(\ker(\tau)) \ltimes \ker(\tau)) \ltimes (\ker(\tau) \times \mathbb{R})$$

such that for $(R_i, r_i, u_i, t_i) \in \mathrm{Gal}(\mathscr{G}_{\mathrm{can}}), i = 1, 2,$ *the group operation on* $\mathrm{Gal}(\mathscr{G}_{\mathrm{can}})$ *is given by*

$$((R_1, r_1, u_1, t_1) \cdot (R_2, r_2, u_2, t_2) = (R_1 \circ R_2, r_1 + R_1(r_2) + t_2 u_1, u_1 + R_1(u_2), t_1 + t_2)$$

Proof: We first find the form of a Galilean transformation $\phi : \mathscr{E}_{\mathrm{can}} \to \mathscr{E}_{\mathrm{can}}$. Recall that since ϕ is an affine map, it has the form $\phi(x,t) = A(x,t) + (r,\sigma)$ where $A : \ker(\tau) \times \mathbb{R} \to \ker(\tau) \times \mathbb{R}$ is $\mathbb{R}$-linear and where $(r,\sigma) \in \ker(\tau) \times \mathbb{R}$. Let us write $A(x,t) = (A_{11}x + A_{12}t, A_{21}x + A_{22}t)$ where $A_{11} \in L(\ker(\tau), \ker(\tau)), A_{12} \in L(\mathbb{R}, \ker(\tau)), A_{21} \in L(\ker(\tau), \mathbb{R})$ and $A_{22} \in L(\mathbb{R}, \mathbb{R})$. By GM3, A_{11} is a g-orthogonal transformation of $\ker(\tau)$. GM2 implies that

$$A_{22}(t_2 - t_1) + A_{21}(x_2 - x_1) = t_2 - t_1, \quad t_1, t_2 \in \mathbb{R}, x_1, x_2 \in \ker(\tau)$$

Thus, taking $x_1 = x_2$, we see that $A_{22} = 1$. This in turn implies that $A_{21} = 0$. Gathering this information shows that a Galilean transformation has the form

$$\phi : \begin{pmatrix} x \\ t \end{pmatrix} \mapsto \begin{bmatrix} R & u \\ 0^t & 1 \end{bmatrix} \begin{pmatrix} x \\ t \end{pmatrix} + \begin{pmatrix} r \\ \sigma \end{pmatrix}$$

where $R \in O(\ker(\tau)), \sigma \in \mathbb{R}$ and $r, u \in \ker(\tau)$. This proves the first part of the proposition. Now, it is easy to see that if $\phi_i, i = 1, 2$ are Galilean transformations given by

$$\phi_i : \begin{pmatrix} x \\ t \end{pmatrix} \mapsto \begin{bmatrix} R_i & u_i \\ 0^t & 1 \end{bmatrix} \begin{pmatrix} x \\ t \end{pmatrix} + \begin{pmatrix} r_i \\ \sigma_i \end{pmatrix}, \quad i = 1, 2.$$

then

$$\phi_1 \circ \phi_2 : \begin{pmatrix} x \\ t \end{pmatrix} \mapsto \begin{bmatrix} R_1 \circ R_2 & u_1 + R_1(u_2) \\ 0^t & 1 \end{bmatrix} \begin{pmatrix} x \\ t \end{pmatrix} + \begin{pmatrix} r_1 + R_1(r_2) + \sigma_2 u_1 \\ \sigma_1 + \sigma_2 \end{pmatrix}.$$

which gives us the desired group operation. ■

4.3 Remarks: (1) It is clear from this proposition that $\mathrm{Gal}(\mathscr{G}_{\mathrm{can}})$ is a 10-dimensional group. This is not altogether obvious from the definition.

(2) The meaning of the appearance of two semi-direct products in the decomposition of $\mathrm{Gal}(\mathscr{G}_{\mathrm{can}})$ should be understood correctly. They arise because $\ker(\tau) \times \mathbb{R}$ is a normal subgroup of $\mathrm{Gal}(\mathscr{G}_{\mathrm{can}})$ and the quotient group itself is a semi-direct product of $O(\ker(\tau))$ by $\ker(\tau)$.

(3) A canonical Galilean transformation may now be written as a composition of one of three basic classes of transformations.

(i) *A shift of origin*

$$\begin{pmatrix} x \\ t \end{pmatrix} \mapsto \begin{pmatrix} x \\ t \end{pmatrix} + \begin{pmatrix} r \\ \sigma \end{pmatrix}$$

for $r \in \ker(\tau), \ \sigma \in \mathbb{R}$.

(ii) *A "rotation" of reference frame*:

$$\begin{pmatrix} x \\ t \end{pmatrix} \mapsto \begin{bmatrix} R & 0 \\ 0^t & 1 \end{bmatrix} \begin{pmatrix} x \\ t \end{pmatrix}$$

for $R \in O(\ker(\tau))$.

(iii) *A (Galilean) velocity boost*

$$\begin{pmatrix} x \\ t \end{pmatrix} \mapsto \begin{bmatrix} \mathrm{id}_{\ker(\tau)} & u \\ 0^t & 1 \end{bmatrix} \begin{pmatrix} x \\ t \end{pmatrix}$$

for $u \in \ker(\tau)$.

The names we have given these fundamental transformations are suggestive. A shift of the origin should be thought of as moving the origin to a new position, and resetting the clock, but maintaining the same orientation in space. A rotation of reference frame means the origin stays in the same place, and uses the same clock but, but rotates the "point of view." The final basic transformation, a velocity boost, means the origin maintains its orientation and uses the same clock, but now moves with a certain velocity with respect to the previous origin.

Subgroups of the canonical Galilean group. In the previous section we showed that the elements of the canonical Galilean group $\mathrm{Gal}(\mathscr{G}_{\mathrm{can}})$ are composed of temporal and spatial transformations, spatial rotations, and velocity boosts. In this section, we study some of the subgroups of $\mathrm{Gal}(\mathscr{G}_{\mathrm{can}})$ more carefully. Below we list the various subgroups and discuss some of their basic properties. Some of what we say here can be found in [Lévy-Leblond 1971].

(i) The set $\mathrm{Gal}_0(\mathscr{G}_{\mathrm{can}}) := \{(R, r, u, t) \in \mathrm{Gal}(\mathscr{G}_{\mathrm{can}}) : t = 0\}$: This is a 9-dimensional subgroup sometimes called the *isochronous* Galilean group. This subgroup, which is the maximal proper subgroup of $\mathrm{Gal}(\mathscr{G}_{\mathrm{can}})$ is also its commutator subgroup.

(ii) The set $\mathrm{Gal}_{\mathrm{ub}}(\mathscr{G}_{\mathrm{can}}) := \{(R, r, u, t) \in \mathrm{Gal}(\mathscr{G}_{\mathrm{can}}) : u = 0\}$: This is a 7-dimensional subgroup which we call the *unboosted* (or no-boost) subgroup.

(iii) The set $\{(R, r, u, t) \in \mathrm{Gal}(\mathscr{G}_{\mathrm{can}}) : R = \mathrm{id}_{\ker(\tau)}\}$: This is a 7-dimensional subgroup sometimes called the *anisotropic* Galilean group.

(iv) The set $\mathrm{E}(\mathscr{G}_{\mathrm{can}}) := \{(R, r, u, t) \in \mathrm{Gal}(\mathscr{G}_{\mathrm{can}}) : u = 0,\ t = 0\}$: This is a 6-dimensional subgroup called the Euclidean subgroup. It consists of space translations and rotations.

(v) The set $\mathrm{Lin}(\mathscr{G}_{\mathrm{can}}) := \{(R, r, u, t) \in \mathrm{Gal}(\mathscr{G}_{\mathrm{can}}) : r = 0,\ t = 0\}$: This is a 6-dimensional subgroup which we call the *linear* subgroup of $\mathrm{Gal}(\mathscr{G}_{\mathrm{can}})$. It is also called the *homogeneous* canonical Galilean group. This subgroup is actually the quotient of $\mathrm{Gal}(\mathscr{G}_{\mathrm{can}})$ by the normal subgroup of the space-time translations mentioned below and can be identified with the semi-direct product of $O(\ker(\tau))$ by $\ker(\tau)$.

(vi) The set $\{(R, r, u, t) \in \mathrm{Gal}(\mathscr{G}_{\mathrm{can}}) : R = \mathrm{id}_{\ker(\tau)}, t = 0\}$: This is a 6-dimensional subgroup identified with the set of straight world lines in $\mathscr{E}_{\mathrm{can}}$ which can be parametrized by giving the intercept r_0 of the line with a given time= constant plane, and the slope u_0 of the line, writing its equation

$$r = r_0 + u_0 t,$$

the physical interpretation of which is clear.

(vii) $\ker(\tau) \times \mathbb{R}$: this is a 4-dimensional subgroup of space-time translations. It can be seen that this is a normal subgroup.

(viii) The set $\{(R, r, u, t) \in \mathrm{Gal}(\mathscr{G}_{\mathrm{can}}) : r = 0, u = 0\}$: This is a 4-dimensional subgroup consisting of rotations and time translations.

(ix) $O(\ker(\tau))$: This is a 3-dimensional subgroup, also called the subgroup of rotations or the orthogonal subgroup.

(x) The set $\{(R, r, u, t) \in \mathrm{Gal}(\mathscr{G}_{\mathrm{can}}) : R = \mathrm{id}_{\ker(\tau)}, r = 0, t = 0\}$: This is a 3-dimensional Abelian subgroup consisting of pure velocity boosts.

(xi) The set $\{(R, r, u, t) \in \mathrm{Gal}(\mathscr{G}_{\mathrm{can}}) : R = \mathrm{id}_{\ker(\tau)}, u = 0, t = 0\}$: This is a 3-dimensional Abelian subgroup consisting of pure space translations.

(xii) $\mathbb{R}$: This is a 1-dimensional subgroup consisting of pure time translations. It is the quotient of $\mathrm{Gal}(\mathscr{G}_{\mathrm{can}})$ by the isochronous Galilean group $\mathrm{Gal}_0(\mathscr{G}_{\mathrm{can}})$ and is obviously Abelian.

Properties of the canonical Galilean group. In this section, we study some fundamental objects associated with the canonical group. We denote the Lie algebra of $\mathrm{Gal}(\mathscr{G}_{\mathrm{can}})$ by $\mathfrak{gal}(\mathscr{G}_{\mathrm{can}})$ and note that as a vector space it is given by $\mathfrak{o}(\ker(\tau)) \oplus \ker(\tau) \oplus \ker(\tau) \oplus \mathbb{R}$. The following result gives the Lie bracket on this Lie algebra.

