Geodesic reduction via frame bundle geometry[1]

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2007/07/26

Abstract

Reduction theory for systems with symmetry deals with the problem of understanding dynamics on a manifold with an action of a Lie group. In geometric mechanics, this problem can be formulated in the Lagrangian, Hamiltonian or affine connection frameworks. While the Lagrangian and Hamiltonian formulations have been well developed, the results obtained in these setups are based on variational principles and symplectic geometry. These methods cannot be used directly in the affine connection formulation unless additional structure is available.

In this thesis, a manifold with an arbitrary affine connection is considered, and the geodesic spray associated with the connection is studied in the presence of a Lie group action. In particular, results are obtained that provide insight into the structure of the reduced dynamics associated with the given invariant affine connection. The geometry of the frame bundle of the given manifold is used to provide an intrinsic description of the geodesic spray. A fundamental relationship between the geodesic spray, the tangent lift and the vertical lift of the symmetric product is obtained, which provides a key to understanding reduction in this formulation.

Contents

Chapter 1

Introduction

The geometry of systems with symmetry has been an active area of research in the last 300 years. The study of manifolds with certain special geometric structure invariant under Lie group action leads to what is known as reduction theory. Such questions arise in, for example, geometric mechanics. In this framework, the presence of symmetry allows the dynamics on a manifold to be studied on a lower dimensional manifold. In mechanics, there are at least three different ways of describing dynamics on a manifold, corresponding to the Lagrangian, Hamiltonian and affine connection formulations respectively. While the reduction theory for Lagrangian and Hamiltonian systems has been well developed (see [22, 1, 7, 6]), the results have been obtained by using variational analysis and symplectic geometry respectively. The main reason behind following this approach is the fact that the dynamics for such systems arises from variational principles which are manifested by symplectic structures in the Hamiltonian framework. However, when the dynamics on a manifold are given in terms of the geodesic equation of an affine connection, we cannot use variational analysis unless additional structure is provided.

1.1. Reduction theory for mechanical systems

In this section we motivate the idea of reduction in mechanics by presenting a simple example. What we say here can be found in [21, 20]. A detailed review of reduction theory will be presented in Chapter 2.

The equations of motion of a particle with charge e and mass m in a magnetic field can be considered as reduced equations coming from dynamics on a larger manifold as we discuss below.

First of all, let B be a closed two-form on $\mathbb{R}^3$ and let $\mathbf{B} = B_x \mathbf{i} + B_y \mathbf{j} + B_Z \mathbf{k}$ be the associated divergence free vector field satisfying

$$\mathbf{i_B}(dx \wedge dy \wedge dz) = B,$$

where $\mathbf{i_B}$ refers to the interior product. The vector field $\mathbf{B}$ can be thought of as a magnetic field. Let the configuration space be given by $M = \mathbb{R}^3 \times \mathbb{S}^1$ with variables (q, θ). The Lie group $G = \mathbb{S}^1$ acts on M in a natural way and $M/G \simeq \mathbb{R}^3$. If $B = \mathbf{d}A$, for some one-form A on $\mathbb{R}^3$, that is, $\mathbf{B} = \nabla \times \mathbf{A}$, where $\mathbf{A}^\flat = A$, we consider the one-form $\omega = A + d\theta$ regarded

as a principal connection one-form. Define a Lagrangian $L : TM \to \mathbb{R}$ as follows:

$$L(q, \dot{q}, \theta, \dot{\theta}) = \frac{1}{2}m||\dot{q}||^2 + \frac{1}{2}(\omega(q, \dot{q}, \theta, \dot{\theta}))^2.$$

Now, the Euler–Lagrange equations corresponding to this Lagrangian are the geodesic equations on M for the Levi–Civita connection corresponding to the metric for which L is the kinetic energy. These equations can be reduced, for example, by using the procedure described in [20], and the reduced dynamics is given by

$$
\begin{aligned}
m\ddot{x} &= \frac{e}{c}(B_z\dot{y} - B_y\dot{z}), \\
m\ddot{y} &= \frac{e}{c}(B_x\dot{z} - B_z\dot{x}), \\
m\ddot{z} &= \frac{e}{c}(B_y\dot{x} - B_x\dot{y}).
\end{aligned}
\tag{1.1.1}
$$

These equations correspond to the Lorentz force law for a particle with charge e and mass m:

$$m\frac{d\mathbf{v}}{dt} = \frac{e}{c}\mathbf{v} \times \mathbf{B},$$

where $\mathbf{v} = (\dot{x}.\dot{y}, \dot{z})$. This therefore provides an example of a physical system whose dynamics can be thought of as the dynamics of a reduced system.

1.2. Contribution of this thesis

In this thesis, we consider an *arbitrary* affine connection on a manifold and provide results that enable us to decompose the reduced geodesic spray corresponding to the affine connection using tools from affine differential geometry only. In other words, we do not use variational methods. In arriving at our results, we come to a deeper understanding of the geometry of bundle of linear frames and its relationship with the geometry of the tangent bundle of the given manifold.

The setup we consider is the following. Let M be a manifold and G a Lie group which acts on M in such a manner that M is the total space of a principal bundle over M/G. Let A be an arbitrary principal connection on this bundle. The Lie group G also acts on the bundle $L(M)$ of linear frames over M via the lifted action. In such a case we say that $L(M)$ is G-compatible. It is known that there is a one-to-one correspondence between principal connections on $L(M)$ and affine connections on M [12]. Let ω be a G-invariant linear connection on $L(M)$ with ∇ the corresponding affine connection on M. The geodesic spray Z corresponding to ∇ is a *second-order vector field* on the tangent bundle TM with the property that the projection of its integral curves correspond to geodesics on M. Thus, to understand how the dynamics evolves under symmetry, Z is the appropriate object to study. Since additional structure is not available, we exploit the geometry of the linear frame bundle in order to completely understand the meaning of the geodesic spray (which classically is only defined in local coordinates). The first significant step in this direction is to provide an intrinsic definition of the geodesic spray that uses frame bundle geometry. We are able to provide such a definition and build on this knowledge to give an

alternate proof of the statement that there is a one-to-one correspondence between torsion-free connections and geodesic sprays. Along the way, we provide proofs to some general statements concerning the space of all principal connections on a principal bundle as well as the space of affine connections on a manifold.

Our investigation of invariant linear connections on the frame bundle leads us to the more general problem of studying invariant connections on arbitrary principal bundles. In this context, we prove a generalization of a result by Wang [35] which characterizes invariant connections on principal bundles. We also study the geometry of the frame bundle of a Lie group and provide an intrinsic derivation of the Euler–Poincaré equations. The derivations found in the literature depend on the choice of a basis for the Lie algebra of the Lie group and this reinforces our belief that frame bundle geometry can provide valuable insight into understanding the dynamics on a manifold with an affine connection.

Moving ahead, we study the notion of geodesic invariance for a distribution on the manifold M and provide an intrinsic proof of a characterization due to Lewis [16] using the symmetric product. We investigate the structure of a G-compatible frame bundle adapted to a principal connection and provide a decomposition in terms of the frame bundle $L(M/G)$ and construct several bundles that help us understand the geometry of the reduced frame bundle.

Next, we turn our attention to understanding the reduced geodesic spray of a given connection. We prove an important relationship between the geodesic spray, the tangent lift and the vertical lift of the symmetric product. This result is of fundamental importance and it enables us to decompose the reduced geodesic spray in terms of objects defined on reduced spaces.

1.3. Contents of the thesis

Below we provide a chapter-by-chapter description of the thesis and state what is new in each chapter.

Chapter 2 In this chapter we review the existing literature relevant to our investigation and present a historical development of reduction theory.

Chapter 3 In this chapter we review some fundamental concepts and definitions that we shall build upon and notation that we shall use in this thesis. We provide a detailed description of the linear frame bundle and how its geometry is related to that of TM.

Chapter 4 Here we characterize the set of all principal connections on a principal bundle and specialize it to the case of the linear frame bundle. In particular, we show that there is a one-to-one correspondence between $(1, 2)$ tensor fields on M and tensorial one-forms on the frame bundle that have a certain property. These results are known but to our knowledge they have not been written down in this form. Next, we provide an intrinsic frame bundle interpretation of the Liouville vector field on TM as well as the geodesic spray corresponding to a linear connection, and define the Ehresmann connection induced by the linear connection. These constructions have not appeared in the literature previously. We prove a weak generalization of a result by Wang [35] which relates invariant principal connections to certain mappings between vector bundles. We also study the geometry of the frame bundle of a Lie group and provide an intrinsic derivation of the Euler–Poincaré equation.

Chapter 5 In this chapter we present the main results of this thesis. We first study the notion of geodesic invariance of a distribution, and provide an alternate proof of a result by Lewis [16] using the linear frame bundle. Next, in the presence of a principal connection A, we provide a decomposition of the frame bundle adapted to the connection similar to the decomposition of the tangent bundle induced by a principal connection. These bundle constructions have not appeared in the literature previously. Next, we prove an important formula that relates the geodesic spray of a connection evaluated at a point to the tangent lift and the vertical lift of the symmetric product. This formula provides insight into the nature of Z and how it is related to the concept of geodesic invariance. Furthermore, it enables us to decompose the reduced geodesic spray into pieces that we understand.

Chapter 6 In this chapter we write down the conclusions we draw based on this investigation and point to certain avenues for further research.

Chapter 2

Literature review

In this chapter, we review the work done in the area of reduction of mechanical systems with symmetry.

2.1. The development of Lagrangian reduction theory

We refer to Cendra, Marsden and Ratiu [6] for what we say in this section. Reduction theory has its origins in the work of Euler, Lagrange, Hamilton, Jacobi, Routh, Poincaré and Lie. Below we survey the progress that has been made in this area over the past 150 years.

One of the earliest contributions in this field is due to Routh [29, 30] who worked on reduction for Abelian groups. Lie [17] discovered several basic structures in symplectic and Poisson geometry and their link with symmetry. Poincaré [28] discovered the generalization of the Euler equations for a rigid body mechanics to general Lie algebras (see also, [33, 3]). Modern reduction theory began with Arnold [2] and Smale [31]. In order to synthesize the Lie algebra reduction methods of Arnold with the methods of Smale on the reduction of cotangent bundle, Marsden and Weinstein [22] developed reduction theory for symplectic manifolds. We describe their construction below.

Let (P, Ω) be a symplectic manifold and let a Lie group G act freely and properly on P by symplectic maps. Assume that this action has an equivariant momentum map $J : P \to \mathfrak{g}^*$. Then, the **symplectic reduced space** $J^{-1}(\mu)/G_\mu =: P_\mu$ is a symplectic manifold in a natural way; the induced symplectic form Ω_μ is determined uniquely by $\pi_\mu^* \Omega_\mu = i_\mu^* \Omega$ where $\pi_\mu : J^{-1}(\mu) \to P_\mu$ is the projection and $i_\mu : J^{-1}(\mu) \to P$ is the inclusion. If the momentum map is not equivariant, Souriau [32] discovered how to centrally extend the group to make it equivariant. Coadjoint orbits were shown to be symplectic reduced space by Marsden and Weinstein [22]. In the reduction construction, if we choose $P = T^*G$, with G acting by cotangent lift, the corresponding space P_μ is identified with the coadjoint orbit $\mathcal{O}_\mu$ through μ together with its coadjoint orbit symplectic structure. Likewise, the Lie-Poisson bracket on $\mathfrak{g}^*$ is inherited from the canonical Poisson structure on T^*G by *Poisson reduction*, that is, by identifying $\mathfrak{g}^*$ with the quotient T^*G/G. This observation is implicit in Lie [17], Kirilov [11], Guillemin and Sternberg [10] and Marsden and Weinstein [24, 23].

2.1.1. Tangent and cotangent bundle reduction. Given a manifold M with a free and proper action of a Lie group G, the simplest case of cotangent bundle reduction is reduction

at zero in which case one chooses $P = T^*M$ and then the reduced space at $\mu = 0$ is given by $P_0 = T^*(M/G)$ with the canonical symplectic form. Another simple case is when G is Abelian. Here, $(T^*M)_\mu \simeq T^*(M/G)$ but the latter has a symplectic structure modified by the curvature of a connection.

The Abelian version of cotangent bundle reduction was developed by Smale [31] and was generalized to the nonabelian case in Abraham and Marsden [1]. Kummer [14] provided an interpretation of Abraham and Marsden's results in terms of a connection, now called the *mechanical connection*. The geometry of this situation was used by Guichardet [9] and Montgomery [25, 26, 27]. Routh reduction may be viewed as the Lagrangian analogue of cotangent bundle reduction.

Tangent and cotangent bundle reduction evolved into a "bundle picture of mechanics". This point of view was developed in Marsden, Montgomery and Ratiu [18] and Montgomery [25]. That work was influenced by the work of Sternberg [34] and Weinstein [36]. The main result of the bundle picture gives a structure to the quotient spaces $(T^*M)/G$ and $(TM)/G$ where G acts by cotangent and tangent lifted actions. The structure of the reduced tangent bundle TM/G forms part of the structure we use in Chapter 5.

2.1.2. Lagrangian versus Hamiltonian reduction. In symplectic and Poisson reduction, the objective is to pass the symplectic form and the Poisson bracket as well as Hamiltonian dynamics to the quotient. In modern Lagrangian reduction it is the variational principles that pass onto the quotient. Of course, the two methodologies are related by the Legendre transform. Below we provide a brief summary of Lagrangian reduction theory.

2.1.3. Routh reduction. Routh reduction for Lagrangian systems is associated with systems having cyclic variables; a modern treatment of the subject may be found in Marsden and Ratiu [19]. An important feature of Routh reduction is that when one drops the Euler-Lagrange equations to the quotient space associated with symmetry and when the momentum map is constrained to a specified value, then the resulting equations are in Euler-Lagrange form not with respect to the Lagrangian itself, but with respect to the *Routhian*.

2.1.4. Euler-Poincaré reduction. Another fundamental case of Lagrangian reduction is that of *Euler-Poincaré reduction*. In this case the configuration manifold is a Lie group G. This case has its origins in the work of Lagrange [15] and Poincaré [28].

The classical Euler-Poincaré equations are as follows. Let ξ^a be coordinates for the Lie algebra $\mathfrak{g}$ of a Lie group G and let C^a_{bd} be the associated structure constants. Let $TG \to \mathbb{R}$ be a given left-invariant Lagrangian and let $\ell : \mathfrak{g} = TG/G \to \mathbb{R}$ be the corresponding reduced Lagrangian. Then the ***Euler-Poincaré*** equations for a curve $\xi(t) \in \mathfrak{g}$ are

$$\frac{d}{dt}\frac{\partial \ell}{\partial \xi^b} = \frac{\partial \ell}{\partial \xi^a}C^a_{db}\xi^d.$$

These equations are equivalent to the Euler-Lagrange equations for L for a curve $g(t) \in G$, where $g(t)^{-1}\dot{g}(t) = \xi(t)$. This is one of the most basic formulations of Lagrangian reduction.

The general formulation of Euler-Poincaré reduction in terms of variational principles was given by Marsden and Scheurle [21, 20]. A modern treatment of this subject is given

in [19]. In Section 4.5.2 we provide a intrinsic derivation of these equations without using variational principles.

2.1.5. Lagrange-Poincaré reduction. Marsden and Scheurle [21, 20] also generalized the Routh theory to the non-Abelian case and introduced the idea of *reducing variational principles*. In Cendra, Marsden and Ratiu [7], the Euler-Poincaré case is extended to arbitrary configuration manifolds. This leads to *Lagrange-Poincaré reduction*.

One of the things that makes the Lagrangian side of reduction theory more interesting is the absence of a general category that is the Lagrangian analogue of Poisson manifolds. Such a category, that of *Lagrange-Poincaré bundles* is developed in Cendra, Marsden and Ratiu [7].

The Lagrangian analogue of the bundle picture is the bundle TM/G, which is a vector bundle over M/G. The equations and variational principles are developed on this space. For $M = G$ this reduces to Euler-Poincaré reduction and for G Abelian, it reduces to the classical Routh procedure. Given a Lagrangian L on TM, it induces a Lagrangian ℓ on TM/G. The resulting equations inherited on this space are the *Lagrange-Poincaré equations* (or the *reduced Euler-Lagrange equations*).

2.2. Relation of thesis to previous work

In the previous sections, we have provided a brief account of some of the main work done in the field of reduction of systems with symmetry. As stated in the introduction, our work differs from the existing literature in that we do not use variational principles and symplectic geometry to study the reduced dynamics. Our treatment is based on an intrinsic formulation of the problem using the geometry of frame bundles.

Chapter 3

Definitions and notation

In this chapter we present some mathematical tools and establish notation that we shall be using in this thesis. In Section 3.1, we review some fundamental concepts related to Lie groups and their actions on manifolds. We follow the treatment in Marsden and Ratiu [19]. In the next section, we introduce locally trivial fiber bundles, and then proceed to define a principal fiber bundle in Section 3.2.1. The notion of a connection on a principal fiber bundle will be central to our investigations and we present a fairly detailed account of the basic geometric structures on principal fiber bundles. Finally, in Section 3.3, we define the bundle of linear frames and introduce the notion of torsion. We follow the classic text of Kobayashi and Nomizu [12] for most of what we say in this section.

3.1. Action of a Lie group on a manifold

3.1.1. Lie groups. A *Lie group* G is a group with a differentiable structure that makes the group multiplication

$$\mu : G \times G \to G, \quad (g, h) \mapsto gh$$

a smooth map. Given $g \in G$, the *left translation map* $L_g : G \to G$ is defined by $L_g(h) = gh$. Similarly, given $h \in G$, the *right translation map* $R_h : G \to G$ is given by $R_h(g) = gh$. The left and right translations commute. That is

$$L_g \circ R_h = R_h \circ L_g, \quad g, h \in G.$$

A vector field X on G is called *left-invariant* if for every $g \in G$, we have $L_g^* X = X$, that is, if

$$T_h L_g(X(h)) = X(gh).$$

A left-invariant vector field on G is uniquely determined by its value at the identity. We denote the left-invariant vector field with value $\xi \in T_e G$ at the identity by X_ξ. The tangent space $T_e G$ to G at e is a vector space and the Lie bracket on $T_e G$ is defined by

$$[\xi, \eta] = [X_\xi, X_\eta](e),$$

where $[X_\xi, X_\eta]$ is the Lie bracket of vector fields. The vector space $T_e G$ equipped with this bracket is called the **Lie algebra of G** and we denote it by $\mathfrak{g}$. The Lie algebra of a Lie group is isomorphic to the set of left-invariant vector fields on G.

