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Potential energy shaping after kinetic energy shaping*

Andrew D. Lewis†

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Abstract

Sufficient conditions are given for the existence of solutions to the partial differential equation for the potential energy shaping that follows kinetic energy shaping.

Keywords. mechanical systems, energy shaping, integrability of partial differential equations.

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1. Introduction

Suppose that we have a mechanical system (not a control system yet) with a configuration manifold Q, a kinetic energy metric $\mathbb{G}$, and a potential energy function V. Equilibria are then points $q_0 \in \mathsf{Q}$ for which $\boldsymbol{d}V(q_0) = 0$, and an equilibrium q_0 is stable if $\operatorname{Hess} V(q_0)$ is positive-definite. If one has dissipation then, provided that the dissipation gives enough "coupling," one can additionally infer asymptotic stability of the equilibrium using the LaSalle Invariance Principle. Moreover, one can get a good grip on the domain of attraction by considering the level sets of the potential function. The point is that a great deal of the system's behaviour follows from knowledge of the character of the potential function V. We refer to the discussion in [Bullo and Lewis 2004, Section 6.2] for precise statements and for references on the somewhat classical subject of stability of equilibria for mechanical systems.

Now, if one has a *control* system and an *unstable* equilibrium q_0, one may wish to stabilise it using feedback. Since the stability properties of mechanical systems are generally easy to understand, one might additionally wish to design the feedback so that the closed-loop system is mechanical. There is a bit of a history to this approach. We do not attempt an exhaustive review of the literature here, but rather an historical one. Potential shaping for fully-actuated systems seems to date to [Takegaki and Arimoto 1981]. The situation in the underactuated case was worked out by van der Schaft [1986], where one sees that "integrability conditions" exist on the set of possible closed-loop potentials. If one additionally allows for shaping of the kinetic energy then the class of systems that can be stabilised using energy shaping is enlarged. The first papers in this direction seem to be [Bloch, Chang,

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†Professor, Department of Mathematics and Statistics, Queen's University, Kingston, ON K7L 3N6, Canada
Email: andrew.lewis@queensu.ca, URL: http://www.mast.queensu.ca/~andrew/

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Leonard, and Marsden 2001, Bloch, Leonard, and Marsden 2000] in the Lagrangian setting and [Ortega, Spong, Gómez-Estern, and Blankenstein 2002] in the Hamiltonian setting. In the Hamiltonian setting energy shaping is related to the extensive work on port controlled Hamiltonian systems and passivity, and we do not attempt a review of this literature here. A geometric formulation of the energy shaping method is given in [Auckly and Kapitanski 2002, Auckly, Kapitanski, and White 2000], and in the later of these papers the integrability of the energy shaping partial differential equations, particularly the partial differential equation for kinetic energy shaping, is discussed. Since these preliminary forays, there have been a multitude of papers exploring specific parts of the method and its applications. Again, we do not attempt a thorough review of this literature.

In this paper we are interested in the problem of potential energy shaping after kinetic energy shaping has taken place. In terms of the stabilisation problem, this is the most important part of the procedure, since it is through the shaping of the potential energy that stability is achieved. In some sense, all other parts of the energy shaping method are simply present to facilitate the shaping of the potential energy. As we shall see, the shaping of the potential energy after one does shaping of the kinetic energy leads naturally to a partial differential equation. The obvious questions are:

1. Does this partial differential equation have solutions?

2. If it does have solutions, what do they look like?

In previous work [Lewis 2004] we answered the second of these questions. In this paper we address the first using the integrability theory for linear partial differential equations developed in [Goldschmidt 1967a]. (We actually use a small extension of the linear theory that follows from the more general nonlinear theory reported in [Goldschmidt 1967b].)

This theory of integrability is rather complicated and it is simply not possible to provide any sort of useful review in this paper; it seems as if there is no way around this. We therefore simply suppose that the reader is familiar with the theory, or is willing to spend the time to learn it. Some places to start include the book of Pommaret [1978], Chapter IX of Bryant, Chern, Gardner, Goldschmidt, and Griffiths [1991], and (particularly for an introduction to involutivity of symbols) Chapter 4 of [Ivey and Landsberg 2006].

We do point out, however, that it is possible to understand the meaning of all of the words in the main result, Theorem 2.3, even if one does not understand at all the Goldschmidt theory of integrability of partial differential equations. Moreover, the conditions of Theorem 2.3 are eminently checkable.

2. Statement of problem and main result

In this section we provide the technical backdrop to the problem, and as quickly as possible reduce the problem to a partial differential equation. We refer to the references for more details on the energy shaping method in general. For background on geometric control theory for mechanical systems in the approach we use here, we refer to [Bullo and Lewis 2004]. We assume the reader to be familiar with this geometric approach.

2.1. Definitions. We shall always assume geometric objects to be at least of class C^∞.

A ***simple mechanical control system*** is a quadruple $\Sigma = (\mathsf{Q}, \mathbb{G}_{\mathrm{ol}}, V_{\mathrm{ol}}, \mathscr{F} = \{F^1, \dots, F^m\})$, where Q is the configuration manifold for the system, $\mathbb{G}_{\mathrm{ol}}$ is the kinetic energy metric, V_{ol} is the potential energy function, and $\mathscr{F}$ is a collection of one-forms on Q. The subscript "ol" means "open-loop." We will also be using closed-loop versions of the kinetic energy metric and potential energy function. The equations governing a simple mechanical control system are

$$\overset{\mathbb{G}_{\mathrm{ol}}}{\nabla}_{\gamma'(t)}\gamma'(t) = -\operatorname{grad} V_{\mathrm{ol}}(\gamma(t)) + \sum_{a=1}^{m} u^a(t)\mathbb{G}_{\mathrm{ol}}^{\sharp} \circ F^a(\gamma(t)),$$

where $\mathbb{G}_{\mathrm{ol}}^{\sharp} \colon \mathsf{T}^*\mathsf{Q} \to \mathsf{T}\mathsf{Q}$ is the musical isomorphism associated with $\mathbb{G}$ (we will also use the other such isomorphism, $\mathbb{G}_{\mathrm{ol}}^{\flat} \colon \mathsf{T}\mathsf{Q} \to \mathsf{T}^*\mathsf{Q}$). We denote by $\mathcal{F}$ the codistribution on Q generated by the one-forms $\mathscr{F}$.