4.4 Lemma: *The following statements hold:*

(i) the exponential map from $\mathfrak{gal}(\mathscr{G}_{\mathrm{can}})$ *to* $\mathrm{Gal}(\mathscr{G}_{\mathrm{can}})$ *is given by*

$$\exp(\hat{\omega}, \beta, \nu, \tau) = (\exp(\hat{\omega}), A_\omega(\beta) + \tau B_\omega(\nu), A_\omega(\nu), \tau),$$

where

$$\exp(\hat{\omega}) = \begin{cases} \mathrm{id}_{\ker(\tau)} + \sin\|\omega\| \frac{\hat{\omega}}{\|\omega\|} + (1 - \cos\|\omega\|) \frac{\hat{\omega}^2}{\|\omega\|^2}, & \omega \neq 0 \\ \mathrm{id}_{\ker(\tau)}, & \omega = 0 \end{cases}$$

and

$$A_\omega = \begin{cases} \mathrm{id}_{\ker(\tau)} + \left(\frac{1-\cos\|\omega\|}{\|\omega\|}\right) \frac{\hat{\omega}}{\|\omega\|} + \left(1 - \frac{\sin\|\omega\|}{\|\omega\|}\right) \frac{\hat{\omega}^2}{\|\omega\|^2}, & \omega \neq 0 \\ \mathrm{id}_{\ker(\tau)}, & \omega = 0 \end{cases}$$

and

$$B_\omega = \begin{cases} \frac{1}{2}\mathrm{id}_{\ker(\tau)} + \left(\frac{\|\omega\| - \sin\|\omega\|}{\|\omega\|^2}\right) \frac{\hat{\omega}}{\|\omega\|} + \left(\frac{\|\omega\|^2 - 2(1-\cos\|\omega\|)}{\|\omega\|^2}\right) \frac{\omega^2}{\|\omega\|^2}, & \omega \neq 0 \\ \frac{1}{2}\mathrm{id}_{\ker(\tau)}, & \omega = 0 \end{cases}$$

(ii) the Lie bracket in $\mathfrak{gal}(\mathscr{G}_{\text{can}})$ *is given by*

$$[(\hat{\omega}_1, \beta_1, \nu_1, \tau_1), (\hat{\omega}_2, \beta_2, \nu_2, \tau_2)] = \\ (\widehat{\omega_1 \times \omega_2}, \omega_1 \times \beta_2 - \omega_2 \times \beta_1 + \tau_2\nu_1 - \tau_1\nu_2, \omega_1 \times \nu_2 - \omega_1 \times \nu_1, 0).$$

Proof: We prove the lemma by faithfully representing $\text{Gal}(\mathscr{G}_{\text{can}})$ in a vector space, in which case the exponential map is the usual one for linear mappings, and the Lie bracket is simply the usual commutator of the linear mappings of this vector space. We let $W = \ker(\tau) \oplus \mathbb{R} \oplus \mathbb{R}$ and for $(R, r, u, t) \in \text{Gal}(\mathscr{G}_{\text{can}})$ define an isomorphism of W by

$$(\mu, \sigma, \xi) \mapsto (R(\mu) + \sigma u + \xi r, \sigma + \xi t, \xi).$$

One readily verifies that this is a homomorphism from $\text{Gal}(\mathscr{G}_{\text{can}})$ to $GL(W)$. To see that the representation is faithful, suppose that

$$(\mu, \sigma, \xi) \mapsto (R(\mu) + \sigma u + \xi r, \sigma + \xi t, \xi) = (\mu, \sigma, \xi)$$

for all $(\mu, \sigma, \xi) \in W$. Then we must have $\sigma + \xi t = 0$, for all $\sigma, \xi \in \mathbb{R}$, implying that $t = 0$. Similarly, $R(\mu) + \sigma u + \xi r = \mu$ for all $(\mu, \sigma, \xi) \in W$ implies that $r = 0, u = 0$, and $R = \text{id}_{\ker(\tau)}$. Thus the representation is faithful. In block matrix form, the representation of (R, r, u, t) on W is

$$\begin{bmatrix} R & u & r \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} \in GL(W).$$

(i) A computation gives

$$\exp \begin{bmatrix} \hat{\omega} & \beta & \nu \\ 0 & 0 & \tau \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \exp(\hat{\omega}) & A_\omega \nu & A_\omega \beta + \tau B_\omega \nu \\ 0 & 0 & \tau \\ 0 & 0 & 1 \end{bmatrix},$$

where

$$\begin{aligned} \exp(\hat{\omega}) &= \text{id}_{\ker(\tau)} + \hat{\omega} + \frac{\hat{\omega}^2}{2!} + \frac{\hat{\omega}^3}{3!} + \cdots \\ A_\omega &= \text{id}_{\ker(\tau)} + \frac{\hat{\omega}^2}{2!} + \frac{\hat{\omega}^3}{3!} + \frac{\hat{\omega}^4}{4!} + \cdots \\ B_\omega &= \frac{\text{id}_{\ker(\tau)}}{2!} + \frac{\hat{\omega}^3}{3!} + \frac{\hat{\omega}^4}{4!} + \frac{\hat{\omega}^5}{5!} + \cdots . \end{aligned}$$

Thus our stated formulae for A_ω and B_ω hold when $\omega = 0$. For $\omega \neq 0$ we recall the following readily verified identities for $\hat{\omega} \in \mathfrak{o}(\ker(\tau))$:

$$\begin{aligned} \hat{\omega}^2 &= \omega \otimes \omega^\flat - \|\omega\|^2 \text{id}_{\ker(\tau)} \\ \hat{\omega}^3 &= -\|\omega\|^2 \hat{\omega}, \end{aligned}$$

where $\omega \otimes \omega^\flat : \ker(\tau) \to \ker(\tau)$ is the linear map defined by

$$(\omega \otimes \omega^\flat)(u) = g(u, \omega)\omega.$$

We use these relations to inductively compute the higher powers of $\hat{\omega}$ and $\hat{\omega}^2$:

$$\hat{\omega}^{2k+1} = (-1)^k \|\omega\|^{2k}\hat{\omega}, \quad k \in \mathbb{N}$$
$$\hat{\omega}^{2(k+1)} = (-1)^k \|\omega\|^{2k}\hat{\omega}^2, \quad k \in \mathbb{N}$$

and after some computations we arrive at

$$\begin{aligned}
\exp(\hat{\omega}) &= \mathrm{id}_{\ker(\tau)} + \sin\|\omega\|\frac{\hat{\omega}}{\|\omega\|} + (1-\cos\|\omega\|)\frac{\hat{\omega}^2}{\|\omega\|^2} \\
A_\omega &= \mathrm{id}_{\ker(\tau)} + \left(\frac{1-\cos\|\omega\|}{\|\omega\|}\right)\frac{\hat{\omega}}{\|\omega\|} + \left(1 - \frac{\sin\|\omega\|}{\|\omega\|}\right)\frac{\hat{\omega}^2}{\|\omega\|^2} \\
B_\omega &= \frac{1}{2}\mathrm{id}_{\ker(\tau)} + \left(\frac{\|\omega\| - \sin\|\omega\|}{\|\omega\|^2}\right)\frac{\hat{\omega}}{\|\omega\|} + \left(\frac{\|\omega\|^2 - 2(1-\cos\|\omega\|)}{\|\omega\|^2}\right)\frac{\omega^2}{\|\omega\|^2}.
\end{aligned}$$

(ii) The above computations show that the Lie subalgebra $\mathfrak{gl}(W)$ of $GL(W)$, generated by $\mathfrak{gal}(\mathscr{G}_{\mathrm{can}})$ using the given representation consists of those matrices of the form

$$\begin{bmatrix} \hat{\omega} & \nu & \beta \\ 0 & 0 & \tau \\ 0 & 0 & 0 \end{bmatrix} \in \mathfrak{gl}(W),$$

where $(\hat{\omega}, \beta, \nu, \tau) \in \mathfrak{gal}(\mathscr{G}_{\mathrm{can}})$. The Lie bracket in $\mathfrak{gal}(\mathscr{G}_{\mathrm{can}})$ is now readily computed using the commutator for linear maps on W, using the fact that $\hat{} : \ker(\tau) \to \mathfrak{o}(\ker(\tau))$ is a Lie algebra isomorphism if the Lie bracket on $\ker(\tau)$ is $\times$. ■

4.2. The abstract Galilean group. In this section, we investigate the structure of the abstract Galilean group and its subgroups.

The structure of the Galilean group. In the usual presentation of Galilean invariant mechanics (e.g. [Souriau 1997]) one considers a spacetime $\mathbb{R}^3 \times \mathbb{R}$ and Galilean invariance is imposed by asking that the system admit the Galilean group as a symmetry group. In this case, the Galilean group naturally breaks down into rotations, Galilean boosts (constant velocity shifts) and temporal origin shifts. In our abstract setting, the Galilean group $\mathrm{Gal}(\mathscr{G})$ does not admit such a decomposition. Note that this is similar to what one sees in an affine Euclidean space where a decomposition of an isometry into rotation and translation is not possible until one chooses an origin about which to measure rotations. However, the presence of an observer in a Galilean spacetime defines, for each instant, an isomorphism from the abstract Galilean group $\mathrm{Gal}(\mathscr{G})$ into the canonical group $\mathrm{Gal}(\mathscr{G}_{\mathrm{can}})$.

4.5 Proposition: *Let $\mathscr{G} = (\mathscr{E}, V, g, \tau)$ be a Galilean spacetime with $\mathscr{O}$ an observer. The following statements hold.*

1. *The mapping from V to $\ker(\tau) \oplus \mathbb{R}$ defined by $v \mapsto (P_{\mathscr{O}}(v), \tau(v))$ is an isomorphism.*
2. *For each $s_0 \in I_{\mathscr{G}}$ the observer at s_0, $\mathscr{O}_{s_0}$, induces a natural isomorphism $\iota_{\mathscr{O}_{s_0}}$ from $\mathrm{Gal}(\mathscr{G})$ to the group $\mathrm{Gal}(\mathscr{G}_{\mathrm{can}})$. Explicitly, if $\psi \in \mathrm{Gal}(\mathscr{G})$ with t_ψ as defined in Lemma 2.18, and if $R_\psi \in O(\ker(\tau))$ and $r_{\psi,\mathscr{O}} \in \ker(\tau)$ satisfy*

$$\psi(x) = (R_\psi(x - \mathscr{O}_{s_0}) + r_{\psi,\mathscr{O}}) + \mathscr{O}_{t_\psi + s_0}, \quad x \in \mathscr{E}(s_0)$$

then

$$\iota_{\mathscr{O}_s}(\psi) = (R_\psi, r_{\psi,\mathscr{O}}, u_{\psi,\mathscr{O}}, t_\psi)$$

where $u_{\psi,\mathscr{O}} = P_{\mathscr{O}}(\psi_V(v_{\mathscr{O}}))$.

Proof: (1) It suffices to show that the mapping $v \mapsto (P_{\mathscr{O}}(v), \tau(v))$ is injective. If $\tau(v) = 0$ then $v \in \ker(\tau)$. Now, if we also have

$$P_{\mathscr{O}}(v) = v - (\tau(v))v_{\mathscr{O}} = 0,$$

we must have $v = 0$, thus the mapping is injective as desired.

