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Gelfand duality for manifolds, and vector and other bundles

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Abstract

In general terms, Gelfand duality refers to a correspondence between a geometric, topological, or analytical category, and an algebraic category. For example, in smooth differential geometry, Gelfand duality refers to the topological embedding of a smooth manifold in the topological dual of its algebra of smooth functions. This is generalised here in two directions. First, the topological embeddings for manifolds are generalised to the cases of real analytic and Stein manifolds, using a unified cohomological argument. Second, this type of duality is extended to vector bundles, affine bundles, and jet bundles by using suitable classes of functions, the topological duals in which the embeddings take their values.

Keywords. Gelfand duality, embedding of manifolds, embedding of bundles, analytic differential geometry

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1. Introduction

In smooth differential geometry, the embedding of a smooth manifold M in the topological dual $\mathrm{C}^\infty(\mathsf{M})'$ of its algebra of smooth functions is well known [e.g., Nestruev 2003, Theorem 7.2]. That the result is true appears to originate as [Milnor and Stasheff 1974, Problem 1-C]. This is also known in the case of Stein manifolds, and seems due to Rossi [1963, Theorem 2.6]. For real analytic manifolds, the first proof of which we are aware is given in the PhD thesis of Jafarpour [2016, Theorem 3.4.4]. In all cases, the idea is a refrain from algebraic geometry: one wishes to understand correspondences between a space and the space of natural functions defined on the space. This general idea is known as Gelfand duality, and can be regarded as providing a full and faithful functor between the geometric category (say, smooth manifolds and mappings) and the algebraic category (say, the opposite category[1] of the category of $\mathbb{R}$-algebras).

In this paper we are concerned with the geometry/algebra correspondence for bundles with algebraic structure, specifically vector, affine, and jet bundles. For vector bundles, certain algebraic correspondences are standard.

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[1]It has to be the opposite category because of the fact that the pull-back of a composition is the reversed composition of the pull-backs.

1. *The "locally free, locally finitely generated" correspondence*: A commonly called upon correspondence between a geometric object and a (sort of) algebraic object is the categorical equivalence between vector bundles and locally free, locally finitely generated sheaves of modules over the sheaf of rings of functions. This is explained in the context of smooth geometry by Ramanan [2005, §2.2] and quickly by Grauert and Remmert [1984, §1.4.2] in the holomorphic case.
2. *The "projective module" correspondence*: A related correspondence that is brought up in the same vein is the correspondence between the modules of sections of vector bundles and finitely generated projective modules over the ring of functions. This is known as the "Serre–Swan Theorem" as it is proved for algebraic vector bundles over affine varieties by Serre [1955] and for vector bundles over compact Hausdorff topological spaces by Swan [1962]. The version for smooth vector bundles came into being at some point, and is given by [Nestruev 2003, Theorem 11.32]. For holomorphic vector bundles over a Stein base, the result was first proved by Forster [1967, Satz 6.7 and 6.8]; see also [Morye 2013, Corollary 3.7].

These well-known geometric/algebraic correspondences for vector bundles are not without their limitations. The correspondence with locally free, locally finitely generated sheaves is quite perfect; indeed, it is rather close to a tautology once one understands the words involved. On the other hand, the correspondence with finitely generated projective modules is not quite tautological. However, it is not uniquely defined, in the sense that the module of sections can be a summand of a module in many different ways. Also, this projective module characterisation is of a different character than the standard Gelfand correspondence for manifolds. Indeed, the two correspondences seem a bit orthogonal. Moreover, both of these correspondences become complicated when the one talks about vector bundles over different base spaces, as the base ring changes for the modules under consideration.

In the paper we address three questions that arise from the preceding discussion.

1. Can the Gelfand duality for manifolds be unified across regularity classes?
2. Does the Serre–Swan Theorem hold for vector bundles with regularity other than what is mentioned above?
3. Is there a full and faithful functor for vector (or other) bundles that more closely resembles the Gelfand duality for manifolds?

As to the first question, we give a proof of Gelfand duality for smooth, real analytic, and Stein manifolds that is "the same" for all cases. It relies on reducing a crucial part of the proof to an argument using the vanishing of sheaf cohomology in the three cases. The results are presented in Section 3, with the main result being Theorem 3.2.

Concerning the second question, we prove as Theorem 6.5 the Serre–Swan Theorem for smooth, real analytic, and Stein base spaces, again using a unified argument.

The answering of these first two questions can be seen as wrapping up some loose ends, and tightening up the presentation of existing results. However, the line of the third question seems unaddressed in the existing literature. In Sections 4 and 5 we answer the question by providing, for a few classes of bundles—vector bundles, affine bundles, and jet bundles—a version of Gelfand duality for these spaces. The geometry/algebra correspondence we give is closely integrated with the standard Gelfand duality for manifolds, which distinguishes it from the existing correspondences for vector bundles. Another feature of our approach to the geometry/algebra correspondence for bundles is that it utilises topological properties of the function spaces involved to make the correspondences homeomorphisms. The key idea is the determination of (1) an appropriate space of functions to play the rôle of the algebra of all functions with desired regularity and (2) the appropriate set of morphisms that makes the correspondence functorial.

Notation. When A is a subset of a set X, we write $A \subseteq X$. If we wish to exclude the possibility that $A = X$, we write $A \subset X$. The identity map on a set X is denoted by id_X.

By $\mathbb{Z}$ we denote the set of integers. We use the notation $\mathbb{Z}_{>0}$ and $\mathbb{Z}_{\geq 0}$ to denote the subsets of positive and nonnegative integers. By $\mathbb{R}$ we denote the sets of real numbers. By $\mathbb{R}_{>0}$ we denote the subset of positive real numbers. By $\mathbb{C}$ we denote the set of complex numbers. We shall work simultaneously with real and complex numbers, and so denote $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ in these cases. We denote by $\mathbb{F}^n$ the n-fold Cartesian product of $\mathbb{F}$.

If R is a ring (a commutative ring with unit) and if U and V and R-modules, we denote by $\mathrm{Hom}_{\mathsf{R}}(\mathsf{U}; \mathsf{V})$ the set of module homomorphisms from U to V. We denote by $\mathsf{V}^* = \mathrm{Hom}_{\mathsf{R}}(\mathsf{V}; \mathsf{R})$ the algebraic dual. If $v \in \mathsf{V}$ and $\alpha \in \mathsf{V}^*$, we will denote the evaluation of α on v at various points by $\alpha(v)$, $\alpha \cdot v$, or $\langle \alpha; v \rangle$, whichever seems most pleasing to us at the moment.