The objective of energy shaping is to find a state feedback $u \colon \mathsf{T}\mathsf{Q} \to \mathbb{R}^m$, defining an $\mathcal{F}$-valued map F on $\mathsf{T}\mathsf{Q}$ by

$$F(v_q) = \sum_{a=1}^{m} u^a(v_q)F^a(q),$$

such that the closed-loop system equations are those of a mechanical system with kinetic energy metric $\mathbb{G}_{\mathrm{cl}}$ and potential energy function V_{cl}. This means that we require that

$$\overset{\mathbb{G}_{\mathrm{ol}}}{\nabla}_{\gamma'(t)}\gamma'(t) + \operatorname{grad} V_{\mathrm{ol}}(\gamma(t)) - \mathbb{G}_{\mathrm{ol}}^{\sharp} \circ F(\gamma'(t)) = \overset{\mathbb{G}_{\mathrm{cl}}}{\nabla}_{\gamma'(t)}\gamma'(t) + \operatorname{grad} V_{\mathrm{cl}}(\gamma(t)).$$

As described in [Lewis 2004], we find the state feedback F in two stages. Define $\Lambda_{\mathrm{cl}} = \mathbb{G}_{\mathrm{ol}}^{\flat} \circ \mathbb{G}_{\mathrm{cl}}^{\sharp}$, noting that this is a vector bundle automorphism of $\mathsf{T}^*\mathsf{Q}$. We first find $F_{\mathrm{kin}} \colon \mathsf{T}\mathsf{Q} \to \mathcal{F}$ with the property that

$$\mathbb{G}_{\mathrm{ol}}^{\sharp} \circ F_{\mathrm{kin}}(\gamma'(t)) = \overset{\mathbb{G}_{\mathrm{cl}}}{\nabla}_{\gamma'(t)}\gamma'(t) - \overset{\mathbb{G}_{\mathrm{ol}}}{\nabla}_{\gamma'(t)}\gamma'(t),$$

and then we find $F_{\mathrm{pot}} \colon \mathsf{Q} \to \mathcal{F}$ with the property that

$$F_{\mathrm{pot}}(\gamma(t)) = \Lambda_{\mathrm{cl}} \circ \boldsymbol{d}V_{\mathrm{cl}}(\gamma(t)) - \boldsymbol{d}V_{\mathrm{ol}}(\gamma(t)).$$

In this paper we assume that some kinetic energy shaping has already been performed and that we have in this manner arrived at a closed-loop kinetic energy metric $\mathbb{G}_{\mathrm{cl}}$. The set of possible closed-loop kinetic energy metrics can be enlarged by allowing the addition of gyroscopic terms in the state feedback. We do not address this explicitly here, but refer to [Blankenstein, Ortega, and van der Schaft 2002, Chang, Bloch, Leonard, Marsden, and Woolsey 2002, Lewis 2004] for various interpretations of the gyroscopic term in the state feedback. For the purposes of the discussion here, suffice it to say that we allow that $\mathbb{G}_{\mathrm{cl}}$ has been achieved by the use of gyroscopic terms in the feedback. All we care about is that we have at hand some open-loop metric $\mathbb{G}_{\mathrm{ol}}$ and some closed-loop metric $\mathbb{G}_{\mathrm{cl}}$.

2.2. Reduction to a partial differential equation. Given two kinetic energy metrics $\mathbb{G}_{\mathrm{ol}}$ and $\mathbb{G}_{\mathrm{cl}}$, we take $\Lambda_{\mathrm{cl}} = \mathbb{G}_{\mathrm{ol}}^{\flat} \circ \mathbb{G}_{\mathrm{cl}}^{\sharp}$ as above. We define a codistribution $\mathcal{F}_{\mathrm{cl}} = \Lambda_{\mathrm{cl}}^{-1}(\mathcal{F})$. For a section F of $\mathcal{F}$ we denote by $F_{\mathrm{cl}} = \Lambda_{\mathrm{cl}}^{-1} \circ F$ the corresponding section of $\mathcal{F}_{\mathrm{cl}}$. We suppose that we have an open-loop potential function V_{ol}. Motivated by our presentation of the energy shaping problem above we have the following definition.

2.1 Definition: A section F of $\mathcal{F}$ is a ***$(\mathbb{G}_{\mathrm{ol}}, \mathbb{G}_{\mathrm{cl}})$-potential energy shaping feedback*** if there exists a function V_{cl} on Q such that

$$F(q) = \Lambda_{\mathrm{cl}} \circ \boldsymbol{d}V_{\mathrm{cl}}(q) - \boldsymbol{d}V_{\mathrm{ol}}(q), \qquad q \in \mathsf{Q}. \qquad \bullet$$

To convert the problem of finding potential energy shaping feedbacks into a partial differential equation in the sense of Goldschmidt [1967a] we use the language of jet bundles. We refer to [Saunders 1989] and [Pommaret 1978, Chapter 2] for background on jet bundles. For a vector bundle $\pi\colon \mathsf{E} \to \mathsf{X}$, we denote by $\mathsf{J}_k\mathsf{E}$ the vector bundle of k-jets of sections of E. If ξ is a section of E we denote by $j_k\xi$ the corresponding section of $\mathsf{J}_k\mathsf{E}$. If $\tau\colon \mathsf{F} \to \mathsf{X}$ is another vector bundle, a ***kth-order linear differential operator*** is a map $\mathscr{D}\colon \Gamma(\mathsf{E}) \to \Gamma(\mathsf{F})$ between the sections of E and F such that there exists a vector bundle map $\Phi\colon \mathsf{J}_k\mathsf{E} \to \mathsf{F}$ over id_{X} for which $\mathscr{D}(\xi) = \Phi(j_k\xi)$.