Next, we define a map $\exp : \mathfrak{g} \to G$ as follows. For $\xi \in \mathfrak{g}$ let Φ_t^ξ be the flow of the left-invariant vector field X_ξ. Then,

$$\exp(\xi) = \Phi_1^\xi(e).$$

Given a Lie group G and its Lie algebra $\mathfrak{g}$, recall that the **adjoint representation** Ad of G on $\mathfrak{g}$ is defined by

$$\mathrm{Ad}_g \eta = T_e L_g R_{g^{-1}} \eta, \quad \eta \in \mathfrak{g}, \ g \in G.$$

3.1.2. Action of a Lie group on a manifold. Let P be a manifold and G a Lie group. A **left action** of G on P is a smooth mapping $\Phi : G \times P \to P$ such that

(i) $\Phi(e, u) = u, \forall u \in P$ and

(ii) $\Phi(g, (\Phi(h, u))) = \Phi(gh, u), \forall g, h \in G$ and $u \in P$.

A **right action** is a map $\Psi : P \times G \to P$ that satisfies $\Psi(u, e) = u$ and $\Psi(\Psi(u, g), h) = \Psi(u, gh)$. We shall sometimes use the notation $g \cdot u = \Phi(g, u)$ and $u \cdot g = \Psi(u, g)$. Given $g \in G$ and $u \in P$, we also define maps $\Phi_g : P \to P$ and $\Phi_u : G \to P$ that satisfy $\Phi_g(u) = \Phi(g, u) = \Phi_u(g)$. We can similar define maps Ψ_g and Ψ_u for right actions.

An action of G on P is called

(i) **transitive** if, for every $u, v \in P$, there exists $g \in G$ such that $g \cdot u = v$,

(ii) **effective** (or **faithful**) if $\Phi_g = \mathrm{id}_P$ implies that $g = e$,

(iii) **free** if $\Phi_g(u) = u$ implies that $g = e$, and

(iv) **proper** if the mapping $\tilde{\Phi} : G \times P \to P \times P$ given by

$$\tilde{\Phi}(g, u) = (u, \Phi(g, u))$$

is proper, that is, the preimage of a compact set under this mapping is a compact set.

The following result proved in [1] gives a sufficient condition for the quotient P/G to be a smooth manifold.

3.1 Proposition: *If $\Phi : G \times P \to P$ is a free and proper action, then P/G is a smooth manifold and the natural projection $\pi_{P/G} : P \to P/G$ is a surjective submersion.*

For each $\xi \in \mathfrak{g}$, the action Φ of G on P induces a vector field ξ_P on P as follows.

$$\xi_P(u) = \left. \frac{d}{dt} \right|_{t=0} \Phi_{\exp \xi t} u.$$

The vector field ξ_P is called the **infinitesimal generator** corresponding to ξ for the action of G on P. If Φ is free and proper, we say that ξ_P is a **vertical vector field** in the sense that, for each $u \in P$, the vector $\xi_P(u)$ projects to zero under the map $T\pi_{P/G}$. Let us denote the set of vertical vector fields on P by $\Gamma(VP)$. The map $\sigma_V : \mathfrak{g} \to \Gamma(VP)$ that takes ξ to ξ_P is a Lie algebra antihomomorphism. That is, for $\xi, \eta \in \mathfrak{g}$, we have

$$\sigma_V([\xi, \eta]) = -[\sigma_V(\xi), \sigma_V(\eta)].$$

For right actions, the map σ_V is a Lie algebra homomorphism.

Let P and Q be manifolds and let G be a Lie group that acts on P by $\Phi_g : P \to P$ and on Q by $\Psi_g : Q \to Q$. A smooth map $f : P \to Q$ is called **G-equivariant** with respect to these actions if for all $g \in G$, we have

$$f \circ \Phi_g = \Psi_g \circ f.$$

3.2. Locally trivial fiber bundles

A **locally trivial fiber bundle** is a 4-tuple (π, P, M, F), where

(i) P, M and F are smooth manifolds;

(ii) $\pi : P \to M$ is a surjective submersion, and

(iii) P is locally trivial, that is, every point $x \in M$ has a neighborhood U such that $\pi^{-1}(U)$ is diffeomorphic to $U \times F$ via a diffeomorphism $\psi : \pi^{-1}(U) \to U \times F$ with the property that $\mathrm{pr}_1 \circ \psi = \pi$ (pr_1 being the projection onto the first component).

We call M the **base space** and F the **standard fiber**. Given $x \in M$, we shall denote by P_x the preimage $\pi^{-1}(x)$ and call it the **fiber over x**. For $u \in \pi^{-1}(x)$ the **fiber through u** is defined as the fiber over $\pi(u)$ and denoted by P_u.

We shall often use the term "fiber bundle" to refer to a locally trivial fiber bundle with the understanding that local triviality is implicit. If (π_1, P_1, M_1, F_1) and (π_2, P_2, M_2, F_2) are locally trivial fiber bundles, a map $f : P_1 \to P_2$ is called a **fiber bundle map** if there exists a map $f_0 : M_1 \to M_2$ such that the following diagram commutes:

$$
\begin{array}{ccc}
P_1 & \xrightarrow{\ f\ } & P_2 \\
\pi_1 \downarrow & & \downarrow \pi_2 \\
M_1 & \xrightarrow[f_0]{} & M_2
\end{array}
$$

In such a case, we say that f is a fiber bundle map **over** f_0. A fiber bundle map is called a **fiber bundle isomorphism** if both f and f_0 are diffeomorphisms.

3.2.1. Principal fiber bundles. Let M be a manifold and G a Lie group. A right principal bundle over M with structure group G consists of a manifold P and an action Φ of G on P satisfying the following conditions:

(i) G acts freely on P on the right.

(ii) M is the quotient space of P by the equivalence relation induced by the action of G on P, and the natural projection $\pi : P \to M$ is differentiable;

(iii) P is locally trivial, that is, every point $x \in M$ has a neighborhood U such that $\pi^{-1}(U)$ is diffeomorphic to $U \times G$. In other words, there is a diffeomorphism $\psi : \pi^{-1}(U) \to U \times G$ such that $\psi(u) = (\pi(u), \phi(u))$ where $\phi : \pi^{-1}(U) \to G$ is a G-equivariant map.

We shall denote a principal fiber bundle as $P(M, G)$. A *left* principal bundle can be defined in a similar way. Unless stated otherwise, throughout this dissertation, a principal fiber bundle will mean a *right* principal bundle.

3.2.2. Associated bundles. Let $P(M, G)$ be a principal fiber bundle and let F be a manifold on which G acts on the left:

$$\Psi : G \times F \to F$$
$$(g, \xi) \mapsto \Psi(g, \xi) =: g \cdot \xi.$$

Define a *right* action of G on $P \times F$ as follows:

$$(P \times F) \times G \to (P \times F)$$
$$((u, \xi), g) \mapsto (\Phi(u, g), \Psi(g^{-1}, \xi))$$
$$:= (ug, g^{-1}\xi).$$

Denote by $E := (P \times F)/G =: P \times_G F$ the quotient of $P \times F$ by G and the projection onto E by $\pi_G : P \times F \to E$. Given $(u, \xi) \in P \times F$, we know that $\pi_G(u, \xi)$ is the equivalence class (defined by the action of G on $P \times F$) containing (u, ξ). We shall denote this equivalence class by $[u, \xi]_G$. Define a map $\pi_E : E \to M$ by

$$\pi_E([u, \xi]_G) = \pi(u).$$

Now, it can be shown that E has a differentiable structure that makes π_E a surjective submersion. Thus, $\pi_E : E \to M$ is a (*locally trivial*) fiber bundle with standard fiber F and we call it the **bundle associated with $P(M, G)$ with standard fiber F**. We shall denote this bundle by $E(M, F, P, G)$. To avoid the use of excessive language, we shall sometimes call $E(M, F, P, G)$ simply the associated bundle whenever the underlying principal fiber bundle and the standard fiber are understood to be $P(M, G)$ and F respectively. This should cause no confusion. The following result is immediate once the notation is understood properly.

3.2 Proposition: *Let $P(M, G)$ be a principal fiber bundle and F a manifold on which G acts on the left. Let $E(M, F, G, P)$ be the associated bundle. For $u \in P$ and $\xi \in F$, we write $[u, \xi]_G := u\xi \in E$. Then each $u \in P$ is a mapping of F onto $F_x = \pi_E^{-1}(x)$ where $x = \pi(u)$ and*

$$(ug)\xi = u(g\xi) \qquad \text{for } g \in G, \ u \in P, \ \xi \in F. \tag{3.2.1}$$

3.3 Remark: For $u \in P$ and $x = \pi(u)$, the map $u : F \to F_x$ is given by $u\xi = [u, \xi]_G$ and thus it is easy to see that $[ug, \xi]_G = [u, g\xi]_G$, which is (3.2.1). We shall use the notation "$u\xi$" and "$[u, \xi]_G$" interchangeably, depending on context.

For *fixed* $\xi \in F$, the map $\Phi_\xi : P \to E$ defined by $\Phi_\xi(u) = u\xi$ is a fiber bundle map over the identity map of M. We call it the **association map** for E. This map will appear frequently in the sequel.

The adjoint bundle. We now introduce a very important associated bundle that we shall use frequently in later chapters. Let $M(M/G, G)$ be a principal bundle, and let $\mathfrak{g}$ be the Lie algebra of G. The Lie group G acts on $\mathfrak{g}$ via the adjoint representation. The **adjoint bundle**, denoted by $\tilde{\mathfrak{g}}$ is the bundle associated with $M(M/G, G)$ with standard fiber $\mathfrak{g}$. A point in the adjoint bundle will be typically represented as $[x, \xi]_G$, where $x \in M$ and $\xi \in \mathfrak{g}$.

Tensorial forms and associated bundles. Let $P(M, G)$ be a principal bundle over M with structure group G. For $u \in P$, let $T_u P$ be the tangent space of P at u. The **vertical space** of P at u is defined as

$$V_u P = \{X \in T_u P \mid T_u \pi(X) = 0\}.$$

It can be seen that $V_u P$ is the set of vectors tangent to the fiber through u. Given a principal fiber bundle $P(M, G)$ and a representation ρ of G on a finite-dimensional vector space V, a **pseudotensorial form of degree r on P of type (ρ, V)** is a V-valued r-form φ on P such that

$$\Phi_g^* \varphi = \rho(g^{-1}) \cdot \varphi, \qquad g \in G,$$

where Φ is the action of G on P. A pseudotensorial form of degree r on P of type (ρ, V) is called a **tensorial form** if it is horizontal in the sense that $\varphi(X_1, \ldots, X_r) = 0$ whenever X_i is vertical for at least one $i \in \{1, \ldots, r\}$.

Now, given $P(M, G)$ and ρ on V, consider the associated bundle $E(M, V, G, P)$ with standard fiber V on which G acts by ρ. A tensorial form φ of degree r of type (ρ, V) can be regarded as an assignment

$$M \ni x \mapsto \tilde{\varphi}_x \in \Lambda^r(T_x^* M).$$

In particular, we define

$$\tilde{\varphi}_x(X_1, \ldots, X_r) = u(\varphi(X_1^*, \ldots, X_r^*)), \quad X_i \in T_x M, \tag{3.2.2}$$

where $u \in P$ is such that $\pi(u) = x$ and X_i^* is any vector at u that projects to X_i, that is $T_u \pi(X_i^*) = X_i$ for each $i = 1, \ldots, r$. Since φ is a V-valued r-form, $\varphi(X_1, \ldots, X_r) \in V$. By Proposition 3.2 we know that $u : V \to \pi_E^{-1}(x)$ and thus the right-hand side of (3.2.2) is in $\pi_E^{-1}(x)$. Skew-symmetry and multilinearity properties are clear. To see that the right-hand side of 3.2.2 is independent of the choice of X_i^*, suppose that $Y_k^* \in T_u P$ is such that $T_u \pi(Y_k^*) = X_k = T_\pi(X_k^*)$ for some fixed k. This means that $X_k^* - Y_k^*$ is vertical. We compute

$$\varphi(X_1^*, \ldots, X_k^*, \ldots, X_r^*) - \varphi(X_1^*, \ldots, Y_k^*, \ldots, X_r^*) = \varphi(X_1^*, \ldots, X_k^* - Y_k^*, \ldots, X_r^*) = 0,$$

since φ is tensorial. This implies that

$$\varphi(X_1^*, \ldots, X_k^*, \ldots, X_r^*) = \varphi(X_1^*, \ldots, Y_k^*, \ldots, X_r^*),$$

which shows that definition of $\tilde{\varphi}_x$ is independent of the choice of X_i^* for each i. Finally, we must also show that the definition is independent of the choice of u. To see this, let $v \in P$ such that $\pi(v) = x$. This means that $v = ua$ for some $a \in G$. Since G acts on V by ρ, by Proposition 3.2 we have

$$(ua)X = u(\rho(a)X), \quad u \in P, \ X \in V.$$

Choose $Z_i^* \in T_{ua} P$ such that $T_{ua} \pi(Z_i^*) = X_i$. We compute

$$\begin{aligned}
(ua)(\varphi(Z_1^*, \ldots, Z_r^*)) &= u(\rho(a)\varphi(Z_1^*, \ldots, Z_r^*)) \\
&= u(\varphi(T_u R_{a^{-1}} Z_1^*, \ldots, T_u R_{a^{-1}} Z_r^*)) \\
&= u(\varphi(X_1^*, \ldots, X_r^*)).
\end{aligned}$$

The last step follows since

$$T_u\pi(T_u R_{a^{-1}} Z_i^*) = T_{ua}(\pi \circ R_{a^{-1}})Z_i^* = T_{ua}\pi(Z_i^*) = X_i = T_u\pi(X_i^*).$$

We have thus shown that $\tilde{\varphi}_x$ is well-defined for each $x \in M$.

Conversely, given an r-multilinear, skew-symmetric mapping $\tilde{\varphi}_x \in \Lambda^r(T_x^* M) \otimes \pi_E^{-1}(x)$ for each $x \in M$, we can define a V-valued tensorial r-form φ by

$$\varphi(\bar{X}_1, \ldots, \bar{X}_r) = u^{-1}\tilde{\varphi}_x(T_u\pi(\bar{X}_1), \ldots, T_u\pi(\bar{X}_r)), \quad \bar{X}_i \in T_u P, \ \pi(u) = x. \qquad (3.2.3)$$

3.4 Example: The above discussion shows that a tensorial 0-form of type (ρ, V) on P can be identified with a section $M \to E$ of $E(M, V, G, P)$. In other words, each V-valued function $f : P \to V$ satisfying $f(ua) = \rho(a^{-1})f(u)$ for $u \in P$ and $a \in G$ can be identified with a section of E. We shall have occasion to use this fact later on.

Let $P(M, G)$ be a principal fiber bundle and ρ a representation of G onto $\mathbb{R}^n$. Let $E(M, \mathbb{R}^n, G, P)$ be the associated bundle with standard fiber $\mathbb{R}^n$ on which G acts through ρ. We shall call this associated bundle a ***vector bundle over M***. Each fiber $\pi_E^{-1}(x), x \in M$ has the structure of a vector space such that (see Proposition 3.2) every $u \in P$ with $\pi(u) = x$ considered as a mapping from $\mathbb{R}^n$ to $\pi_E^{-1}(x)$ is a linear isomorphism. Restating this in our notation, this means that given $[u, \xi]_G, [u, \xi_1]_G, [u, \xi_2]_G \in \pi_E^{-1}(x)$ where $\pi(u) = x$ and $c \in \mathbb{R}$, the vector space structure is given by

$$c[u, \xi]_G = [u, c\xi]_G, \quad \text{and} \ \ [u, \xi_1]_G + [u, \xi_2]_G = [u, \xi_1 + \xi_2]_G.$$

It clear from Proposition 3.2 that vector addition and scalar multiplication are well-defined operations. That this definition is equivalent to the "usual" definition of a vector bundle is not immediate here. We shall touch upon this issue in Section 3.3.1.

3.2.3. Connections in principal bundles. Let $P(M, G)$ be a principal bundle over M with structure group G. A ***principal connection*** on $P(M, G)$ is a distribution HP on P such that, for each $u \in P$,

  (i) $T_u P = H_u P \oplus V_u P$,

  (ii) $H_{\Phi_g(u)}P = T_u\Phi_g H_u P$ for $u \in P$ and $g \in G$, and

  (iii) $H_u P$ depends differentiably on u.

We call $H_u P$ the ***horizontal subspace*** at u and represent the horizontal and vertical parts of a vector $X \in T_u P$ by $\mathrm{hor}(X)$ and $\mathrm{ver}(X)$ respectively. Given a principal connection on $P(M, G)$, we define a $\mathfrak{g}$-valued one-form ω on P as follows:

$$\sigma_V(\omega(X))(u) = \mathrm{ver}(X), \quad X \in T_u P$$

where σ_V is the Lie algebra homomorphism defined in Section 3.1. We call ω the ***connection one-form***. It is easy to see that $X \in T_u P$ is horizontal if and only if $\omega(X) = 0$. The following result records the fundamental properties of a connection one-form[12].

3.5 Proposition: *The connection one-form ω corresponding to a connection HP on P satisfies the following conditions:*

(i) $\omega(\mu_M(u)) = \mu$ *for all $\mu \in \mathfrak{g}$ and $u \in P$;*

(ii) $\omega(T_u\Phi_g X) = \mathrm{Ad}_{g^{-1}}\omega(X)$.

Conversely, given a $\mathfrak{g}$-valued one-form ω on P satisfying (i) and (ii), there exists a unique principal connection HP with connection form ω.

It can be shown that the connection one-form ω is a tensorial one-form of type $(\mathrm{Ad}(G), \mathfrak{g})$, where $\mathrm{Ad}(G)$ corresponds to the adjoint representation of G on $\mathfrak{g}$. The projection $\pi : P \to M$ induces a linear map $T_u\pi : T_uP \to T_xM$ where $\pi(u) = x$. If a principal connection is given, $T_u\pi_M$ maps the horizontal space H_uM isomorphically onto T_xM. Given a vector field X on M, the **horizontal lift** of X is the unique horizontal vector field X^h on P that projects to X.

Next, given an interval $[a, b] \subset \mathbb{R}$ and a curve $c : [a, b] \to M$, we define a **horizontal lift** of c to be a C^1 curve $c^h : [a, b] \to P$ with the property that $\pi(c^h(t)) = c(t)$ for all $t \in [a, b]$ and $\dot{c}^h(t) \in H_{c^h(t)}P$. The following result proved in [12] shows the uniqueness property of the horizontal lift.

3.6 Proposition: *Let $c : [0, 1] \to M$ be a curve in M and $u_0 \in P$ such that $\pi(u_0) = c_0$, for $c_0 \in M$. Then there exists a unique horizontal lift c^h of c which passes through u_0.*

Let $c : [0, 1] \to M$ be a C^1 curve in M such that $c(t_0) = c_0$ and $c(t_1) = c_1$. The **parallel transport** along c is defined as the map $\tau_c^{t_0, t_1} : P_{c_0} \to P_{c_1}$ given by

$$\tau_c^{t_0, t_1}(u_0) = c^h(t_1)$$

where $\pi(u_0) = c_0$. The parallel transport map is actually an isomorphism between P_{c_0} and P_{c_1} since it commutes with the action of G on P.