By $\mathfrak{S}_k$ we denote the permutation group of $\{1, \dots, k\}$. For $k, l \in \mathbb{Z}_{\geq 0}$, we denote by $\mathfrak{S}_{k,l}$ the subset of $\mathfrak{S}_{k+l}$ consisting of permutations σ satisfying

$$\sigma(1) < \cdots < \sigma(k), \quad \sigma(k+1) < \cdots < \sigma(k+l).$$

By $\mathrm{T}^k(\mathsf{V})$ we denote the k-fold tensor algebra of V. By $\mathrm{S}^k(\mathsf{V})$ we denote the k-fold symmetric tensor product of V with itself, and we think of this as a subset of $\mathrm{T}^k(\mathsf{V})$. For $A \in \mathrm{S}^k(\mathsf{V})$ and $B \in \mathrm{S}^l(\mathsf{V})$, we define the symmetric tensor product of A and B to be

$$A \odot B = \sum_{\sigma \in \mathfrak{S}_{k,l}} \sigma(A \otimes B).$$

We shall adopt the notation and conventions of smooth differential geometry of [Abraham, Marsden, and Ratiu 1988]. We shall also make use of real analytic and holomorphic differential geometry. There are no useful textbook references dedicated to real analytic differential geometry, but the book of Cieliebak and Eliashberg [2012] contains much of what we shall need. For complex geometry, we refer to [Wells Jr. 2008]. Throughout the paper, manifolds are connected, second countable, Hausdorff manifolds. The assumption of connectedness can be dispensed with but is convenient as it allows one to not have to worry about manifolds with components of different dimensions and vector bundles with fibres of different dimensions.

We shall work with regularity classes $r \in \{\infty, \omega, \mathrm{hol}\}$, "$\infty$" meaning smooth, "$\omega$" meaning real analytic, and "hol" meaning holomorphic. We shall use $\mathbb{F} = \mathbb{R}$ when working in the smooth and real analytic settings, and use $\mathbb{F} = \mathbb{C}$ when working in the holomorphic setting. In the holomorphic case, we work with Stein manifolds and with vector bundles over Stein manifolds. When we are being careful, as in stating theorems, we will be sure to state this clearly. However, in discussions, we will sometimes make statements about holomorphic geometry that are only true for Stein manifolds or for vector bundles over Stein manifolds, while not explicitly mentioning that the Stein assumption is being made. For this reason, it is probably best to always assume the Stein assumption is being made in the background when it is not being made in the foreground.

We denote by $\mathrm{C}^r(\mathsf{M};\mathsf{N})$ the set of mappings from a manifold M to a manifold N of class C^r. By $\mathrm{Diff}^r(\mathsf{M})$ we denote the set of C^r-diffeomorphisms of M. When $\mathsf{N} = \mathbb{F}$, we denote by $\mathrm{C}^r(\mathsf{M}) = \mathrm{C}^r(\mathsf{M};\mathbb{F})$ the set of scalar-valued functions of class C^r. We denote by 1_{M} the constant function with value 1 on a manifold M. By $\mathrm{d}f$ we denote the differential of f. By $\mathscr{L}_X f$ we denote the Lie derivative of a function f with respect to a vector field X.

By $\pi_{\mathsf{TM}}\colon \mathsf{TM} \to \mathsf{M}$ we denote the tangent bundle of M (the holomorphic tangent bundle in the holomorphic case). If $\Phi \in \mathrm{C}^r(\mathsf{M};\mathsf{N})$, we denote by $T\Phi\colon \mathsf{TM} \to \mathsf{TN}$ the derivative of Φ. By $T_x\Phi$ we denote the restriction of $T\Phi$ to $\mathsf{T}_x\mathsf{M}$.

Let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a vector bundle of class C^r. We shall sometimes denote the fibre over $x \in \mathsf{M}$ by E_x, noting that this has the structure of an $\mathbb{F}$-vector space. By $\mathbb{F}_{\mathsf{M}} = \mathsf{M} \times \mathbb{F}$, we denote the trivial line bundle. If $A \subseteq \mathsf{M}$, we denote by $\mathsf{E}|A = \pi^{-1}(A)$. If $\mathsf{S} \subseteq \mathsf{M}$ is a submanifold, then $\mathsf{E}|\mathsf{S}$ is a vector bundle over S. By $\Gamma^r(\mathsf{E})$ we denote the set of sections of E of class C^r. If $\pi\colon \mathsf{E} \to \mathsf{M}$ and $\theta\colon \mathsf{F} \to \mathsf{N}$ are C^r-vector bundles, a C^r-vector bundle mapping from E to F is a pair (Φ, Φ_0) of C^r-mappings making the diagram

$$\begin{tikzcd} \mathsf{E} \arrow[r, "\Phi"] \arrow[d, "\pi"'] & \mathsf{F} \arrow[d, "\theta"] \\ \mathsf{M} \arrow[r, "\Phi_0"'] & \mathsf{N} \end{tikzcd}$$

commute, and such that $\Phi_x \in \mathrm{Hom}_{\mathbb{F}}(\mathsf{E}_x; \mathsf{F}_{\Phi_0(x)})$, where $\Phi_x = \Phi|\mathsf{E}_x$. We denote by $\mathrm{VB}^r(\mathsf{E};\mathsf{F})$ the set of vector bundle mappings from E to F.

By $\mathscr{C}^r_{\mathsf{M}}$ we denote the sheaf of C^r-functions on M and by $\mathscr{G}^r_{\mathsf{E}}$ we denote the sheaf of C^r-sections of E, thought of as an $\mathscr{C}^r_{\mathsf{M}}$-module.

We shall often make use of the fact that, for the manifolds we consider, there are always globally defined coordinate functions.

1.1 Lemma: (Existence of globally defined coordinate functions) *Let $r \in \{\infty, \omega, \mathrm{hol}\}$ and let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, as appropriate. If M is a C^r-manifold, Stein when $r = \mathrm{hol}$, then, for any $x \in \mathsf{M}$, there exists a chart $(\mathcal{U}, \phi)$ for M about x whose coordinate functions $\chi^1, \dots, \chi^n$ are restrictions to $\mathcal{U}$ of globally defined functions of class C^r.*

Proof: The hypotheses ensure that there is a proper C^r-embedding $\iota_{\mathsf{M}}\colon \mathsf{M} \to \mathbb{F}^N$ for some suitable $N \in \mathbb{Z}_{>0}$; this is a result of Whitney [1936] in the smooth case, Grauert [1958] in the real analytic case, and Remmert [1954] in the holomorphic case. Define $\chi^1, \dots, \chi^N \in \mathrm{C}^r(\mathsf{M})$ by

$$\iota_{\mathsf{M}}(x) = (\chi^1(x), \dots, \chi^N(x)), \qquad x \in \mathsf{M}.$$

