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Tautological control systems*

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Abstract

This paper addresses the problem of feedback invariance in geometric control theory by overviewing a framework that is inherently feedback-invariant. Crucial to the coherence of the method are suitable topologies on spaces of vector fields. The formulation also makes reference to the theory of presheaves and sheaves. While the work is addressed squarely at a fundamental problem of control theory, the technical background for the methods expounded upon are sometimes very complex. For this reason, the intent of the paper is to be expository, rather than technical.

1. Introduction

In this quite lengthy introduction to the paper, we describe the problems in control theory that our framework addresses. We do this first with an example that sharply focuses the issues, and then devolve into a more rambling discussion of the problem of feedback-invariance and modelling of control systems.

1.1. An elementary example. Let us begin the paper with an example that illustrates the sorts of issues the methodology of this paper is intended to address. Consider two control-affine systems in $\mathbb{R}^3$ with two inputs:

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), & \qquad \dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= x_3(t)u_1(t), & \qquad \dot{x}_2(t) &= x_3(t) + x_3(t)u_1(t), \\ \dot{x}_3(t) &= u_2(t), & \qquad \dot{x}_3(t) &= u_2(t). \end{aligned}$$

It is easy to see that the trajectories for the two systems are identical. One also easily sees, using standard Jacobian linearisation [Isidori 1995, page 172], [Khalil 2001, §12.2], [Nijmeijer and van der Schaft 1990, Proposition 3.3], [Sastry 1999, page 236], and [Sontag 1998, Definition 2.7.14], that the linearisation of the system on the left is not controllable, while that for the system on the right is controllable.

The example suggests the possibility of at least two things: (1) classical linearisation is not feedback-invariant; (2) the classical linear controllability test is not feedback-invariant.[1] Both things, in fact, are true. This example has been chosen as being possibly the simplest situation where lack of feedback-invariance of classical control theoretic constructions is

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[1]For the purposes of this article, let us say that a construction is ***feedback-invariant*** if, for systems having the same trajectories, the construction gives the same results.

manifested. An astute reader can probably figure out how to kludge the computations so that both systems are controllable, e.g., linearise the system on the left, not about the control $(0, 0)$, but about the control $(1, 0)$. This, however, begs the question, "What are the proper feedback-invariant definitions for linearisation and linear controllability?" The answer to even this simple question is not as easy as one might wish, and is only given in [Lewis 2016] after a suitable framework is developed. We do not have space here to describe the proper formulation, but will content ourselves with the following whetting of the reader's appetite: One has to allow the possibility that the linearisation of a time-invariant control system about an equilibrium point is a *time-varying* linear system.

1.2. The problem of feedback-invariance. Now, if linearisation is not feedback-invariant, one cannot help but doubt the feedback-invariance of the myriad far more complicated constructions that are made in control theory. Indeed, the fact is that very few of the things done in the field of nonlinear control theory are feedback-invariant. This has led to the study of "feedback equivalence" of control systems. The idea of this venture is that one tries to characterise ways of recognising when two systems are, in some sense, actually the same system. The prototypical example of this is feedback linearisation, where one tries to determine when a system is a linear system in disguise [Jakubczyk and Respondek 1980]. The fact is, however, that this is an exceedingly difficult task in any level of generality, and existing results hold only in very restricted cases.

We note that the lack of feedback-invariance in any modelling framework will make it extremely difficult, perhaps impossible, to address the fundamental structural problems in control theory, such as controllability, stabilisability, and optimality. Each of these problems has to do with very specific ingredients of a system, e.g., controllability and stabilisation have to do with trajectories of a system, and optimality has to do with trajectories of the extended system, with cost included in the state. If one is overlaying additional structure on the system—specifically, a parameterisation of the control set as in the example of Section 1.1—one will very likely end up studying this additional structure, even though this is not the intention. A very clear instance of this is the approach to controllability of control-affine systems using Lie algebras generated by indeterminates that correspond to the drift and control vector fields, e.g., [Bianchini and Stefani 1993, Kawski 1990, Kawski 1999, Kawski 2006, Sussmann 1983, Sussmann 1987]. This technology has been used to establish many of the most useful results in controllability theory. That is to say, using this machinery one can give many very useful sufficient or necessary conditions for local controllability. However, as a device for understanding controllability in a comprehensive way, this approach will simply not work, just because the first step is establishing a bijection from the drift and control vector fields with a set of indeterminates, a process which is inherently dependent on a specific parameterisation of the control set, cf. the example of Section 1.1.