Given a principal bundle $P(M, G)$ and a connection one-form ω, the **curvature form of ω** is a $\mathfrak{g}$-valued two-form Ω on P defined by

$$\Omega(u)(X, Y) = \mathrm{d}\omega(\mathrm{hor}(X), \mathrm{hor}(Y)), \quad X, Y \in T_uP.$$

The curvature form satisfies the **Cartan structure equation**:

$$\Omega(X, Y) = \mathrm{d}\omega(X, Y) - [\omega(X), \omega(Y)].$$

Here we have suppressed the point at which Ω is evaluated. Notice that, if X, Y are vector fields taking values in HP then the structure equation becomes

$$\Omega(X, Y) = \mathrm{d}\omega(X, Y).$$

We also have

$$\mathrm{d}\omega(X, Y) = \mathscr{L}_X(\omega(Y)) - \mathscr{L}_Y(\omega(X)) - \omega([X, Y]) = -\omega([X, Y])$$

since X and Y are horizontal. Hence we deduce the fact that the curvature form Ω measures the lack of integrability of the horizontal distribution HP. In other words, HP is integrable if and only if Ω is zero.

3.2.4. Connections in vector bundles. Given a connection in a principal bundle $P(M, G)$, we can define the notion of parallel transport in the associated bundle $E(M, F, G, P)$ with standard fiber F. Given $w \in E$, choose $u \in P$ and $\xi \in F$ such that $\pi_G(u\xi) = w$. The **vertical subspace** $V_w E$ at w is the set of vectors tangent to the fiber through w. For fixed ξ, the association map Φ_ξ (defined in 3.2.2) maps P_u onto E_w. The **horizontal subspace** $H_w E$ at w is defined as the image of $H_u P$ under the map $T_u \Phi_\xi$. This definition is easily seen to be independent of the choice of u and ξ. We also have $T_w E = H_w E \oplus V_w E$. A differentiable curve in E is *horizontal* if the the tangent vector to the curve at each point lies in the horizontal subspace at that point. Given a curve c in M, the horizontal lift of c onto E is a horizontal curve c^h with the property that $\pi_E \circ c^h = c$. If $c : [t_0, t_1] \to M$ is a curve such that $c(t_0) = x_0$ and $c(t_1) = x_1$ and if $w_0 \in E$ has the property that $\pi_E(w_0) = x_0$, then there exists a unique horizontal lift c^h passing through w_0. Parallel translation along c is a map $\tau_c^{t_0, t_1} : \pi_E^{-1}(x_0) \to \pi_E^{-1}(x_1)$ defined by $\tau_c^{t_0, t_1}(w_0) = c^h(t_1)$, where c^h is the unique horizontal lift of c passing through w_0.

Let us now consider an associated bundle $E(M, \mathbb{R}^n, G, P)$ over M with standard fiber $\mathbb{R}^n$. By definition, it is a vector bundle over M. Let c be a curve in M and σ be a section of E along c, so that $\pi_E(\sigma(c(t))) = c(t)$ for all t. For fixed t, the **covariant derivative** $\nabla_{\dot{c}(t)}\sigma$ **of** σ with respect to $\dot{c}(t)$ is defined by

$$\nabla_{\dot{c}(t)}\sigma = \left.\frac{d}{ds}\right|_{s=0} \tau_c^{t+s,\, t}(\sigma(c(t+s))).$$

The covariant derivation $\nabla_{\dot{c}}(t)\sigma$ defines a section of E along c.

If $X \in T_x M$ and σ is a section of E defined in a neighborhood of x, the *covariant derivative of σ with respect to X* is defined as follows. Let c be a curve in M such that $c(t_0) = x$ and $\dot{c}(t_0) = X$ for some $t_0 \in \mathbb{R}$. Then

$$\nabla_X \sigma = \nabla_{\dot{c}(t_0)}\sigma.$$

It is easy to see that this definition is independent of the choice of the curve c. Next, if σ is a section of E and X is a vector field on M, the *covariant derivative of σ with respect to X* is defined to be the section of E given by

$$\nabla_X \sigma(x) = \nabla_{X(x)}\sigma.$$

The covariant derivative satisfies the following properties.

3.7 Proposition: *Let X and Y be vector fields on M, σ and μ be sections of E, and f a real-valued function on M. Then*

(i) $\nabla_{X+Y}\sigma = \nabla_X \sigma + \nabla_Y \sigma$,

(ii) $\nabla_X(\sigma + \mu) = \nabla_X \sigma + \nabla_X \mu$,

(iii) $\nabla_{fX}\sigma = f\nabla_X \sigma$, *and*

(iv) $\nabla_X(f\sigma) = f\nabla_X \sigma + (\mathscr{L}_X f)\sigma$.

Given a vector bundle $E(M, \mathbb{R}^n, G, P)$ over M, to each section $\sigma : M \to E$, we associate a function $f_\sigma : P \to \mathbb{R}^n$ as follows.

$$f_\sigma(v) = v^{-1}(\sigma(\pi(v))), \quad v \in P.$$

The function f_σ is G-equivariant. That is, for $g \in G$, we have $f_\sigma \circ \Phi_g = \rho(g^{-1}) \cdot f_\sigma$. There is a one-to-one correspondence between G-equivariant functions $f : P \to \mathbb{R}^n$ and sections of E. Given a section σ, we call f_σ the corresponding function on P.

3.8 Proposition: *Let $\sigma : M \to E$ be a section and f_σ the corresponding function, and let X be a vector field on M. Let ω be a principal connection on $P(M,G)$ and ∇ the induced covariant derivative on E. Then $\mathscr{L}_{X^h} f_\sigma$ is the function corresponding to the section $\nabla_X \sigma$.*

Proof: For fixed $x \in M$ and $u \in P$ such that $\pi(u) = x$, let c be a curve in M with the property that $c(0) = x$ and $\dot{c}(0) = X(x)$. Let c^h be the horizontal lift of c through u. We have

$$\mathscr{L}_{X^h} f_\sigma(u) = \frac{d}{dt}(f_\sigma(c^h(t)))\Big|_{t=0} = \frac{d}{dt}\left((c^h(t))^{-1}\sigma(c(t))\right)\Big|_{t=0}.$$

Thus,

$$u(\mathscr{L}_{X^h} f_\sigma(u)) = \frac{d}{dt}\left(u(c^h(t))^{-1}\sigma(c(t))\right)\Big|_{t=0}.$$

To prove the Proposition, it suffices to show that

$$\tau_c^{t,0}(\sigma(c(t))) = u \circ (c^h(t))^{-1}(\sigma(c(t))).$$

We set $\xi = (c^h(t))^{-1}(\sigma(c(t)))$. Then, $c^h(t)\xi$ is a horizontal curve in E, and

$$\tau_c^{t,0}(c^h(t)\xi) = c^h(0)\xi = u\xi = u \circ (c^h(t))^{-1}(\sigma(c(t))).$$

This completes the proof. ∎

3.3. The bundle of linear frames

Let M be an n-dimensional manifold. A ***linear frame at x*** is an ordered basis $u = (X_1, \ldots, X_n)$ for the tangent space $T_x M$ at $x \in M$. Let

$$L_x(M) = \{u \mid u \text{ is a linear frame at } x\}$$

and write

$$L(M) = \bigcup_{x \in M} L_x(M).$$

Define a map $\pi_M : L(M) \to M$ by $\pi_M(u) = x$ if u is a linear frame at x. The general linear group $GL(n; \mathbb{R})$ acts on $L(M)$ on the *right* in the following manner. If $a = (a^i_j) \in GL(n; \mathbb{R})$ and $u = (X_1, \ldots, X_n) \in L_x(M)$, we define $\Phi^L : L(M) \times GL(n; \mathbb{R}) \to L(M)$ by

$$(u, a) \mapsto (ua) := (a^j_1 X_j, \ldots, a^j_n X_j).$$

Rather than using the elaborate notation, we write $\Phi^L(u, a) = R_a(u)$ which is appropriate for right actions. This action is also free and proper. This means that the quotient $L(M)/GL(n; \mathbb{R})$ possesses a differentiable structure and can be identified with the manifold M. Next, we show that $\pi_M : L(M) \to M$ satisfies the local-triviality condition for a principal fiber bundle. Let (U, ϕ) be a chart for M with local coordinates $(x^1, \ldots, x^n)$. Every frame $u \in L_x(M)$, $x \in U$, can be uniquely expressed as

$$u = \left(X^k_1 \frac{\partial}{\partial x^k}, \ldots, X^k_n \frac{\partial}{\partial x^k}\right),$$

where (X_i^k) is an invertible matrix. If we write $X_i = X_i^k \frac{\partial}{\partial x^k}$, the map $\psi : \pi_M^{-1}(U) \to U \times GL(n; \mathbb{R})$ given by

$$(X_1, \ldots, X_n) \mapsto (x, (X_j^k)),$$

is an isomorphism of principal fiber bundles. We can therefore use coordinates (x^i, X_j^k) on $\pi_M^{-1}(U)$ and define a differentiable structure on $L(M)$. It is also clear that the map

$$(X_1, \ldots, X_n) \mapsto (X_j^k)$$

satisfies $R_b(X_1, \ldots, X_n) = (Y_1, \ldots, Y_n)$ where $Y_i = b_i^j X_j$ and $b \in GL(n, \mathbb{R})$ and thus defines a local bundle chart for $L(M)$. We have thus shown that $L(M)(M, GL(n; \mathbb{R}))$ is a principal fiber bundle. We call it the **bundle of linear frames**.

There is another equivalent way to think about a linear frame. A linear frame $u = (X_1, \ldots, X_n)$ at $x \in M$ can be regarded as an isomorphism $u : \mathbb{R}^n \to T_x M$ as follows. If $(e_1, \ldots, e_n)$ is the standard basis for $\mathbb{R}^n$, the map u is given by

$$c^i e_i \mapsto c^i X_i, \qquad c^i \in \mathbb{R}.$$

The right action of $GL(n; \mathbb{R})$ on $L(M)$ is interpreted as follows. Consider $a = (a_j^i) \in GL(n; \mathbb{R})$ as a linear transformation of $\mathbb{R}^n$ which acts on $\mathbb{R}^n$ by matrix multiplication. Then $ua = R_a(u) : \mathbb{R}^n \to T_x M$ is the composition of the following two maps:

$$\mathbb{R}^n \xrightarrow{a} \mathbb{R}^n \xrightarrow{u} T_x M.$$

3.3.1. The tangent bundle as an associated bundle. Let $\tau_M : TM \to M$ be the tangent bundle of M. Recall that $GL(n; \mathbb{R})$ acts on $\mathbb{R}^n$ on the left by $(a, \xi) \mapsto a\xi$ (this is simply matrix vector multiplication). Given a manifold M, we write $E = L(M) \times_{GL(n;\mathbb{R})} \mathbb{R}^n$ and construct the bundle $E(M, \mathbb{R}^n, GL(n; \mathbb{R}), L(M))$ associated with $L(M)(M, GL(n; \mathbb{R}))$ with standard fiber $\mathbb{R}^n$. It is clear that this is a vector bundle over M in the sense of the definition given in Section 3.2.2. We have the following result.

3.9 Lemma: *The bundles $E(M, \mathbb{R}^n, GL(n; \mathbb{R}), L(M))$ and $\tau_M : TM \to M$ are naturally isomorphic as vector bundles over M. In particular, there exists a natural vector bundle isomorphism from E to TM over the identity mapping of M.*

Proof: Following the discussion at the end of Section 3.3, we think of a frame $u \in L_x(M)$ as an isomorphism $u : \mathbb{R}^n \to T_x M$. Thus for $\xi \in \mathbb{R}^n$, we have $u\xi \in T_x M$ where $x = \pi(u)$. We also know that $[u, \xi]_G \in \pi_E^{-1}(x)$. Now, define a map $\iota : E \to TM$ by

$$[u, \xi]_G \mapsto u\xi.$$

To see that this is well-defined, for $a \in G$, consider $[ua, a^{-1}\xi]_G$ (which is equal to $[u, \xi]_G$). We have

$$\iota([ua, a^{-1}\xi]_G) = (ua)(a^{-1}\xi).$$

The right-hand side is the composition of the following maps on ξ

$$\mathbb{R}^n \ni \xi \xmapsto{a^{-1}} \mathbb{R}^n \xrightarrow{a} \mathbb{R}^n \xrightarrow{u} T_x M,$$

and therefore $\iota([u, \xi]_G) = \iota([ua, a^{-1}\xi]_G)$.

Next, given $v \in T_x M$, we claim that $\iota^{-1}(v) = [u, u^{-1}(v)]_G$ for any $u \in L_x(M)$. First we show that this statement is independent of the choice of u. Suppose that $\tilde{u} \in L_x(M)$, then $\tilde{u} = R_b(u) = ub$ for some $b \in GL(n; \mathbb{R})$. Thus we have

$$[\tilde{u}, \tilde{u}^{-1}(v)]_G = [ub, (ub)^{-1}(v)]_G = [ub.b^{-1}u^{-1}(v)]_G = [u, u^{-1}(v)]_G.$$

Thus ι maps each fiber of E isomorphically to a fiber of TM. From the discussion on associated vector bundles it is also clear that ι is linear. Finally, $\tau_M(\iota([u, \xi]_G) = x$ and thus we conclude that ι is a vector bundle isomorphism between E and TM over the identity on M. ∎

3.10 Remarks: (i) Notice that we shall use the notation "$u\xi$" to represent two different objects in the sequel. In Proposition 3.2 "$u\xi$" represents the image of ξ under the map $u : F \to \pi_E^{-1}(x)$. Let us call this the "first" definition. In the proof of Lemma 3.9, we have used it to represent the image of ξ under the map $u : \mathbb{R}^n \to T_x M$. Call this the "second" definition. For the associated bundle E considered in Lemma 3.9, the natural fiber $F = \mathbb{R}^n$ and thus according to the "first" definition we have $u : F = \mathbb{R}^n \to \pi_E^{-1}(x)$. Lemma 3.9 shows that $\pi_E^{-1}(x)$ is naturally isomorphic to $T_x M$ for every $x \in M$ and thus that the "first" and the "second" definitions are really the same (up to a natural isomorphism).

(ii) The associated vector bundle construction described in this section is actually a special case of a general construction for arbitrary vector bundles. In the section on associated vector bundles, we presented a definition of a vector bundle over a manifold M. We now show how this definition is equivalent to the standard definition of a vector bundle. So suppose $\pi_E : E \to M$ is a vector bundle in the usual sense, with fiber E_x over each $x \in M$. For $x \in M$, define $P_x := L(\mathbb{R}^n, E_x) = \{u : \mathbb{R}^n \to E_x | u$ is a linear isomorphism $\}$ and set

$$P = \bigcup_{x \in M} L(\mathbb{R}^n, E_x).$$

It is easy to see that $GL(n; \mathbb{R})$ acts on P on the right and that this action is free and proper. Thus, $P(M, GL(n; \mathbb{R}))$ is a principal fiber bundle. Next, consider the usual left action of $GL(n; \mathbb{R})$ on $\mathbb{R}^n$ and form the associated bundle with total space $\tilde{E} := (P \times \mathbb{R}^n)/GL(n; \mathbb{R})$ associated with $P(M, GL(n; \mathbb{R}))$ with standard fiber $\mathbb{R}^n$. It can be seen that this associated bundle $\tilde{E}(M, \mathbb{R}^n, GL(n; \mathbb{R}), P)$ is isomorphic to $\pi_E : E \to M$. That is, there exists a bundle isomorphism from $\tilde{E}$ to E over the identity map of M. This justifies why it makes sense to define vector bundles the way we have done in this thesis.

(iii) One can think of the bundle $T_s^r(TM)$ of (r, s) tensors on M as an associated bundle as well. Observe that $GL(n; \mathbb{R})$ acts on $\mathbb{R}^n$ by $(a, \xi) \mapsto A\xi$ and thus it also acts on $T_s^r(\mathbb{R}^n)$ (the (r, s) tensor space of $\mathbb{R}^n$) on the left by push-forward. That is

$$GL(n; \mathbb{R}) \times T_s^r(\mathbb{R}^n) \to T_s^r(\mathbb{R}^n)$$
$$(a, t) \mapsto a_* t.$$

It can be seen that the fibers of $T_s^r(TM)$ are isomorphic to the fibers of the bundle $E(M, T_s^r(\mathbb{R}^n), GL(n; \mathbb{R}), L(M))$ associated with $L(M)(M, GL(n; \mathbb{R}))$ with standard fiber $T_s^r(\mathbb{R}^n)$ where $E = (L(M) \times T_s^r(\mathbb{R}^n))/GL(n; \mathbb{R})$.

3.3.2. Linear connections. Let $L(M)(M, GL(n; \mathbb{R}))$ be the bundle of linear frames of M where $n = \dim(M)$. Denote the canonical projection by $\pi_M : L(M) \to M$.

3.11 Definition: A principal connection in the bundle $L(M)(M, GL(n; \mathbb{R}))$ of linear frames over M is called a **_linear connection of M_**.

The **_canonical form_** of $L(M)$ is the $\mathbb{R}^n$-valued one-form $\theta : TL(M) \to \mathbb{R}^n$ define by

$$\theta(X) = u^{-1}(T_u \pi_M X)), \quad X \in T_u L(M),$$

where $u \in L(M)$ is considered as a linear isomorphism $u : \mathbb{R}^n \to T_{\pi_M(u)} M$ as before.

3.12 Proposition: _The canonical form θ of $L(M)$ is a tensorial one-form of type $(GL(n; \mathbb{R}), \mathbb{R}^n)$. It corresponds to the identity transformation of $T_x M$ at each $x \in M$._

Proof: Note that $GL(n; \mathbb{R})$ acts on $\mathbb{R}^n$ by $(a, \xi) \mapsto \rho(a)\xi = a\xi$, $a \in GL(n; \mathbb{R})$ and thus we write $(ua) : \mathbb{R}^n \to T_x M$, $x = \pi_M(u)$ such that $(ua)\xi = u(a\xi)$ as usual. Let $X \in T_u L(M)$ and $a \in GL(n; \mathbb{R})$. Then $T_{ua} R_a X \in T_{ua} L(M)$. We now compute

$$(R_a^* \theta) = \theta(T_{ua} R_a X) = (ua)^{-1}(T_{ua} \pi_M(T_{ua} R_a X))$$
$$= a^{-1} u^{-1}(\theta(T_u \pi_M X)) = a^{-1} \theta(X),$$

which shows that θ is pseudo-tensorial. Now, let $X \in T_u L(M)$ be vertical. Then $\theta(X) = u^{-1}(T_u \pi_M X) = 0$ and thus θ is tensorial.