(2) To prove this part of the proposition, we first assign to each $(R, r, u, t) \in \mathrm{Gal}(\mathscr{G}_{\mathrm{can}})$ a Galilean mapping ψ, and we show that the construction implies that $(R, r, u, t) = (R_\psi, r_\psi, u_\psi, t_\psi)$, thus showing that $\iota_{\mathscr{O}_{s_0}}$ is invertible. Given $(R, r, u, t) \in \mathrm{Gal}_{\mathrm{can}}(\mathscr{G})$, we define a map $\psi : \mathscr{E} \to \mathscr{E}$ by

$$\psi(x) = tv_{\mathscr{O}} + (R(x - \mathscr{O}_{\pi_{\mathscr{G}}(x)}) + (\pi_{\mathscr{G}}(x) - s_0)u + r) + \mathscr{O}_{\pi_{\mathscr{G}}(x)}. \tag{4.1}$$

We now show that this mapping is Galilean. First we show that it is affine. For $v \in V$ we compute

$$\begin{aligned}
\psi(v + \mathscr{O}_{s_0}) - \psi(\mathscr{O}_{s_0}) &= tv_{\mathscr{O}} + (R(v + \mathscr{O}_{s_0} - \mathscr{O}_{\tau(v)+s_0}) + ((\tau(v) + s_0) - s_0)u + r) \\
&\quad + \mathscr{O}_{\tau(v)+s} - (tv_{\mathscr{O}} + r + \mathscr{O}_{s_0}) \\
&= R(v + \mathscr{O}_{s_0} - (\tau(v)v_{\mathscr{O}} + \mathscr{O}_{s_0})) + \tau(v)(u + v_{\mathscr{O}}) \\
&= R(v - \tau(v)v_{\mathscr{O}}) + \tau(v)(u + v_{\mathscr{O}}) \\
&= R(P_{\mathscr{O}}(v)) + \tau(v)(u + v_{\mathscr{O}}).
\end{aligned} \tag{4.2}$$

Thus the map $v \mapsto \psi(v + \mathscr{O}_{s_0}) - \psi(\mathscr{O}_{s_0})$ is linear, so ψ is affine. Similarly, we calculate

$$\begin{aligned}
\tau(\psi(x_1) - \psi(x_2)) &= \tau(tv_{\mathscr{O}} + \mathscr{O}_{\pi_{\mathscr{G}}(x_1)}) - \tau(tv_{\mathscr{O}} + \mathscr{O}_{\pi_{\mathscr{G}}(x_2)}) \\
&= t + \pi_{\mathscr{G}}(x_1) - (t + \pi_{\mathscr{G}}(x_2)) \\
&= \tau(x_1 - x_2).
\end{aligned}$$

So GM2 is satisfied. Next, for $s_0 \in I_{\mathscr{G}}$, consider $y_1, y_2 \in \mathscr{E}(s)$. We compute

$$\begin{aligned}
\psi(y_1) - \psi(y_2) &= tv_{\mathscr{O}} + R(y_1 - \mathscr{O}_{s_0}) + (s_0 - s_0)u + r + \mathscr{O}_{s_0} \\
&\quad -(tv_{\mathscr{O}} + R(y_2 - \mathscr{O}_{s_0}) + (s_0 - s_0)u + r + \mathscr{O}_{s_0}) \\
&= R(y_1 - y_2)
\end{aligned}$$

So ψ satisfies GM3. Next we show that $(R, r, u, t) = (R_\psi, r_\psi, u_\psi, t_\psi)$. By restricting ψ to $\mathscr{E}(s_0)$ we get

$$\begin{aligned}
(\psi|\mathscr{E}(s_0))(x) &= tv_{\mathscr{O}} + (R(x - \mathscr{O}_{s_0}) + r) + \mathscr{O}_{s_0} \\
&= (R(x - \mathscr{O}_{s_0}) + r) + \mathscr{O}_{t+s_0}
\end{aligned}$$

However, the definition of R_ψ and r_ψ gives

$$R(x - \mathscr{O}_{s_0}) + r = R_\psi(x - \mathscr{O}_{s_0}) + r_{\psi,\mathscr{O}}$$

for each $x \in \mathscr{E}(s_0)$. Taking $x = \mathscr{O}_{s_0}$ gives $r = r_{\psi,\mathscr{O}}$ from which it follows that $R = R_\psi$. Also, for $x \in \mathscr{E}(s_0)$, we have

$$\psi_{I_{\mathscr{G}}}(s_0) = \pi_{\mathscr{G}}(\psi(x)) = t + s_0.$$

From Lemma 2.18, it follows that $t = t_\psi$. From (4.2) we also have

$$P_{\mathscr{O}}(\psi_V(v_{\mathscr{O}})) = R(P_{\mathscr{O}}(v_{\mathscr{O}})) + \tau(v_{\mathscr{O}})u = u$$

using the fact that $P_{\mathscr{O}}(v_{\mathscr{O}}) = 0$. This shows that $u = u_{\psi,\mathscr{O}}$. We have now shown that for every $(R, r, u, t) \in \mathrm{Gal}(\mathscr{G}_{\mathrm{can}})$ there is a Galilean mapping ψ such that $\iota_{\mathscr{O}_{s_0}}(\psi) = (R, r, u, t)$. Thus we have shown that $\iota_{\mathscr{O}_{s_0}}$ is surjective. Next we show that it is injective. For this, let $\tilde{\psi} \in \mathrm{Gal}(\mathscr{G})$ be such that, for $x \in \mathscr{E}$,

$$\tilde{\psi}(x) = tv_{\mathscr{O}} + (R(x - \mathscr{O}_{s_0}) + (\pi_{\mathscr{G}}(x) - s_0)u + r) + \mathscr{O}_{\pi_{\mathscr{G}}(x)}.$$

that is, suppose

$$\iota_{\mathscr{O}_{s_0}}(\tilde{\psi}) = (R, r, u, t) = \iota_{\mathscr{O}_{s_0}}(\psi).$$

We shall show that $\tilde{\psi} = \psi$. Since $\tilde{\psi}(\mathscr{O}_{s_0}) = \psi(\mathscr{O}_{s_0})$, using (4.2) this will follow if we can show that $\tilde{\psi}_V = \psi_V$. As in (i) we note that $V \simeq \ker(\tau) \oplus \mathbb{R}$ and the pre-image of (u, t) under this isomorphism is $u + tv_{\mathscr{O}}$. We also write

$$\psi_V(u + tv_{\mathscr{O}}) = A_{11}(u) + A_{12}(t) + (A_{21}(u) + A_{22}(t))v_{\mathscr{O}}$$

for linear mappings $A_{11} : \ker(\tau) \to \ker(\tau), A_{12} : \mathbb{R} \to \ker(\tau), A_{21} : \ker(\tau) \to \mathbb{R}$, and $A_{22} : \mathbb{R} \to \mathbb{R}$. The property GM2 of Galilean mappings implies that ψ_V has $\ker(\tau)$ as an invariant subspace. Thus $A_{21} = 0$. We next calculate

$$\begin{aligned}\tau(\psi_V(tv_{\mathscr{O}})) &= \tau(\psi(tv_{\mathscr{O}} + \mathscr{O}_{s_0})) - \tau(\psi(\mathscr{O}_{s_0})) \\ &= \tau(tv_{\mathscr{O}} + \mathscr{O}_{s_0}) - \tau(\mathscr{O}_{s_0}) = t.\end{aligned}$$

This gives $A_{22} = t$. With $t = 0$ property GM3 implies that $A_{11} \in O(\ker(\tau))$. Thus we have

$$\psi_V(u + tv_{\mathscr{O}}) = \tilde{R}(u) + t(\tilde{u} + v_{\mathscr{O}}) \tag{4.3}$$

for some $\tilde{R} \in O(\ker(\tau))$ and $\tilde{u} \in \ker(\tau)$. Since $\psi|\mathscr{E}(s) = \tilde{\psi}|\mathscr{E}(s)$ we have

$$\psi_V(u) = \tilde{\psi}(u + \mathscr{O}_{s_0}) - \tilde{\psi}(\mathscr{O}_{s_0}) = R_\psi(u),$$

giving $\tilde{R} = R$. From (4.2) we also have

$$P_{\mathscr{O}}(\psi_V(v_{\mathscr{O}}) = \tilde{u}$$

from which we get $u = \tilde{u}$. This shows that $\psi_V = \tilde{\psi}_V$ thus showing that if $\psi_1, \psi_2 \in \mathrm{Gal}(\mathscr{G})$ satisfy $\iota_{\mathscr{O}_{s_0}}(\psi_1) = \iota_{\mathscr{O}_{s_0}}(\psi_2)$, then $\psi_1 = \psi_2$. Therefore $\iota_{\mathscr{O}_{s_0}}$ is injective.

Finally we show that $\iota_{\mathscr{O}_{s_0}}$ is a homomorphism. We let $\psi_1, \psi_2 \in \mathrm{Gal}(\mathscr{G})$ and denote $\iota_{\mathscr{O}}(\psi_i) = (R_i, r_i, u_i, t_i), i = 1, 2$. We also let $\iota_{\mathscr{O}_{s_0}}(\psi_1 \circ \psi_2) = (R_{12}, r_{12}, u_{12}, t_{12})$. First, we compute

$$\begin{aligned}
(\psi_1 \circ \psi_2)_V(v) &= (\psi_1 \circ \psi_2)(v + \mathscr{O}_{s_0}) - (\psi_1 \circ \psi_2)(\mathscr{O}_{s_0}) \\
&= \psi_1(\psi_2(v + \mathscr{O}_{s_0})) - \psi_1(\psi_2(\mathscr{O}_{s_0})) \\
&= \psi_1(tv_{\mathscr{O}} + R_2(v + \mathscr{O}_{s_0} - \mathscr{O}_{\tau(v)+s_0}) + \tau(v)u_2 + r_2 + \mathscr{O}_{\tau(v)+s_0}) \\
&\quad - \psi_1(tv_{\mathscr{O}} + r + \mathscr{O}_{s_0}) \\
&= \psi_1((tv_{\mathscr{O}} + r) + R_2(P_{\mathscr{O}}(v)) + \tau(v)(u_2 + v_{\mathscr{O}}) + \mathscr{O}_{s_0}) \\
&\quad - \psi_1(tv_{\mathscr{O}} + r_2 + \mathscr{O}_{s_0}) \\
&= \psi_{1,V}(R_2(P_{\mathscr{O}}(v)) + \tau(v)(u_2 + v_{\mathscr{O}})) \\
&= \psi_{1,V}(\psi_{2,V}(v))
\end{aligned}$$

From this we deduce that

$$R_{12}(u) + t(u_{12} + v_{\mathscr{O}}) = R_1 \circ R_2(u) + t(u_1 + R_1(u_2) + v_{\mathscr{O}})$$

for each $(u, t) \in \ker(\tau) \times \mathbb{R}$. Thus we have

$$R_{12} = R_1 \circ R_2, \quad u_{12} = u_1 + R_1(u_2).$$

Next we have

$$\psi_1 \circ \psi_2(\mathscr{O}_{s_0}) = r_{12} + \mathscr{O}_{t_{12}+s_0}. \tag{4.4}$$

Also,

$$\psi_2(\mathscr{O}_{s_0}) = r_2 + \mathscr{O}_{t_2+s_0}$$

Therefore

$$\begin{aligned}
\psi_1 \circ \psi_2(\mathscr{O}_{s_0}) &= \psi_1(r_2 + \mathscr{O}_{t_2+s_0}) \\
&= R_1(r_2 + \mathscr{O}_{t_2+s_0} - \mathscr{O}_{t_2+s_0}) + t_2u_1 + r_1 + \mathscr{O}_{s_0+t_2+t_1} \\
&= R_1r_2 + t_2u_1 + r_1 + \mathscr{O}_{s_0+t_1+t_2}
\end{aligned}$$

comparing this to (4.2.3) we get

$$r_{12} = R_1r_2 + t_2u_1 + r_1, \quad t_{12} = t_1 + t_2.$$

Thus we have shown that the group action defined on $\mathrm{Gal}(\mathscr{G}_{\mathrm{can}})$ agrees with that on $\mathrm{Gal}(\mathscr{G})$ under the bijection $\iota_{\mathscr{O}_{s_0}}$. ■

Next, given a motion Ψ in a Galilean spacetime $\mathscr{G} = (\mathscr{E}, V, g, \tau)$, we explore how the quantities $r_{\Psi,\mathscr{O}}$ and $u_{\Psi,\mathscr{O}}$ change when we use a different observer.