The usual manner in which issues like this are addressed is to prove that a methodology is feedback-invariant. Such verifications are virutally never carried out, however. There are at least three reasons for this: (1) the importance of feedback-invariance is not universally recognised; (2) the verification of the feedback-invariance of a given control theoretic methodology will be very difficult, probably impossible; (3) most control theoretic constructions will fail such a verification, and to point this out is unlikely to be a priority.

1.3. "Physical" versus "mathematical" modelling. The approach we describe in this paper is different than the usual approach to feedback-invariance. Rather than work with the usual class of control theoretic models, i.e., the "$\dot{x} = F(x,u)$" models, and try to give constructions for these that are feedback-invariant, we simply cast aside the usual models, replacing them with what we call "tautological[2] control systems." This philosophy has a connection with the philosophy of differential geometry, so let us describe this as we believe it provides a useful context that will be familiar to many readers.

In a control-theoretic model, one typically has "states" and "controls." When dealing with a concrete application, these normally have a physical meaning that one wishes to carefully account for, just as should be the case. Thus states may be "position," "current," "quantity of reactant," etc., and controls may be, "force," "voltage," "angle of valve opening," etc. It has come to be realised that the labelling of states in a general modelling framework is a bad idea, because it obfuscates structure. This leads one to "geometric control theory," where states reside in a differentiable manifold, and one agrees to work with constructions that do not depend on specific choices of coordinates for the manifold. One would like to do the same for controls. First, we mention that the fact that this is a desirable thing to do is, for the most part, a foreign idea in the practice of control theory, even geometric control theory. But a few ideas exist for trying to remove the dependence on control parameterisation in the way that a differentiable manifold removes dependence on state parameterisation. First of all, there is the theory of differential inclusions, where one simply prescribes the subset of each tangent space in which tangent vectors to trajectories must lie. This is a very appealing theory in this respect. However, it suffers from being highly unstructured, e.g., one has to do a lot of work to even show that a trajectory exists. Another way of accounting for control parameterisation is to "bundle" the controls with the states in a differentiable manifold, [Brockett 1977, Willems 1979]. This picture of a control system is very appealing to a differential geometer, but it has shortcomings for a control theoretician, since the typical nature of control sets, e.g., with boundaries and corners, does not give rise to a smooth manifold structure.

1.4. What is in the paper and what is not. With the above as motivation, we now say that the idea of our approach is to eliminate the explicit parameterisation of the system model by a control set. This appears like a simple idea. However, it is one that is difficult to adhere to faithfully, since it is more difficult than expected to not revert back to the "$\dot{x} = F(x,u)$" models with which one is comfortable. Moreover, the approach relies in a fundamental way on placing topologies on spaces of vector fields. This is well understood in the smooth category, and has been used often in control theory, e.g., the chronological calculus of Agrachev and Gamkrelidze [1978] (described nicely in the book [Agrachev and Sachkov 2004]). In our work, this is extended to the real analytic category, and the method also applies equally well to the finitely differentiable and Lipschitz categories. The latter are fairly easily achieved. However, the extension to the real analytic case is nontrivial. This is done in a restricted setting in [Agrachev and Gamkrelidze 1978], but this topology is usefully described for the first time only in [Jafarpour and Lewis 2014b]. The point is that, while the basic idea of our approach is fairly easily described, the technicalities are

[2]In this paper we will not have the space to fully motivate where the term "tautological" comes from. Perhaps the best way to describe this here is to say that it refers to the fact that these systems have no extraneous structure.

substantial. In this paper we shall explain the idea behind the framework, omitting the technicalities almost entirely, instead giving references. We refer to [Lewis 2014] for an expansive description of the methodology, without the full details of the topologies on the spaces of vector fields that underpin the method.