Now, for $x \in \mathsf{M}$, $T_x\iota_{\mathsf{M}}$ is injective, and so there exists $j_1, \dots, j_n \in \{1, \dots, N\}$ such that $(\mathrm{d}\chi^{j_1}(x), \dots, \mathrm{d}\chi^{j_n}(x))$ is a basis for $\mathsf{T}^*_x\mathsf{M}$. In some neighbourhood $\mathcal{U}$ of x, we will have linear independence of $(\mathrm{d}\chi^{j_1}(x), \dots, \mathrm{d}\chi^{j_n}(x))$, and so $\chi^{j_1}, \dots, \chi^{j_n}$ are coordinate functions on $\mathcal{U}$. ■

Finally, we mention that for the topological assertions in our main results, we make use of topologies on spaces of sections of C^r-vector bundles. We do not go into detail about what these topologies are, as a detailed understanding of this is not material to the important points we are making in this paper. However, there may be some applications of our results here that will benefit from a detailed understanding of the topologies involved. We refer to [Lewis 2023] for details in the real analytic case. Also in that work some words are said about how to simplify the tools in the real analytic case to the smooth and holomorphic cases; the holomorphic case is particularly simple, as the topology in this case is the topology of uniform convergence on compact sets. Alternatively, one can use the smooth topology on holomorphic mappings since holomorphic mappings form a closed subset of smooth mappings [Kriegl and Michor 1997, Theorem II.8.2].

2. Functions on vector, affine, and jet bundles

While vector bundles are most commonly encountered in differential geometry, for what we do in this paper it is most natural to work with affine bundles. The reason for this is that it is affine functions (not linear functions) that we will use to characterise Gelfand duality for vector bundles. Therefore, we prefer to work with affine bundles, where affine functions are most naturally defined. Also, the affine structure of jet bundles makes it possible to extend our notions of Gelfand duality from affine bundles to jet bundles.

2.1. Affine bundles.

We assume the reader is familiar with the notion of an affine space modelled on a vector space [Berger 1987, Chapter 2]. If A is an affine space modelled on a vector space V, then $\mathsf{A}^{*,\mathrm{aff}}$ denotes the ***affine dual*** of A, by which we mean the set of affine maps from A to the field over which V is defined. We shall adopt the notation for linear duals, and write $\langle\lambda; a\rangle = \lambda(a)$ for the evaluation of $\lambda \in \mathsf{A}^{*,\mathrm{aff}}$ on $a \in \mathsf{A}$.

An affine bundle is the extension of the idea of an affine space to differential geometry, in the same way as a vector bundle is an extension to differential geometry of a vector space.

2.1 Definition: (Affine bundle) Let $r \in \{\infty, \omega, \mathrm{hol}\}$ and let $\mathbb{F} = \mathbb{R}$ if $r \in \{\infty, \omega\}$ and let $\mathbb{F} = \mathbb{C}$ if $r = \mathrm{hol}$. Let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a C^r-vector bundle. A ***C^r-affine bundle*** over M modelled on E is a C^r-fibre bundle $\beta\colon \mathsf{B} \to \mathsf{M}$ with the following structure:

(i) there exists a C^r-fibre bundle mapping $\alpha\colon \mathsf{E} \times_{\mathsf{M}} \mathsf{B} \to \mathsf{B}$ such that $a + e \triangleq \alpha(e, a)$ makes B_x into an affine space modelled on E_x for each $x \in \mathsf{M}$;

(ii) for each $x \in \mathsf{M}$, there exists a C^r-local trivialisation $\tau\colon \beta^{-1}(\mathcal{U}) \to \mathcal{U} \times \mathbb{F}^k$ for which $\mathrm{pr}_2 \circ (\tau|\mathsf{B}_y)\colon \mathsf{B}_y \to \mathbb{F}^k$ is an isomorphism of affine spaces for each $y \in \mathcal{U}$. •

Let us flesh out the meaning of the second condition concerning local trivialisations. Suppose that $(\mathcal{U}, \phi)$ is a vector bundle chart for the model vector bundle $\pi\colon \mathsf{E} \to \mathsf{M}$ associated with the affine bundle $\beta\colon \mathsf{B} \to \mathsf{M}$. Then, possibly after shrinking $\mathcal{U}$, there is a local trivialisation $\tau\colon \beta^{-1}(\mathcal{U}_0) \to \mathcal{U}_0 \times \mathbb{F}^k$ satisfying the second condition in the definition. We also have a local trivialisation $\lambda\colon \pi^{-1}(\mathcal{U}_0) \to \mathcal{U}_0 \times \mathbb{F}^k$ given by the vector bundle chart that is a vector bundle mapping. This means that the representation in these local trivialisations of the mapping α providing the affine structure is

$$(\boldsymbol{x}, (\boldsymbol{e}, \boldsymbol{a})) \mapsto (\boldsymbol{x}, \boldsymbol{a} + \boldsymbol{e}).$$

Thus, locally, the affine bundle looks like the product of an open set with the affine space $\mathbb{F}^k$.

2.2 Definition: (Affine bundle map) Let $r \in \{\infty, \omega, \mathrm{hol}\}$ and let $\mathbb{F} = \mathbb{R}$ if $r \in \{\infty, \omega\}$ and let $\mathbb{F} = \mathbb{C}$ if $r = \mathrm{hol}$. If $\beta_1 \colon \mathsf{B}_1 \to \mathsf{M}_1$ and $\beta_2 \colon \mathsf{B}_2 \to \mathsf{M}_2$ are C^r-affine bundles, then a ***C^r-affine bundle map*** between these affine bundles is a C^r-map $\Phi \colon \mathsf{B}_1 \to \mathsf{B}_2$ for which there exists a C^r-map $\Phi_0 \colon \mathsf{M}_1 \to \mathsf{M}_2$ such that the diagram

$$\begin{tikzcd} \mathsf{B}_1 \arrow[r, "\Phi"] \arrow[d, "\beta_1"'] & \mathsf{B}_2 \arrow[d, "\beta_2"] \\ \mathsf{M}_1 \arrow[r, "\Phi_0"'] & \mathsf{M}_2 \end{tikzcd}$$

commutes and with the property that $\Phi|\mathsf{B}_{1,x} \colon \mathsf{B}_{1,x} \to \mathsf{B}_{2,\Phi_0(x)}$ is an affine map. If Φ is a C^r-diffeomorphism we say it is an ***affine bundle isomorphism***.

We denote by $\mathrm{AB}^r(\mathsf{B}_1; \mathsf{B}_2)$ the set of C^r-affine bundle mappings from B_1 to B_2. •

We let $\Gamma^r(\mathsf{B})$ denote the set of sections of an affine bundle B. Being fibre bundles, affine bundles are entitled to the possession of local sections. Also, just being fibre bundles, they are not *a priori* entitled the possession of global sections. However, one feels that they are really a lot like vector bundles, and so should possess as many global sections as their model vector bundles. Indeed, if an affine bundle $\beta \colon \mathsf{B} \to \mathsf{M}$ possesses *one* section σ, then $\sigma + \xi$ is also a section for any section ξ of the model vector bundle $\pi \colon \mathsf{E} \to \mathsf{M}$. Thus the question of the character of the set of sections of an affine bundle really boils down to the existence of one section.