Now we assume that $\mathcal{F}$ is a vector bundle; this is a necessary regularity assumption. We think of $\mathsf{Q}_{\mathbb{R}} \triangleq \mathsf{Q}\times\mathbb{R}$ as a trivial vector bundle over Q with canonical projection $\pi\colon \mathsf{Q}_{\mathbb{R}} \to \mathsf{Q}$. Let us denote a typical section of $\mathsf{Q}_{\mathbb{R}}$ by V, as these will be potential functions in our setup. We identify sections of $\mathsf{Q}_{\mathbb{R}}$ with functions on Q in the obvious way; namely, given a function V the corresponding section is $q \mapsto (q, V(q))$. Define an $\mathsf{T}^*\mathsf{Q}$-valued differential operator $\mathscr{D}_{\boldsymbol{d}}$ on $\mathsf{Q}_{\mathbb{R}}$ by $\mathscr{D}_{\boldsymbol{d}}(V) = \boldsymbol{d}V$. Since $\mathscr{D}_{\boldsymbol{d}}$ is a first-order differential operator, there is a vector bundle map $\Phi_{\boldsymbol{d}}\colon \mathsf{J}_1\mathsf{Q}_{\mathbb{R}} \to \mathsf{T}^*\mathsf{Q}$ such that $\mathscr{D}_{\boldsymbol{d}}(V)(q) = \Phi_{\boldsymbol{d}}(j_1V(q))$ for every $q \in \mathsf{Q}$. Explicitly, in coordinates, this mapping is given by $(q^i, V, V_{,j}) \mapsto (q^i, V_{,j})$, where $V_{,j}$ means the partial derivative of V with respect to q^j, thinking of this as a coordinate for $\mathsf{J}_1\mathsf{Q}_{\mathbb{R}}$.

Let us abbreviate $\alpha_{\mathrm{cl}} = \Lambda_{\mathrm{cl}}^{-1}\circ\boldsymbol{d}V_{\mathrm{ol}}$. We let $\pi_{\mathcal{F}_{\mathrm{cl}}}\colon \mathsf{T}^*\mathsf{Q} \to \mathsf{T}^*\mathsf{Q}/\mathcal{F}_{\mathrm{cl}}$ the canonical projection onto the quotient vector bundle.

We then define

$$\mathsf{R}_{\mathrm{pot}} = \{p \in \mathsf{J}_1\mathsf{Q}_{\mathbb{R}} \mid \pi_{\mathcal{F}_{\mathrm{cl}}} \circ \Phi_{\boldsymbol{d}}(p) = \pi_{\mathcal{F}_{\mathrm{cl}}} \circ \alpha_{\mathrm{cl}}(q),\ \pi_1(p) = q\},$$

where $\pi_1\colon \mathsf{J}_1\mathsf{Q}_{\mathbb{R}} \to \mathsf{Q}$ is the canonical projection. The subset $\mathsf{R}_{\mathrm{pot}}$ is now a partial differential equation in the sense of Goldschmidt [1967a], and so we are in a position to apply the integrability theory from that paper. Moreover, the following obvious result establishes the explicit correspondence between solutions to $\mathsf{R}_{\mathrm{pot}}$ and potential energy shaping feedbacks.

2.2 Proposition: *A section F of $\mathcal{F}$ is a $(\mathbb{G}_{\mathrm{ol}}, \mathbb{G}_{\mathrm{cl}})$-potential energy shaping feedback if and only if $F = \Lambda_{\mathrm{cl}} \circ \boldsymbol{d}V - \boldsymbol{d}V_{\mathrm{ol}}$ for a solution V to $\mathsf{R}_{\mathrm{pot}}$.*

2.3. Statement and discussion of main result. Before we state the result we first introduce some convenient notation. If $\Lambda \subset \mathsf{T}^*\mathsf{Q}$ is a subbundle we let $\mathsf{I}(\Lambda)$ be the ideal generated by Λ in the set $\bigwedge(\mathsf{T}^*\mathsf{Q})$ of exterior forms on Q. For $k \in \mathbb{Z}$ let $\mathsf{I}_k(\Lambda) = \mathsf{I}(\Lambda) \cap \bigwedge^k(\mathsf{T}^*\mathsf{Q})$.

With this notation, we have the following result.