For each $x \in M$, the linear map $\tilde{\theta}_x : T_x M \to T_x M$ corresponding to θ is given by

$$\tilde{\theta}_x(X) = u(\theta(X^*)), \quad X \in T_x M, \ \pi_M(u) = x,$$

where $X^* \in T_u L(M)$ is such that $T_u \pi_M(X^*) = X$. Using the definition of θ we get

$$\tilde{\theta}_x(X) = u(u^{-1} T_u \pi_M(X^*)) = X.$$

This is what we wished to show. ∎

For $x \in M$, let U be a neighborhood of x in M with local coordinates $(x^1, \ldots, x^n)$. We denote the vector field $\frac{\partial}{\partial x^i}$ by X_i. Every linear frame at a point $x \in U$ can be expressed uniquely by

$$(X_1^i(X_i)_x, \ldots, X_n^i(X_i)_x),$$

where X_i^j is an invertible $n \times n$ matrix. The pair (x^i, X_k^j) is a coordinate system in $\pi^{-1}(U) \subset L(M)$. Let Y_k^j be the inverse matrix of X_k^j and $(e_1, \ldots, e_n)$ be the standard basis for $\mathbb{R}^n$. In the coordinate system (x^i, X_k^j), write the canonical form $\theta = \theta^i e_i$, where $\theta^1, \ldots, \theta^n$ are one-forms on M. Then we have $\theta^i = Y_j^i dx^j$. Next, let $\{E_i^j\}$ be the standard basis for $\mathfrak{gl}(n, \mathbb{R})$. We can write the connection form ω of a linear connection of M with respect to this basis as $\omega = \omega_j^i E_i^j$, where ω_j^i are one-forms on M. Let σ be the section of $L(M)$ over U

which assigns to each $x \in U$ the linear frame $((X_1)_x, \ldots, (X_n)_x)$. Define a $\mathfrak{gl}(n, \mathbb{R})$-valued one-form ω_U on U by $\omega_U = \sigma^*\omega$. We define n^3 functions Γ^i_{jk} on U by

$$\omega_U = \Gamma^i_{jk} dx^j \otimes E^k_i.$$

The functions Γ^i_{jk} are called the ***Christoffel symbols*** of the linear connection with respect to the coordinate system $(x^1, \ldots, x^n)$. Using these Christoffel symbols, the connection form ω can be reconstructed as follows [12].

3.13 Proposition: *The connection form $\omega = \omega^i_j E^j_i$ is given in terms of the local coordinate system (x^i, X^j_k) by*

$$\omega^i_j = Y^i_k(dX^k_j + \Gamma^k_{ml} X^l_j dx^m), \quad i, j = 1, \ldots, n.$$

We can also express the Christoffel symbols in terms of the covariant derivative.

3.14 Proposition: *Let $(x^1, \ldots, x^n)$ be a local coordinate system on a manifold M with a linear connection. Set $X_i = \frac{\partial}{\partial x^i}$, $i = 1, \ldots, n$. Then the Christoffel symbols Γ^i_{jk} of the connection with respect to $(x^1, \ldots, x^n)$ are defined by*

$$\nabla_{X_i} X_j = \Gamma^k_{ij} X_k.$$

Given a linear connection Γ of M, we associate with each $\xi \in \mathbb{R}^n$ a horizontal vector field $B(\xi)$ on $L(M)$ as follows. For each $u \in L(M)$, $(B(\xi))_u$ is the unique horizontal vector at u with the property that $T_u \pi_M(B(\xi)_u) = u\xi$. We shall call $B(\xi)$ the ***standard horizontal vector field corresponding to ξ***. Note that this vector field is only defined in the presence of a linear connection of M.

3.15 Proposition: *The standard horizontal vector fields have the following properties:*

(i) if θ is the canonical form of $L(M)$, then $\theta(B(\xi)_u) = \xi$ for each $\xi \in \mathbb{R}^n$ and $u \in L(M)$;

(ii) $T_u R_a(B(\xi)_u) = (B(a^{-1}\xi))_{ua}$, $a \in GL(n; \mathbb{R}), \xi \in \mathbb{R}^n$;

(iii) if $\xi \neq 0$, then $B(\xi)$ never vanishes.

The following result provides a representation of a standard horizontal vector field in local coordinates.

3.16 Proposition: *Given $\xi = \xi^i e_i \in \mathbb{R}^n$, the standard horizontal vector field $B(\xi)$ corresponding to ξ is represented in local coordinates as*

$$B(\xi)_u = \xi^i X^p_i \frac{\partial}{\partial x^p} - \Gamma^k_{ml} X^l_j (\xi^i X^m_i) \frac{\partial}{\partial X^k_j}.$$

Proof: The proof is straightforward. In our local coordinate system for $L(M)$, a frame $u \in L_x(M)$ can be thought of as a map $\mathbb{R}^n \to T_x M$ that takes $e^i \in \mathbb{R}^n$ to $X^k_i X_k$. Thus, we have $u\xi = \xi^i (X^p_i \frac{\partial}{\partial x^p})\big|_x$. Now, an arbitrary vector field V on $L(M)$ can be written as

$$V(u) = \lambda^p \frac{\partial}{\partial x^p}\bigg|_u + \Lambda^k_j \frac{\partial}{\partial X^k_j}\bigg|_u.$$

If V is the standard horizontal vector field corresponding to ξ, then we must have (for each $u \in \pi_M^{-1}(U)$) that $T\pi_M(V(u)) = u\xi$ and $\omega(V(u)) = 0$. The second condition implies that

$$\Lambda_j^k + \Gamma_{pl}^k X_j^l \lambda^p = 0.$$

That is, $\Lambda_j^k = -\Gamma_{pl}^k X_j^l \lambda^p$. The first condition implies that $\lambda^p = \xi^i X_i^p$. ∎

Standard horizontal vector fields have the following homogeneity property [12].

3.17 Proposition: *Let $A \in \mathfrak{gl}(n, \mathbb{R})$, and let $B(\xi)$ be the standard horizontal vector field corresponding to $\xi \in \mathbb{R}^n$. Then,*

$$[A_{L(M)}, B(\xi)] = B(A\xi).$$

3.18 Proposition: *Let $\mathfrak{T}(M)$ be the algebra of tensor fields on M. Let X and Y be vector fields on M. Then the covariant derivative has the following properties:*

  (i) $\nabla_X : \mathfrak{T}(M) \to \mathfrak{T}(M)$ is a type-preserving derivation;

  (ii) ∇_X commutes with every contraction;

  (iii) $\nabla_X f = L_X f$ for every function $f : M \to \mathbb{R}$;

  (iv) $\nabla_{X+Y} = \nabla_X + \nabla_Y$;

  (v) $\nabla_{fX} K = f \cdot \nabla_X K$ for every function f on M and $K \in \mathfrak{T}(M)$.

As a consequence of this result and Proposition 3.7 we have the following result.

3.19 Proposition: *If X, Y and Z are vector fields on M, then*

  (i) $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z$,

  (ii) $\nabla_{X+Y} Z = \nabla_X Z + \nabla_Y Z$,

  (iii) $\nabla_{fX} Y = f \cdot \nabla_X Y$ for every $f \in C^\infty(M)$, and

  (iv) $\nabla_X(fY) = f \cdot \nabla_X Y + (\mathscr{L}_X f)Y$ for every $f \in C^\infty(M)$.

This result thus shows that, given a linear connection of M, there exists a map $\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ that has properties (i)-(iv). The operator ∇ satisfying the properties listed in Proposition 3.19 is called an ***affine connection*** on M. There is a one-to-one correspondence between linear connections on $L(M)$ and affine connections on M [12].

Given an (r, s) tensor field K on M, the ***covariant differential*** ∇K of K is an $(r, s+1)$ tensor field defined by

$$(\nabla K)(X_1, \ldots, X_s; X) = (\nabla_X K)(X_1, \ldots, X_s).$$

Thus both sides of the above expression are $(r, 0)$ tensor fields. We only write the vector field arguments since those are the only ones involved in the definition.

3.3.3. Torsion. Given a manifold M with a linear connection ω, let ∇ be the corresponding connection on TM. We represent the set of smooth vector fields on M by $\Gamma(TM)$. The ***torsion tensor field*** is the $(1, 2)$ tensor field T given by

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y], \quad X, Y \in \Gamma(TM). \tag{3.3.1}$$

Corresponding to this tensor field, there exists an $\mathbb{R}^n$-valued tensorial two-form Θ called the **torsion form** defined by

$$\Theta(\widetilde{X}_u, \widetilde{Y}_u) = u^{-1} T(T_u \pi_M \widetilde{X}_u, T_u \pi_M(\widetilde{Y}_u)).$$

Given a connection ∇ with torsion T, we can define a connection $\widetilde{\nabla}$ by

$$\widetilde{\nabla}_X Y = \nabla_X Y - \frac{1}{2} T(X, Y).$$

The connection $\widetilde{\nabla}$ is torsion-free.

3.3.4. Geodesics. A smooth curve $c : \mathbb{R} \to M$ on a manifold M with a linear connection is called a **geodesic** if $\nabla_{\dot{c}(t)} \dot{c}(t) = 0$ for all $t \in \mathbb{R}$. A useful characterization of geodesics is given by the following result [12].

3.20 Proposition: *A curve on M is a geodesic if and only if it is the projection of an integral curve of a standard horizontal field of $L(M)$.*

We shall use this result to derive the Euler–Poincaré equations in the next chapter.

Chapter 4

The geometry of the linear frame bundle

In this chapter, we study the geometry of the linear frame bundle $L(M)$ of a manifold M in detail and explore the relationship between $TL(M)$ and TTM. In Section 4.1, we characterize the space of all principal connections on a principal bundle and use this characterization to describe the space of all affine connections on a manifold. In Section 4.2 we provide precise intrinsic definitions of the Liouville vector field and the geodesic spray on TM respectively. In the next section, we describe an Ehresmann connection on TM corresponding to an affine connection on M and provide a proof of the statement that there is a one-to-one correspondence between geodesic sprays and torsion-free connections. In Section 4.4, we provide a characterization of invariant principal connections on principal bundles. Finally, we study the geometry of the frame bundle of a Lie group and provide an intrinsic derivation of the Euler–Poincaré equations.

4.1. The space of linear connections of M

Let $P(M, G)$ be a principal fiber bundle. The following result characterizes the set of all principal connections on P.

4.1 Proposition: *Let ω be a principal connection one-form on a principal bundle $P(M, G)$ and let α be a tensorial one-form of type $(\mathrm{Ad}(G), \mathfrak{g})$ on P. Then $\bar{\omega} := \omega + \alpha$ defines a new principal connection on P. Conversely, given any two principal connection forms ω and $\bar{\omega}$ respectively, the object $\alpha := \bar{\omega} - \omega$ is a tensorial one-form of type $(\mathrm{Ad}(G), \mathfrak{g})$ on P.*

Proof: The proof is straightforward. For $\mu \in \mathfrak{g}$, let $\mu_P(u)$ be the infinitesimal generator corresponding to μ at $u \in P$. Since α is tensorial, it vanishes on vertical vectors. Thus

$$\bar{\omega}(\mu_P(u)) = \omega(\mu_P(u)) + 0 = \mu.$$

Also, since α is pseudotensorial, for $a \in G$, we have $\alpha(T_u \Phi_a(X)) = \mathrm{Ad}_{a^{-1}} \alpha(X)$ for $X \in T_u P$. Thus,

$$\bar{\omega}(T_u \Phi_a(X)) = (\omega + \alpha)(T_u \Phi_a(X)) = \mathrm{Ad}_{a^{-1}} \omega(X) + \mathrm{Ad}_{a^{-1}} \alpha(X) = \mathrm{Ad}_{a^{-1}}(\bar{\omega}(X)).$$

The one-form $\bar{\omega}$ therefore satisfies the two properties of a connection one-form given in Proposition 3.5. Conversely, given two principal connections ω and $\bar{\omega}$, it is easy to see that $\omega - \bar{\omega}$ is a pseudotensorial one-form of type $(\mathrm{Ad}(G), \mathfrak{g})$. If $X \in T_u P$ is a vertical vector, by definition, we have $(\omega(X))_P(u) = X$ and $(\bar{\omega}(X))_P(u) = X$. Thus, we must have $\omega(X) = \bar{\omega}(X)$ and thus $\omega - \bar{\omega}$ vanishes on vertical vectors, and is therefore tensorial. This concludes the proof. ∎

The following result provides a relationship between the horizontal distributions corresponding to two given principal connections.

4.2 Proposition: *Suppose that ω and $\bar{\omega}$ are two distinct connection one-forms on a principal bundle $P(M, G)$ with horizontal distributions HP and $\overline{HP}$ respectively. If α is the unique tensorial one-form on P of type $(\mathrm{Ad}(G), \mathfrak{g})$ such that $\alpha = \bar{\omega} - \omega$, then, for each $u \in P$,*

$$H_u P = \overline{H_u P} + \overline{V_u P}$$

where $\overline{V_u P} = \{\overline{X} \in T_u P \mid (\alpha(\overline{X}))_P(u)\} \subset V_u P$.

Proof: Let $\overline{X} \in T_u P$, we have

$$\mathrm{hor}(\overline{X}) - \overline{\mathrm{hor}}(\overline{X}) = (\overline{X} - (\omega(\overline{X}))_P(u)) - (\overline{X} - (\bar{\omega}(\overline{X}))_P(u))$$
$$= ((\bar{\omega} - \omega)(\overline{X})_P(u) = (\alpha(\overline{X}))_P(u).$$

This proves the result. ∎

We shall use the above result in the next section to study the Ehresmann connection corresponding to a principal connection on M. The following result provides a correspondence between tensorial one-forms on $L(M)$ and $(1, 2)$ tensor fields on M.

4.3 Proposition: *There is a one-to-one correspondence between tensorial one-forms of type $(\mathrm{Ad}(GL(n, \mathbb{R})), \mathfrak{gl}(n, \mathbb{R}))$ on $L(M)$ and $(1, 2)$ tensor fields on M.*

Proof: Since TM is the bundle associated with $L(M)$ with standard fiber $\mathbb{R}^n$, for each $(1, 2)$ tensor field S on M, we can define a map $\alpha_S : TL(M) \to \mathfrak{gl}(n, \mathbb{R})$ as follows. Let $\widetilde{X} \in T_u L(M)$, for $u \in L(M)$ and let $\eta \in \mathbb{R}^n$. Then

$$u\left(\alpha_S(u)(\widetilde{X})\eta\right) = S(T_u \pi_M(\widetilde{X}), u\eta). \tag{4.1.1}$$

Since $\alpha(\widetilde{X}) \in \mathfrak{gl}(n, \mathbb{R})$, the product $\alpha(\widetilde{X})\eta \in \mathbb{R}^n$. We now show that α_S is a tensorial form of type $(\mathrm{Ad}(GL(n, \mathbb{R})), \mathfrak{gl}(n, \mathbb{R}))$. For $a \in GL(n, \mathbb{R})$, let $\widetilde{Y} = T_u \Phi_a \widetilde{X} \in T_{ua} L(M)$. Then, using the definition (4.1.1), we get

$$(ua)\left(\alpha_S(ua)(\widetilde{Y})\eta\right) = S(T_{ua} \pi_M(\widetilde{Y}), ua\eta) = u\left(\alpha_S(u)(\widetilde{X})(a\eta)\right)$$

from which we get

$$\alpha_S(ua)(\widetilde{Y}) = a^{-1}\alpha_S(u)(\widetilde{X})a,$$

which means that α_S is pseudotensorial. Next, if $\widetilde{X} \in V_u L(M)$, it is easy to see that $\alpha_S(\widetilde{X}) = 0$, which shows that α_S is tensorial.

Conversely, given a tensorial one-form $\alpha : TP \to \mathfrak{gl}(n, \mathbb{R})$, we can define a $(1,2)$ tensor field S_α as follows.

$$S_\alpha(X, Y) = u\left(\alpha(\widetilde{X})(u^{-1}Y)\right), \quad X, Y \in T_xM, \ \pi_M(u) = x,$$

where $\widetilde{X}_u \in T_uL(M)$ has the property that $T_u\pi_M(\widetilde{X}) = X$. This completes the proof. ∎

4.4 Corollary: *Let ω and $\bar{\omega}$ be linear connections of M and let ∇ and $\overline{\nabla}$, respectively, be the corresponding covariant derivatives. Let $\alpha = \bar{\omega} - \omega$. Then, for vector fields X and Y on M, we have*

$$\overline{\nabla}_X Y = \nabla_X Y + S_\alpha(X, Y)$$

where S_α is the (1,2) tensor field on M corresponding to α.

This result,therefore, characterizes the set of all affine connections on the manifold M.

4.2. First-order geometry

Given a manifold M of dimension n, the tangent bundle TM is a manifold of dimension $2n$. In this section we study the geometry of the tangent bundle TTM of TM and relate it to the geometry of $TL(M)$.

4.2.1. Tangent and vertical lifts. If X is a vector field on M we can define a unique vector field $\widetilde{X}$ on $L(M)$ corresponding to X as follows. Let ϕ_t^X be the flow of X. The **tangent lift** X^T is a vector field on TM defined by

$$X^T(v_x) = \left.\frac{d}{dt}\right|_{t=0} T\phi_t^X(v_x).$$

Let $u \in L_x(M)$ and $\xi \in \mathbb{R}^n$ be such that $u\xi = v_x$. For ξ fixed, recall that the association map $\Phi_\xi : L(M) \to TM$ is given by $\Phi_\xi u = u\xi$. The flow of X^T defines a curve u_t in $L(M)$ by $u_t = T_x\phi_t^X \cdot u$. That is,

$$\Phi_\xi u_t = (T_x\phi_t^X \circ \Phi_\xi)u.$$

The map $\widetilde{\Phi}_t(u) = u_t$ defines a flow on $L(M)$. The corresponding vector field is called the **natural lift** $\widetilde{X}$ of X onto $L(M)$. Thus, we have

$$X^T(v_x) = T_u\Phi_\xi\widetilde{X}(u).$$

Given $v_x, w_x \in T_xM$, the **vertical lift** of w at v is defined by

$$\mathrm{vlft}_{v_x}(w_x) = \left.\frac{d}{dt}\right|_{t=0} (v_x + tw_x).$$

The **canonical almost tangent structure** on M is a $(1,1)$ tensor field J_M on TM given by

$$J_M(W_{v_x}) = \mathrm{vlft}_{v_x}(T\tau_M(W_{v_x})), \quad W_{v_x} \in T_{v_x}TM.$$

Now, we define a vertical vector field Δ on TM, called the **Liouville vector field** (also sometimes called the **dilation vector field**) as follows:

$$\Delta(v_x) = T_u\Phi_\xi(\mathrm{id}_{n\times n})_{L(M)}(u), \quad u\xi = v_x,$$

where $(\mathrm{id}_{n\times n})_{L(M)}$ is the infinitesimal generator corresponding to $\mathrm{id}_{n\times n} \in \mathfrak{gl}(n,\mathbb{R})$ for the action of $GL(n,\mathbb{R})$ on $L(M)$. The vector field Δ is an example of a vertical vector field that is not a vertical lift [8]. The following is usually taken to be the definition of the dilation vector field.