2.3 Proposition: (Existence of sections of affine bundles) *Let $r \in \{\infty, \omega, \mathrm{hol}\}$ and let $\mathbb{F} = \mathbb{R}$ if $r \in \{\infty, \omega\}$ and let $\mathbb{F} = \mathbb{C}$ if $r = \mathrm{hol}$. Let $\beta \colon \mathsf{B} \to \mathsf{M}$ be a C^r-affine bundle modelled on the C^r-vector bundle $\pi \colon \mathsf{E} \to \mathsf{M}$ and suppose that M is Stein if $r = \mathrm{hol}$. Then $\Gamma^r(\mathsf{B}) \neq \varnothing$.*

Proof: To begin, we claim that there is an open cover $(\mathcal{U}_a)_{a \in A}$ such that the sheaves $\mathscr{G}^r_{\mathsf{E}}|(\cap_{a \in F} \mathcal{U}_a)$, $F \subseteq A$ finite, are acyclic. First of all, in the smooth and real analytic cases, this is true for *every* open cover. In the smooth case, this follows from the vanishing of the sheaf cohomology for sheaves of modules over $\mathscr{C}^\infty_{\mathsf{M}}$ ([Wells Jr. 2008, Proposition 2.3.11], along with [Wells Jr. 2008, Examples 2.3.4(d, e)] and [Wells Jr. 2008, Proposition 2.3.5]). In the real analytic case, we note that $\mathscr{G}^\omega_{\mathsf{E}}$ is coherent by the Oka Coherence Theorem (see [Grauert and Remmert 1984, Theorem 2.5.2] in the holomorphic case; the same proof works in the real analytic case. Thus the assertion follows from Cartan's Theorem B [Cartan 1957, Proposition 6]. In the holomorphic case, we can assume that the open sets are Stein (e.g., we can take the open sets $\mathcal{U}_a$, $a \in A$, to be preimages under an holomorphic chart of a polydisk in $\mathbb{C}^n$). Then, since finite intersections of Stein open sets are Stein (by [Demailly 2012, Proposition I.6.20(c)]), it follows from Cartan's Theorem B that $\mathscr{G}^r_{\mathsf{E}}|(\cap_{a \in F} \mathcal{U}_a)$ is acyclic for every finite $F \subseteq A$.

About any $x \in \mathsf{M}$ there is a neighbourhood $\mathcal{U}$ so that $\mathsf{E}|\mathcal{U} \simeq \mathcal{U} \times \mathbb{F}^k$, $\mathsf{B}|\mathcal{U} \simeq \mathcal{U} \times \mathbb{F}^k$, and the affine structure on the fibres is the standard one. Therefore, there are local sections $\sigma_1, \dots, \sigma_{k+1} \in \Gamma^r_{\mathcal{U}}(\mathsf{B})$ such that any local section of $\mathsf{B}|\mathcal{U}$ is an affine combination of these, i.e.,

$$\sigma \in \Gamma^r_{\mathcal{U}}(\mathsf{B}) \implies \sigma(x) = \sum_{a=1}^{k+1} f^a(x)\sigma_a(x), \quad f^1, \dots, f^{k+1} \in \mathrm{C}^r(\mathcal{U}),\ \sum_{a=1}^{k+1} f^a(x) = 1,\ x \in \mathcal{U}.$$

Therefore, there exists an open cover $\mathscr{U} = (\mathcal{U}_a)_{a\in A}$ for M such that, for each $a \in A$, we have local generators $(\sigma_{ai})_{i\in I}$ for the affine bundle B (as above) on $\mathcal{U}_a$. In the holomorphic case, we assume that the open sets $\mathcal{U}_a$ are Stein, as can be done without loss of generality. The index set I can be taken to be the same for all open sets by our assumption that M is connected. For $a \in A$, fix $i_0 \in I$ and denote $\sigma_{a0} = \sigma_{ai_0}$. For $x \in \mathcal{U}_a$, we have $\mathsf{B}_x = \sigma_{a0}(x) + \mathsf{E}_x$. If $\mathcal{U}_a \cap \mathcal{U}_b \neq \varnothing$, then we have $\sigma_{a0}(x) - \sigma_{b0}(x) \in \mathsf{E}_x$ for $x \in \mathcal{U}_a \cap \mathcal{U}_b$. Said otherwise,

$$\sigma_{a0}|\mathcal{U}_a \cap \mathcal{U}_b - \sigma_{b0}|\mathcal{U}_a \cap \mathcal{U}_b \in \mathscr{G}^r_{\mathsf{E}}(\mathcal{U}_a \cap \mathcal{U}_b).$$

Denote $\xi_{ab} \in \mathscr{G}^r_{\mathsf{E}}(\mathcal{U}_a \cap \mathcal{U}_b)$ by

$$\xi_{ab} = \sigma_{a0}|\mathcal{U}_a \cap \mathcal{U}_b - \sigma_{b0}|\mathcal{U}_a \cap \mathcal{U}_b,$$

and note that

$$\xi_{ac}|\mathcal{U}_a \cap \mathcal{U}_b \cap \mathcal{U}_c = \xi_{ab}|\mathcal{U}_a \cap \mathcal{U}_b \cap \mathcal{U}_c + \xi_{bc}|\mathcal{U}_a \cap \mathcal{U}_b \cap \mathcal{U}_c.$$

Thus we have a Čech 1-cocycle $(\xi_{ab})_{a,b\in A} \in \check{\mathrm{Z}}^1(\mathscr{U}; \mathscr{G}^r_{\mathsf{E}})$. As we have seen in the first paragraph of the proof, $\mathrm{H}^1(\mathsf{M}; \mathscr{G}^r_{\mathsf{E}}) = 0$. By Leray's Theorem ([Ramanan 2005, Theorem 4.5.3]), the Čech cohomology $\check{\mathrm{H}}^1(\mathscr{U}; \mathscr{G}^r_{\mathsf{E}})$ vanishes. We thus have a 1-coboundary $(\eta_a)_{a\in A} \in \check{\mathrm{B}}^1(\mathscr{U}, \mathscr{G}^r_{\mathsf{E}})$ such that

$$\eta_b|\mathcal{U}_a \cap \mathcal{U}_b - \eta_a|\mathcal{U}_a \cap \mathcal{U}_b = \xi_{ab}, \qquad a, b \in A.$$

Let $\sigma_a \in \Gamma^r_{\mathcal{U}_a}(\mathsf{B})$ be given by $\sigma_a = \sigma_{a0} + \eta_a$ and note that

$$\sigma_a|\mathcal{U}_a \cap \mathcal{U}_b = (\xi_{a0} + \eta_a)|\mathcal{U}_a \cap \mathcal{U}_b = (\xi_{b0} + \eta_b)|\mathcal{U}_a \cap \mathcal{U}_b = \sigma_b|\mathcal{U}_a \cap \mathcal{U}_b.$$

Since the sheaf of sections of B is a sheaf, there exists $\sigma \in \Gamma^r(\mathsf{B})$ such that $\sigma|\mathcal{U}_a = \sigma_a$, $a \in A$. Thus σ is the section we are after. ■

We have the following two corollaries that will be useful.