2. The main idea

In this section we outline, using fairly imprecise language, the main ideas behind tautological control systems. As we shall see, there are two parts of this. First, we consider "ordinary" control systems, and extract from these those elements that will form the essence of our so-called tautological framework. Second, we make use of the notion of a presheaf of vector fields. This allows us to assign vector fields on open subsets of the state manifold, and not just globally. For "ordinary" control systems, this assignment is done in the obvious way, by restriction. However, for other sorts of systems, being able to prescribe a system using only local data can have advantages.

2.1. Extracting the essence of a control system. In this section we essentially establish a dictionary from "ordinary" control systems to tautological control systems. This dictionary, like an actual dictionary, is not precise. Some parts of it we will make precise during the course of the paper, and other we will not, instead referring to other sources.

Let us suppose that we have a control system $\Sigma = (\mathsf{M}, F, \mathcal{C})$, meaning that M is a manifold of some prescribed regularity (say, smooth or real analytic), $\mathcal{C}$ is the control set, and F describes the dynamics, in the sense that trajectories are absolutely continuous curves $t \mapsto \xi(t) \in \mathsf{M}$ satisfying

$$\xi'(t) = F(\xi(t), \mu(t))$$

for some control $t \mapsto \mu(t) \in \mathcal{C}$. Of course, there needs to be some technical conditions on F to ensure that trajectories exist for reasonable controls. As a minimum, one needs something like: (1) the vector field F^u defined by $F^u(x) = F(x,u)$ is at least continuously differentiable, and maybe smooth or real analytic; (2) $\mathcal{C}$ should be a topological space, and the derivatives of F with respect to x should be jointly continuous functions of x and u. Issues such as this are discussed in the setting of locally convex topologies in the paper [Jafarpour and Lewis 2014a] to appear in these same proceedings. In any case, there are two elements of this model that we wish to pull out. First of all, Σ defines a set of vector fields

$$\mathscr{F}_\Sigma = \{F^u \mid u \in \mathcal{C}\}.$$

Second, to define a trajectory for Σ, one first specifies an open-loop control $t \mapsto \mu(t)$. This open-loop control then defines a time-varying vector field F^μ by $(t,x) \mapsto F(x,\mu(t))$. A trajectory is then an integral curve of this time-varying vector field. If, for example, F satisfies the conditions above and the control μ is locally essentially bounded, i.e., takes values in a compact subset of $\mathcal{C}$ on compact subsets of time, then the vector field F^μ will satisfy the hypotheses of the Carathéodory existence and uniqueness theorem for integral curves [Sontag 1998, Theorem 54].

Now, using only these two elements of "ordinary" control systems, let us see if we can fashion a methodology for eliminating the parameterisation by the control set $\mathcal{C}$.

First of all, rather than working with the set of vector fields $\mathscr{F}_\Sigma$, we instead work simply with a subset of vector fields, denoted by $\mathscr{F}$. We may ask that these vector fields have a prescribed regularity, and in this work we allow for smooth, Lipschitz, finitely differentiable, and real analytic dependence on state. This is the easy part. The difficult part is mimicking the effects of specifying an open-loop control $t \mapsto \mu(t)$. As we saw above, this defines a time-varying vector field F^μ. In the case where one simply has a family of vector fields $\mathscr{F}$, one needs to specify a curve $t \mapsto X_t \in \mathscr{F}$ in the family of vector fields. The technical issues that arise are that one needs to be able to describe properties of this curve that ensure that the resulting time-varying vector field $(t, x) \mapsto X_t(x)$ is nice enough to possess integral curves. It turns out that the way to do this is to ask that the curve $t \mapsto X_t$ be "measurable" and "integrable" in an appropriate topology on the space of vector fields. It is precisely this that we do *not* describe in this paper, referring instead to [Jafarpour and Lewis 2014a, Jafarpour and Lewis 2014b, Jafarpour and Lewis 2016].

2.2. The "presheaf of vector fields" point of view. From the preceding section, we see that subsets of vector fields will feature prominently in our framework. We break with the common approach in our work by talking only about families of vector fields assigned locally. Somewhat precisely (we will remove the "somewhat" below), to each open subset $\mathcal{U} \subseteq \mathsf{M}$ we assign a subset $\mathscr{F}(\mathcal{U})$ of vector fields, and we require that, if open sets $\mathcal{U}$ and $\mathcal{V}$ satisfy $\mathcal{V} \subseteq \mathcal{U}$, then the restrictions of vector fields from $\mathscr{F}(\mathcal{U})$ to $\mathcal{V}$ are members of $\mathscr{F}(\mathcal{V})$. Such a construction is called a "presheaf of sets of vector fields."