4.6 Proposition: *Let Ψ be a motion in a Galilean spacetime $\mathscr{G} = (\mathscr{E}, V, g, \tau)$ and let $\mathscr{O}$ and $\tilde{\mathscr{O}}$ be two observers. For $s_0 \in I_{\mathscr{G}}$, , let $(R_\Psi(t), r_{\Psi,\mathscr{O}}(t), u_{\Psi,\mathscr{O}}(t), t_\Psi)$ and $(R_\Psi(t), \tilde{r}_{\Psi,\tilde{\mathscr{O}}}(t), \tilde{u}_{\Psi,\tilde{\mathscr{O}}}(t), t_\Psi)$ be the images of Ψ_t under the isomorphisms between $\mathrm{Gal}(\mathscr{G})$ and $\mathrm{Gal}(\mathscr{G}_{\mathrm{can}})$ induced by $\mathscr{O}$ and $\tilde{\mathscr{O}}$ respectively. Then,*

$$\begin{aligned}
\tilde{r}_{\Psi,\tilde{\mathscr{O}}}(t) &= r_{\Psi,\mathscr{O}}(t) + (\mathrm{id}_{\ker(\tau)} - R_\Psi(t))(\mathscr{O}_{s_0} - \tilde{\mathscr{O}}_{s_0}) + t_\Psi(v_{\mathscr{O}} - v_{\tilde{\mathscr{O}}}) \\
\tilde{u}_{\Psi,\tilde{\mathscr{O}}}(t) &= u_{\Psi,\mathscr{O}}(t) + (\mathrm{id}_{\ker(\tau)} - R_\Psi(t))(v_{\mathscr{O}} - v_{\tilde{\mathscr{O}}})
\end{aligned}$$

Proof: For $x \in \mathscr{E}(s_0)$ we have

$$\begin{aligned}
\Psi_t(x) &= R_\Psi(t)(x - \tilde{\mathscr{O}}_{s_0}) + \tilde{r}_{\Psi,\tilde{\mathscr{O}}}(t) + \tilde{\mathscr{O}}_{s_0+t_\Psi} \\
&= R_\Psi(t)(x - \mathscr{O}_{s_0}) + r_{\Psi,\mathscr{O}}(t) + \mathscr{O}_{s_0+t_\Psi}
\end{aligned}$$

This means

$$R_\Psi(t)(x - \tilde{\mathscr{O}}_{s_0}) - R_\Psi(t)(x - \mathscr{O}_{s_0}) + (\tilde{r}_{\Psi,\tilde{\mathscr{O}}}(t) - r_{\Psi,\mathscr{O}}(t)) + (\tilde{\mathscr{O}}_{t_\Psi+s_0} - \mathscr{O}_{t_\Psi+s_0}) = 0.$$

for all $x \in \mathscr{E}(s_0)$. Letting $x = \tilde{\mathscr{O}}_{s_0}$ we get

$$\begin{aligned}
\tilde{r}_{\Psi,\tilde{\mathscr{O}}}(t) &= r_{\Psi,\mathscr{O}}(t) + R_\Psi(t)(\tilde{\mathscr{O}}_{s_0} - \mathscr{O}_{s_0}) + (\mathscr{O}_{t_\Psi+s_0} - \tilde{\mathscr{O}}_{t_\Psi+s_0}) \\
&= r_{\Psi,\mathscr{O}}(t) + R_\Psi(t)(\tilde{\mathscr{O}}_{s_0} - \mathscr{O}_{s_0}) + (\mathscr{O}_{s_0} - \tilde{\mathscr{O}}_{s_0}) + t(v_{\mathscr{O}} - v_{\tilde{\mathscr{O}}})
\end{aligned}$$

which is the required expression.

Now, by definition

$$\begin{aligned}
\tilde{u}_{\Psi,\tilde{\mathscr{O}}}(t) &= P_{\tilde{\mathscr{O}}}(\Psi_{t,V}(v_{\tilde{\mathscr{O}}})) \\
&= \Psi_{t,V}(v_{\tilde{\mathscr{O}}}) - v_{\tilde{\mathscr{O}}}.
\end{aligned}$$

Similarly,

$$u_{\Psi,\mathscr{O}}(t) = \Psi_{t,V}(v_{\mathscr{O}}) - v_{\mathscr{O}}.$$

Therefore,

$$\begin{aligned}
\tilde{u}_{\Psi,\tilde{\mathscr{O}}}(t) - u_{\Psi,\mathscr{O}}(t) &= \Psi_{t,V}(v_{\tilde{\mathscr{O}}} - v_{\mathscr{O}}) + v_{\mathscr{O}} - v_{\tilde{\mathscr{O}}} \\
&= R_\Psi(t)(v_{\tilde{\mathscr{O}}} - v_{\mathscr{O}}) + (v_{\mathscr{O}} - v_{\tilde{\mathscr{O}}}) \\
&= (\mathrm{id}_{\ker(\tau)} - R_\Psi(t))(v_{\mathscr{O}} - v_{\tilde{\mathscr{O}}})
\end{aligned}$$

which gives the expression for $\tilde{u}_{\Psi,\tilde{\mathscr{O}}}$. ■

Subgroups of the abstract Galilean group. In this section we wish to characterize the various subgroups of the abstract Galilean group $\mathrm{Gal}(\mathscr{G})$. First of all we recall that the action of V on $\mathscr{E}$ is affine so GM1 is satisfied. It is easy to see that GM2 and GM3 are also satisfied and that V is a subgroup of $\mathrm{Gal}(\mathscr{G})$. The following result is easily verified.

4.7 Lemma: *Let $\mathscr{G} = (\mathscr{E}, V, g, \tau)$ be a Galilean spacetime and let $\mathrm{Gal}(\mathscr{G})$ be the Galilean group. The following are subgroups of $\mathrm{Gal}(\mathscr{G})$:*

(i) The set $\mathrm{Gal}_0(\mathscr{G}) := \{\psi \in \mathrm{Gal}(\mathscr{G}) : \psi_{I_{\mathscr{G}}} = \mathrm{id}_{I_{\mathscr{G}}}\}$

(ii) The set $\mathrm{Gal}_{\mathrm{an}}(\mathscr{G}) := \{\psi \in \mathrm{Gal}(\mathscr{G}) : \psi_V|_{\ker(\tau)} = \mathrm{id}_{\ker(\tau)}\}$

(iii) The set $\mathrm{Gal}_{0\mathrm{b}}(\mathscr{G}) := \mathrm{Gal}_{\mathrm{an}}(\mathscr{G}) \cap \mathrm{Gal}_0(\mathscr{G})$

(iv) V.

4.8 Remarks: (i) Following the discussion on the subgroups of the canonical Galilean group, we call $\mathrm{Gal}_0(\mathscr{G})$ the *isochronous* Galilean group.

(ii) The notation used here is not altogether obvious. It is meant to be consistent with the the one used for the subgroups of the canonical Galilean group. $\mathrm{Gal}_{\mathrm{an}}(\mathscr{G})$ therefore denotes the *anisotropic* Galilean group.

The following result shows that these subgroups have an additional property.

4.9 Lemma: *Let $\mathscr{G} = (\mathscr{E}, V, g, \tau)$ be a Galilean spacetime and let $\mathrm{Gal}(\mathscr{G})$ be the Galilean group. The subgroups $V, \mathrm{Gal}_0(\mathscr{G})$ and $\mathrm{Gal}_{\mathrm{an}}(\mathscr{G})$ and are normal subgroups of $\mathrm{Gal}(\mathscr{G})$.*

Proof: First, let us show that V is a normal subgroup. For this, let $\psi \in \mathrm{Gal}(\mathscr{G})$ and $\phi \in V$. We wish to show that $\psi \circ \phi \circ \psi^{-1} \in V$. Choose $x \in \mathscr{E}$. Since V acts on $\mathscr{E}$ by translations, there exists some $v_\psi \in V$ such that

$$\psi \circ \phi \circ \psi^{-1}(x) = \psi \circ (v_\psi + \psi^{-1}(x)).$$

This means that

$$\begin{aligned}\psi \circ \phi \circ \psi^{-1}(x) - x &= \psi \circ \phi \circ \psi^{-1}(x) - \psi(\psi^{-1}(x)) \\ &= \psi(v_\psi + \psi^{-1}(x)) - \psi(\psi^{-1}(x)) \\ &= \psi_V(v_\psi)\end{aligned}$$

which implies that $\psi \circ \phi \circ \psi^{-1}(x) = x + \psi_V(v_\psi)$, thus showing that $\psi \circ \phi \circ \psi^{-1} \in V$. This is what we wanted to show.

Next, consider $x \in \mathscr{E}$. It can be shown that for $\psi_1, \psi_2 \in \mathrm{Gal}(\mathscr{G})$, we have $(\psi_1 \circ \psi_2)_{I_{\mathscr{G}}} = \psi_{1,I_{\mathscr{G}}}\psi_{2,I_{\mathscr{G}}}$. To see this we recall that for any $\psi \in \mathrm{Gal}(\mathscr{G})$, the definition of $\psi_{I_{\mathscr{G}}}$ gives

$$\begin{aligned}(\psi_1 \circ \psi_2)_{I_{\mathscr{G}}}(\pi_{\mathscr{G}}(x)) = \pi_{\mathscr{G}}(\psi_1(\psi_2(x))) &= \psi_{1,I_{\mathscr{G}}}(\pi_{\mathscr{G}}(\psi_2(x))) \\ &= \psi_{1,I_{\mathscr{G}}}(\psi_{2,I_{\mathscr{G}}}(\pi_{\mathscr{G}}(x))\end{aligned}$$

which is what we want. Next, for $\phi \in \mathrm{Gal}_0(\mathscr{G})$, we have

$$(\psi \circ \phi \circ \psi^{-1})_{I_{\mathscr{G}}} = \psi_{I_{\mathscr{G}}} \circ \psi_{I_{\mathscr{G}}}^{-1} = \mathrm{id}_{I_{\mathscr{G}}}$$

This shows that $\psi \circ \phi \circ \psi^{-1} \in \mathrm{Gal}_0(\mathscr{G})$ thus showing that $\mathrm{Gal}_0(\mathscr{G})$ is a normal subgroup. Finally, consider $\phi \in \mathrm{Gal}_{\mathrm{an}}(\mathscr{G})$. We need only show that $(\psi \circ \phi \circ \psi^{-1})_V = \mathrm{id}_{\ker(\tau)}$. However, this follows since $(\psi_1 \circ \psi_2)_V = \psi_{1,V} \circ \psi_{2,V}$, $\forall \psi_1, \psi_2 \in \mathrm{Gal}(\mathscr{G})$ and the proof is complete. ■

4.10 Corollary: *Let $\mathscr{G} = (\mathscr{E}, V, g, \tau)$ be a Galilean spacetime. The groups* $\ker(\tau)$ *and* $\mathrm{Gal}_{0\mathrm{b}}(\mathscr{G})$ *are normal subgroups of* $\mathrm{Gal}(\mathscr{G})$.