2.4 Corollary: (Isomorphism of affine bundle with model vector bundle) *Let $r \in \{\infty, \omega, \mathrm{hol}\}$, and let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a C^r-vector bundle and $\beta\colon \mathsf{B} \to \mathsf{M}$ be a C^r-affine bundle modelled on E. Assume that M is Stein when $r = \mathrm{hol}$. Then there exists a C^r-affine bundle isomorphism $\Psi\colon \mathsf{B} \to \mathsf{E}$ over id_{M}.*

Proof: We first note that, if there exists a C^r-section of $\beta\colon \mathsf{B} \to \mathsf{M}$, then the lemma holds. Indeed, suppose that we have a C^r-section $\sigma\colon \mathsf{M} \to \mathsf{B}$. Then one readily verifies that the mapping

$$\begin{aligned} \widehat{\Psi}\colon \mathsf{E} &\to \mathsf{B} \\ e_x &\mapsto \alpha(\sigma(x), e_x) \end{aligned}$$

is a C^r-affine bundle isomorphism, the isomorphism on fibres being that where $\sigma(x)$ serves as the "origin" for the affine space B_x. We take $\Psi = \widehat{\Psi}^{-1}$. ■

For the next corollary, we denote by $\mathsf{J}^m\mathsf{B}$ the bundle of m-jets of an affine bundle B.

2.5 Corollary: (Sections of affine bundles with prescribed jet at a point) *Let $r \in \{\infty, \omega, \mathrm{hol}\}$, and let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a C^r-vector bundle and $\beta\colon \mathsf{B} \to \mathsf{M}$ be a C^r-affine bundle modelled on E. Assume that M is additionally Stein when $r = \mathrm{hol}$. If $\Xi \in \mathsf{J}^m\mathsf{B}$, then there exists $\sigma \in \Gamma^r(\mathsf{B})$ satisfying $j_m\sigma(\beta(b)) = \Xi$.*

Proof: A generalisation of this is proved for vector bundles as Lemma 2.1 in [Lewis 2023]. For affine bundles, the result is then a consequence of Corollary 2.4. ■

2.2. Jet bundles. We will consider various sorts of jet bundles in this paper. We refer to [Saunders 1989] and [Kolář, Michor, and Slovák 1993, §12] as useful references. Here we shall mainly introduce the notation we use.

Throughout this section, we let $r \in \{\infty, \omega, \mathrm{hol}\}$ and let $\mathbb{F} = \mathbb{R}$ if $r \in \{\infty, \omega\}$ and let $\mathbb{F} = \mathbb{C}$ if $r = \mathrm{hol}$.

2.2.1. Jets of sections of a fibred manifold. Let $r \in \{\infty, \omega, \mathrm{hol}\}$. We consider a C^r-fibred manifold $\rho\colon \mathsf{X} \to \mathsf{M}$ (i.e., a surjective submersion) and C^r-local sections of this manifold. For $p \in \mathsf{X}$ and $m \in \mathbb{Z}_{\geq 0}$, we denote by $\mathsf{J}^m_p\mathsf{X}$ the set of m-jets of local sections that take the value p at $x = \rho(p)$. Thus an element of $\mathsf{J}^m_p\mathsf{X}$ is an equivalence class of local sections taking the value p at x and whose first m-derivatives (say, in a fibred chart) agree at x. For a C^r-local section σ defined in a neighbourhood of $x \in \mathsf{M}$, we denote by $j_m\sigma(x) \in \mathsf{J}^m_{\sigma(x)}\mathsf{X}$ the m-jet of σ. We denote by $\mathsf{J}^m\mathsf{X} = \mathring{\cup}_{p\in\mathsf{X}} \mathsf{J}^m_p\mathsf{X}$ the bundle of m-jets of local sections. Note that $\mathsf{J}^0\mathsf{X} \simeq \mathsf{X}$. For $m, l \in \mathbb{Z}_{\geq 0}$ with $m \geq l$, we denote by $\rho^m_l\colon \mathsf{J}^m\mathsf{X} \to \mathsf{J}^l\mathsf{X}$ the projection, and we abbreviate $\rho_m \triangleq \rho \circ \rho^m_0\colon \mathsf{J}^m\mathsf{X} \to \mathsf{M}$. This defines $\mathsf{J}^m\mathsf{X}$ as a fibred manifold over M.

Let us denote by $\mathsf{V}\mathsf{X} = \ker(T\rho) \subseteq \mathsf{T}\mathsf{X}$ the vertical bundle of the projection, and let $\nu = \pi_{\mathsf{T}\mathsf{X}}|\mathsf{V}\mathsf{X}$. We claim that $\rho^m_{m-1}\colon \mathsf{J}^m\mathsf{X} \to \mathsf{J}^{m-1}\mathsf{X}$ is an affine bundle modelled on $\rho^*_{m-1}\mathrm{S}^m(\mathsf{T}^*\mathsf{M}) \otimes (\rho^{m-1}_0)^*\mathsf{V}\mathsf{X}$. The affine structure is defined as follows. Let $p \in \mathsf{X}$, let $x = \rho(p)$, let $\alpha^1, \dots, \alpha^m \in \mathsf{T}^*_x\mathsf{M}$, and let $v \in \mathsf{V}_p\mathsf{X}$. Let $f^1, \dots, f^m$ be C^r-functions defined near x, vanishing at x, and satisfying $\mathrm{d}f^j(x) = \alpha^j$, $j \in \{1, \dots, m\}$. Let $(y,t) \mapsto \omega(y,t) \in \mathsf{X}$ be a mapping defined near $(x,0)$ and satisfying $\omega(x,t) = p$ for all t, $\rho \circ \omega(y,t) = y$ for all (y,t), and $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\omega(x,t) = v$.[2] Then we define the affine action by

$$j_k\sigma(x) + \alpha^1 \odot \cdots \odot \alpha^m \otimes v = j_k(\sigma_\omega)(x),$$

where σ_ω is the section $\sigma_\omega(x) = \omega(x, f^1(x)\cdots f^m(x))$. Thus we have the diagram

$$0 \longrightarrow \rho^*_{m-1}\mathrm{S}^m(\mathsf{T}^*\mathsf{M}) \otimes (\rho^{m-1}_0)^*\mathsf{V}\mathsf{X} \longrightarrow \mathsf{J}^m\mathsf{X} \longrightarrow \mathsf{J}^{m-1}\mathsf{X} \longrightarrow 0$$

Also note that $\rho^m_0\colon \mathsf{J}^m\mathsf{X} \to \mathsf{X}$ is an affine bundle modelled on $\nu_m\colon \mathsf{J}^m\mathsf{V}\mathsf{X} \to \mathsf{X}$.