2.3 Theorem: *Let $\Sigma = (\mathsf{Q}, \mathbb{G}_{\mathrm{ol}}, V_{\mathrm{ol}}, \mathscr{F})$ be an analytic simple mechanical control system and let $\mathbb{G}_{\mathrm{cl}}$ be an analytic Riemannian metric. Let $p_0 \in \mathsf{R}_{\mathrm{pot}}$ and let $q_0 = \pi_1(p_0)$. Assume that q_0 is a regular point for $\mathcal{F}$ and that $\mathcal{F}_{\mathrm{cl}}$ is integrable in a neighbourhood of q_0. Then the following statements are equivalent:*

- *(i) there exists a neighbourhood $\mathcal{U}$ of q_0 and an analytic $(\mathbb{G}_{\mathrm{ol}}, \mathbb{G}_{\mathrm{cl}})$-potential energy shaping feedback F defined on $\mathcal{U}$ which satisfies $\Phi_{\boldsymbol{d}}(p_0) = F_{\mathrm{cl}}(q_0) + \alpha_{\mathrm{cl}}(q_0)$;*
- *(ii) there exists a neighbourhood $\mathcal{U}$ of q_0 such that $\boldsymbol{d}\alpha_{\mathrm{cl}}(q) \in \mathsf{I}(\mathcal{F}_{\mathrm{cl},q})$ for each $q \in \mathcal{U}$.*

2.4 Remarks: 1. The regularity condition on $\mathcal{F}$ in the hypotheses can be thought of as being of secondary importance. For example, the subset of Q where this condition holds is open and dense in the C^∞ case, and possesses a complement that is a strict analytic subset of Q in the analytic case. That being said, if one finds oneself at a point where this regularity assumption does not hold, then the theorem is of no help.

2. One of the interesting features of the theorem is that it shows, or more precisely its proof shows, that there are "compatibility conditions"—namely the conditions that $\mathcal{F}_{\mathrm{cl}}$ be integrable and that $\boldsymbol{d}\alpha_{\mathrm{cl}} \in \Gamma(\mathsf{I}_2(\mathcal{F}_{\mathrm{cl}}))$—needed to ensure the existence, even locally, of $(\mathbb{G}_{\mathrm{ol}}, \mathbb{G}_{\mathrm{cl}})$-potential energy shaping feedback. It then becomes of interest to know which closed-loop kinetic energy metrics allow for potential energy shaping, since it can be expected that there will be some for which the compatibility conditions are not met. Indeed, the condition that $\boldsymbol{d}\alpha_{\mathrm{cl}} \in \Gamma(\mathsf{I}(\mathcal{F}_{\mathrm{cl}}))$ is one that will generally not hold.

3. Of course, Theorem 2.3 does not give solutions to the potential energy shaping partial differential equation. What it does, in actuality, is indicate that Taylor series solutions can be constructed "order-by-order." This is the idea behind the theory of formal integrability. •

To illustrate the hypotheses of the theorem, let us show how the potential shaping result of van der Schaft [1986] follows from it.

2.5 Corollary: *Let $\Sigma = (\mathsf{Q}, \mathbb{G}, V_{\mathrm{ol}}, \mathscr{F})$ be an analytic simple mechanical control system, let p_0 and let $q_0 = \pi_1(p_0)$. Assume the following:*

(i) q_0 is a regular point for $\mathcal{F}$;

(ii) $\mathcal{F}$ is integrable in a neighbourhood of q_0.

Then there exists an $(\mathbb{G}_{\mathrm{ol}}, \mathbb{G}_{\mathrm{ol}})$-potential energy shaping feedback F defined on $\mathcal{U}$ which satisfies $\Phi_{\boldsymbol{d}}(p_0) = F(q_0)$.

Proof: This follows immediately since $\mathcal{F}_{\mathrm{cl}} = \mathcal{F}$ and since $\boldsymbol{d}\alpha_{\mathrm{cl}} = 0$ by virtue of the fact that $\mathbb{G}_{\mathrm{cl}} = \mathbb{G}_{\mathrm{ol}}$. ■

2.6 Remark: The converse of the corollary is true as well. That is to say, the only potential shaping feedbacks are those with values in the largest involutive codistribution contained in $\mathcal{F}$. This is proved by van der Schaft [1986]. It is not known to the author whether the compatibility condition of integrability of $\mathcal{F}_{\mathrm{cl}}$ in Theorem 2.3 is necessary as well as sufficient for the existence of potential energy shaping feedbacks. The author conjectures that this is so, under suitable regularity assumptions on the codistributions involved. •

2.4. Conclusions and open problems. Since our proof is somewhat technical we strictly relegate it to the end of the paper and give our closing remarks here for convenience.

Theorem 2.3 gives sufficient conditions for the existence of potential energy shaping feedback after one has already done kinetic energy shaping. We refer to [Lewis 2004, Proposition 7] for a discussion of the nature of the set of *all* potential energy shaping feedbacks after one has found an initial one (for example, using Theorem 2.3).

Much basic work remains to be done on the problem of energy shaping. Let us list some of the fundamental open problems.

1. *Describe the set of achievable closed-loop kinetic energy metrics.* This is discussed by [Auckly and Kapitanski 2002, Auckly, Kapitanski, and White 2000]. However, a fully functional (or even elegant) geometric description of the solutions of the kinetic energy shaping partial differential equations has so far eluded researchers in this area. Also, the matter of what can be additionally achieved with the addition to the problem of gyroscopic forces has not been even touched upon in a fundamental way.
2. *Describe the set of all closed-loop potentials achievable by allowing the closed-loop potentials to vary over their achievable set.* This is the "holy grail" of the stabilisation problem, since in the stabilisation problem one wants to know whether the set of closed-loop potentials contains one for which Hess $V_{\mathrm{cl}}(q_0)$ is positive-definite at a desired equilibrium q_0.
3. *Apply the theory to nontrivial examples.* In particular, as is explained in [Lewis 2004], the case where $\mathcal{F}$ has codimension 1 is degenerate, although this case seems to be the source of many of the problems used to illustrate the method of energy shaping [Acosta, Ortega, Astolfi, and Mahindrakar 2005]. Examples where $\mathcal{F}$ has codimension greater than 1 are therefore of real interest if one is to understand the method and its limitations.