4.5 Proposition: *The vector field Δ is generated by the flow $\Phi_t^\Delta(v_x) = e^t v_x$.*

Proof: The proof is obvious from our definition of Δ. ∎

A vector field $S : TM \to TTM$ is called a **second-order vector field** if $T\tau_M \circ S = \mathrm{id}_{TM}$. A second-order vector field S has the property $J_M \circ S = \Delta$.

4.2.2. The geodesic spray of an affine connection.
In this section we define an important second-order vector field associated with a given affine connection.

Given a linear connection ω on M, for *fixed* $\xi \in \mathbb{R}^n$, let $\Phi_\xi : L(M) \to TM$ be the association map. We define a second-order vector field $Z : TM \to TTM$ as follows.

$$Z(v) = T_u\Phi_\xi(B(\xi)_u), \qquad v \in TM, \tag{4.2.1}$$

where, $u \in L_{\tau_M(v)}(M)$ and $\xi \in \mathbb{R}^n$ are such that $u\xi = v$, and $B(\xi)$ is the standard horizontal vector field corresponding to ξ for the linear connection Γ associated with ∇. We have the following result.

4.6 Proposition: *The map Z defined in (4.2.1) is a second-order vector field on TM. The coordinate expression for Z, in terms of the canonical tangent bundle coordinates (x^i, v^i) is given by*

$$Z = v^i \frac{\partial}{\partial x^i} - \Gamma^i_{jk}v^j v^k \frac{\partial}{\partial v^i} \tag{4.2.2}$$

Proof: We first show that Z as given by (4.2.1) is well-defined. For a given $v \in TM$, we write $x := \tau_M(v)$. Suppose that $u' \in L_x(M)$ and $\xi' \in \mathbb{R}^n$ are such that $u'\xi' = v = u\xi$. Then, we must have $u' = ua$ for some $a \in GL(n;\mathbb{R})$. Consequently, $\xi' = a^{-1}\xi$. We compute

$$\begin{aligned}
T_{ua}\Phi_{a^{-1}\xi}(B(a^{-1}\xi)_{ua}) &= T_{ua}\Phi_{a^{-1}\xi}(T_u R_a(B(\xi)_u) \\
&= T_u(\Phi_{a^{-1}\xi} \circ R_a)B(\xi)_u \\
&= T_u\Phi_\xi(B(\xi)_u),
\end{aligned}$$

where the first equality follows from the properties of a standard horizontal vector field. Let us now show that Z is a second-order vector field. We have

$$\begin{aligned}
T\tau_M(Z(v)) = T\tau_M(T_u\Phi_\xi B(\xi)_u) &= T_u(\tau_M \circ \Phi_\xi)B(\xi)_u \\
&= T_u(\pi_M)(B(\xi)_u) = u\xi = v
\end{aligned}$$

as desired.

It now remains to be shown that the coordinate representation of Z is as given in (4.2.2), but this follow directly from Proposition 3.16, and the definition of Φ_ξ. ∎

The geodesic spray satisfies the following homogeneity property.

4.7 Proposition: $[\Delta, Z] = Z$.

Proof: This follows directly from Proposition 3.17. ∎

4.3. Ehresmann connections induced by a linear connection

An ***Ehresmann connection*** on a locally trivial fiber bundle $\pi : P \to M$ is a complement HP to $VP := \ker(T\pi)$ in TP. Given a second-order vector field S on TM, the kernel of the map $(\mathrm{id}_{TTM} + \mathcal{L}_S J_M)$ defines an Ehresmann connection $HTM(S)$ on $\tau : TM \to M$ [8, 13]. In natural coordinates (x, v) for TM, we can write a second-order vector field S as

$$S = v^i \frac{\partial}{\partial x^i} + S^i(x, v) \frac{\partial}{\partial v^i}.$$

It can be verified that a local basis for $HTM(S)$ is given by the vector fields

$$\mathrm{hlft}\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial x^i} + \frac{1}{2} \frac{\partial S^j}{\partial v^i} \frac{\partial}{\partial v^j}, \qquad i = 1, \ldots, n. \tag{4.3.1}$$

If Z is the geodesic spray corresponding to a linear connection, in local coordinates the Ehresmann connection $HTM(Z)$ associated with Z is spanned by the vector fields

$$\mathrm{hlft}\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial x^i} - \frac{1}{2}(\Gamma_{ik}^j + \Gamma_{ki}^j)v^k \frac{\partial}{\partial v^j}, \qquad i = 1, \ldots, n$$

where Γ_{ik}^j are the Christoffel symbols for the connection. We have the following result which is a consequence of Proposition 4.1.

4.8 Proposition: *Let ω be a linear connection on $L(M)$ and denote the horizontal distribution corresponding to the connection by $HL(M)$. For $v_x \in T_x M$, define*

$$\overline{H_{v_x} TM} = T_u \Phi_\xi H_u L(M).$$

Then, $\overline{HTM}$ is an Ehresmann connection on $\tau_M : TM \to M$. Furthermore, if α_T is the tensorial one-form of type $(\mathrm{Ad}(GL(n, \mathbb{R})), \mathfrak{gl}(n, \mathbb{R}))$ corresponding to the torsion tensor field T, we have

$$\overline{H_{v_x} TM} - H_{v_x} TM(Z) = \{T_u \Phi_\xi(\alpha_T(\tilde{X}_u))_{L(M)}(u) | \tilde{X}_u \in T_u L(M)\}.$$

4.9 Corollary: *Let ∇ be a connection on M with torsion T and let ω be the corresponding linear connection on $L(M)$. The connection $\overline{\nabla}$ defined by*

$$\overline{\nabla}_X Y = \nabla_X Y - \frac{1}{2} T(X, Y)$$

has zero torsion. Let $\bar{\omega}$ be the linear connection corresponding to $\overline{\nabla}$. Then, the geodesic sprays Z and $\bar{Z}$ corresponding to ω and $\bar{\omega}$ respectively are equal. In other words, there is a one-to-one correspondence between geodesic sprays and torsion-free connections.

Proof: It is clear that $\overline{\nabla}$ is torsion-free. We also have

$$\bar{\omega} - \omega = -\frac{1}{2}\alpha_T,$$

where α_T is the tensorial one-form of type $(\mathrm{Ad}(GL(n,\mathbb{R}))$ on $L(M)$. Using Proposition 4.8 we get

$$B(\xi)_u - \overline{B}(\xi)_u = \frac{1}{2}\alpha_T(\widetilde{X}_u))_{L(M)}(u),$$

where $\widetilde{X}_u \in T_u L(M)$ is such that $T_u\pi_M(\widetilde{X}_u) = u\xi$. Thus,

$$
\begin{aligned}
Z(u\xi) - \overline{Z}(u\xi) &= \frac{1}{2}T_u\Phi_\xi\alpha_T(\widetilde{X}_u))_{L(M)}(u) \\
&= \frac{1}{2}\left.\frac{d}{dt}\right|_{t=0}\Phi_\xi(u\exp(t\alpha_T(\widetilde{X}_u)) \\
&= \frac{1}{2}\mathrm{vlft}_{u\xi}\left(u\left(\alpha_T(\widetilde{X}_u)\xi\right)\right) \\
&= \frac{1}{2}\mathrm{vlft}_{u\xi}\left(T(T_u\pi_M(\widetilde{X}_u), u\xi)\right) \\
&= 0
\end{aligned}
$$

since T is skew-symmetric. Hence $Z = \overline{Z}$. ∎

4.4. Invariant principal connections

In this section, we study connections on principal fiber bundles that are invariant under the action of a certain Lie group and derive a useful characterization of such connections. We first describe the setup we shall consider.

4.10 Definition: Let $P(M, G)$ be a principal fiber bundle and let K be a Lie group. The bundle $P(M, G)$ is called **K-compatible** if the following hold.

  (i) K acts smoothly on P and M (through actions Φ^P and Φ^M respectively) such that the map $\pi : P \to M$ is equivariant with respect to these actions. That is,

$$\pi(\Phi_k^P(u)) = \Phi_k^M\pi(u), \quad u \in P, \ k \in K.$$

  (ii) Φ^M is free and proper.

A K-compatible principal fiber bundle will typically be denoted by a pair $(P(M,G), K)$.

We shall often write $\Phi_k^P(u) = ku$ and $\Phi_k^M(x) = kx$, etc. This should cause no confusion. A principal connection ω on P is **K-invariant** if $\Phi_k^*\omega = \omega$ for every $k \in K$. We shall denote the horizontal distribution corresponding to ω by HP. Given $X \in \mathfrak{k}$, denote by X_P the infinitesimal generator corresponding to X (for the action Φ^P). The following is easy to verify.

4.11 Proposition: *If ω is K-invariant, the following statements hold:*

  (i) $\mathscr{L}_{X_P}\omega = 0$ for every $X \in \mathfrak{k}$;

  (ii) HP is K-invariant,

  (iii) The parallel transport map of ω is K-equivariant.

4.12 Remark: In general, if Y is a vector field on a manifold M, a vector-valued one-form α on M is **Y-invariant**, if $\mathscr{L}_Y \alpha = 0$. Thus, if ω is K-invariant, it is X_P-invariant for every $X \in \mathfrak{k}$.

Let K be a Lie group acting on a principal fiber bundle $P(M, G)$ as a group of automorphisms. Let u_0 be a point in P, which we choose as a reference point. Every element $k \in K$ induces a transformation $\tilde{k}$ of M. Let

$$J := \{k \in K \mid \tilde{k}(\pi(u_0)) = \pi(u_0)\}.$$

J is a closed subgroup of K, and we call it the **isotropy subgroup of K at $x_0 := \pi(u_0)$**. We can define a homomorphism $\lambda : J \to G$ as follows. For each $j \in J$, $ju_0 \in \pi^{-1}(x_0)$ and thus is of the form $ju_0 = u_0 a$, for some $a \in G$. Define $\lambda(j) = a$. This is easily seen to be a homomorphism (we refer the reader to [12] for details). We say that K acts **fiber-transitively** if, for any two fibers of P, there is an element of K which maps one fiber into the other, that is, the action of K on M is transitive. We now recall a theorem by Wang [35].

4.13 Theorem: (Wang) *If a (connected) Lie group K is a fiber-transitive automorphism group of a principal fiber bundle $P(M, G)$ and if J is the isotropy subgroup of K at $x_0 = \pi(u_0)$, $u_0 \in P$, then there is a one-to-one correspondence between the set of K-invariant connections in P and the set of linear mappings $\Lambda : \mathfrak{k} \to \mathfrak{g}$ satisfying*

(i) $\Lambda|_{\mathfrak{j}} = T_e \lambda$ and

(ii) $\Lambda \circ \mathrm{Ad}(j) = \mathrm{Ad}(\lambda(j)) \circ \Lambda, \quad j \in J$.

The correspondence is given by

$$\Lambda(u_0)(X) = \omega_{u_0}(X_P(u_0)), \quad X \in \mathfrak{k}.$$

Now, notice that the map Λ in Wang's theorem defines a left-invariant $\mathfrak{g}$-valued one-form on K as follows. Define $A_\Lambda : TK \to \mathfrak{g}$ by

$$A_\Lambda(k)(v_k) = \Lambda \circ \theta(v_k), \quad v_k \in T_k K,$$

where θ is the *canonical 1-form* on K. Also, for fixed u_0, we define a map $\Phi_{u_0} : K \to P$ given by $\Phi_{u_0}(k) = k u_0$. Then, there is a left-invariant $\mathfrak{g}$-valued one-form A_ω on K defined by $A_\omega = \Phi_{u_0}^* \omega$, and we can define $\Lambda_A := A_\omega|_{\mathfrak{k}}$. We can now restate Wang's theorem as follows.

4.14 Theorem: *If a (connected) Lie group K is a fiber-transitive automorphism group of a principal fiber bundle $P(M, G)$ and if J is the isotropy subgroup of K at $x_0 = \pi(u_0)$, $u_0 \in P$, then there is a one-to-one correspondence between the set of K-invariant connections in P and the set of left-invariant $\mathfrak{g}$-valued 1-forms A on K satisfying the following two conditions:*

(i) $A|_{\mathfrak{j}} = T_e \lambda$;

(ii) $A \circ \mathrm{Ad}(j) = \mathrm{Ad}(\lambda(j))A, \quad j \in J$.

The map Λ, and thus the left-invariant form A_Λ in the theorem depends on the choice of the point $u_0 \in P$, as well as on the assumption that K acts fiber-transitively. We now prove a generalization of Wang's result. Define an action $\Phi^{\mathfrak{k}}$ of $K \times G$ on $P \times \mathfrak{k}$ as follows:

$$\Phi^{\mathfrak{k}}_{(k,g)}(u, \xi) = (kug, \mathrm{Ad}_k \xi).$$

Let us show that this action is free. Assume that $\Phi^{\mathfrak{k}}_{(k,g)}(u,\xi) = (u,\xi)$. Then we must have $kug = u$ and $\mathrm{Ad}_k\xi = \xi$. That is,

$$\pi(kug) = k\pi(ug) = k\pi(u) = \pi(u),$$

which implies that $k = e_K$, and thus $g = e_G$ since the action of G on P is free. Thus we conclude that $\Phi^{\mathfrak{k}}$ is free. Notice that we can identify M/K with $P/(K \times G)$ by the map

$$[\pi(u)]_K \longmapsto [u]_{K \times G}.$$

Also, we can identify $(P \times \mathfrak{k})/(K \times G)$ with $(M \times \mathfrak{k})/K$ by the map

$$[u,\xi]_{K \times G} \longmapsto [\pi(u),\xi]_K.$$

Next, define an action $\Phi^{\mathfrak{g}}$ of $K \times G$ on $P \times \mathfrak{g}$ by

$$\Phi^{\mathfrak{g}}_{(k,g)}(u,X) = (kug, \mathrm{Ad}_{g^{-1}}X).$$

This action is again seen to be free. Now, if $P(M,G)$ is K-compatible, P/K is a differentiable manifold, and there is an induced action of G on P/K given by

$$P/K \times G \to P/K$$
$$([u]_K, g) \mapsto [ug]_K.$$

Thus, G acts on $P/K \times \mathfrak{g}$ as follows

$$(P/K \times \mathfrak{g}) \times G \to P/K \times \mathfrak{g}$$
$$(([u]_K, X), g) \longmapsto ([ug]_K, \mathrm{Ad}_{g^{-1}}X).$$

Notice that we can identify $(P \times \mathfrak{g})/K \times G$ with $(P/K \times \mathfrak{g})/G$ by the map

$$[u,X]_{K \times G} \longmapsto [[u]_K, X]_G.$$

Let us denote by $\tilde{k}$ the adjoint bundle $(M \times \mathfrak{k})/K \to M/K$ and by $\tilde{\mathfrak{g}}$ the adjoint bundle $(P/K \times \mathfrak{g})/G \to (P/K)/G$. Let us also define the map

$$\Lambda_0 : M/K \to (P/K)/G$$
$$[\pi(u)]_K \to [[u]_K]_G.$$

We have the following result.

4.15 Proposition: *Let $(P(M,G), K)$ be a K-compatible principal fiber bundle. Then for each K-invariant connection in P there exists a vector bundle mapping $\tilde{\Lambda} : \tilde{\mathfrak{k}} \to \tilde{\mathfrak{g}}$ over the mapping Λ_0. The correspondence is given by*

$$\tilde{\Lambda}[\pi(u),\xi]_K = [[u]_K, \Lambda(u)\xi]_G,$$

where Λ at a point u is given by Wang's theorem.

Proof: Let ω be a K-invariant connection on P. Let $u_0, u_1 \in P$ be such that $u_1 \in \pi^{-1}(\pi(u_0))$. Then, it is clear that

$$\Lambda(u_1)(X) = \omega(u_0 A)(X_P(uA)) = \mathrm{Ad}(A^{-1})\Lambda(u_0)(X),$$

where $A \in G$ is such that $u_1 = u_0 A$.

Next, for $u_2 \in \mathrm{Orb}_K(u_0)$ (where orb denotes the orbit through u_0), we have,

$$\Lambda(k u_0)(X) = \Lambda(u_0)(\mathrm{Ad}(k^{-1})X), \quad k \in K.$$

To see that $\tilde{\Lambda}$ is well-defined, we compute

$$\begin{aligned}
\tilde{\Lambda}([k\pi(ug), \mathrm{Ad}_k \xi]_K) = \tilde{\Lambda}([\pi(kuA), \mathrm{Ad}_k \xi]_K) &= [[kuA]_K, \Lambda(kuA)(\mathrm{Ad}_k \xi)]_G \\
&= [[uA]_K, \Lambda(uA)(\mathrm{Ad}_{k^{-1}}(\mathrm{Ad}_k \xi))]_G \\
&= [[uA]_K, \mathrm{Ad}_{A^{-1}}\Lambda(u)(\xi)]_G \\
&= [[u]_K, \Lambda(u)(\xi)]_G. \qquad (4.4.1)
\end{aligned}$$

∎

This result provides a characterization of K-invariant principal connections on K-compatible principal bundles. As we shall see in Chapter 5, this is a setup in which invariant affine connections on manifolds can be studied.

4.5. The frame bundle $L(G)$ of a Lie group G

In this section we study the linear frame bundle corresponding to a Lie group G and provide an intrinsic derivation of the Euler–Poincaré equation.

Consider an n-dimensional Lie group G with a left-invariant affine connection ∇. We look at the frame bundle $L(G)(G, GL(n;\mathbb{R}))$ of G. The canonical projection is denoted by $\pi_G : L(G) \to G$.

If we left-trivialize the tangent bundle TG, we have the following result.