2.2.2. Jets of mappings of manifolds. Let $r \in \{\infty, \omega, \mathrm{hol}\}$ and let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, as appropriate. We next work with C^r-manifolds M and N, and jets of C^r-mappings from M to N. For $(x,y) \in \mathsf{M} \times \mathsf{N}$, we denote by $\mathsf{J}^m_{(x,y)}(\mathsf{M};\mathsf{N})$ the set of m-jets of mappings that map x to y. Thus these are equivalence classes of mappings from M to N that map x to y and whose first

[2] One can think of ω as being a family of sections that map x to p.

m-derivatives agree (in a chart, for instance). For $\Phi \in \mathrm{C}^r(\mathsf{M};\mathsf{N})$, $j_m\Phi(x) \in \mathsf{J}^m_{(x,\Phi(x))}(\mathsf{M};\mathsf{N})$ denotes the m-jet of Φ at x. We denote

$$\mathsf{J}^m(\mathsf{M};\mathsf{N}) = \mathop{\mathring{\bigcup}}_{(x,y)\in\mathsf{M}\times\mathsf{N}} \mathsf{J}^m_{(x,y)}(\mathsf{M};\mathsf{N}),$$

which is the bundle of m-jets of mappings from M to N. Note that $\mathsf{J}^0(\mathsf{M};\mathsf{N}) \simeq \mathsf{M}\times\mathsf{N}$. For $m, l \in \mathbb{Z}_{\geq 0}$ with $m \geq l$, we denote $\rho^m_l \colon \mathsf{J}^m(\mathsf{M};\mathsf{N}) \to \mathsf{J}^l(\mathsf{M};\mathsf{N})$ as the natural projection. We abbreviate $\rho_m \triangleq \mathrm{pr}_1 \circ \rho^m_0 \colon \mathsf{J}^m(\mathsf{M};\mathsf{N}) \to \mathsf{M}$.

In the special case where $\mathsf{N} = \mathbb{F}$, i.e., when we are dealing with jets of functions, we denote by $\mathsf{T}^{*m}_x\mathsf{M} = \mathsf{J}^m_{(x,0)}(\mathsf{M};\mathbb{F})$ the m-jets of functions that have the value 0 at x. We note that this is an $\mathbb{F}$-algebra by

$$j_mf(x) + j_mg(x) = j_m(f+g)(x), \quad a(j_mf(x)) = j_m(af)(x),$$
$$(j_mf(x))\cdot(j_mg(x)) = j_m(fg)(x).$$

We can then think of $\mathsf{J}^m_{(x,y)}(\mathsf{M};\mathsf{N})$ as the set of $\mathbb{F}$-algebra homomorphisms from $\mathsf{T}^{*m}_y\mathsf{N}$ to $\mathsf{T}^{*m}_x\mathsf{M}$ by

$$j_m\Phi(x)(j_mg(y)) = j_m(\Phi^*g)(x), \qquad j_k\Phi(x) \in \mathsf{J}^m_{(x,y)}(\mathsf{M};\mathsf{N}),\ j_kg(y) \in \mathsf{T}^{*m}_y\mathsf{N}.$$

One can verify that this correspondence is a bijection, and it allows us to think of jets of mappings in a concrete algebraic context.

To understand some of the structure of jet bundles of mappings, it is convenient to treat such jets as a special case of the jets of the fibred manifold $\rho\colon \mathsf{M}\times\mathsf{N} \to \mathsf{M}$, with $\rho = \mathrm{pr}_1$. In this case, $\Gamma^r(\mathsf{M}\times\mathsf{N}) \simeq \mathrm{C}^r(\mathsf{M};\mathsf{N})$ by the observation that $\sigma(x) = (x, \Phi_\sigma(x))$ for $\sigma \in \Gamma^r(\mathsf{M}\times\mathsf{N})$ and for the associated mapping $\Phi_\sigma \in \mathrm{C}^r(\mathsf{M};\mathsf{N})$. In like manner, we identify $\mathsf{J}^m(\mathsf{M}\times\mathsf{N})$ and $\mathsf{J}^m(\mathsf{M};\mathsf{N})$ by

$$j_m\sigma(x) = (x, j_m\Phi_\sigma(x)).$$

We use $\mathsf{J}^m(\mathsf{M};\mathsf{N})$ to denote the space of m-jets in this setting, although it is sometimes convenient to think of this as $\mathsf{J}^m(\mathsf{M}\times\mathsf{N})$, and we shall work with both ways of thinking things, depending on which is most convenient. We shall use $\mathsf{J}^0(\mathsf{M};\mathsf{N})$, noting that this is simply identified with $\mathsf{M}\times\mathsf{N}$. Note that $\mathsf{V}(\mathsf{M}\times\mathsf{N}) \simeq 0 \oplus \mathsf{TN}$. An important specialisation that happens in this case of the jet bundle of a fibred manifold comes about because the fibred manifold is trivial (and not just merely trivialisable). Because of this $\rho^m_0\colon \mathsf{J}^m(\mathsf{M};\mathsf{N}) \to \mathsf{J}^0(\mathsf{M}\times\mathsf{N})$ is a *vector bundle*, not just an affine bundle. Indeed, it is isomorphic to the vector bundle $\rho^*_m\mathsf{T}^{*m}\mathsf{M}\otimes(\mathrm{pr}_2\circ\rho^m_0)^*\mathsf{TN}$. Note, however, that while $\mathsf{J}^m(\mathsf{M};\mathsf{N})$ is a vector bundle, it is not the jet bundle of sections of a vector bundle. Now, for $(x,y) \in \mathsf{J}^0(\mathsf{M};\mathsf{N})$ and $m \in \mathbb{Z}_{>0}$, we define $\epsilon_m\colon \mathrm{S}^m(\mathsf{T}^*_x\mathsf{M})\otimes\mathsf{T}_y\mathsf{N} \to \mathsf{J}^m_{(x,y)}(\mathsf{M};\mathsf{N})$ by

$$\epsilon_m(\mathrm{d}f^1(x)\odot\cdots\odot\mathrm{d}f^m(x)\otimes Y(y))(j_kg(y)) = j_m(f^1\cdots f^m(\mathscr{L}_Yg(y)))(x),$$

where $f^1,\dots,f^m$ are C^r-functions on M defined near x and which vanish at x, and where Y is a C^r-vector field on N defined near y. Note that, in writing this formula, we are defining an m-jet of mappings as an algebra homomorphism from $\mathsf{T}^{*m}_y\mathsf{N}$ to $\mathsf{T}^{*m}_x\mathsf{M}$. This then gives rise to the short exact sequence

$$0 \longrightarrow \rho^*_{m-1}\mathrm{S}^m(\mathsf{T}^*_x\mathsf{M})\otimes(\mathrm{pr}_2\circ\rho^{m-1}_0)^*\mathsf{TN} \xrightarrow{\epsilon_m} \mathsf{J}^m(\mathsf{M};\mathsf{N}) \xrightarrow{\rho^m_{m-1}} \mathsf{J}^{m-1}(\mathsf{M};\mathsf{N}) \longrightarrow 0 \qquad (2.1)$$