3. Proof of main result

In this section we prove Theorem 2.3. Our proof relies on the formal theory of partial differential equations developed for linear equations by Goldschmidt [1967a]. This theory has a strongly algebraic character. As part of our presentation, therefore, there is an algebraic component. To the reader unfamiliar with the theory of Goldschmidt [1967a] (and others), the algebraic part of the proof might seem completely unmotivated.

3.1. The algebraic part of the proof. If V is a $\mathbb{R}$-vector space, we denote by $\otimes_{j=1}^{k}\mathsf{V}^*$, $S^k(\mathsf{V}^*)$, and $\bigwedge^k(\mathsf{V}^*)$ the vector spaces of $(0,k)$-tensors, symmetric $(0,k)$-tensors, and skew-symmetric $(0,k)$-tensors, respectively, on V. By $\bigwedge(\mathsf{V}^*)$ we denote the set of all skew-symmetric tensors.

We suppose that V is a finite-dimensional $\mathbb{R}$-vector space with V^* its dual space. We let E and F be subspaces of V with the property that $\mathsf{V} = \mathsf{E} \oplus \mathsf{F}$. We denote the corresponding decomposition of V^* as $\mathsf{V}^* = \mathsf{E}^* \oplus \mathsf{F}^*$. The decomposition also induces direct sum decompositions of the vector spaces $\otimes_{j=1}^{k}\mathsf{V}^*$, $S^k(\mathsf{V}^*)$, and $\bigwedge^k(\mathsf{V}^*)$. For example, we have

$$\textstyle\bigwedge^2(\mathsf{V}^*) = \bigwedge^2(\mathsf{E}^*) \oplus \bigwedge^2(\mathsf{F}^*) \oplus (\mathsf{E}^* \otimes \mathsf{F}^*)$$

and

$$S^2(\mathsf{V}^*) = S^2(\mathsf{E}^*) \oplus S^2(\mathsf{F}^*) \oplus (\mathsf{E}^* \otimes \mathsf{F}^*).$$

(One can think of these decompositions as they manifest themselves for skew-symmetric (resp. symmetric) matrices. If one writes such a matrix in four blocks, one has two skew-symmetric (resp. symmetric) diagonal blocks, and the two off-diagonal blocks differ by a transpose, so it suffices to determine only one of these.) Using these decompositions we have natural inclusions of, for example, $S^k(\mathsf{F}^*)$ and $\bigwedge^k(\mathsf{F}^*)$ in $S^k(\mathsf{V}^*)$ and $\bigwedge^k(\mathsf{V}^*)$, respectively. Moreover, the images of these inclusions have natural complements, so there are also natural projections onto the subspaces. We shall take all of these inclusions and projections for granted in our discussions to follow. We shall also use the decomposition $\mathsf{V} = \mathsf{E} \oplus \mathsf{F}$, and the induced decompositions of the tensor algebra, to identify quotients with complements. We will do all of this without explicit indication.

We denote by $\sigma \colon \mathsf{V}^* \to \mathsf{E}^*$ the projection onto the first factor. Explicitly,

$$\sigma(\alpha)(u_1) = \alpha(u_1 \oplus 0), \qquad u_1 \in \mathsf{E}.$$

We let $\Delta \colon S^2(\mathsf{V}^*) \to \mathsf{V}^* \otimes \mathsf{V}^*$ be the canonical inclusion and define $\sigma_1 \colon S^2(\mathsf{V}^*) \to \mathsf{V}^* \otimes \mathsf{E}^*$ by $\sigma_1 = (\mathrm{id}_{\mathsf{V}^*} \otimes \sigma) \circ \Delta$. Thus σ_1 is the first prolongation of σ. Explicitly,

$$\sigma_1(B)(u_1 \oplus u_2, v_1) = B(u_1 \oplus u_2, v_1 \oplus 0)$$

for $u_1, v_1 \in \mathsf{E}$ and $u_2 \in \mathsf{F}$. Since B is symmetric this is equivalent to

$$\sigma_1(B)(u_1 \oplus u_2, v_1) = B(u_1 \oplus 0, v_1 \oplus 0). \tag{3.1}$$

We shall be interested in the kernels and cokernels of the maps σ and σ_1. For σ we have the following result.

3.1 Lemma: *We have* $\ker(\sigma) = \mathsf{F}^*$ *and* $\operatorname{coker}(\sigma) = 0$.

Proof: This is obvious. ■

To state the analogous result for σ_1 we need some notation. For a subspace Λ of V^*, let $\mathsf{I}(\Lambda)$ be the ideal generated in $\bigwedge(\mathsf{V}^*)$ by Λ and let $\mathsf{I}_2(\Lambda) = \mathsf{I}(\Lambda) \cap \bigwedge^2(\mathsf{V}^*)$.

We have the following result for σ_1.

3.2 Lemma: *We have* $\ker(\sigma_1) = S^2(\mathsf{F}^*)$ *and* $\operatorname{coker}(\sigma_1) = \mathsf{I}_2(\mathsf{E}^*)$.