4.16 Proposition: $L(G)$ *is diffeomorphic to* $G \times L(\mathbb{R}^n, \mathfrak{g})$ *as a principal fiber bundle.*

Proof: To see this, notice that corresponding to each pair $(g, A) \in G \times L(\mathbb{R}^n, \mathfrak{g})$ there exists an isomorphism $u : \mathbb{R}^n \to T_g G$ as follows:

$$u(x) = T_e L_g(Ax), \quad x \in \mathbb{R}^n.$$

Conversely, given an isomorphism $u : \mathbb{R}^n \to T_g G$, such that $\pi_G(u) = g$, we can use the left trivialization of $T_g G$ to define $A \in L(\mathbb{R}^n, \mathfrak{g})$

$$A(x) = T_g L_{g^{-1}} u(x).$$

The result now follows. ∎

Unless stated otherwise, we shall use the identification $L(G) \simeq G \times L(\mathbb{R}^n, \mathfrak{g})$ throughout this section. If $u \in L_g(G)$ is the unique frame corresponding to $(g, A) \in G \times L(\mathbb{R}^n, \mathfrak{g})$, for $v \in T_g G$ we shall often write

$$(g, A)^{-1}(v) := A^{-1} T_g L_{g^{-1}}(v).$$

4.5.1. The connection 1-form of a left-invariant affine connection on G. In this section, we compute the linear connection 1-form corresponding to a left-invariant affine connection ∇ on G. We apply Proposition 3.8 to the case when $P = L(G)$ is a Lie group. Note that a left-invariant affine connection on G defines a bilinear map $S : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$. Thus, for any two left-invariant vector fields X_ξ and X_η, we have

$$\nabla_{X_\xi} X_\eta(g) = X_{S(\xi, \eta)}(g).$$

We also define the map $S^\flat : \mathfrak{g} \to L(\mathfrak{g}, \mathfrak{g})$ by $S^\flat(\xi)\eta = S(\xi, \eta)$.

4.17 Proposition: *Let G be a Lie group with a left-invariant affine connection ∇. Then the corresponding linear connection 1-form $\omega : T(G \times L(\mathbb{R}^n, \mathfrak{g})) \to \mathfrak{gl}(n; \mathbb{R})$ is given by*

$$\omega(X, Y) = A^{-1}Y + A^{-1}S^\flat(T_g L_{g^{-1}} X)A, \quad (X, Y) \in T_{(g,A)}(G \times L(\mathbb{R}^n, \mathfrak{g})).$$

Proof: We first prove a lemma.

1 Lemma: *Let X_ξ be a left-invariant vector field on G. Then the horizontal lift X_ξ^* of X_ξ is given by*

$$X_\xi^*(g, A) = (X_\xi(g), -S^\flat(\xi)A).$$

Proof: Let X_ξ and X_η be left-invariant vector fields on G. We define $f_\eta(g, A) = (g, A)^{-1} X_\eta(g) = A^{-1}\eta$. Let us denote the integral curve of the horizontal lift X_ξ^* passing through (g, A) be $(g(t), A(t))$. Using Proposition 3.8 we have

$$\frac{d}{dt} f_\eta(g(t), A(t))|_{t=0} = (g(t), A(t))^{-1}(\nabla_{X_\xi} X_\eta(g))|_{t=0} = A^{-1}S(\xi, \eta).$$

Also, we have

$$\frac{d}{dt} f_\eta(g(t), A(t))|_{t=0} = \frac{d}{dt}(A(t)^{-1}\eta)|_{t=0}.$$

Now, differentiating the equality $A(t)(A(t)^{-1}\eta) = \eta$ with respect to t, we get

$$0 = T_{A^{-1}\eta}\Phi_{A(t)}\frac{d}{dt}(A(t)^{-1}\eta)|_{t=0} + T_A \Phi^{A(t)^{-1}\eta}\dot{A}(t)|_{t=0}.$$

That is,

$$A(0)(A^{-1}S(\xi, \eta)) + \dot{A}(0)A(0)^{-1}\eta = 0.$$

We thus conclude that $\dot{A}(0) = -S^\flat(\xi)A$. In other words, the horizontal lift X_ξ^* is given by $X_\xi^*(g, A) = (X_\xi(g), -S^\flat(\xi)A)$. This is what we wished to prove.   ▼

Now, given an arbitrary vector $(X, Y) \in T_{(g, A)}L(G)$, we have

$$\mathrm{hor}(X, Y) = X^*(g, A) = (X, -S^\flat(T_g L_{g^{-1}} X)A).$$

In other words,

$$\omega(X, Y)_{L(G)}(g, A) = (0, Y + S^\flat(T_g L_{g^{-1}} X))A).$$

We know that the infinitesimal generator corresponding to an element $a \in \mathfrak{gl}(n; \mathbb{R})$ at a point (g, A) is given by

$$\frac{d}{ds}(g, Ae^{as})|_{s=0} = (0, Aa).$$

We thus conclude that $\omega(X, Y) = A^{-1}Y + A^{-1}S^\flat(T_g L_{g^{-1}} X)A$.   ■

4.5.2. Geodesics on G. In this section, we recover the classical geodesic equation of a left-invariant connection on a Lie group using the language of linear frame bundles. This equation is known as the Euler–Poincaré equation. The derivations of this equation that are found in the literature (see, for example, [5]) are based on a choice of a basis for the Lie algebra of G. Our approach therefore provides a more satisfactory intrinsic derivation independent of the choice of a basis.

4.18 Proposition: *Let $g : \mathbb{R} \to G$ be the geodesic with the initial condition $X \in T_g G$ and let $\xi(t) = g(t)^{-1}\dot{g}(t)$. Then $\dot{\xi}(t) = -S(\xi(t), \xi(t))$.*

Proof: Let $v = (g, A)^{-1}(X) = A^{-1}(T_g L_{g^{-1}} X)$. Consider the standard horizontal vector field corresponding to v. That is, $B(v)_{(g,A)} \in \mathrm{hor}_{(g,A)}(G \times L(\mathbb{R}^n, \mathfrak{g}))$ such that $T_{(g,A)}\pi_G(B(v)_{(g,A)}) = (g, A)(v)$. From our previous computations, it is clear that

$$B(v)_{(g,A)} = (X_\xi(g), -S^\flat(Av)A),$$

where $\xi = T_g L_{g^{-1}} X$. Let us find the integral curve of $B(v)$. We need a curve $(g(t), A(t))$ such that

$$(g'(t), A'(t)) = (X_{A(t)v}(g(t)), -S^\flat(A(t)v)A(t)).$$

Thus, we want to find a curve $A(t)$ such that

$$A'(t)A(t)^{-1} = -S^\flat(A(t)v). \tag{4.5.1}$$

A geodesics $t \mapsto g(t)$ therefore satisfy

$$g'(t) = T_e L_{g(t)} A(t)v,$$

where $A(t)$ satisfies (4.5.1). Now, $\xi(t) = g(t)^{-1}\dot{g}(t) = A(t)v$. Thus,

$$\dot{\xi}(t) = \dot{A}(t)v = -S^\flat(A(t)v)A(t)v = -S(\xi(t), \xi(t)),$$

which gives us the result. ∎

Chapter 5

Geodesic Reduction

In this section we consider an arbitrary affine connection ∇ on the total space of a principal bundle $\pi_{M/G} : M \to M/G$, and compute the reduced geodesic equation. In Section 5.1, we study the notion of geodesic invariance of a distribution, and using frame bundle geometry, provide an intrinsic proof of a characterization of geodesic invariance given by Lewis [16] in terms of the symmetric product. In the next section, we consider the bundle of linear frames adapted to a given principal connection, and construct several bundles that help provide insight into the structure of the reduced frame bundle. In Section 5.3 we begin by proving an important relationship between the geodesic spray, the tangent lift and the vertical lift of the symmetric product. Next, we explore the structure of the reduced geodesic spray by decomposing it using a principal connection on $M(M/G, G)$ and an induced principal connection on $TM(TM/G, G)$. We are able to provide meaning to the various terms obtained in this decomposition.

5.1. Geodesic invariance

We recall the notion of geodesic invariance.

5.1 Definition: A distribution D on a manifold M with an affine connection ∇ is called **geodesically invariant** if for every geodesic $c : [a, b] \to M$, $\dot{c}(a) \in D_{c(a)}$ implies that $\dot{c}(t) \in D_{c(t)}$ for all $t \in [a, b]$.

It turns out that geodesic invariance can be characterized by studying a certain product on the set of vector fields on M. Let M be a manifold with a connection ∇. Given $X, Y \in \Gamma(TM)$, the **symmetric product** $\langle X : Y \rangle$ is the vector field defined by

$$\langle X : Y \rangle = \nabla_X Y + \nabla_Y X. \qquad (5.1.1)$$

Given a distribution D on M, we represent by $\Gamma(D)$ the set of vector fields taking values in D. The following result, proved by Lewis [16], provides infinitesimal tests for geodesic invariance and gives the geometric meaning of the symmetric product.

5.2 Theorem: (Lewis) *Let D be a distribution on a manifold M with a connection ∇. The following are equivalent.*

  (i) D is geodesically invariant;
 (ii) $\langle X : Y \rangle \in \Gamma(D)$ for every $X, Y \in \Gamma(D)$;

(iii) $\nabla_X X \in \Gamma(D)$ *for every* $X \in \Gamma(D)$.

We give a proof of this theorem below. Thus, for geodesically invariant distributions, the symmetric product plays the role that the Lie bracket plays for integrable distributions.

Now, given a p-dimensional distribution D on an n-dimensional manifold M with a linear connection, we say that a frame $u \in L_x(M)$ is **D-adapted** if $u|_{\mathbb{R}^p} : \mathbb{R}^p \oplus \mathbb{R}^{n-p} \to D_x$ is an isomorphism. Let $L(M, D)$ be the collection of D-adapted frames. We observe that $L(M, D)$ is invariant under the subgroup of $GL(n, \mathbb{R})$ consisting of those automorphisms which leave $\mathbb{R}^p$ invariant. It turns out that $L(M, D)$ is a subbundle of $L(M)$.

We have the following result.

5.3 Proposition: *The distribution D is geodesically invariant if and only if, for each $\xi \in \mathbb{R}^p$, $B(\xi \oplus 0)|_{L(M,D)}$ is a vector field on $L(M, D)$.*

Proof: We first prove the "if" statement. Suppose that $B(\xi \oplus 0)$ is a vector field on $L(M, D)$ and let $c : \mathbb{R} \to L(M)$ be its integral curve passing through $\bar{u} \in L(M, D)$. Then, we know that $x(t) := \pi_M(c(t))$ is the unique geodesic with the initial condition $\bar{u}\xi \in D$. We must show that $\dot{x}(t) \in D_{x(t)}$ for all t. We have

$$\dot{x}(t) = T\pi_M(B(\xi \oplus 0)_{c(t)}) = c(t)(\xi \oplus 0).$$

Since $B(\xi \oplus 0)$ is a vector field on $L(M, D)$, we must have $c(t) \in L(M, D)$ for all t. Thus, we have $\dot{x}(t) \in D_{x(t)}$ for all t. The "only if" part of the statement can be proved by reversing this argument. ∎

An immediate consequence of this result is the following.

5.4 Corollary: *A distribution D is geodesically invariant if and only if the geodesic spray Z is tangent to the submanifold D of TM.*

We are now in a position to provide a proof of Theorem 5.2 using frame bundle geometry.

Proof of Theorem 5.2: (i) $\implies$ (ii) Suppose that D is geodesically invariant, and let $X_1, X_2 \in \Gamma(D)$. Then, we know that the corresponding functions $f_{X_i} : L(M, D) \to \mathbb{R}^p \oplus \mathbb{R}^{n-p}$, $i = 1, 2$, take values in $\mathbb{R}^p$. Also,

$$(X_i)^h(u) = c_i^j B(e_j \oplus 0)_u, \qquad u \in L(M, D),$$

where c_i^j are functions on $L(M)$ and $\{e_j\}_{j=1,\ldots,p}$ is the standard basis for $\mathbb{R}^p$. We have

$$f_{(\nabla_{X_1} X_2 + \nabla_{X_2} X_1)} = \mathscr{L}_{(X_1)^h} f_{X_2} + \mathscr{L}_{(X_2)^h} f_{X_1} = c_1^j \mathscr{L}_{B(e_j \oplus 0)} f_{X_2} + c_2^k \mathscr{L}_{B(e_k \oplus 0)} f_{X_1}.$$

Since f_{X_i}, $i = 1, 2$, are $\mathbb{R}^p$-valued functions on $L(M, D)$ and $B(e_j \oplus 0)|_{L(M,D)}$, $j = 1, \ldots, p$, are vector fields on $L(M, D)$ because the distribution is assumed to be geodesically invariant, we conclude that the function $f_{(\nabla_{X_1} X_2 + \nabla_{X_2} X_1)} : L(M, D) \to \mathbb{R}^p \oplus \mathbb{R}^{n-p}$ takes its values in $\mathbb{R}^p$. This proves (ii).

(ii) $\implies$ (iii) This follows directly from the definition of the symmetric product.

(iii) $\implies$ (i) Assume that $\nabla_X X \in \Gamma(D)$ for every $X \in \Gamma(D)$. This implies that the function $\mathscr{L}_{X^h} f_X : L(M, D) \to \mathbb{R}^p \oplus \mathbb{R}^{n-p}$ takes values in $\mathbb{R}^p$. Once again, we can write $X^h = C^i B(e_i \oplus 0)$ for some functions C^i. This implies that $B(e_i \oplus 0)|_{L(M,D)}$ must be a vector field on $L(M, D)$. ∎

The above result shows that it is possible to check for geodesic invariance by looking at vector fields $B(e_i \oplus 0)$ on the bundle $L(M, A)$.

5.2. Frame bundle adapted to a principal connection

We now consider the following setup. Let M be an n-dimensional manifold with a connection ∇ (and corresponding linear connection ω) and let G be a p-dimensional Lie group with a (left) free and proper action Φ on M. In other words, $M(M/G, G)$ is a principal fiber bundle. We denote the canonical projection by $\pi_{M/G} : M \to M/G$. The Lie group G acts on $L(M)$ on the left via the **lifted action** $\Phi^{L(M)} : G \times L(M) \to L(M)$ defined as follows:

$$\Phi^{L(M)}(g, u) = T_{\pi_M(u)} \Phi_g \circ u.$$

Now let A be a principal connection on the bundle $\pi_{M/G} : M \to M/G$. This defines a distribution HM complementary to the vertical distribution VM. Next, consider the linear frame bundle $L(M)$. We know that the tangent bundle TM is a bundle associated with $L(M)$ with standard fiber $\mathbb{R}^n$. A frame $u \in L(M)$ is called **A-adapted** if $u|_{\mathbb{R}^{n-p}}$ is an isomorphism onto $H_x M$ and $u|_{\mathbb{R}^p}$ is an isomorphism onto $V_x M$. Denote by $L(M, A)$, the collection of A-adapted linear frames. A frame $u \in L_x(M, A)$, is a map $u : \mathbb{R}^{n-p} \oplus \mathbb{R}^p \to T_x M$. It therefore induces, for each $y \in M/G$, a map $\tilde{u} : \mathbb{R}^{n-p} \to T_y(M/G)$ given by

$$\tilde{u}(\eta) = T_x \pi_{M/G}(u(\eta \oplus 0)), \quad \pi_G(x) = y, \ u \in L_x(M).$$

We must verify that this is well-defined. To see this, we compute

$$
\begin{aligned}
T_{g \cdot x} \pi_{M/G}(\Phi_g^{L(M)} u(\eta \oplus 0)) &= T_{g \cdot x} \pi_{M/G}(T\Phi_g u(\eta \oplus 0)) \\
&= T_{g \cdot x}(\pi_{M/G} \circ \Phi_g) u(\eta \oplus 0) \\
&= T_x \pi_{M/G}(u(\eta \oplus 0)).
\end{aligned}
$$

This thus defines a map $f_G : L(M, A) \to L(M/G)$ as follows:

$$L_x(M, A) \ni u \mapsto \tilde{u} \in L_{\pi_{M/G}(x)}(M/G).$$

Now, $L(M, A)$ is a subbundle of $L(M)$ with structure group given by

$$H = \left\{ \left(\begin{array}{cc} a & 0 \\ 0 & c \end{array} \right) \middle| a \in GL(n - p; \mathbb{R}), c \in GL(p; \mathbb{R}) \right\}.$$

It is easy to see that the map $H \to GL(n - p; \mathbb{R})$ given by

$$\left(\begin{array}{cc} a & 0 \\ 0 & c \end{array} \right) \mapsto a$$

is a Lie group homomorphism. We denote it by the same symbol f_G. This should not cause any confusion. We have the following result.

5.5 Proposition: *The map $f_G : L(M, A) \to L(M/G)$ is a principal bundle homomorphism over π_G.*

Proof: We must verify that $f_G(uB) = f_G(u)f_G(B)$, $B \in H$, so that the following diagram commutes:

$$\begin{array}{ccc} L(M, A) & \xrightarrow{\ f_G\ } & L(M/G) \\ \pi_M \downarrow & & \downarrow \bar{\pi}_{M/G} \\ M & \xrightarrow{\ \pi_{M/G}\ } & M/G \end{array}$$

where $\bar{\pi}_{M/G}$ is the natural projection from $L(M/G)$ to M/G. Let

$$B = \left(\begin{array}{cc} a & 0 \\ 0 & c \end{array} \right).$$

By definition, for $\xi \in \mathbb{R}^{n-p}$, we have

$$f_G(uB)\xi = T_x\pi_{M/G}(uB(\eta \oplus 0)) = T_x\pi_{M/G}(ua(\eta \oplus 0)) = f_G(u)f_G(B)\xi,$$

which is what we wanted to show. ∎

5.2.1.   The reduced frame bundle. A principal connection on a principal bundle $M(M/G, G)$ induces a vector bundle isomorphism between the reduced tangent bundle TM/G and the Whitney sum $T(M/G) \oplus \tilde{\mathfrak{g}}$ of bundles over M/G. This decomposition of TM/G can be used to decompose the dynamics into its horizontal and vertical parts. In this section, we study the geometry of the reduced linear frame bundle $L(M)/G$ and the reduced bundle $L(M, A)/G$ associated to a principal connection. We construct several bundles that allow us to better understand the structure of these reduced bundles.

Let G be a Lie group and $(L(M), M, GL(n, \mathbb{R})), G)$ a G-compatible principal fiber bundle. The action of G on $L(M)$ is assumed to the the lift of the action of G on M. This action is induced by the **tangent lift** Φ^T of the action Φ of G on M onto TM as defined earlier. We can define a map $[\tau_M]_G : TM/G \to M/G$ as follows:

$$[\tau_M]_G([v]_G) = [\tau_M(v)]_G, \quad [v]_G \in TM/G.$$

The following result proved in [7] shows that TM/G is a vector bundle.

5.6 Proposition: $[\tau_M]_G : TM/G \to M/G$ *is a vector bundle over M/G and the fiber $(TM/G)_x$ over $x \in M/G$ is isomorphic to T_yM, for each y for which $x = [y]_G$.*

If A is a principal connection on the bundle $\pi_{M/G} : M \to M/G$, we can decompose the bundle TM/G into its horizontal and vertical parts [19].