The rationale for making constructions such as this may be initially difficult to grasp. Here we point out three reasons for using this structure.

1. Sometimes control theoretic constructions are easily made locally, but global analogues are not so easily understood. Here is an example. A smooth or real analytic distribution is a subset $\mathsf{D} \subseteq \mathsf{TM}$ such that, for each $x \in \mathsf{M}$, there exists a neighbourhood $\mathcal{N}_x$ of x and a family $\mathscr{X}_x$ of smooth or real analytic, respectively, vector fields on $\mathcal{N}_x$ such that

   $$\mathsf{D}_y \triangleq \mathsf{D} \cap \mathsf{T}_y\mathsf{M} = \operatorname{span}_{\mathbb{R}}(X(y)|\ X \in \mathscr{X}_x).$$

   The existence, locally, of plenty of vector fields taking values in the distribution follows from the definition. However, the question of whether there are many smooth or real analytic, respectively, vector fields X for which $X(x) \in \mathsf{D}_x$ is not so trivial. In the smooth case, such vector fields can be constructed using partitions of unity. However, in the real analytic case, the existence of globally defined vector fields only follows from nontrivial sheaf theoretic constructions, including, but not limited to, a deployment of Cartan's Theorem A [Cartan 1957].

2. The presheaf point of view is the natural one for defining germs. It is to be imagined that many (all?) important local properties of a real analytic system about $x \in \mathsf{M}$ are contained in the germ of the system at x. This is a statement that will not come as a surprise. However, adopting the presheaf formalism makes consideration of such matters an integral part of the framework.

3. The third reason for adopting the presheaf formalism is that it aids in addressing questions that come up even in routine control theory. We shall not delve into this here because of space limitations. However, we can say the following as a teaser [Lewis 2014,

Proposition 5.3(iii)]. Given an "ordinary" control system $\Sigma = (\mathsf{M}, F, \mathcal{C})$ as above, one associates a natural differential inclusion

$$\mathscr{X}_\Sigma(x) = \{F(x,u) \mid u \in \mathcal{C}\}.$$

A natural question, then, is: given two control systems Σ_1 and Σ_2 with $\mathscr{X}_{\Sigma_1} = \mathscr{X}_{\Sigma_2}$, is it true that $\Sigma_1 = \Sigma_2$? The answer is, "No," and an understanding of the connection between a system and its associated differential inclusion is related to the sheaf of vector fields associated to a system.

The point of the preceding discussion is this: while we expect that many readers will doubt the value of the presheaf formalism that is a part of the tautological control system framework, it is nonetheless the case that this formalism is not hollow in control theoretic terms.

3. The basics of tautological control theory

Having exhausted more than half of the space available in the paper for motivation, let us now be precise, and define clearly the basic elements of tautological control theory. Our intent here is to show that there is a enough structure in the tautological control system framework to do control theory. There is a great deal more that can be, and has been, said about even this basic material [Lewis 2014].

As we have mentioned, we deal with a broad range of classes of regularity. If $m \in \mathbb{Z}_{\geq 0}$, by ***class*** $\mathbf{C}^{\boldsymbol{m}}$ we mean the usual thing, i.e., m-times continuously differentiable. By ***class*** $\mathbf{C}^{\boldsymbol{m}+\mathbf{lip}}$ we mean class C^m, and the mth derivative is locally Lipschitz. As usual, ***class*** $\mathbf{C}^{\boldsymbol{\infty}}$ means smooth, i.e., infinitely differentiable, and ***class*** $\mathbf{C}^{\boldsymbol{\omega}}$ means real analytic. We shall often write: Let $m \in \mathbb{Z}_{\geq 0}$, let $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m+m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. This has the obvious meaning: $r = \omega$ if $\nu = \omega$ and $r = \infty$ otherwise. The set of vector fields of class C^ν on a manifold M of class C^r is denoted by $\Gamma^\nu(\mathsf{TM})$.

3.1. Presheaves of sets of vector fields.

The first construction we make is a formal definition of the notion of a presheaf mentioned above.