Proof: We claim that $\ker(\tau) = \mathrm{Gal}_0(\mathscr{G}) \cap V$. To see this, let $\phi \in \mathrm{Gal}_0(\mathscr{G}) \cap V$. This means that for every $x \in \mathscr{E}$, there exists some $v_\phi \in V$ such that $\phi(x) = x + v_\phi$ *and* $\phi_{I_{\mathscr{G}}} = \mathrm{id}_{I_{\mathscr{G}}}$. By definition, this means that

$$\begin{aligned}\phi_{I_{\mathscr{G}}}(\pi_{\mathscr{G}}(x)) &= \pi_{\mathscr{G}}(\phi(x)) \\ &= \pi_{\mathscr{G}}(x) + \tau(v_\phi) = \pi_{\mathscr{G}}(x)\end{aligned}$$

which gives $\tau(v_\phi) = 0$. The result now follows from the fact that the intersection of two normal subgroups is itself normal. ■

4.11 Remarks: (i) The subgroups defined in the above propositions are *natural* subgroups of $\mathrm{Gal}(\mathscr{G})$ because they do not depend on any particular choice of points in $\mathscr{E}$ or vectors in V. By making these choices, additional subgroups of $\mathrm{Gal}(\mathscr{G})$ can be defined. We shall, however, not deal with this situation.

(ii) Similar to the case of the isochronous canonical Galilean group $\mathrm{Gal}_0(\mathscr{G}_{\mathrm{can}})$, the subgroup $\mathrm{Gal}_0(\mathscr{G})$ is a normal subgroup of $\mathrm{Gal}(\mathscr{G})$.

Next, we look at the quotients of $\mathrm{Gal}(\mathscr{G})$ by these normal subgroups. We recall that the quotient G/H of a group G by its normal subgroup H is also a group.

4.12 Proposition: *Let $\mathscr{G} = (\mathscr{E}, V, g, \tau)$ be a Galilean spacetime and let the subgroups* $\mathrm{Gal}_0(\mathscr{G}), \mathrm{Gal}_{\mathrm{an}}(\mathscr{G})$ *and* $\mathrm{Gal}_{0\mathrm{b}}(\mathscr{G})$ *be as before. The following statements hold.*

(i) $\mathrm{Gal}(\mathscr{G})/\mathrm{Gal}_0(\mathscr{G}) \simeq \mathbb{R}$.

(ii) $\mathrm{Gal}(\mathscr{G})/V = \{\psi_V : \psi \in \mathrm{Gal}(\mathscr{G})\}$.

(iii) $\mathrm{Gal}(\mathscr{G})/\mathrm{Gal}_{0\mathrm{b}}(\mathscr{G}) \simeq O(\ker(\tau)) \times \mathbb{R}$.

(iv) $\mathrm{Gal}(\mathscr{G})/\mathrm{Gal}_{\mathrm{an}}(\mathscr{G}) \simeq O(\ker(\tau))$.

Proof: (i) Let $\psi \in \mathrm{Gal}(\mathscr{G})$. Recall that there exists a unique $t_\psi \in \mathbb{R}$ such that $\psi_{I_{\mathscr{G}}}(s) = s + t_\psi$. Now, the orbit of ψ is given by

$$\mathrm{orb}_{\mathrm{Gal}_0(\mathscr{G})}(\psi) = \{\psi \circ \phi : \phi \in \mathrm{Gal}_0(\mathscr{G})\}.$$

Since $\mathrm{Gal}_0(\mathscr{G})$ consists of "instant-preserving" Galilean maps, we know that for any $\tilde{\psi} \in \mathrm{orb}_{\mathrm{Gal}_0(\mathscr{G})}(\psi)$, we have $\tilde{\psi}_{I_{\mathscr{G}}} = \psi_{I_{\mathscr{G}}}$. We therefore have the canonical projection

$$\begin{aligned}\pi_{\mathrm{Gal}_0(\mathscr{G})} : \mathrm{Gal}(\mathscr{G}) &\to \mathrm{Gal}(\mathscr{G})/\mathrm{Gal}_0(\mathscr{G}) \\ \psi &\mapsto t_\psi\end{aligned}$$

This proves the first part.

(ii) For $\psi \in \mathrm{Gal}(\mathscr{G})$ the orbit of ψ under V is

$$\mathrm{orb}_V(\psi) = \{\psi \circ \phi : \phi \in V\}$$

Notice that for $\phi \in V$,

$$\phi_V(w) = \phi(w+x) - \phi(x) = (w+x) + v_\phi - (x + v_\phi) = w, \ \ \forall w \in V.$$

That is, $\phi_V = \mathrm{id}_V$. For any $\tilde{\psi} \in \mathrm{orb}_V(\psi)$ we therefore have $\tilde{\psi}_V = \psi_V$. Thus, we have the canonical projection

$$\begin{aligned} \pi_V : \mathrm{Gal}(\mathscr{G}) &\to \mathrm{Gal}(\mathscr{G})/V \\ \psi &\mapsto \psi_V \end{aligned}$$

which is what we wanted to show.

(iii) In a similar manner, for $\psi \in \mathrm{Gal}(\mathscr{G})$, we form the orbit $\mathrm{orb}_{\mathrm{Gal}_{0\mathrm{b}}(\mathscr{G})}(\psi)$ and observe that for any $\tilde{\psi} \in \mathrm{orb}_{\mathrm{Gal}_{0\mathrm{b}}(\mathscr{G})}(\psi)$, we have $R_{\tilde{\psi}} = R_\psi$ and $t_{\tilde{\psi}} = t_\psi$. The canonical projection is thus written as

$$\begin{aligned} \pi_{\mathrm{Gal}_{0\mathrm{b}}(\mathscr{G})} : \mathrm{Gal}(\mathscr{G}) &\to \mathrm{Gal}(\mathscr{G})/\mathrm{Gal}_0(\mathscr{G}) \\ \psi &\mapsto (R_\psi, t_\psi) \end{aligned}$$

as desired.

(v) It is easy to see that given $\psi \in \mathrm{Gal}(\mathscr{G})$, for any $\tilde{\psi} \in \mathrm{orb}_{\mathrm{Gal}_{\mathrm{an}}(\mathscr{G})}(\psi)$, we have $R_{\tilde{\psi}} = R_\psi$ and therefore the projection map is given by

$$\begin{aligned} \pi_{\mathrm{Gal}_{\mathrm{an}}(\mathscr{G})} : \mathrm{Gal}(\mathscr{G}) &\to \mathrm{Gal}(\mathscr{G})/\mathrm{Gal}_{\mathrm{an}}(\mathscr{G}) \\ \psi &\mapsto R_\psi \end{aligned}$$

This completes the proof. ■

4.3. Total velocities. Consider a rigid motion Ψ in a Galilean spacetime $\mathscr{G} = (\mathscr{E}, V, g, \tau)$. We first notice that $\dot{\Psi}_t \in T_{\Psi_t}\mathrm{Gal}(\mathscr{G})$. Therefore $T_{\Psi_t}L_{\Psi_t^{-1}}(\dot{\Psi}_t) \in \mathfrak{gal}(\mathscr{G})$ and $T_{\Psi_t}R_{\Psi_t^{-1}}(\dot{\Psi}_t) \in \mathfrak{gal}(\mathscr{G})$ (here L_{Ψ_t} and R_{Ψ_t} are as usual the left and right translation maps on $\mathrm{Gal}(\mathscr{G})$). We shall call these respectively, the total body velocity and the total spatial velocity of the rigid motion Ψ. The following result provides the decomposition of total spatial and body velocities in the presence of an observer.

4.13 Proposition: *Let $\mathscr{G} = (\mathscr{E}, V, g, \tau)$ be a Galilean spacetime with $\mathscr{O}$ an observer and Ψ a rigid motion. For $s_0 \in I_{\mathscr{G}}$, let $\iota_{\mathscr{O}_{s_0}}(\Psi_t) = (R_\Psi(t), r_{\Psi,\mathscr{O}}(t), u_{\Psi,\mathscr{O}}(t), t)$ Then, the following statements hold.*

(i) The image of $T_{\Psi_t}L_{\Psi_t^{-1}}(\dot{\Psi}_t) \in \mathfrak{gal}(\mathscr{G})$ under the isomorphism of the Lie algebras induced by $\iota_{\mathscr{O}_{s_0}}$ is

$$(\hat{\Omega}_\Psi(t), V_{\Psi,\mathscr{O}}(t) - R_\Psi^{-1}(t)u_{\Psi,\mathscr{O}}(t), R_\Psi^{-1}(t)\dot{u}_{\Psi,\mathscr{O}}(t), 1) \in \mathfrak{gal}(\mathscr{G}_{\mathrm{can}}).$$

(ii) The image of $T_{\Psi_t}(R_{\Psi_t^{-1}}\dot{\Psi}_t) \in \mathfrak{gal}(\mathscr{G})$ under the isomorphism of Lie algebra induced by $\iota_{\mathscr{O}_{s_0}}$ is

$$(\hat{\omega}_\Psi(t), v_{\Psi,\mathscr{O}}(t) - t(\dot{u}_{\Psi,\mathscr{O}}(t) + u_{\Psi,\mathscr{O}}(t) \times \omega_\Psi(t)), \dot{u}_{\Psi,\mathscr{O}}(t) + u_{\Psi,\mathscr{O}}(t) \times \omega_\Psi(t), 1) \in \mathfrak{gal}(\mathscr{G}_{\mathrm{can}}).$$

where $V_{\Psi,\mathscr{O}}(t) = R_\Psi^{-1}(t)\dot{r}_{\Psi,\mathscr{O}}(t)$ and $v_{\Psi,\mathscr{O}}(t) = \dot{r}_{\Psi,\mathscr{O}}(t) + r_{\Psi,\mathscr{O}}(t) \times \omega_\Psi(t)$.

Proof: We use the representation of $\mathrm{Gal}(\mathscr{G}_{\mathrm{can}})$ in the vector space W as described earlier. We then compute

$$\begin{bmatrix} R & u & r \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} R^{-1} & -R^{-1}u & R^{-1}(tu-r) \\ 0 & 1 & -t \\ 0 & 0 & 1 \end{bmatrix}.$$

With this expression both parts follow from direct computation. ■

We call the components in (i) as respectively the *total body angular velocity*, the *total body linear velocity* and the *total body acceleration* and denote them by $\Omega_\Psi^{\mathrm{total}}, V_{\Psi,\mathscr{O}}^{\mathrm{total}}$ and $A_{\Psi,\mathscr{O}}^{\mathrm{total}}$ respectively. Similarly, we call the components in (ii) as respectively the *total spatial angular velocity*, the *total spatial linear velocity* and the *total spatial acceleration* and represent them by $\omega_\Psi^{\mathrm{total}}, v_{\Psi,\mathscr{O}}^{\mathrm{total}}$ and $a_{\Psi,\mathscr{O}}^{\mathrm{total}}$ respectively . The next result shows how these velocities change with the observer.