5.7 Lemma: *The map $\alpha_A : TM/G \to T(M/G) \oplus \tilde{\mathfrak{g}}$ given by*

$$\alpha_A([v_x]_G) = T\pi_{M/G}(v_x) \oplus [x, A(v_x)]_G$$

is a well-defined vector bundle isomorphism.

The second component of the map α_A will be denoted by $\rho_A : TM/G \to \tilde{\mathfrak{g}}$. Now, consider the vector bundle $[\tau_M]_G : TM/G \to M/G$ and define

$$(L(M)_{/G})_{[x]_G} := L(\mathbb{R}^n, [\tau_M]_G^{-1}([x]_G)), \qquad [x]_G \in M/G$$

to be the set of linear isomorphisms. For each $x \in M$, $L(M)_{/G[x]_G}$ is isomorphic to $L_x M$. Let $L(M)_{/G} := \bigcup_{[x]_G}(L(M)_{/G})_{[x]_G}$. The natural projection maps $u_{/G} \in (L(M)_{/G})_{[x]_G} \mapsto [x]_G$. Similarly, if A is a principal connection on the bundle $\pi_{M/G} : M \to M/G$, we can define $L(M, A)_{/G}$.

Next, we consider the adjoint bundle $\tau_{\mathfrak{g}} : (M \times \mathfrak{g})/G \to M/G$ and define $\tilde{\mathscr{G}} := \bigcup_{[x]_G} \tilde{\mathscr{G}}_{[x]_G}$, where

$$\tilde{\mathscr{G}}_{[x]_G} = L(\mathbb{R}^p, \tau_{\mathfrak{g}}^{-1}([x]_G))$$

is the set of linear isomorphisms.

Let $L(M/G) \times_{M/G} \tilde{\mathscr{G}}$ be the bundle over M/G with the fiber being the direct product of the fibers of the bundles $L(M/G)$ and $\tilde{\mathscr{G}}$ respectively. The structure group H of $L(M, A)$ also acts on $L(M/G) \times_{M/G} \tilde{\mathscr{G}}$ on the right as follows.

$$(\bar{u}, \tilde{u}) \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} \mapsto (\bar{u}a, \tilde{u}c).$$

This fibered product bundle is a principal bundle over M/G with structure group H. Finally, we note that $T(M/G)$ is a bundle associated with $L(M/G)(M/G, \mathbb{R}^{n-p})$, with standard fiber $\mathbb{R}^{n-p}$.

We also consider the bundle $\pi_{/G} : L(M)/G \to M/G$. The next result provides a decomposition of $L(M)_{/G}$ similar to the one given in Lemma 5.7.

5.8 Proposition: *The following statements hold.*

(i) $L(M)_{/G}(M/G, GL(n; \mathbb{R}))$ *and* $\tilde{\mathscr{G}}(M/G, GL(p; \mathbb{R}))$ *are principal fiber bundles.*

(ii) *The bundles* $L(M)_{/G}$ *and* $L(M)/G$ *are isomorphic.*

(iii) $[\tau_M]_G : TM/G \to M/G$ *is a bundle associated with* $L(M)_{/G}(M/G, GL(n; \mathbb{R}))$ *with standard fiber* $\mathbb{R}^n$, *and* $\tilde{\mathfrak{g}}$ *is a bundle associated with* $\tilde{\mathscr{G}}$ *with standard fiber* $\mathbb{R}^p$.

(iv) *Consider the map* $\alpha_A^L : L(M, A)_{/G} \to L(M/G) \times_{M/G} \tilde{\mathscr{G}}$ *given by*

$$u_{/G} \mapsto (f_G(u_{/G}), \tilde{u}_{/G}),$$

where f_G *is the principal bundle homomorphism defined in Section 5.2 and* $\tilde{u}_{/G} \in \tilde{\mathscr{G}}_{[\pi_M(u)]_G}$ *is such that*

$$\tilde{u}_{/G}(\xi) = [\pi_M(u), A(u|_{\mathbb{R}^p}(0 \oplus \xi))]_G.$$

Then, α_A^L *is a principal bundle isomorphism over the identity mapping on* M/G.

Sketch of proof: (i) The structure group $GL(n, \mathbb{R})$ acts on $L(M)_{/G}$ by composition on the right.

$$L(M)_{/G} \times GL(n, \mathbb{R}) \to L(M)_{/G}$$
$$(u_{/G}, a) \mapsto u_{/G}a, \qquad u_{/G} \in (L(M)_{/G})_{[x]_G},\ a \in GL(n, \mathbb{R}).$$

It can be seen that $(L(M)_{/G})/GL(n,\mathbb{R})$ is isomorphic to M/G. The other case with $\tilde{\mathscr{G}}$ can be similarly worked out.

(ii) The map $\iota_G : L(M)/G \to L(M)_{/G}$ defined by

$$\iota_G([u]_G)\xi = [u\xi]_G.$$

is an isomorphism over the identity map on M/G.

(iii) This is straightforward.

(iv) A chase through the various definitions gives us this result. ∎

5.3. The reduced geodesic spray

In this section, we study how the geodesic spray of a connection behaves under the action of G. We first prove an important result that will enable us to understand the nature of the reduced geodesic spray.

5.9 Proposition: *Let $v \in T_xM$ for some $x \in M$, and X_v be an arbitrary vector field that has the value v at x. Then,*

$$Z(v) = (X_v)^T(v) - \text{vlft}_v(\nabla_{X_v}X_v(x)). \qquad (5.3.1)$$

Proof: Using local coordinates x^i around x in M, we write $X_v = X_v{}^i \frac{\partial}{\partial x^i}$. Then,

$$(X_v)^T(v) = v^i \frac{\partial}{\partial x^i} + v^j \frac{\partial X_v{}^i}{\partial x^j} \frac{\partial}{\partial v^i},$$

and

$$\nabla_{X_v}X_v(x) = \left(\frac{\partial X_v{}^i}{\partial x^j} X_v{}^j + \Gamma^i_{jk} X_v{}^j X_v{}^k \right) \frac{\partial}{\partial x^i} = \left(\frac{\partial X_v{}^i}{\partial x^j} v^j + \Gamma^i_{jk} v^j v^k \right) \frac{\partial}{\partial x^i}.$$

So

$$\text{vlft}_v(\nabla_{X_v}X_v(x)) = \left(\frac{\partial X_v{}^i}{\partial x^j} v^j + \Gamma^i_{jk} v^j v^k \right) \frac{\partial}{\partial v^i}.$$

Thus,

$$(X_v)^T(v) - \text{vlft}_v(\nabla_{X_v}X_v(x)) = v^i \frac{\partial}{\partial x^i} + v^j \frac{\partial X_v{}^i}{\partial x^j} \frac{\partial}{\partial v^i} - \left(\frac{\partial X_v{}^i}{\partial x^j} v^j + \Gamma^i_{jk} v^j v^k \right) \frac{\partial}{\partial v^i}$$

$$= v^i \frac{\partial}{\partial x^i} - \Gamma^i_{jk} v^j v^k \frac{\partial}{\partial v^i} = Z(v).$$

∎

5.10 Remark: Notice that, even though each of the two terms $X_v^T(v)$ and $\mathrm{vlft}_v(\nabla_{X_v} X_v)$ depends on the extension X_v, terms that depend on the derivative of X_v cancel in the expression for Z.

5.3.1. Decomposition of the reduced geodesic spray. In this section, we generalize the notion of a second-order vector field. Let us first remark that, if $M(M/G, G)$ is a principal fiber bundle, then so is $TM(TM/G, G)$ (the action Φ^T of G on TM being the tangent lift of the action of G on M). We denote the canonical projection by the map $\pi_{TM/G} : TM \to TM/G$. Furthermore, G acts on TTM by the tangent lift of Φ^T. We denote by $\overline{T\tau_M} : TTM/G \to TM/G$ the map given by

$$\overline{T\tau_M}([W_{v_x}]_G) = [T\tau_M(W_{v_x})]_G.$$

It is easy to see that this map is well-defined. We know from the previous section that a principal connection A on the bundle $M(M/G, G)$ induces a vector bundle isomorphism $\alpha_A : TM/G \to T(M/G) \oplus \tilde{\mathfrak{g}}_{M/G}$. Similarly, a principal connection $\hat{A}$ on TM induces an isomorphism $\alpha_{\hat{A}} : TTM/G \to T(TM/G) \oplus \tilde{\mathfrak{g}}_{TM/G}$. Thus, if A and $\hat{A}$ are chosen, there is a vector bundle isomorphism

$$TTM/G \simeq TT(M/G) \oplus_A T\tilde{\mathfrak{g}}_{M/G} \oplus_{\hat{A}} \tilde{\mathfrak{g}}_{TM/G}.$$

We now consider the decomposition of TTM/G using a principal connection A on $\pi_{M/G} : M \to M/G$ as follows.

5.11 Lemma: *The object $\hat{A} = \tau_M^* A$ is a principal connection on $\pi_{TM/G} : TM \to TM/G$.*

Proof: Let ξ_{TM} be the infinitesimal generator on TM corresponding to $\xi \in \mathfrak{g}$. Then, we have

$$T\tau_M(\xi_{TM}(v_x)) = \frac{d}{dt}\tau_M \circ T\Phi_{\exp\xi t}(v_x)\Big|_{t=0} = \frac{d}{dt}\Phi_{\exp\xi t}\tau_M(v_x)\Big|_{t=0} = \xi_M(x).$$

Therefore, we get

$$\hat{A}(\xi_{TM}(v_x)) = A(T\tau_M(\xi_{TM}(v_x))) = A(\xi_M(x)) = \xi.$$

Next, given $W_{v_x} \in T_{v_x}TM$ and $g \in G$, we compute

$$\hat{A}(T\Phi_g^T(W_{v_x})) = A(T\tau_M(T\Phi_g^T(W_{v_x})) = A(T\Phi_g T\tau_M(W_{v_x})) = \mathrm{Ad}_g A(T\tau_M(W_{v_x})).$$

This shows that

$$\hat{A}(T\Phi_g^T(W_{v_x})) = \mathrm{Ad}_g \hat{A}(W_{v_x})$$

which gives us the result.   ∎

The connection $\hat{A}$ has the following useful property.

5.12 Corollary: *If $S : TM \to TTM$ is a second-order vector field, then*

$$\hat{A}(S(v_x)) = [v_x, A(v_x)]_G \in \tilde{\mathfrak{g}}_{TM/G}.$$

In other words, if we choose connections A and $\hat{A}$ on $M(M/G, G)$ and $TM(TM/G, G)$ respectively, studying a second-order vector field such as the geodesic spray reduces to studying the $TT(M/G) \oplus T\tilde{\mathfrak{g}}$ components, since the $\tilde{\mathfrak{g}}_{TM/G}$ component is completely determined by A itself. We now define the reduced geodesic spray.

5.13 Proposition: *Let ω be a G-invariant linear connection on $L(M)$ and ∇ the corresponding connection on M. The object $\overline{Z} : TM/G \to TTM/G$ defined by*

$$\overline{Z}([v_x]_G) = [Z(v_x)]_G = [T_u\Phi_\xi B(\xi)_u]_G.$$

*is well-defined. We call $\overline{Z}$ the **reduced geodesic spray**.*

Proof: We first prove a lemma.

1 Lemma: *The standard horizontal vector fields corresponding to a G-invariant linear connection ω on $L(M)$ are G-invariant. That is,*

$$B(\xi)_{g.u} = T_u\Phi_g^{L(M)} B(\xi)_u, \qquad u \in L(M), \ \xi \in \mathbb{R}^n.$$

Proof: Let $X = u\xi \in T_xM$. Then, $g \cdot u\xi = T_x\Phi_g(v_x)$. By definition, $B(\xi)_{g.u}$ is the unique horizontal vector that projects to $T_x\Phi_g v_x$. On the other hand, since ω is G-invariant, the vector $T_u\Phi_g^{L(M)} B(\xi)_u$ is horizontal and it projects to $T_u\Phi_g(v_x)$. By the uniqueness of horizontal lift at a point, we must have

$$B(\xi)_{g.u} = T_u\Phi_g^{L(M)} B(\xi)_u,$$

which proves the lemma.   ▼

Since $L(M)$ is G-compatible, the association map $\Phi_\xi : L(M) \to TM$ corresponding to $\xi \in \mathbb{R}^n$ is G-equivariant. That is,

$$\Phi_\xi \circ \Phi_g^{L(M)} = \Phi_g^T \circ \Phi_\xi.$$

Now, for $g \in G$, we compute

$$\begin{aligned}
[Z(T_x\Phi_g(v_x))]_G &= [T_{g\cdot u}\Phi_\xi B(\xi)_{g\cdot u}]_G \\
&= [T_{g\cdot u}\Phi_\xi T_u\Phi_g^{L(M)} B(\xi)_u]_G = [T_{v_x}\Phi_g^T T_u\Phi_\xi B(\xi)_u]_G \\
&= [T_u\Phi_g^T Z(v_x)]_G.
\end{aligned}$$

■

Now, since G acts on TTM via the lifted action, we can define a map $\overline{T\pi_{TM/G}} : TTM/G \to T(T(M/G))$ as follows.

$$\overline{T\pi_{TM/G}}[W_{v_x}]_G = T\pi_{TM/G}(W_{v_x}).$$

This is well-defined since given any $g \in G$, we have $\pi_{TM/G} \circ T\Phi_g^T = \pi_{TM/G}$. By abuse of notation, we shall sometimes use the maps $T\pi_{TM/G}$ and $\overline{T\pi_{TM/G}}$ interchangeably.

We have the following result.

5.14 Proposition: *Let $S_Z : T(M/G) \to TT(M/G)$ be the map defined by*

$$S_Z(\bar{X}) = T(T\pi_{M/G} \circ \pi_{TM/G})\overline{Z}([\bar{X}^h(x)]_G), \quad \bar{X} \in T_{[x]_G}(M/G),$$

where $\bar{X}^h$ is an invariant horizontal vector field that projects to $\bar{X}$ at $x \in M$. The following statements hold.

(i) *S_Z is a second-order vector field on $T(M/G)$;*

(ii) *$S_Z(\bar{X}) = \bar{X}^T(\bar{X}) - \mathrm{vlft}_{\bar{X}}T\pi_{M/G}(\nabla_{\bar{X}^h}\bar{X}^h)$, where, by abuse of notation, $\bar{X}$ is a vector field on M/G which has a value $\bar{X}$ at $[x]_G \in M/G$.*

Proof: (i) We compute

$$\begin{aligned}
T\tau_{M/G}S_Z(\bar{X}) &= T(\tau_{M/G} \circ T\pi_{M/G} \circ Z(\bar{X}^h(x)) \\
&= T\pi_{M/G}T\tau_M(Z(\bar{X}^h(x))) \\
&= T\pi_{M/G}(\bar{X}^h(x)) = \bar{X}.
\end{aligned}$$

(ii) Let $\Phi_t^{\bar{X}^h}$ and $\Phi_t^{\bar{X}}$ be the flows of $\bar{X}^h$ and $\bar{X}$ respectively. We have

$$\begin{aligned}
TT\pi_{M/G}(\bar{X}^h)^T(\bar{X}^h(x)) &= \frac{d}{dt}\Big|_{t=0}(T\pi_{M/G} \circ T\Phi_t^{X^h}(\bar{X}^h(x))) \\
&= \frac{d}{dt}\Big|_{t=0}(T(\pi_{M/G} \circ \Phi_t^{X^h})(\bar{X}^h(x))) \\
&= \frac{d}{dt}\Big|_{t=0}(\frac{d}{ds}\Big|_{s=0}(\pi_{M/G} \circ \Phi_t^{X^h})(\Phi_s^{X^h}(x))) \\
&= \frac{d}{dt}\Big|_{t=0}\frac{d}{ds}\Big|_{s=0}(\pi_{M/G} \circ \Phi_{t+s}^{X^h}(x)) \\
&= \frac{d}{dt}\Big|_{t=0}\frac{d}{ds}\Big|_{s=0}(\Phi_{t+s}^X([x]_G) = \bar{X}^T(\bar{X}).
\end{aligned}$$

Next, we look at

$$\begin{aligned}
TT\pi_{M/G}\mathrm{vlft}_{\bar{X}^h(x)}(\nabla_{\bar{X}^h}\bar{X}^h(x)) &= \frac{d}{dt}(tT\pi_{M/G}\nabla_{\bar{X}^h}\bar{X}^h(x) + T\pi_{M/G}\bar{X}^h(x))\Big|_{t=0} \\
&= \frac{d}{dt}(tT\pi_{M/G}\nabla_{\bar{X}^h}\bar{X}^h(x) + \bar{X}([x]_G))\Big|_{t=0} \\
&= \mathrm{vlft}_{\bar{X}([x]_G)}T\pi_{M/G}\nabla_{\bar{X}^h}\bar{X}^h(x).
\end{aligned}$$

This finally gives us

$$S_Z(\bar{X}) = \bar{X}^T(\bar{X}) - \mathrm{vlft}_{\bar{X}}(T\pi_{M/G}\nabla_{\bar{X}^h}\bar{X}^h(x)). \qquad \blacksquare$$

The idea here is that we use principal connections A and τ_M^*A on $M(M/G, G)$ and $TM(TM/G, G)$, respectively, to write the reduced geodesic spray corresponding to an invariant linear connection as a map from $T(M/G) \oplus \tilde{\mathfrak{g}}$ to $TT(M/G) \oplus T\tilde{\mathfrak{g}}$. The map S_Z gives us one component of this decomposition.

Next, we define a map $P_Z : \tilde{\mathfrak{g}} \to TT(M/G)$ as follows

$$P_Z([x, \xi]_G) = TT\pi_{M/G} T\pi_{TM/G} \overline{Z}([\xi_L^V(x)]_G).$$

where ξ_L^V is the left-invariant vector field on M that satisfies $\xi_L^V(x) = \xi_M(x)$. We must verify that this is well-defined. To see this, notice that $[g \cdot x, \text{Ad}_g \xi]_G = [x, \xi]_G$. Next, we have

$$
\begin{aligned}
(\text{Ad}_g \xi)_M (g \cdot x) &= \frac{d}{dt} \Phi_{\exp(\text{Ad}_g \xi)t}(g \cdot x)\Big|_{t=0} \\
&= \frac{d}{dt} \Phi(g(\exp \xi t)g^{-1}, g \cdot x)\Big|_{t=0} \\
&= T_x \Phi_g \xi_M(x) = \xi_L^V(g \cdot x).
\end{aligned}
$$

Let us denote $\tilde{\xi} := [x, \xi]_G$. Using (5.3.1), we get

$$
\begin{aligned}
P_Z(\tilde{\xi}) = TT\pi_{M/G} Z(\xi_L^V(x)) &= TT\pi_{M/G}((\xi_L^V)^T(\xi_L^V(x)) - TT\pi_{M/G}\text{vlft}_{\xi_L^V(x)}(\nabla_{\xi_L^V}\xi_L^V(x)) \\
&= -\text{vlft}_{\mathbf{0}}(T\pi_{M/G}\nabla_{\xi_L^V}\xi_L^V(x)).
\end{aligned}
$$

We write $\mathcal{S}(\tilde{\xi}, \tilde{\xi}) = (T\pi_{M/G}\nabla_{\xi_L^V}\xi_L^V(x))$. Since ∇ is G-invariant, this map is well-defined.