2.2.3. Jets of sections of a vector bundle. Let $r \in \{\infty, \omega, \mathrm{hol}\}$ and let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a C^r-vector bundle. For $x \in \mathsf{M}$ and $m \in \mathbb{Z}_{\geq 0}$, $\mathsf{J}^m_x\mathsf{E}$ denotes the set of m-jets of sections of E at x. For a C^r-section ξ defined in some neighbourhood of x, $j_k\xi(x) \in \mathsf{J}^m_x\mathsf{E}$ denotes the m-jet of ξ. We denote by $\mathsf{J}^m\mathsf{E} = \mathring{\cup}_{x\in\mathsf{M}} \mathsf{J}^m_x\mathsf{E}$ the bundle of m-jets. For $m, l \in \mathbb{Z}_{\geq 0}$ with $m \geq l$, we denote by $\pi^m_l\colon \mathsf{J}^m\mathsf{E} \to \mathsf{J}^l\mathsf{E}$ the projection. Note that $\mathsf{J}^0\mathsf{E} \simeq \mathsf{E}$. We abbreviate $\pi_m \triangleq \pi \circ \pi^m_0\colon \mathsf{J}^m\mathsf{E} \to \mathsf{M}$, and note that $\mathsf{J}^m\mathsf{E}$ has the structure of a vector bundle over M, with addition and scalar multiplication defined by

$$j_m\xi(x) + j_m\eta(x) = j_m(\xi + \eta)(x), \quad a(j_m\xi(x)) = j_m(a\xi)(x)$$

for sections ξ and η and for $a \in \mathbb{F}$.

For $x \in \mathsf{M}$ and $m \in \mathbb{Z}_{>0}$, define $\epsilon_m\colon \mathrm{S}^m(\mathsf{T}^*_x\mathsf{M}) \otimes \mathsf{E}_x \to \mathsf{J}^m_x\mathsf{E}$ by

$$\epsilon_m(\mathrm{d}f^1(x) \odot \cdots \odot \mathrm{d}f^m(x) \otimes \xi(x)) = j_m(f^1 \cdots f^m\xi), \tag{2.2}$$

where $f^1, \dots, f^m$ are locally defined C^r-functions around x that vanish at x. One can easily show that we then have the following short exact sequence:

$$0 \longrightarrow \pi^*_{m-1}\mathrm{S}^m(\mathsf{T}^*\mathsf{M}) \otimes \pi^*_{m-1}\mathsf{E} \xrightarrow{\epsilon_m} \mathsf{J}^m\mathsf{E} \xrightarrow{\pi^m_{m-1}} \mathsf{J}^{m-1}\mathsf{E} \longrightarrow 0 \tag{2.3}$$

2.2.4. Jets of sections of an affine bundle. We also will talk about jet bundles of affine bundles. Thus we let $r \in \{\infty, \omega, \mathrm{hol}\}$ and let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a C^r-vector bundle with $\beta\colon \mathsf{B} \to \mathsf{M}$ a C^r-affine bundle modelled on E. The affine bundle, being a fibred manifold, is entitled to its jet bundles. Then we have vector bundles $\pi_m\colon \mathsf{J}^m\mathsf{E} \to \mathsf{M}$ and affine bundles $\beta_m\colon \mathsf{J}^m\mathsf{B} \to \mathsf{M}$, $m \in \mathbb{Z}_{>0}$. We let $\mathsf{VB} = \ker(T\beta) \subseteq \mathsf{TB}$ be the vertical bundle, which is a vector bundle $\nu\colon \mathsf{VB} \to \mathsf{B}$ over B. Note that $\mathsf{VB} \simeq \beta^*\mathsf{E}$. We note that $\beta^m_{m-1}\colon \mathsf{J}^m\mathsf{B} \to \mathsf{J}^{m-1}\mathsf{B}$ is an affine bundle modelled according to the short exact sequence

$$0 \longrightarrow \beta^*_{m-1}\mathrm{S}^m(\mathsf{T}^*\mathsf{M}) \otimes \beta^*_{m-1}\mathsf{E} \xrightarrow{\epsilon_m} \mathsf{J}^m\mathsf{B} \xrightarrow{\beta^m_{m-1}} \mathsf{J}^{m-1}\mathsf{B} \longrightarrow 0 \tag{2.4}$$

We claim that $\mathsf{J}^m\mathsf{B}$ is an affine bundle modelled on $\mathsf{J}^m\mathsf{E}$. Indeed, if $\alpha\colon \mathsf{E} \times_{\mathsf{M}} \mathsf{B} \to \mathsf{B}$ is the affine bundle structure for B, then the affine bundle structure for $\mathsf{J}^m\mathsf{B}$ is

$$\begin{aligned} j_m\alpha\colon \mathsf{J}^m\mathsf{E} \times_{\mathsf{M}} \mathsf{J}^m\mathsf{B} &\to \mathsf{J}^m\mathsf{B} \\ (j_m\xi(x), j_m\sigma(x)) &\mapsto j_m(\alpha(\xi, \sigma))(x). \end{aligned} \tag{2.5}$$

Given a section $\sigma \in \Gamma^r(\mathsf{B})$, we have the C^r-affine bundle isomorphism

$$\begin{aligned} \iota_\sigma\colon \mathsf{B} &\to \mathsf{E} \\ b &\mapsto b - \sigma(b) \end{aligned}$$

from Corollary 2.4. In terms of sections, this gives an isomorphism

$$\begin{aligned} \hat{\iota}_\sigma\colon \Gamma^r(\mathsf{B}) &\to \Gamma^r(\mathsf{E}) \\ \gamma &\mapsto \gamma - \sigma \end{aligned}$$

of $\mathbb{F}$-affine spaces. This then induces an isomorphism

$$\begin{aligned} j_m\iota_\sigma\colon \mathsf{J}^m\mathsf{B} &\to \mathsf{J}^m\mathsf{E} \\ j_m\gamma(x) &\mapsto j_m\gamma(x) - j_m\sigma(x) \end{aligned}$$

of affine bundles.