4.14 Proposition: *Let Ψ be a rigid motion in a Galilean spacetime $\mathscr{G} = (\mathscr{E}, V, g, \tau)$ and let $\mathscr{O}$ and $\tilde{\mathscr{O}}$ be two observers. For fixed $s_0 \in I_{\mathscr{G}}$, let $V_{\Psi,\mathscr{O}}, V_{\Psi,\tilde{\mathscr{O}}}, v_{\Psi,\mathscr{O}}$ and $v_{\Psi,\tilde{\mathscr{O}}}$ be as defined in Proposition 4.13. Then,*

$$\begin{aligned}
\tilde{v}_{\Psi,\tilde{\mathscr{O}}}(t) &= v_{\Psi,\mathscr{O}}(t) + (\mathscr{O}_{t+s_0} - \tilde{\mathscr{O}}_{t+s_0}) \times \omega_\Psi(t) + v_{\mathscr{O}} - v_{\tilde{\mathscr{O}}} \\
\tilde{V}_{\Psi,\tilde{\mathscr{O}}}(t) &= V_{\Psi,\mathscr{O}}(t) + (\mathscr{O}_{s_0} - \tilde{\mathscr{O}}_{s_0}) \times \Omega_\Psi(t) + R_\Psi^{-1}(t)(v_{\mathscr{O}} - v_{\tilde{\mathscr{O}}}) \\
\tilde{v}_{\Psi,\tilde{\mathscr{O}}}^{\mathrm{total}}(t) &= v_{\Psi,\mathscr{O}}^{\mathrm{total}}(t) + (\mathscr{O}_{s_0} - \tilde{\mathscr{O}}_{s_0}) \times \omega_\Psi(t) + v_{\mathscr{O}} - v_{\tilde{\mathscr{O}}} \\
\tilde{V}_{\Psi,\tilde{\mathscr{O}}}^{\mathrm{total}}(t) &= V_{\Psi,\mathscr{O}}^{\mathrm{total}}(t) + (\mathscr{O}_{s_0} - \tilde{\mathscr{O}}_{s_0}) \times \Omega_\Psi(t) + v_{\mathscr{O}} - v_{\tilde{\mathscr{O}}} \\
\tilde{a}_{\Psi,\tilde{\mathscr{O}}}^{\mathrm{total}}(t) &= a_{\Psi,\mathscr{O}}^{\mathrm{total}}(t) + (v_{\mathscr{O}} - v_{\tilde{\mathscr{O}}}) \times \omega_\Psi(t) \\
\tilde{A}_{\Psi,\tilde{\mathscr{O}}}^{\mathrm{total}}(t) &= A_{\Psi,\mathscr{O}}^{\mathrm{total}}(t) + (v_{\mathscr{O}} - v_{\tilde{\mathscr{O}}}) \times \Omega_\Psi(t)
\end{aligned}$$

Proof:

$$\begin{aligned}
\tilde{v}_{\Psi,\tilde{\mathscr{O}}}(t) &= \dot{r}_{\Psi,\tilde{\mathscr{O}}}(t) + \tilde{r}_{\Psi,\tilde{\mathscr{O}}}(t) \times \omega_\Psi(t) \\
&= \dot{r}_{\Psi,\mathscr{O}}(t) + \dot{R}_\Psi(t)(\tilde{\mathscr{O}}_{s_0} - \mathscr{O}_{s_0}) + v_{\mathscr{O}} - v_{\tilde{\mathscr{O}}} + r_{\Psi,\mathscr{O}}(t) \times \omega_\Psi(t) \\
&\quad + ((R_\Psi(t) - \mathrm{id}_{\ker(\tau)}(\tilde{\mathscr{O}}_{s_0} - \mathscr{O}_{s_0})) \times \omega_\Psi(t) + t(v_{\mathscr{O}} - v_{\tilde{\mathscr{O}}}) \times \omega_\Psi(t) \\
&= v_{\Psi,\mathscr{O}}(t) + \hat{\omega}_\Psi(t) \circ R_\Psi(t)(\tilde{\mathscr{O}}_{s_0} - \mathscr{O}_{s_0}) + (R_\Psi(t)(\tilde{\mathscr{O}}_{s_0} - \mathscr{O}_{s_0})) \times \omega_\Psi(t) \\
&\quad + (\mathscr{O}_{s_0} - \tilde{\mathscr{O}}_{s_0}) \times \omega_\Psi(t) + t(v_{\mathscr{O}} - v_{\tilde{\mathscr{O}}}) \times \omega_\Psi(t) \\
&= v_{\Psi,\mathscr{O}}(t) + (\mathscr{O}_{t+s_0} - \tilde{\mathscr{O}}_{t+s_0}) \times \omega_\Psi(t) + v_{\mathscr{O}} - v_{\tilde{\mathscr{O}}}
\end{aligned}$$

Next,

$$\begin{aligned}
\tilde{V}_{\Psi,\tilde{\mathscr{O}}}(t) &= R_\Psi^{-1}(t)(\dot{r}_{\Psi,\tilde{\mathscr{O}}}(t)) \\
&= R_\Psi^{-1}(t)(\dot{r}_{\Psi,\mathscr{O}}(t) + \dot{R}_\Psi(t)(\tilde{\mathscr{O}}_{s_0} - \mathscr{O}_{s_0}) + (v_{\mathscr{O}} - v_{\tilde{\mathscr{O}}}) \\
&= V_{\Psi,\mathscr{O}}(t) + \Omega_\Psi(t) \times (\tilde{\mathscr{O}}_{s_0} - \mathscr{O}_{s_0}) + R^{-1}(t)(v_{\mathscr{O}} - v_{\tilde{\mathscr{O}}}),
\end{aligned}$$

which is the required expression. Now,

$$\begin{aligned}
\tilde{v}^{\text{total}}_{\Psi,\tilde{\mathscr{O}}}(t) &= \tilde{v}_{\Psi,\tilde{\mathscr{O}}}(t) - t(\dot{\tilde{u}}_{\Psi,\tilde{\mathscr{O}}}(t) + \tilde{u}_{\Psi,\tilde{\mathscr{O}}} \times \omega_\Psi(t)) \\
&= \tilde{v}_{\Psi,\tilde{\mathscr{O}}}(t) - t(\dot{u}_{\Psi,\mathscr{O}}(t) + u_{\Psi,\mathscr{O}}(t) \times \omega_\Psi(t)) \\
&\quad + t\dot{R}_\Psi(t)(v_{\mathscr{O}} - v_{\tilde{\mathscr{O}}}) - t(\mathrm{id}_{\ker(\tau)} - R_\Psi(t))(v_{\mathscr{O}} - v_{\tilde{\mathscr{O}}}) \times \omega_\Psi(t) \\
&= v^{\text{total}}_{\Psi,\mathscr{O}}(t) + (\mathscr{O}_{t+s_0} - \tilde{\mathscr{O}}_{t+s_0}) \times \omega_\Psi(t) + v_{\mathscr{O}} - v_{\tilde{\mathscr{O}}} + \\
&\quad - t(v_{\mathscr{O}} - v_{\tilde{\mathscr{O}}}) \times \omega_\Psi(t) \\
&= v^{\text{total}}_{\Psi,\mathscr{O}}(t) + (\mathscr{O}_{s_0} - \tilde{\mathscr{O}}_{s_0}) \times \omega_\Psi(t) + v_{\mathscr{O}} - v_{\tilde{\mathscr{O}}}
\end{aligned}$$

which gives us what we wanted to show. Next,

$$\begin{aligned}
\tilde{V}^{\text{total}}_{\Psi,\tilde{\mathscr{O}}}(t) &= \tilde{V}_{\Psi,\tilde{\mathscr{O}}}(t) - R_\Psi^{-1}(t) u_{\Psi,\tilde{\mathscr{O}}}(t) \\
&= \tilde{V}_{\Psi,\tilde{\mathscr{O}}}(t) - R_\Psi^{-1}(t)\left(u_{\Psi,\mathscr{O}}(t) + (\mathrm{id}_{\ker(\tau)} - R_\Psi(t))(v_{\mathscr{O}} - v_{\tilde{\mathscr{O}}})\right) \\
&= V_{\Psi,\mathscr{O}}(t) + (\mathscr{O}_{s_0} - \tilde{\mathscr{O}}_{s_0}) \times \Omega_\Psi(t) + R_\Psi^{-1}(t)(v_{\mathscr{O}} - v_{\tilde{\mathscr{O}}}) - R_\Psi^{-1}(t) u_{\Psi,\mathscr{O}}(t) \\
&\quad + (\mathrm{id}_{\ker(\tau)} - R_\Psi^{-1}(t))(v_{\mathscr{O}} - v_{\tilde{\mathscr{O}}}) \\
&= V^{\text{total}}_{\Psi,\mathscr{O}}(t) + (\mathscr{O}_{s_0} - \tilde{\mathscr{O}}_{s_0}) \times \Omega_\Psi(t) + v_{\mathscr{O}} - v_{\tilde{\mathscr{O}}}
\end{aligned}$$

Finally, we compute the acceleration,

$$\begin{aligned}
\tilde{a}^{\text{total}}_{\Psi,\tilde{\mathscr{O}}}(t) &= \dot{u}_{\Psi,\tilde{\mathscr{O}}}(t) + u_{\Psi,\tilde{\mathscr{O}}}(t) \times \omega_\Psi(t) \\
&= \dot{u}_\Psi(t) - \dot{R}_\Psi(t)(v_{\mathscr{O}} - v_{\tilde{\mathscr{O}}}) + u_{\Psi,\mathscr{O}}(t) \times \omega_\Psi(t) \\
&\quad (\mathrm{id}_{\ker(\tau)} - R_\Psi(t))(v_{\mathscr{O}} - v_{\tilde{\mathscr{O}}}) \times \omega_\Psi(t) \\
&= a^{\text{total}}_{\Psi,\mathscr{O}}(t) - \hat{\omega}_\Psi(t) R_\Psi(t)(v_{\mathscr{O}} - v_{\tilde{\mathscr{O}}}) + (\mathrm{id}_{\ker(\tau)} - R_\Psi(t))(v_{\mathscr{O}} - v_{\tilde{\mathscr{O}}}) \times \omega_\Psi(t) \\
&= a^{\text{total}}_{\Psi,\mathscr{O}}(t) + (v_{\mathscr{O}} - v_{\tilde{\mathscr{O}}}) \times \omega_\Psi(t)
\end{aligned}$$

and

$$\begin{aligned}
\tilde{A}^{\text{total}}_{\Psi,\tilde{\mathscr{O}}}(t) &= R_\Psi^{-1}(t)(\dot{\tilde{u}}_{\Psi,\tilde{\mathscr{O}}}(t)) \\
&= R_\Psi^{-1}(t)\left(\dot{u}_{\Psi,\mathscr{O}}(t) - \dot{R}_\Psi(t)(v_{\mathscr{O}} - v_{\tilde{\mathscr{O}}})\right) \\
&= A^{\text{total}}_{\Psi,\mathscr{O}}(t) - \hat{\Omega}(t)(v_{\mathscr{O}} - v_{\tilde{\mathscr{O}}}).
\end{aligned}$$

This finishes the proof. ■

5. Dynamics of rigid bodies in the presence of an observer

In Chapter 3, we formulated rigid body dynamics in an observer independent way. In this chapter, we shall explore the effect of introducing an observer to this formulation. In Section 5.1, we show that in the presence of an observer the body and spatial velocities defined in Chapter 3, project to the corresponding total velocities. In the next section, we show that the momenta also project to the well known quantities in the presence of the observer. Finally, in Section 5.3, we illustrate how we recover the familiar Euler equations for a rigid body.