3.1 Definition: Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m+m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let M be a manifold of class C^r. A ***presheaf of sets of*** $\mathbf{C}^{\boldsymbol{\nu}}$***-vector fields*** is an assignment to each open set $\mathcal{U} \subseteq \mathsf{M}$ a subset $\mathscr{F}(\mathcal{U})$ of $\Gamma^\nu(\mathsf{T}\mathcal{U})$ with the property that, for open sets $\mathcal{U}, \mathcal{V} \subseteq \mathsf{M}$ with $\mathcal{V} \subseteq \mathcal{U}$, the map

$$\begin{aligned} r_{\mathcal{U},\mathcal{V}} \colon \mathscr{F}(\mathcal{U}) &\to \Gamma^\nu(\mathsf{T}\mathcal{V}) \\ X &\mapsto X|\mathcal{V} \end{aligned}$$

takes values in $\mathscr{F}(\mathcal{V})$. Elements of $\mathscr{F}(\mathcal{U})$ are called ***local sections*** of $\mathscr{F}$ over $\mathcal{U}$. •

3.2 Definition: Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m+m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let M be a manifold of class C^r. A presheaf $\mathscr{F}$ of sets of C^ν-vector fields is a ***sheaf of sets of*** $\mathbf{C}^{\boldsymbol{\nu}}$***-vector fields*** if, for every open set $\mathcal{U} \subseteq \mathsf{M}$, for every open cover $(\mathcal{U}_a)_{a \in A}$ of $\mathcal{U}$, and for every choice of local sections $X_a \in \mathscr{F}(\mathcal{U}_a)$ satisfying $X_a|\mathcal{U}_a \cap \mathcal{U}_b = X_b|\mathcal{U}_a \cap \mathcal{U}_b$, there exists $X \in \mathscr{F}(\mathcal{U})$ such that $X|\mathcal{U}_a = X_a$ for every $a \in A$. •

The condition in the definition is called the ***gluing condition***. Readers familiar with sheaf theory will note the absence of the other condition, sometimes called the separation condition, normally placed on a presheaf in order for it to be a sheaf: it is automatically satisfied for presheaves of sets of vector fields.

Let us give some examples of presheaves and sheaves.

3.3 Examples: 1. Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let M be a manifold of class C^r. The presheaf of *all* vector fields of class C^ν is denoted by $\mathscr{G}^\nu_{\mathsf{TM}}$. Thus $\mathscr{G}^\nu_{\mathsf{TM}}(\mathcal{U}) = \Gamma^\nu(\mathsf{T}\mathcal{U})$ for every open set $\mathcal{U}$. Presheaves such as this are extremely important in the "normal" applications of sheaf theory.

2. Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m+m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let M be a manifold of class C^r. If $\mathscr{X} \subseteq \Gamma^\nu(\mathsf{TM})$ is any family of vector fields on M, then we can define an associated presheaf $\mathscr{F}_{\mathscr{X}}$ of sets of vector fields by

$$\mathscr{F}_{\mathscr{X}}(\mathcal{U}) = \{X|\mathcal{U} \mid\ X \in \mathscr{X}\}.$$

Note that $\mathscr{F}(\mathsf{M})$ is necessarily equal to $\mathscr{X}$, and so we shall typically use $\mathscr{F}(\mathsf{M})$ to denote the set of globally defined vector fields giving rise to this presheaf. A presheaf of this sort will be called ***globally generated***. It is fairly easy to see that presheaves of this sort are almost never sheaves [Lewis 2014, Example 4.3–2]. •

3.2. Tautological control systems.

Our definition of a tautological control system is relatively straightforward, given the definitions of the preceding section.

3.4 Definition: Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m+m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required.

(i) A ***C^ν-tautological control system*** is a pair $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$, where M is a manifold of class C^r whose elements are called ***states*** and where $\mathscr{F}$ is a presheaf of sets of C^ν-vector fields on M.

(ii) A tautological control system $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ is ***complete*** if $\mathscr{F}$ is a sheaf and is ***globally generated*** if $\mathscr{F}$ is globally generated. •

This is a pretty featureless definition, sorely in need of some connection to control theory. Let us begin to build this connection by pointing out the manner in which more common constructions give rise to tautological control systems, and vice versa.