Proof: By definition, if $B \in \ker(\sigma_1)$ then

$$B(u_1 \oplus 0, v_1 \oplus 0) = 0, \qquad u_1, v_1 \in \mathsf{E}.$$

Thus $B \in S^2(\mathsf{F}^*)$. We have $\mathrm{image}(\sigma_1) = S^2(\mathsf{E}^*)$ by (3.1). Since

$$\mathsf{V}^* \otimes \mathsf{E}^* = (\mathsf{E}^* \otimes \mathsf{E}^*) \oplus (\mathsf{F}^* \otimes \mathsf{E}^*) = S^2(\mathsf{E}^*) \oplus \textstyle\bigwedge^2(\mathsf{E}^*) \oplus (\mathsf{F}^* \otimes \mathsf{E}^*)$$

and since

$$\mathsf{I}_2(\mathsf{E}^*) = \textstyle\bigwedge^2(\mathsf{E}^*) \oplus (\mathsf{F}^* \otimes \mathsf{E}^*),$$

it follows that $\operatorname{coker}(\sigma_1) = \mathsf{I}_2(\mathsf{E}^*)$ as claimed. ■

Let us define $\mathsf{G} = \mathsf{F}^* \subset \mathsf{V}^*$ and $\mathsf{G}_1 = S^2(\mathsf{F}^*) \subset S^2(\mathsf{V}^*)$. We note that, by definition of σ_1, G_1 is the first prolongation of G. The following lemma gives an important property of the subspace G.

3.3 Lemma: *The subspace* G *is an involutive subspace of* $\mathsf{V}^* \simeq \mathsf{V}^* \otimes \mathbb{R}$.

Proof: Let $\{v_1, \dots, v_n\}$ be a basis for V with the property that $\{v^1, \dots, v^m\}$ forms a basis for F^*. We claim that this basis is quasi-regular which will show involutivity. We have

$$\begin{aligned} \mathsf{G} &= \ker(\sigma) = \mathsf{F}^* \simeq \mathbb{R}^m, \\ \mathsf{G}_{v_1} &= \{\alpha \in \mathsf{G} \mid \ v_1 \lrcorner\, \alpha = 0\} \simeq \mathbb{R}^{m-1}, \\ \mathsf{G}_{v_1,v_2} &= \{\alpha \in \mathsf{G} \mid \ v_1 \lrcorner\, \alpha = v_2 \lrcorner\, \alpha = 0\} \simeq \mathbb{R}^{m-2}, \\ &\ \vdots \\ \mathsf{G}_{v_1,\dots,v_{m-1}} &= \{\alpha \in \mathsf{G} \mid \ v_1 \lrcorner\, \alpha = \dots = v_{m-1} \lrcorner\, \alpha = 0\} \simeq \mathbb{R}. \end{aligned}$$

Note that $\mathsf{G}_{v_1,\dots,v_k} = \{0\}$ for $k \geq m$. We have

$$\dim(\mathsf{G}) + \sum_{j=1}^{m-1} \dim(\mathsf{G}_{v_1,\dots,v_j}) = \sum_{j=1}^{m} j = \tfrac{1}{2} m(m+1) = \dim(\mathsf{G}_1),$$

giving involutivity, as desired. ■

(In fact, of course, every subspace of $\mathsf{V}^* \otimes \mathbb{R}$ is involutive, and this may be proved exactly as was the lemma.)

3.2. The rest of the proof. The sufficiency for existence of potential energy shaping feedbacks of the condition $\boldsymbol{d}\alpha_{\mathrm{cl}} \in \Gamma(\mathsf{I}_2(\mathcal{F}_{\mathrm{cl}}))$ is the most difficult part of the theorem. To prove this we prove the formal integrability of the partial differential equation $\mathsf{R}_{\mathrm{pot}}$. It then follows from a general result of Malgrange [1972a, 1972b] that analytic solutions exist as stated.

Following [Pommaret 1978, Corollary 2.4.9] we have the following result which was proved by Goldschmidt [1967b]. We denote by $\rho_1(\mathsf{R}_{\mathrm{pot}}) \subset \mathsf{J}_2\mathsf{Q}_{\mathbb{R}}$ the first prolongation of $\mathsf{R}_{\mathrm{pot}}$.

3.4 Theorem: *The partial differential equation* $\mathsf{R}_{\mathrm{pot}}$ *is formally integrable in a neighbourhood of* q_0 *if*

(i) it has an involutive symbol at every point in that neighbourhood, if

(ii) the first prolongation of the symbol is a vector bundle, and if

(iii) $\rho_1(\mathsf{R}_{\mathrm{pot}})$ *projects surjectively onto* $\mathsf{R}_{\mathrm{pot}}$ *in that neighbourhood.*

In the remainder of the proof we will let $\mathcal{E}$ be a subbundle of $\mathsf{T}^*\mathsf{Q}$ which is complementary to $\mathcal{F}_{\mathrm{cl}}$: $\mathsf{T}^*\mathsf{Q} = \mathcal{E} \oplus \mathcal{F}_{\mathrm{cl}}$. Moreover, we shall also assume that $\mathcal{E}$ is integrable. This is possible in a neighbourhood $\mathcal{U}_1$ of q_0 since q_0 is a regular point for $\mathcal{F}_{\mathrm{cl}}$. (Explicitly, since $\mathcal{F}_{\mathrm{cl}}$ is integrable, choose coordinates $(q^1, \dots, q^n)$ in a neighbourhood $\mathcal{U}_1$ of q_0 such that $\mathcal{F}_{\mathrm{cl},q} = \operatorname{span}_{\mathbb{R}}(\mathrm{d}q^1(q), \dots, \mathrm{d}q^m(q))$ for each $q \in \mathcal{U}_1$. Then define $\mathcal{E}_q = \operatorname{span}_{\mathbb{R}}(\mathrm{d}q^{m+1}(q), \dots, \mathrm{d}q^n(q))$ for each $q \in \mathcal{U}_1$.) We denote the projection onto $\mathcal{E}$ by $P_{\mathcal{E}} \colon \mathsf{T}^*\mathsf{Q} \to \mathcal{E}$.