5.1. Angular and linear velocities. In this section we shall investigate the effect of an observer on the angular and linear body and spatial velocities defined in Chapter 3.

Angular velocities. Recall that given a rigid motion Ψ in a Galilean spacetime $\mathscr{G} = (\mathscr{E}, V, g, \tau)$, the spatial angular velocity is defined as the map $\omega : \mathbb{R} \to \ker(\tau)$ satisfying

$$\hat{\omega}(t) = \dot{R}_\Psi(t) R_\Psi^{-1}(t)$$

and the body angular velocity is the map $\Omega : \mathbb{R} \to \ker(\tau)$ satisfying

$$\hat{\Omega}(t) = R_\Psi^{-1}(t) \dot{R}_\Psi(t)$$

It is clear that angular velocities remain the same even in the presence of an observer.

Linear velocities. Recall that given a motion Ψ in a Galilean spacetime, the body linear velocity is the map $V_\Psi^b : \mathscr{E} \times \mathbb{R} \to V_{\mathscr{G}}$ given by

$$V_\Psi^b(x,t) = \Psi_{t,V}^{-1}(\frac{d}{dt}(\Psi_t(x))$$

and the spatial linear velocity is the map $v_\Psi^s : \mathscr{E} \times \mathbb{R} \to V_{\mathscr{G}}$ given by

$$v_\Psi^s(x,t) = -\Psi_{t,V}(\frac{d}{dt}(\Psi_t^{-1}(x)))$$

Let's see what these velocities look like in the presence of an observer. We look at body linear velocity first.

5.1 Proposition: *Let Ψ be a motion in a Galilean spacetime $\mathscr{G} = (\mathscr{E}, V, g, \tau)$ and let $\mathscr{O}$ be an observer. Then,*

$$P_{\mathscr{O}}(V_\Psi^b(x,t)) = V_{\Psi,\mathscr{O}}^{\text{total}}(t)$$

Proof: We know that for each instant $s_0 \in I_{\mathscr{G}}$, there exists an isomorphism $\iota_{\mathscr{O}_{s_0}}$ such that for a motion Ψ, we have

$$\iota_{\mathscr{O}_{s_0}}(\Psi_t) = (R_\Psi(t), r_{\Psi,\mathscr{O}}(t), u_{\Psi,\mathscr{O}}(t), t).$$

Also,

$$\Psi_t(x) = R_\Psi(t)(x - \mathscr{O}_{\pi_{\mathscr{G}}(x)}) + (\pi_{\mathscr{G}}(x) - s_0) u_{\Psi,\mathscr{O}}(t) + r_{\Psi,\mathscr{O}}(t) + \mathscr{O}_{\pi_{\mathscr{G}}(x)+t}$$

Now, for $x \in \mathscr{O}$, we have

$$\Psi_t(x) = r_{\Psi,\mathscr{O}}(t) + \mathscr{O}_{\pi_{\mathscr{G}}(x)+t}$$

so we have

$$\frac{d}{dt}(\Psi_t(x)) = \dot{r}_{\Psi,\mathscr{O}}(t) + v_{\mathscr{O}}$$

Next,

$$\begin{aligned}
\Psi_{t,V}^{-1}(\frac{d}{dt}(\Psi_t(x))) &= \Psi_{t,V}^{-1}(\dot{r}_{\Psi,\mathscr{O}}(t) + v_{\mathscr{O}}) \\
&= \Psi_t^{-1}(\dot{r}_{\Psi,\mathscr{O}}(t) + v_{\mathscr{O}} + \mathscr{O}_{s_0}) - \Psi_t^{-1}(\mathscr{O}_{s_0}) \\
&= R_\Psi^{-1}(t)(\dot{r}_{\Psi,\mathscr{O}}(t) + v_{\mathscr{O}} + \mathscr{O}_{s_0} - \mathscr{O}_{1+s_0}) - R_\Psi^{-1}(t)u_{\Psi,\mathscr{O}}(t) \\
&\quad + R_\Psi^{-1}(t)(tu_{\Psi,\mathscr{O}}(t) - r_{\Psi,\mathscr{O}}(t)) + (1-t)v_{\mathscr{O}} + \mathscr{O}_{s_0} \\
&\quad - R_\Psi^{-1}(t)(tu_{\Psi,\mathscr{O}}(t) - r_{\Psi,\mathscr{O}}(t)) + tv_{\mathscr{O}} - \mathscr{O}_{s_0} \\
&= R_\Psi^{-1}(t)\dot{r}_{\Psi,\mathscr{O}}(t) - R_\Psi^{-1}(t)u_{\Psi,\mathscr{O}}(t) + v_{\mathscr{O}} \\
&= V_{\Psi,\mathscr{O}}(t) - R_\Psi^{-1}(t)u_{\Psi,\mathscr{O}}(t) + v_{\mathscr{O}}
\end{aligned}$$

From this, the result follows. ■

Let's look at spatial linear velocity now.

5.2 Proposition: *Let Ψ be a motion in a Galilean spacetime $\mathscr{G} = (\mathscr{E}, V, g, \tau)$ and let $\mathscr{O}$ be an observer. Then,*

$$P_{\mathscr{O}}(v_\Psi^s(x,t)) = v_{\Psi,\mathscr{O}}^{\text{total}}(t)$$

Proof: For $x \in \mathscr{O}$, we compute,

$$\Psi_t^{-1}(x) = R_\Psi^{-1}(t)(tu_{\Psi,\mathscr{O}}(t) - r_{\Psi,\mathscr{O}}(t)) - tv_{\mathscr{O}} + \mathscr{O}_{s_0}$$

Therefore,

$$\begin{aligned}
\frac{d}{dt}(\Psi_t^{-1}(x)) &= R_\Psi^{-1}(t)(u_{\Psi,\mathscr{O}}(t) + t\dot{u}_{\Psi,\mathscr{O}}(t) - \dot{r}_{\Psi,\mathscr{O}}(t)) \\
&\quad + (-R_\Psi^{-1}(t)\dot{R}_\Psi(t)R_\Psi^{-1}(t))(tu_{\Psi,\mathscr{O}}(t) - r_{\Psi,\mathscr{O}}(t))) - v_{\mathscr{O}} \\
&= R_\Psi^{-1}(t)(u_{\Psi,\mathscr{O}}(t) + t\dot{u}_{\Psi,\mathscr{O}}(t) - \dot{r}_{\Psi,\mathscr{O}}(t)) \\
&\quad - R_\Psi^{-1}(t)\hat{\omega}_\Psi(t)(tu_{\Psi,\mathscr{O}}(t) - r_{\Psi,\mathscr{O}}(t)) - v_{\mathscr{O}} \\
&= R_\Psi^{-1}(t)u_{\Psi,\mathscr{O}}(t) + R_\Psi^{-1}(t)[t(\dot{u}_{\Psi,\mathscr{O}}(t) + u_\Psi(t) \times \omega_\Psi(t))] \\
&\quad - R_\Psi^{-1}(t)(\dot{r}_{\Psi,\mathscr{O}}(t) + r_{\Psi,\mathscr{O}}(t) \times \omega_\Psi(t)) - v_{\mathscr{O}} \\
&= R_\Psi^{-1}(t)u_\Psi(t) - R_\Psi^{-1}(t)(v_{\Psi,\mathscr{O}}(t) - t(\dot{u}_{\Psi,\mathscr{O}}(t) + u_{\Psi,\mathscr{O}}(t) \times \omega_\Psi(t))) - v_{\mathscr{O}}
\end{aligned}$$

Let's call the last expression as $\dot{\Psi}_t^{-1}(x)$. Now, we compute

$$\begin{aligned}
-\Psi_{t,V}(\frac{d}{dt}(\Psi_t^{-1}(x)) &= \Psi_t(-\dot{\Psi}_t^{-1}(x) + \mathscr{O}_{s_0}) - \Psi_t(\mathscr{O}_{s_0}) \\
&= R_\Psi(t)(-\dot{\Psi}_t^{-1}(x) + \mathscr{O}_{s_0} - \mathscr{O}_{1+s_0}) + u_{\Psi,\mathscr{O}}(t) + r_{\Psi,\mathscr{O}}(t) + tv_{\mathscr{O}} \\
&\quad + \mathscr{O}_{1+s_0} - r_{\Psi,\mathscr{O}}(t) - tv_{\mathscr{O}} - \mathscr{O}_{s_0} \\
&= R_\Psi(t)(-\dot{\Psi}_t^{-1}(x) - v_{\mathscr{O}}) + u_{\Psi,\mathscr{O}}(t) + v_{\mathscr{O}} \\
&= R_\Psi(t)\left(R_\Psi^{-1}(t)(v_{\Psi,\mathscr{O}}(t) - t(\dot{u}_{\Psi,\mathscr{O}}(t) + u_{\Psi,\mathscr{O}}(t) \times \omega_\Psi(t))\right) \\
&\quad - u_{\Psi,\mathscr{O}}(t) + u_{\Psi,\mathscr{O}}(t) + v_{\mathscr{O}} \\
&= v_{\Psi,\mathscr{O}}(t) - t(\dot{u}_{\Psi,\mathscr{O}}(t) + u_{\Psi,\mathscr{O}}(t) \times \omega_\Psi(t)) + v_{\mathscr{O}}
\end{aligned}$$

From this the result follows. ■

We notice that in the presence of an observer, the spatial linear and body linear velocities project onto the total spatial linear and total body linear velocities respectively.

5.2. Spatial and body momenta. We let $\mathscr{G} = (\mathscr{E}, V, g, \tau)$ be a Galilean spacetime with Ψ a rigid motion, $\mathscr{O}$ an observer and $(\mathcal{B}, \mu)$ a rigid body with $\mathcal{B} \in \mathscr{E}(s_0)$. We wish to see how our definitions of spatial and body momenta look like when we have an observer $\mathscr{O}$. In such a case, we have the following result.