3.5 Examples: One of the topics of interest to us will be the relationship between our notion of tautological control systems and the more common notions of control systems. We begin here by making some more or less obvious associations.

1. Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m+m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\Sigma = (\mathsf{M}, F, \mathcal{C})$ be a C^ν-control system. To this control system we associate the C^ν-tautological control system $\mathfrak{G}_\Sigma = (\mathsf{M}, \mathscr{F}_\Sigma)$ by

$$\mathscr{F}_\Sigma(\mathcal{U}) = \{F^u|\mathcal{U} \in \Gamma^\nu(\mathsf{T}\mathcal{U}) \mid\ u \in \mathcal{C}\}.$$

The presheaf of sets of vector fields in this case is of the globally generated variety, as in Example 3.3–2. According to Example 3.3–2 we should generally not expect tautological control systems such as this to be *a priori* complete.

2. Let us consider a means of going from a large class of tautological control systems to a control system. Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. We suppose that we have a C^ν-tautological control system $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ where the presheaf $\mathscr{F}$ is globally generated. We define a C^ν-control system $\Sigma_{\mathfrak{G}} = (\mathsf{M}, F_{\mathscr{F}}, \mathcal{C}_{\mathscr{F}})$ as follows. We take $\mathcal{C}_{\mathscr{F}} = \mathscr{F}(\mathsf{M})$, i.e., the control set is our family of globally defined vector fields and the topology is that induced from $\Gamma^\nu(\mathsf{TM})$. We define

$$\begin{aligned} F_{\mathscr{F}} \colon \mathsf{M} \times \mathcal{C}_{\mathscr{F}} &\to \mathsf{TM} \\ (x, X) &\mapsto X(x). \end{aligned}$$

Note that $F_{\mathscr{F}}^X = X$, and so this is somehow the identity map in disguise (this is one reason why our systems can be thought of as "tautological").

3. Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\mathscr{X} \colon \mathsf{M} \twoheadrightarrow \mathsf{TM}$ be a differential inclusion. If $\mathcal{U} \subseteq \mathsf{M}$ is open, we denote

$$\Gamma^\nu(\mathscr{X}|\mathcal{U}) = \{X \in \Gamma^\nu(\mathsf{T}\mathcal{U}) \mid \ X(x) \in \mathscr{X}(x), \ x \in \mathcal{U}\}.$$

One should understand, of course, that we may very well have $\Gamma^\nu(\mathscr{X}|\mathcal{U}) = \varnothing$. This might happen for two reasons.

   (a) First, the differential inclusion may lack sufficient regularity to permit even local sections of the prescribed regularity.

   (b) Second, even if it permits local sections, there may be be problems finding sections defined on "large" open sets, because there may be global obstructions. One might anticipate this to be especially problematic in the real analytic case, where the specification of a vector field locally determines its behaviour globally by the Identity Theorem, cf. [Gunning 1990, Theorem A.3].

   This caveat notwithstanding, we can go ahead and define a tautological control system $\mathfrak{G}_{\mathscr{X}} = (\mathsf{M}, \mathscr{F}_{\mathscr{X}})$ with $\mathscr{F}_{\mathscr{X}}(\mathcal{U}) = \Gamma^\nu(\mathscr{X}|\mathcal{U})$. It is fairly easy to show that $\mathfrak{G}_{\mathscr{X}}$ is complete [Lewis 2014, Example 5.2–3]. The sheaf $\mathscr{F}_{\mathscr{X}}$ is not often globally generated since it is, indeed, a sheaf, cf. Example 3.3–2.

4. Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Note that there is also associated to any C^ν-tautological control system $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ a differential inclusion $\mathscr{X}_{\mathfrak{G}}$ by

$$\mathscr{X}_{\mathfrak{G}}(x) = \{X(x) \mid \ [X]_x \in \mathscr{F}_x\},$$

where $\mathscr{F}_x$ is the stalk of $\mathscr{F}$ at x, i.e., the set of germs of vector fields from $\mathscr{F}$ at x. •

3.3. Open-loop systems and open-loop subfamilies. Next we turn to characterising the analogue of the time-varying vector fields F^μ from Section 2.1 that correspond to a choice of open-loop control μ.