Note that this gives a corresponding direct sum decomposition $\mathsf{TQ} = \operatorname{coann}(\mathcal{F}_{\mathrm{cl}}) \oplus \operatorname{coann}(\mathcal{E})$. We shall use this decomposition below without explicit reference. As we did in Section 3.1, we shall also suppose that this decomposition gives rise to decompositions of the tensor algebra, and we shall use these decompositions to give explicit inclusions and projections from and onto various subspaces of tensors.

If we identify $\mathcal{E}$ and $\mathsf{T}^*\mathsf{Q}/\mathcal{F}_{\mathrm{cl}}$ in the natural way then $P_{\mathcal{E}}$ is just the representation of $\pi_{\mathcal{F}_{\mathrm{cl}}}$ under this identification. Let us define $\Phi_{\mathrm{pot}}\colon \mathsf{J}_1\mathsf{Q}_{\mathbb{R}} \to \mathcal{E}$ by $\Phi_{\mathrm{pot}} = P_{\mathcal{E}} \circ \Phi_{\boldsymbol{d}}$. Thus we have

$$\mathsf{R}_{\mathrm{pot}} = \{p \in \mathsf{J}_1\mathsf{Q}_{\mathbb{R}} \mid \Phi_{\mathrm{pot}}(p) = P_{\mathcal{E}} \circ \alpha_{\mathrm{cl}}(q),\ q = \pi_1(p)\}.$$

Let us first determine the symbol $\mathsf{G}(\mathsf{R}_{\mathrm{pot}})$ for $\mathsf{R}_{\mathrm{pot}}$.

3.5 Lemma: *We have* $\mathsf{G}(\mathsf{R}_{\mathrm{pot}}) = \mathcal{F}_{\mathrm{cl}}$.

Proof: Note that $\mathsf{G}(\mathsf{R}_{\mathrm{pot}}) = \ker(\sigma(\Phi_{\mathrm{pot}}))$, where $\sigma(\Phi_{\mathrm{pot}})$ is the symbol of Φ_{pot}. A direct computation using the definition of Φ_{pot} gives

$$\sigma(\Phi_{\mathrm{pot}})(\alpha)(u_1) = \alpha(u_1 \oplus 0).$$

By Lemma 3.1 our claim about $\mathsf{G}(\mathsf{R}_{\mathrm{pot}})$ follows. ■

From the preceding lemma and Lemma 3.3 we know that $\mathsf{G}(\mathsf{R}_{\mathrm{pot}})$ is involutive.

Let us give the first prolongation of $\mathsf{G}(\mathsf{R}_{\mathrm{pot}})$, which we denote by $\rho_1(\mathsf{G}(\mathsf{R}_{\mathrm{pot}}))$.

3.6 Lemma: *We have* $\rho_1(\mathsf{G}(\mathsf{R}_{\mathrm{pot}})) = S^2(\mathcal{F}_{\mathrm{cl}})$.

Proof: We make the following observations:

1. $\mathsf{G}(\mathsf{R}_{\mathrm{pot}})_q$ is the kernel of the map σ used in Section 3.1 if we take $\mathsf{V}^* = \mathsf{T}^*_q\mathsf{Q}$ and $\mathsf{F}^* = \mathcal{F}_{\mathrm{cl},q}$;
2. the map σ_1 in Section 3.1 is the first prolongation of σ.

An application of Lemma 3.2 gives the result. ■

We can then see that the first prolongation of $\mathsf{G}(\mathsf{R}_{\mathrm{pot}})$ is a vector bundle on the open subset $\mathcal{U}_2$ of Q on which $\mathcal{F}_{\mathrm{cl}}$ is a vector bundle. Clearly $q_0 \in \mathrm{int}(\mathcal{U}_2)$.

We have now to verify only the third of the hypotheses of Theorem 3.4.

Let $\mathsf{K} = \mathrm{coker}(\sigma_1(\Phi_{\mathrm{pot}}))$ and denote by τ the canonical projection from $\mathsf{T}^*\mathsf{Q} \otimes \mathcal{E}$ to K.

Following [Pommaret 1978, page 69] we have the following commutative and exact diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & & \\
 & & S^2(\mathsf{T}^*\mathsf{Q}) \arrow[r, "\sigma_1(\Phi_{\mathrm{pot}})"] \arrow[d] & \mathsf{T}^*\mathsf{Q} \otimes \mathcal{E} \arrow[r, "\tau"] \arrow[d] & \mathsf{K} \arrow[r] & 0 \\
0 \arrow[r] & \rho_1(\mathsf{R}_{\mathrm{pot}}) \arrow[r] \arrow[d] & \mathsf{J}_2\mathsf{Q}_{\mathbb{R}} \arrow[r, "\rho_1(\Phi_{\mathrm{pot}})"] \arrow[d] & \mathsf{J}_1\mathcal{E} \arrow[d] & & \\
0 \arrow[r] & \mathsf{R}_{\mathrm{pot}} \arrow[r] & \mathsf{J}_1\mathsf{Q}_{\mathbb{R}} \arrow[r, "\Phi_{\mathrm{pot}}"] \arrow[d] & \mathcal{E} \arrow[d] & & \\
 & & 0 & 0 & &
\end{tikzcd}$$

All unmarked arrows are either canonical inclusions or canonical projections. We define a map κ from $\mathsf{R}_{\rm pot}$ to K as follows. Let $p \in \mathsf{R}_{\rm pot}$ project to $q \in \mathsf{Q}$. Then $\Phi_{\rm pot}(p) = P_{\mathcal{E}} \circ \alpha_{\rm cl}(q)$. Choose $p' \in \mathsf{J}_2\mathsf{Q}_{\mathbb{R}}$ projecting to p and define $\omega = \rho_1(\Phi_{\rm pot})(p') \in \mathsf{J}_1\mathcal{E}$. Then, by commutativity of the diagram, ω projects to $\Phi_{\rm pot}(p) = P_{\mathcal{E}} \circ \alpha_{\rm cl}(q) \in \mathcal{E}$. We then take $\kappa(p) = \tau(\omega - j_1\alpha_{\rm cl}(q))$. One can easily show that this definition of κ is independent of the choice of p'. Pommaret [1978, Theorem 2.4.1] shows that p lies in the image of the projection of $\rho_1(\mathsf{R}_{\rm pot})$ to $\mathsf{R}_{\rm pot}$ if and only if $\kappa(p) = 0$.