5.3 Proposition: *Let $\mathscr{G} = (\mathscr{E}, V, g, \tau)$ be a Galilean spacetime with Ψ a rigid motion, $\mathscr{O}$ an observer and $(\mathcal{B}, \mu)$ a rigid body with $\mathcal{B} \in \mathscr{E}(s_0)$and let $m_{\Psi,\mathcal{B}}, \ell_{\Psi,\mathcal{B}}, M_{\Psi,\mathcal{B}}$ and $L_{\Psi,\mathcal{B}}$ be as defined in Section 3.4. For $s_0 \in I_{\mathscr{G}}$, let $\iota_{\mathscr{O}_{s_0}}(\Psi_t) = (R_\Psi(t), r_{\Psi,\mathscr{O}}(t), u_{\Psi,\mathscr{O}}(t), t)$. Then the following statements hold.*

(i) $P_{\mathscr{O}}(m_{\Psi,\mathcal{B}}(t)) = \mu(\mathcal{B})\dot{r}_{\Psi,\mathscr{O}}(t)$

(ii) $P_{\mathscr{O}}(\ell_{\Psi,\mathcal{B}}(t)) = \mathbb{I}_c(R_\Psi^{-1}(t)\omega_\Psi(t))$

(iii) $P_{\mathscr{O}}(M_{\Psi,\mathcal{B}}(t)) = \mu(\mathcal{B})V^{\text{total}}_{\Psi,\mathscr{O}}(t)$

(iv) $P_{\mathscr{O}}(L_{\Psi,\mathcal{B}}(t)) = R_\Psi^{-1}(t)(\mathbb{I}_c(R_\Psi^{-1}(t)\omega_\Psi(t))$

Proof: (i) We compute,

$$\begin{aligned} P_{\mathscr{O}}(m_{\Psi,\mathcal{B}}(t)) &= \mu(\mathcal{B})P_{\mathscr{O}}(\frac{d}{dt}(\Psi_t(x_c)) \\ &= \mu(\mathcal{B})P_{\mathscr{O}}(\dot{r}_{\Psi,\mathscr{O}}(t) + v_{\mathscr{O}}) \\ &= \mu(\mathcal{B})\dot{r}_{\Psi,\mathscr{O}}(t) \end{aligned}$$

where we have used the computations carried out in Proposition 5.1.

(ii) and (iv) are just definitions. Since these quantities reside in $\ker(\tau)$, the projection $P_{\mathscr{O}}$ is the identity map on $\ker(\tau)$.

To see (iii), we compute

$$\begin{aligned} P_{\mathscr{O}}(M_{\Psi,\mathcal{B}}(t)) &= \mu(\mathcal{B})P_{\mathscr{O}}(\Psi^{-1}_{t,V}(\frac{d}{dt}(\Psi_t(x_c))) \\ &= \mu(\mathcal{B})P_{\mathscr{O}}(\Psi^{-1}_{t,V}(\dot{r}_{\Psi,\mathscr{O}}(t) + v_{\mathscr{O}})) \\ &= \mu(\mathcal{B})P_{\mathscr{O}}(R_\Psi^{-1}(t)\dot{r}_{\Psi,\mathscr{O}}(t) - R_\Psi^{-1}(t)u_{\Psi,\mathscr{O}}(t) + v_{\mathscr{O}}) \\ &= \mu(\mathcal{B})V^{\text{total}}_{\Psi,\mathscr{O}}(t) \end{aligned}$$

as desired. ■

5.3. Euler equations for a rigid body. Finally, we look at the Galilean Euler equations in the presence of an observer. We shall see that the equations given in Section 3.4 reduce to the familiar equations of motion for a rigid body.

5.4 Proposition: *Let Ψ be a rigid motion in a Galilean spacetime $\mathscr{G} = (\mathscr{E}, V, g, \tau)$ and $(\mathcal{B}, \mu)$ a rigid body with $\mathcal{B} \subset \mathscr{E}(s_0)$ and $\mathscr{O}$ be an observer. The Galilean–Euler equations are given by*

$$\begin{aligned} \dot{R}_\Psi(t) &= R_\Psi(t) \circ \hat{\Omega}_\Psi(t) \\ \ddot{r}_{\Psi,\mathscr{O}}(t) &= 0 \\ \mathbb{I}_c(\dot{\Omega}_\Psi(t)) &= \mathbb{I}_c(\Omega_\Psi(t)) \times \Omega_\Psi(t) \end{aligned}$$

Proof: The first equation is simply a definition. It is also independent of the observer so remains the same. Next, using the computations carried out in Proposition 5.1, we have

$$\dot{x}_c(t) = \dot{r}_{\Psi,\mathscr{O}}(t) + v_{\mathscr{O}}$$

from which the second equation follows. The third equation is the usual Galilean Euler equation for the body angular momentum and remains unchanged in the presence of an observer. ■

Discussion. The Galilean–Euler equations given in Proposition 5.4 coincide with the usual Euler equations for a rigid body, i.e. they generate the correct motion of the rigid body. It is interesting to note that in the presence of an observer, the expression for the spatial angular momentum computed using our definition differs from the classical one. However, the equations of motion for the body are the same in both cases. We note that the Galilean boosts take no part in the motion and can therefore take arbitrary values. Here too, we are not able to recover the differential equation for the body linear momentum. We would like to get a better understanding of why this is the case.

6. Conclusions and future scope

In this chapter we summarize the results of our investigation and outline the scope of future work.

6.1. Conclusions. This project aimed at studying rigid body mechanics in abstract Galilean spacetimes. This is a considerable generalization of the classical or "canonical" setup of studying the motion of a rigid body in which the motion takes place in $\mathbb{R}^3$ and the rigid motions essentially constitute $SE(3)$. In such a treatment, it is not apparent to which spaces the velocities and the momenta rightfully belong, since many identifications of spaces are made. Our treatment sorts this problem out in a natural way. We defined rigid motions and the fundamental quantities associated with it. We then formulated the dynamics of a rigid body mechanics in an observer-independent manner. Our treatment is more general than the classical one on several counts. First of all, we have treated body and spatial momenta associated with a rigid body in motion as fundamental quantities which can be defined without requiring additional structure. Equations of motion for a rigid body undergoing rigid motion are derived in this setting. Although we do not learn anything new by looking at these Galilean-Euler equations in the abstract setting, we do come to an understanding of the role played by the Galilean structure when we study them in the presence of an observer.

Next, the structure of the Galilean group is made clear in the presence of an observer. It is shown that an observes induces an isomorphism between the Galilean group and the canonical Galilean group. Here, we have considered rigid motions with arbitrary Galilean boosts. We have also studied the notion of a rigid body carefully and discussed in detail, the properties of the inertia tensor. The inertia tensor is an important object in the description of the dynamics of a rigid body. It is then shown that we recover the familiar quantities and equations for the motion of a rigid body in a Galilean spacetime in the presence of

an observer. Although the observer plays no part in the motion of the rigid body itself, it facilitates the study of the equations of motion.

We now summarize our results below.

(1) We have provided observer-independent definitions for spatial and body angular velocities (Section 3.2) and spatial and body linear velocities (Section 3.2)

(2) we have presented a definition of a rigid body in Section 3.3 and investigated the properties of the inertia tensor of the rigid body.

(3) We have provided observer-independent definitions for spatial linear and angular momenta (Section 3.4) and body linear and angular momenta (Section 3.4).

(4) We have derived observer-independent equations of motion for a rigid body in an abstract Galilean spacetime (Section 3.4). These equations are considerably general that the Euler equations for a rigid body derived in the classical literature.

(5) We have shown that in the presence of an observer, the group $\mathrm{Gal}(\mathscr{G})$ decomposes into rotations, translations, Galilean boosts and temporal origin shifts (Proposition 4.5).

(6) In Section 4.2, we have characterized the various subgroup of $\mathrm{Gal}(\mathscr{G})$ and shown how they are related to the subgroups of $\mathrm{Gal}(\mathscr{G}_{\mathrm{can}})$. In Section 4.3, we have introduced total velocities associated with a rigid motion.

(7) In Section 5.1, we have shown that in the presence of an observer, the body and spatial velocities project to the corresponding total velocities. The momenta also project in a similar manner.

(8) Finally, in Section 5.3, we show that Galilean-Euler equations derived in section 3.4 reduce to the usual Euler equations for a rigid body.

6.2. Future scope. After bringing our work to this point, many questions still remain unanswered. In our treatment, we have not fully understood the role of velocity boosts in the motion of the rigid body and the meaning of the fictitious accelerations that arise when we consider total velocities of $\mathrm{Gal}(\mathscr{G})$ is also not apparent. We would also like to come to a complete understanding of the physical implications of this formulation.

In Proposition 4.5, we have constructed an isomorphism $\iota_{\mathscr{O}_{s_0}}$ between $\mathrm{Gal}(\mathscr{G})$ and $\mathrm{Gal}(\mathscr{G}_{\mathrm{can}})$ induced by an observer $\mathscr{O}$. We believe that this is the most general isomorphism between these groups in the following sense. Given *any* isomorphism I between $\mathrm{Gal}(\mathscr{G})$ and $\mathrm{Gal}(\mathscr{G}_{\mathrm{can}})$, there exists an observer $\tilde{\mathscr{O}}$ and an instant $\tilde{s}_0$ such that $I = \iota_{\tilde{\mathscr{O}}_{\tilde{s}_0}}$. Our efforts are currently directed towards showing that this is indeed true.

It is interesting to note that although our expressions for body and spatial angular momentum in the presence of an observer are different from those in the literature, they produce the same equations of motion. It would therefore be worthwhile to compare various quantities in the usual setting with those in ours.

A more advanced extension of the problem is to consider relativistic mechanics in the framework of Minkowski spacetimes. A more general treatment of mechanics can be given in this setting.

Acknowledgment

I am grateful to my supervisor Dr. Andrew Lewis for introducing me to the beautiful world of Galilean spacetimes. I wish to thank him for providing constant support and motivation and most importantly, for being always accessible for discussion.

References

Arnol′d, V. I. [1978] *Mathematical Methods of Classical Mechanics*, translated by K. Vogtmann and A. Weinstein, number 60 in Graduate Texts in Mathematics, Springer-Verlag: New York/Heidelberg/Berlin, ISBN: 0-387-96890-3, New edition: [Arnol′d 1989].

— [1989] *Mathematical Methods of Classical Mechanics*, translated by K. Vogtmann and A. Weinstein, 2nd edition, number 60 in Graduate Texts in Mathematics, Springer-Verlag: New York/Heidelberg/Berlin, ISBN: 978-0-387-96890-2, First edition: [Arnol′d 1978].

Artz, R. E. [1981] *Classical mechanics in Galilean space-time*, Foundations of Physics, An International Journal Devoted to the Conceptual Bases and Fundamental Theories of Modern Physics, **11**(9-10), pages 679–697, ISSN: 0015-9018, DOI: 10.1007/BF00726944.

Berger, M. [1987] *Geometry I*, Universitext, Springer-Verlag: New York/Heidelberg/Berlin, ISBN: 978-3-540-11658-5.

Lévy-Leblond, J.-M. [1971] *Galilei group and Galilean invariance*, in *Group Theory and its Applications*, volume 2, edited by E. M. Loebl, pages 221–299, Academic Press: New York, NY.

Souriau, J.-M. [1997] *Structure of Dynamical Systems, A Symplectic View of Physics*, translated by C. H. Cushman-de Vries, number 149 in Progress in Mathematics, Birkhäuser: Boston/Basel/Stuttgart, ISBN: 978-0-8176-3695-1.