We first introduce some notation. Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let M be a C^r-manifold and let $\mathbb{T} \subseteq \mathbb{R}$ be an interval. We denote by $\mathrm{LI}\Gamma^\nu(\mathbb{T}; \mathsf{TM})$ the set of mappings $X \colon \mathbb{T} \to \Gamma^\nu(\mathsf{TM})$ that are measurable and locally integrable. Said a little more precisely,

$$\mathrm{LI}\Gamma^\nu(\mathbb{T}; \mathsf{TM}) = \mathrm{L}^1_{\mathrm{loc}}(\mathbb{T}; \Gamma^\nu(\mathsf{TM})),$$

i.e., the locally integrable mappings into the locally convex space $\Gamma^\nu(\mathsf{TM})$. The technicalities required to make this precise we will not deal with here, instead referring to [Jafarpour and Lewis 2014b]. Now, for a C^ν-tautological control system $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$, we denote

$$\mathrm{LI}\Gamma^\nu(\mathbb{T}; \mathscr{F}(\mathcal{U})) = \{X \colon \mathbb{T} \to \mathscr{F}(\mathcal{U}) \mid\ X \in \mathrm{LI}\Gamma^\nu(\mathbb{T}; \mathsf{T}\mathcal{U})\},$$

for $\mathbb{T} \subseteq \mathbb{R}$ an interval and $\mathcal{U} \subseteq \mathsf{M}$ open.

3.6 Definition: Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m+m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ be a C^ν-tautological control system. An ***open-loop system*** for $\mathfrak{G}$ is a triple $\mathfrak{G}_{\mathrm{ol}} = (X, \mathbb{T}, \mathcal{U})$ where

(i) $\mathbb{T} \subseteq \mathbb{R}$ is an interval called the ***time-domain***;

(ii) $\mathcal{U} \subseteq \mathsf{M}$ is open;

(iii) $X \in \mathrm{LI}\Gamma^\nu(\mathbb{T}; \mathscr{F}(\mathcal{U}))$. •

In order to see how we should think about an open-loop system, let us consider this notion in the special case of control systems.

3.7 Example: Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m+m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\Sigma = (\mathsf{M}, F, \mathcal{C})$ be a C^ν-control system with $\mathfrak{G}_\Sigma$ the associated C^ν-tautological control system. If μ is locally essentially bounded, then we have the associated open-loop system $\mathfrak{G}_{\Sigma,\mu} = (F^\mu, \mathbb{T}, \mathsf{M})$ defined by

$$F^\mu(t)(x) = F(x, \mu(t)), \qquad t \in \mathbb{T},\ x \in \mathsf{M}.$$

In order for this to be an open-loop system in the precise sense defined above, we need technical conditions on the control system. It turns out that the proper condition is that the control set $\mathcal{C}$ should be a topological space and the mapping $\mathcal{C} \ni u \mapsto F^u \in \Gamma^\nu(\mathsf{TM})$ should be continuous in the locally convex topology of $\Gamma^\nu(\mathsf{TM})$. Then, according to [Jafarpour and Lewis 2016, Proposition 5.2], this is an open-loop system for the tautological control system $\mathfrak{G}_\Sigma$. •

Generally one might wish to place a restriction on the set of open-loop systems one will use. For tautological control systems we do this as follows.

3.8 Definition: Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ be a C^ν-tautological control system. An ***open-loop subfamily*** for $\mathfrak{G}$ is an assignment, to each interval $\mathbb{T} \subseteq \mathbb{R}$ and each open set $\mathcal{U} \subseteq \mathsf{M}$, a subset $\mathscr{O}_{\mathfrak{G}}(\mathbb{T}, \mathcal{U}) \subseteq \mathrm{LI}\Gamma^\nu(\mathbb{T}; \mathscr{F}(\mathcal{U}))$ with the property that, if $(\mathbb{T}_1, \mathcal{U}_1)$ and $(\mathbb{T}_2, \mathcal{U}_2)$ are such that $\mathbb{T}_1 \subseteq \mathbb{T}_2$ and $\mathcal{U}_1 \subseteq \mathcal{U}_2$, then

$$\{X|\mathbb{T}_1 \times \mathcal{U}_1 \mid\ X \in \mathscr{O}_{\mathfrak{G}}(\mathbb{T}_2, \mathcal{U}_2)\} \subseteq \mathscr{O}_{\mathfrak{G}}(\mathbb{T}_1, \mathcal{U}_1). \qquad \bullet$$

Common sorts of open-loop subfamilies might correspond, for ordinary control systems, to restricting controls to be locally essentially bounded, compact-valued, or piecewise constant.