Let us give a way of explicitly constructing κ. In the construction we identify sections of $\mathsf{Q}_{\mathbb{R}}$ with functions on Q in the obvious way. Let $p \in \mathsf{R}_{\rm pot}$. We let V be a section of $\mathsf{Q}_{\mathbb{R}}$ such that $j_1V(q) = p$. Thus $P_{\mathcal{E}} \circ \boldsymbol{d}V(q) = P_{\mathcal{E}} \circ \alpha_{\rm cl}(q)$. Then define $p' = j_2V(q)$ so that

$$\rho_1(\Phi_{\rm pot})(p') = j_1(P_{\mathcal{E}} \circ \boldsymbol{d}V)(q)$$

by definition of $\rho_1(\Phi_{\rm pot})$. Then we have

$$\kappa(p) = \tau(j_1(P_{\mathcal{E}} \circ \boldsymbol{d}V)(q) - j_1(P_{\mathcal{E}} \circ \alpha_{\rm cl})(q)).$$

To complete the proof, let $(q^1, \dots, q^n)$ be coordinates for which

$$\mathcal{F}_{{\rm cl},q} = \operatorname{span}_{\mathbb{R}}(\mathrm{d}q^1(q), \dots, \mathrm{d}q^m(q)), \quad \mathcal{E}_q = \operatorname{span}_{\mathbb{R}}(\mathrm{d}q^{m+1}(q), \dots, \mathrm{d}q^n(q))$$

in some neighbourhood of q_0. Let us adopt the convention that indices in the set $\{1, \dots, n\}$ will be denoted by j and k, indices in the set $\{1, \dots, m\}$ will be denoted by a and b, and indices in the set $\{m+1, \dots, n\}$ with be denoted by r and s. In these coordinates we write $\alpha_{\rm cl} = \alpha_j \mathrm{d}q^j$ so that

$$\boldsymbol{d}\alpha_{\rm cl} = \Big(\frac{\partial \alpha_k}{\partial q^j} - \frac{\partial \alpha_j}{\partial q^k}\Big)\mathrm{d}q^j \wedge \mathrm{d}q^k, \qquad j < k.$$

The assumption that $\boldsymbol{d}\alpha_{\rm cl} \in \Gamma(\mathsf{I}_2(\mathcal{F}_{\rm cl}))$ is equivalent in coordinates to

$$\frac{\partial \alpha_r}{\partial q^s} = \frac{\partial \alpha_s}{\partial q^r}, \qquad r, s \in \{m+1, \dots, n\}. \tag{3.2}$$

We also have

$$P_{\mathcal{E}} \circ \boldsymbol{d}V = \frac{\partial V}{\partial q^r}\mathrm{d}q^r, \quad P_{\mathcal{E}} \circ \alpha_{\rm cl} = \alpha_r \mathrm{d}q^r.$$

Now let V be such that $j_1V(q_0) = p_0$. Since τ is the projection onto the cokernel of $\sigma_1(\Phi_{\rm pot})$, we have $\kappa(p) = 0$ if and only if

$$j_1(P_{\mathcal{E}} \circ \boldsymbol{d}V)(q_0) - j_1(P_{\mathcal{E}} \circ \alpha_{\rm cl})(q_0) \in \operatorname{image}(\sigma_1(\Phi_{\rm pot})) = S^2(\mathcal{E}).$$

In jet bundle coordinates this condition reads

$$\frac{\partial^2 V}{\partial q^r \partial q^s}(q_0) - \frac{\partial \alpha_s}{\partial q^r}(q_0) = \frac{\partial^2 V}{\partial q^s \partial q^r}(q_0) - \frac{\partial \alpha_r}{\partial q^s}(q_0), \qquad r, s \in \{m+1, \dots, n\}.$$

However, this identity does indeed hold by (3.2). This shows that, if $\boldsymbol{d}\alpha_{\rm cl} \in \Gamma(\mathsf{I}_2(\mathcal{F}_{\rm cl}))$, then there exists a $(\mathbb{G}_{\rm ol}, \mathbb{G}_{\rm cl})$-potential energy shaping feedback F, defined in a neighbourhood of q_0, such that $p_0 = F_{\rm cl}(q_0) + \alpha_{\rm cl}(q_0)$.

The converse is straightforward. We use the coordinates $(q^1, \dots, q^n)$ from above. If F is a $(\mathbb{G}_{\rm ol}, \mathbb{G}_{\rm cl})$-potential energy shaping feedback defined in a neighbourhood $\mathcal{U}$ of q_0, then there exists a function $V \colon \mathcal{U} \to \mathbb{R}$ such that $\frac{\partial V}{\partial q^r}(q) = \alpha_r(q)$, $r \in \{m+1, \dots, n\}$, for all $q \in \mathcal{U}$. It then immediately follows by differentiating with respect to q^s, $s \in \{m+1, \dots, n\}$, that (3.2) holds on $\mathcal{U}$. But this is exactly the condition that $\boldsymbol{d}\alpha_{\rm cl} \in \Gamma(\mathsf{I}_2(\mathcal{F}_{\rm cl}))$.

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