Next, we define $R_Z : T(M/G) \to T\tilde{\mathfrak{g}}$ by

$$R_Z(\bar{X}) = T\rho_A T\pi_{TM/G} Z(\bar{X}^h(x)).$$

Then, using (5.3.1), we calculate

$$T\rho_A T\pi_{TM/G} Z(\bar{X}^h(x)) = T\rho_A T\pi_{TM/G}\left((\bar{X}^h)^T(\bar{X}^h(x) - \text{vlft}_{\bar{X}^h(x)}(\nabla_{\bar{X}^h}\bar{X}^h(x))\right).$$

Let us look at the first term on the right-hand side.

$$
\begin{aligned}
T\rho_A T\pi_{TM/G}\left((\bar{X}^h)^T(\bar{X}^h(x)\right) &= T\rho_A T\pi_{TM/G}\frac{d}{dt}\Big|_{t=0} T\Phi_t^{\bar{X}^h}(\bar{X}^h(x)) \\
&= \frac{d}{dt}\Big|_{t=0} \rho_A([T\Phi_t^{\bar{X}^h}(\bar{X}^h(x))]_G) = 0,
\end{aligned}
$$

since $T\Phi_t^{\bar{X}^h}(\bar{X}^h(x))$ is horizontal and ρ_A vanishes on horizontal vectors. Also,

$$
\begin{aligned}
T\rho_A T&\pi_{TM/G}\text{vlft}_{\bar{X}^h(x)}(\nabla_{\bar{X}^h}\bar{X}^h(x)) \\
&= \frac{d}{dt}\Big|_{t=0} \rho_A \pi_{TM/G}\left(t\nabla_{\bar{X}^h}\bar{X}^h(x)) + \bar{X}^h(x)\right) \\
&= \frac{d}{dt}\Big|_{t=0}\left(t\rho_A \circ \pi_{TM/G}(\nabla_{\bar{X}^h}\bar{X}^h(x) + \rho_A \circ \pi_{TM/G}(\bar{X}^h(x))\right) \\
&= \frac{d}{dt}\Big|_{t=0}\left(t\rho_A \circ \pi_{TM/G}(\nabla_{\bar{X}^h}\bar{X}^h(x) + 0\right),
\end{aligned}
$$

and thus we get

$$R_Z(\bar{X}) = -\text{vlft}_0(\rho_A \circ \pi_{TM/G}(\nabla_{\bar{X}^h}\bar{X}^h(x))).$$

If HM is geodesically invariant, then $\nabla_{\bar{X}^h}\bar{X}^h$ is horizontal, and thus $R_Z = 0$.

Finally, we define $U_Z : \tilde{\mathfrak{g}} \to T\tilde{\mathfrak{g}}$ by

$$U_Z(\tilde{\xi}) = T\rho_A T\pi_{TM/G} Z(\xi_L^V(x)),$$

and a calculation similar to the one performed above shows that

$$U_Z(\tilde{\xi}) = -\mathrm{vlft}_\xi \rho_A(\pi_{TM/G}(\nabla_{\xi_L^V}\xi_L^V(x))).$$

The following lemma is useful.

5.15 Lemma: *The map* $\nabla^A : \Gamma(T(M/G)) \times \Gamma(\tilde{\mathfrak{g}}) \to \Gamma(\tilde{\mathfrak{g}})$ *given by*

$$\nabla_{\bar{X}}^A \tilde{\xi}([x]_G) = \rho_A \pi_{TM/G}\left(\left\langle \bar{X}^h : \xi_L^V \right\rangle(x)\right), \quad [x]_G \in (M/G)$$

defines a vector bundle connection on the bundle $\tilde{\mathfrak{g}}$.

Proof: Let $f : M/G \to \mathbb{R}$ be a differentiable function. Define $f^h : M \to \mathbb{R}$ by $f^h = \pi_{M/G}^* f$. Therefore, $(f\bar{X})^h = f^h\bar{X}^h$. We compute

$$\begin{aligned}
\nabla_{f\bar{X}}^A \tilde{\xi} &= \rho_A\pi_{TM/G}\left(\left\langle f^h\bar{X}^h : \xi_L^V \right\rangle\right) \\
&= \rho_A\pi_{TM/G}\left(f^h\nabla_{\bar{X}^h}\xi_L^V + f^h\nabla_{\xi_L^V}\bar{X}^h + (\mathscr{L}_{\xi_L^V}f^h)\bar{X}^h\right) \\
&= f\nabla_{\bar{X}}^A\tilde{\xi},
\end{aligned}$$

since $(\mathscr{L}_{\xi_L^V}f^h)\bar{X}^h = 0$. The property $\nabla_{\bar{X}}^A f\tilde{\xi} = f\nabla\bar{X}\tilde{\xi} + (\mathscr{L}_{\bar{X}}f)\tilde{\xi}$ can be proved similarly. ∎

We now state the main result of this section.

5.16 Theorem: *Let* $Z_h : T(M/G) \oplus \tilde{\mathfrak{g}} \to TT(M/G)$ *be the map defined by*

$$Z_h(\bar{X} \oplus \tilde{\xi}) = TT\pi_{M/G}\overline{Z}[\bar{X}^h(x) + \xi_L^V(x)]_G,$$

where $\bar{X}^h$ *is an invariant horizontal vector field that projects to* $\bar{X}$ *at* $x \in M$, *and* ξ_L^V *is the left-invariant vertical vector field with value* $\xi_M(x)$ *at* $x \in M$.

Let $Z_v : T(M/G) \oplus \tilde{\mathfrak{g}} \to T\tilde{\mathfrak{g}}$ *be the map defined by*

$$Z_v(\bar{X} \oplus \tilde{\xi}) = T\rho_A\overline{Z}([\bar{X}^h(x) + \xi_L^V(x))]_G,$$

where $\bar{X}^h$ *and* ξ_L^V *are defined as above. The following statements hold.*

(i) $Z_h(\bar{X} \oplus \tilde{\xi}) = S_Z(\bar{X}) - \mathrm{vlft}_{\bar{X}}\mathscr{S}(\tilde{\xi},\tilde{\xi}) - \mathrm{vlft}_{\bar{X}}\left(T\pi_{M/G}\left\langle \bar{X}^h : \xi_L^V \right\rangle\right).$

(ii) $Z_v(\tilde{X} \oplus \tilde{\xi}) = R_Z(\bar{X}) + U_Z(\tilde{\xi}) - \mathrm{vlft}_\xi\left(\nabla_{\bar{X}}^A\tilde{\xi}([x]_G)\right).$

Proof: (i) Let us compute

$$\begin{aligned}
TT&\pi_{M/G}Z(\bar{X}^h(x) + \xi_L^V(x)) \\
&= TT\pi_{M/G}((\bar{X}_h + \xi_L^V)^T)(\bar{X}^h + \xi_L^V(x)) - \mathrm{vlft}_{\bar{X}^h(x)}(T\pi_{M/G}\nabla_{\bar{X}^h+\xi_L^V}(X^h + \xi_L^V))) \\
&= TT\pi_{M/G}(\bar{X}^h)^T(\xi_M(x)) + \bar{X}^T(\bar{X}([x]_G) - \mathrm{vlft}_{\bar{X}^h(x)}(T\pi_{M/G}\nabla_{\bar{X}^h}\bar{X}^h)) \\
&\quad - \mathrm{vlft}_{\bar{X}^h(x)}(\mathscr{S}(\tilde{\xi},\tilde{\xi})) - \mathrm{vlft}_{\bar{X}^h(x)}\left(T\pi_{M/G}\left\langle \bar{X}^h : \xi_L^V \right\rangle\right) \\
&= TT\pi_{M/G}(\bar{X}^h)^T(\xi_L^V(x)) + S_Z(\bar{X}) - \mathrm{vlft}_{\bar{X}^h(x)}(\mathscr{S}(\tilde{\xi},\tilde{\xi})) \\
&\quad - \mathrm{vlft}_{\bar{X}^h(x)}\left(T\pi_{M/G}\left\langle \bar{X}^h : \xi_L^V \right\rangle(x)\right).
\end{aligned}$$

We also have

$$TT\pi_{M/G}(\bar{X}^h + \xi_L^V)^T(\xi_L^V(x)) = \frac{d}{dt}\Big|_{t=0} T\pi_{M/G}\xi_L^V(\Phi_t^{\bar{X}^h}(x)) = 0.$$

This gives us the first part.

(ii) This follows from a computation similar to that in part (i). ∎

5.17 Remark: The fact that the right-hand sides of Z_h and Z_v respectively are independent of the extensions follows from G-invariance of ω and the definition of $\overline{Z}$.

5.3.2. Discussion. The horizontal part of the reduced geodesic spray therefore consists of three terms. The map S_Z is a second-order vector field on $T(M/G)$. The term $S(\tilde{\xi}, \tilde{\xi})$ can be interpreted in the following manner. Recall that the second fundamental form corresponding to the vertical distribution is a map $\mathcal{S} : \Gamma(VM) \times \Gamma(VM) \to HM$ defined by

$$\mathcal{S}(v_x, w_x) = \mathrm{hor}\,(\nabla_X Y)\,, \quad v_x,\ w_x \in V_x M.$$

where X and Y are extensions of v_x and w_x respectively. In view of this, we have

$$\mathcal{S}(\xi_M(x), \xi_M(x)) = \left(\mathscr{S}(\tilde{\xi}, \tilde{\xi})\right)^h(x)$$

Now, the vertical distribution VM is geodesically invariant if and only if $\mathcal{S}$ is skew-symmetric. Hence, if VM is geodesically invariant, we have $\mathscr{S}(\tilde{\xi}, \tilde{\xi}) = 0$.

The last term is related to the curvature of the horizontal distribution, at least in the case when M is a Riemannian manifold with an invariant Riemannian metric, the chosen affine connection is the Levi-Civita connection corresponding to this metric, and A is the mechanical connection as we show below.

Let (M, k) be a Riemannian manifold and G be a Lie group that acts freely and properly on G, so that $\pi_{M/G} : M \to M/G$ is a principal bundle. Suppose that the Riemannian metric k is invariant under G. The mechanical connection corresponding to k is a principal connection on $\pi_{M/G} : M \to M/G$ determined by the condition that the horizontal subbundle is orthogonal to the vertical subbundle VM with respect to the metric. We denote by A the connection one-form corresponding to this connection. We also let ∇ be the Levi-Civita connection corresponding to k.

5.18 Lemma: *The following holds*

$$k\left(\left\langle \bar{X}^h : \xi_L^V \right\rangle(x), \bar{Y}^h(x)\right) = k\left(((B_A(\bar{X}^h(x), \bar{Y}^h(x))_M, \xi_L^V(x)\right)$$

where $\bar{X}^h$ and $\bar{Y}^h$ are invariant horizontal vector fields on M, and B_A is the curvature form corresponding to A.

Proof: Recall that if X, Y and Z are vector fields on M, the Koszul formula is given by

$$2k(\nabla_X Y, Z) = \mathscr{L}_X(k(Y, Z)) + \mathscr{L}_Y(k(X, Z)) - \mathscr{L}_Z(k(X, Y)) + k([X, Y], Z)$$
$$- k([X, Z], Y) - k([Y, Z], X).$$

We therefore have (using the Koszul formula twice and adding the two results)

$$2k\left(\left\langle \bar{X}^h : \xi_L^V \right\rangle(x), \bar{Y}^h(x)\right) = 2\mathscr{L}_{\bar{X}^h}(k\left(\bar{Y}^h(x), \xi_L^V(x)\right) + 2\mathscr{L}_{\xi_L^V}(k\left(\bar{X}^h(x), \bar{Y}^h(x)\right)$$
$$- 2\mathscr{L}_{\bar{Y}^h}(k\left(\bar{X}^h(x), \xi_L^V(x)\right) - 2k\left([\bar{X}^h, \bar{Y}^h](x), \xi_L^V(x)\right)$$
$$- 2k\left([\xi_L^V, \bar{Y}^h](x), \bar{X}^h(x)\right).$$

Now, the first and the third terms respectively on the right-hand side are clearly zero (by the definition of the mechanical connection). The second term is zero since the function $k\left(\bar{x}^h(x), \bar{Y}^h(x)\right)$ is constant along the invariant vertical vector field ξ_L^V. The fifth term is also zero since the Lie bracket $[\xi_L^V, \bar{Y}^h]$ is a vertical vector field. Thus, we get

$$k\left(\left\langle \bar{X}^h : \xi_L^V \right\rangle(x), \bar{Y}^h(x)\right) = k\left([\bar{X}^h, \bar{Y}^h](x), \xi_L^V(x)\right).$$

By the Cartan structure formula, we have

$$[\bar{X}^h, \bar{Y}^h] = [\bar{X}, \bar{Y}]^h - \left(B_A(\bar{X}^h, \bar{Y}^h)\right)_M(x).$$

Therefore,

$$k\left(\left\langle \bar{X}^h : \xi_L^V \right\rangle(x), \bar{Y}^h(x)\right) = k\left(((B_A(\bar{X}^h(x), \bar{Y}^h(x))_M, \xi_L^V(x)\right). \qquad \blacksquare$$

The vertical part of the reduced geodesic spray consists of the map R_Z which vanishes identically if the horizontal distribution corresponding to the principal connection A is geodesically invariant, and can be thought of as the fundamental form corresponding to the horizontal distribution. Lewis [16] has shown that if both HM and VM are geodesically invariant, then the corresponding linear connection restricts to the subbundle $L(M, A)$. The term $U_Z(\tilde{\xi})$ is essentially the Euler–Poincaré term, and the last term corresponds to a connection on $\tilde{\mathfrak{g}}$.

Chapter 6

Conclusions and future work

In this thesis we have investigated the geometry of the linear frame bundle in detail and explained how reduction of a manifold with an arbitrary affine connection under Lie group action can be achieved. It is our belief this way of looking at reduction from the point of view of frame bundle geometry is just the first step towards a complete understanding of the interrelationship between an arbitrary affine connection on a manifold M, an arbitrary principal connection on the bundle $\pi_{M/G} : M \to M/G$ and the reduced dynamics on M/G. In this chapter, we summarize our conclusions and point to some avenues for further research in this area.

6.1. Conclusions

The geometry of the linear frame bundle provides us with a key to understanding the geodesic spray associated with a linear connection. Using an arbitrary principal connection, we first obtain a key formula that relates the geodesic spray, the tangent lift and the vertical lift of the symmetric product. Using this relationship we are able to decompose the reduced geodesic spray into its horizontal and vertical parts. The horizontal part of the reduced spray consists of a second-order vector field on $T(M/G)$. We also get a "curvature" term which exactly corresponds to the curvature of the chosen principal connection in the case of a Riemannian manifold with an invariant Riemannian metric, and a term which has the same flavor as the second-fundamental form corresponding to the vertical distribution.

The vertical part of the reduced geodesic spray consists of an "Euler–Poincaré" term, another term involving the vector bundle connection induced by the principal connection chosen, and another term which is zero if the horizontal distribution of the principal connection is geodesically invariant.

6.2. Future work

The geometry of nonholonomic systems with symmetry has been an active area of research in the last decade or so, yet the geometric picture is far from complete. The first rigorous attempt in this regard was made by Bloch, Krishnaprasad, Marsden and Murray [4]. They investigate the geometry of constrained Lagrangian systems under symmetry in the framework of principal bundle geometry and reduction using the Lagrange-d'Alembert principle in mechanics and variational analysis.

We can consider the following setup. Let M be a Riemannian manifold with a Riemannian metric that is invariant under the action of a Lie group G which acts freely and properly on M. Let D be a G-invariant distribution. Denote the Levi-Civita affine connection associated with k by $\overset{k}{\nabla}$. The Lagrange-d'Alembert principle allows us to conclude that the constrained geodesics $c(t) \in Q$ satisfy

$$\overset{k}{\nabla}_{c'(t)} c'(t) \in D^{\perp}_{c(t)}, \qquad c'(t) \in D_{c(t)}$$

Sometimes these conditions are written as

$$\overset{k}{\nabla}_{c'(t)} c'(t) = \lambda(c(t))$$
$$P^{\perp}(c'(t)) = 0$$

where λ is a section of $\mathcal{D}^{\perp}$ and $P^{\perp} : TM \to TM$ is the projection onto $\mathcal{D}^{\perp}$. Lewis [16] has shown that the trajectories $C : \mathbb{R} \to M$ satisfying the constraints are actually unconstrained geodesics of an affine connection $\tilde{\nabla}$ defined by $\tilde{\nabla}_X Y = \overset{k}{\nabla}_X Y + (\nabla_X P^{\perp})(Y)$. He calls this connection $\tilde{\nabla}$ restricted to $\mathcal{D}$ as the *constraint affine connection*. We can use our approach to study the geodesic spray of the constraint connection in the presence of a principal connection on $\pi_G : M \to M/G$. In the case when $\mathcal{D}$ is non-integrable, this will lead to reduction of a nonholonomic system with symmetry. This is still an open problem if there are no other assumptions on the constraint distribution. We can generalize the problem even further by considering an arbitrary affine connection instead of the Riemannian connection.

In the course of our investigations, we have uncovered the meaning of various geometric objects from a frame bundle point of view. However, we still do not understand what the canonical almost tangent structure J_M on TM and the Lie derivative $\mathscr{L}_Z J_M$ mean in terms of frame bundle geometry.

We can also consider the problem of finding conditions under which a distribution is geodesically invariant using the setup of partial differential equations. More precisely, let M be a manifold with an affine connection, and let D be a distribution on M with the corresponding projection $P : TM \to D$. Given ∇, we would like to find conditions on P such that D is geodesically invariant. Now, let $\tilde{D}$ be a chosen complement of D in TM and $\tilde{P} : TM \to \tilde{D}$ be the natural projection. Define the generalized second fundamental form $S_{D,\tilde{D}} : \Gamma(D) \times \Gamma(D) \to \tilde{D}$ by

$$S_{D,\tilde{D}}(X,Y) = \tilde{P}(\nabla_{PX} PY).$$

Lewis [16] has shown that D is geodesically invariant if and only if $S_{D,\tilde{D}}$ is skew-symmetric for every choice of $\tilde{D}$. In the course of our investigations, we have discovered that the PDE governing this problem is given by

$$P^* \mathrm{Sym}(\nabla P) = 0.$$

It would be interesting to find the solution of this equation using the geometric theory of partial differential equations.

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