3.4. Trajectories. With the concept of open-loop system just developed, it is relatively easy to provide a notion of a trajectory for a tautological control system.

3.9 Definition: Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \text{lip}\}$, let $\nu \in \{m+m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ be a C^ν-tautological control system and let $\mathscr{O}_{\mathfrak{G}}$ be an open-loop subfamily for $\mathfrak{G}$.

(i) For a time-domain $\mathbb{T}$, an open set $\mathcal{U} \subseteq \mathsf{M}$, and for $X \in \mathscr{O}_{\mathfrak{G}}(\mathbb{T}, \mathcal{U})$, an ***$(X, \mathbb{T}, \mathcal{U})$-trajectory*** for $\mathscr{O}_{\mathfrak{G}}$ is a curve $\xi \colon \mathbb{T} \to \mathcal{U}$ such that $\xi'(t) = X(t, \xi(t))$.

(ii) For a time-domain $\mathbb{T}$ and an open set $\mathcal{U} \subseteq \mathsf{M}$, a ***$(\mathbb{T}, \mathcal{U})$-trajectory*** for $\mathscr{O}_{\mathfrak{G}}$ is a curve $\xi \colon \mathbb{T} \to \mathcal{U}$ such that $\xi'(t) = X(t, \xi(t))$ for some $X \in \mathscr{O}_{\mathfrak{G}}(\mathbb{T}, \mathcal{U})$.

(iii) A ***trajectory*** for $\mathscr{O}_{\mathfrak{G}}$ is a curve that is a $(\mathbb{T}, \mathcal{U})$-trajectory for $\mathscr{O}_{\mathfrak{G}}$ for some time-domain $\mathbb{T}$ and some open set $\mathcal{U} \subseteq \mathsf{M}$. •

Sometimes one wishes to keep track of the fact that, associated with a trajectory is an open-loop system. The following notion is designed to capture this.

3.10 Definition: Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \text{lip}\}$, let $\nu \in \{m+m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ be a C^ν-tautological control system and let $\mathscr{O}_{\mathfrak{G}}$ be an open-loop subfamily for $\mathfrak{G}$. A ***referenced $\mathscr{O}_{\mathfrak{G}}$-trajectory*** is a pair (X, ξ) where $X \in \mathscr{O}_{\mathfrak{G}}(\mathbb{T}; \mathcal{U})$ and ξ is a $(\mathbb{T}, \mathcal{U})$ trajectory for $\mathscr{O}_{\mathfrak{G}}$. •

4. Omissions and future work

In this paper we have mainly tried to motivate our introduction of what we call tautological control systems, and have provided the basic definitions required in our framework to establish it as a viable means of doing control theory. The presentation has a number of important omissions. First of all, we have omitted the essential characterisations of the locally convex topologies required to make sense of what we call "open-loop systems," which subsequently are required to define trajectories. For this background, we refer to [Jafarpour and Lewis 2014b]. We have also omitted a lot of basic control theory that can be done in the tautological control system framework. For example, we have seen in Example 3.5 that there is some sort of correspondence between "ordinary" control systems and tautological control systems. The matter of making these connections precise becomes quite technical. In particular, establishing that there is a correspondence between trajectories in the two formulations requires some effort. This is done in [Lewis 2014, §5.5].

We have claimed that one of the advantages of the tautological control system framework is that it is feedback-invariant. This is actually something that can be proved, and this is discussed in [Lewis 2014, §5.6].

While a lot has been done in the tautological control system framework to establish a basic foundation for control theory, much remains to be done. In doing so, it is our hope that the feedback-invariant character of the tautological control system formulation will allow one to understand fundamental structural issues in a way that has hitherto not been possible. For example, the rôle of locally convex topologies in control theory has already allowed for a deep understanding of the structure of "ordinary" control systems, particularly those that are real analytic [Jafarpour and Lewis 2014a, Jafarpour and Lewis 2016].

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