2.3. Functions on vector and affine bundles. As we have indicated, a key ingredient in our extensions of the well-known forms of Gelfand duality in differential geometry is the characterisation of appropriate classes of functions. In this section we consider linear and affine functions.

Let $r \in \{\infty, \omega, \mathrm{hol}\}$ and let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ as appropriate. Let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a vector bundle of class C^r and let $\beta\colon \mathsf{B} \to \mathsf{M}$ be a C^r-affine bundle modelled on E. Note that $\beta^*\colon \mathrm{C}^r(\mathsf{M}) \to \mathrm{C}^r(\mathsf{B})$ is an homomorphism of $\mathbb{F}$-algebras. This then gives $\mathrm{C}^r(\mathsf{B})$ the structure of a $\mathrm{C}^r(\mathsf{M})$-module with multiplication $f \cdot F = (\beta^* f)F$ for $f \in \mathrm{C}^r(\mathsf{M})$ and $F \in \mathrm{C}^r(\mathsf{B})$. Now is a good time to mention that we will be a little bit sloppy and write either of

$$f \cdot F, \quad fF, \quad \beta^* fF$$

for the same thing, whichever seems to best illustrate what we are doing at the moment. Note that we have a short exact sequence of $\mathrm{C}^r(\mathsf{M})$-modules

$$0 \longrightarrow \mathrm{C}^r(\mathsf{M}) \xrightarrow{\beta^*} \mathrm{C}^r(\mathsf{B}) \longrightarrow \mathrm{C}^r(\mathsf{B})/\beta^*\mathrm{C}^r(\mathsf{M}) \longrightarrow 0 \tag{2.6}$$

Let us now introduce two particular classes of functions.

2.6 Definition: (Fibre-linear and fibre-affine functions) Let $r \in \{\infty, \omega, \mathrm{hol}\}$, let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a vector bundle of class C^r, and let $\beta\colon \mathsf{B} \to \mathsf{M}$ be a C^r-affine bundle modelled on E.

(i) A function $F \in \mathrm{C}^r(\mathsf{E})$ is ***fibre-linear*** if, for each $x \in \mathsf{M}$, $F|\mathsf{E}_x$ is a linear function.

(ii) A function $F \in \mathrm{C}^r(\mathsf{B})$ is ***fibre-affine*** if, for each $x \in \mathsf{M}$, $F|\mathsf{B}_x$ is an affine function.

We denote by $\mathrm{Lin}^r(\mathsf{E})$ (resp. $\mathrm{Aff}^r(\mathsf{B})$) the set of C^r-fibre-linear functions on E (resp. C^r-fibre-affine functions on B). •

Let us give some elementary properties of the sets of fibre-linear and fibre-affine functions.

2.7 Lemma: (Properties of sets of fibre-linear and fibre-affine functions) *Let $r \in \{\infty, \omega, \mathrm{hol}\}$, let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a vector bundle of class C^r, and let $\beta\colon \mathsf{B} \to \mathsf{M}$ be an affine bundle of class C^r modelled on E. Then the following statements hold:*

(i) $\mathrm{Lin}^r(\mathsf{E})$ and $\mathrm{Aff}^r(\mathsf{B})$ are submodules of the $\mathrm{C}^r(\mathsf{M})$-modules $\mathrm{C}^r(\mathsf{E})$ and $\mathrm{C}^r(\mathsf{B})$, respectively;

(ii) for $F \in \mathrm{Lin}^r(\mathsf{E})$, there exists $\lambda_F \in \Gamma^r(\mathsf{E}^*)$ such that

$$F(e) = \langle \lambda_F \circ \pi(e); e\rangle, \qquad e \in \mathsf{E},$$

and, moreover, the map $F \mapsto \lambda_F$ is an isomorphism of $\mathrm{C}^r(\mathsf{M})$-modules;

(iii) for $F \in \mathrm{Aff}^r(\mathsf{B})$, there exists $\alpha_F \in \Gamma^r(\mathsf{B}^{*,\mathrm{aff}})$ such that

$$F(b) = \langle \alpha_F \circ \beta(b); b\rangle, \qquad b \in \mathsf{B},$$

and, moreover, the map $F \mapsto \alpha_F$ is an isomorphism of $\mathrm{C}^r(\mathsf{M})$-modules;

(iv) the short exact sequence (2.6) induces a short exact sequence

$$0 \longrightarrow \mathrm{C}^r(\mathsf{M}) \xrightarrow{\beta^*} \mathrm{Aff}^r(\mathsf{B}) \longrightarrow \mathrm{Lin}^r(\mathsf{E}) \longrightarrow 0 \tag{2.7}$$

of $\mathrm{C}^r(\mathsf{M})$-modules;

(v) there is a splitting $\tau \in \mathrm{Hom}_{\mathrm{C}^r(\mathsf{M})}(\mathrm{Lin}^r(\mathsf{E}); \mathrm{Aff}^r(\mathsf{B}))$ *of the preceding short exact sequence.*

Proof: (i) Let $F \in \mathrm{Aff}^r(\mathsf{B})$ and $f \in \mathrm{C}^r(\mathsf{M})$. Then

$$f \cdot F(e) = (f \circ \beta(e))F(e),$$

and so $f \cdot F$ is fibre-affine since a scalar multiple of an affine function is an affine function. Also, since the pointwise sum of affine functions is an affine function, we conclude that $\mathrm{Aff}^r(\mathsf{B})$ is indeed a submodule of $\mathrm{C}^r(\mathsf{E})$. Of course, the same sort of reasoning applies to fibre-linear functions.

(ii) This merely follows by definition of the dual bundle E^*.

(iii) This merely follows by definition of the affine dual bundle $\mathsf{B}^{*,\mathrm{aff}}$.

(iv) First note that $\beta^*(\mathrm{C}^r(\mathsf{M})) \subseteq \mathrm{Aff}^r(\mathsf{B})$. Indeed, elements of $\beta^*(\mathrm{C}^r(\mathsf{M}))$ are constant on fibres of E. Thus they are affine with zero linear part. Now the assertion holds since an element of $\mathrm{Aff}^r(\mathsf{B})/\beta^*\mathrm{C}^r(\mathsf{M})$ consists of fibre-affine functions that differ by a function that is constant on fibres. Affine functions differing by a constant have the same linear part, and so we conclude that elements of $\mathrm{Aff}^r(\mathsf{B})/\beta^*\mathrm{C}^r(\mathsf{M})$ are naturally identified with functions that are linear on fibres. That is,

$$\mathrm{Aff}^r(\mathsf{B})/\beta^*\mathrm{C}^r(\mathsf{M}) \simeq \mathrm{Lin}^r(\mathsf{E}),$$

as claimed.

(v) We note that, by Corollary 2.4, the bundles B and E are isomorphic as affine bundles. Thus, if we can prove this part of the lemma for the affine bundle E, it will follow for the affine bundle B. The result is clear for the affine bundle E, however, since, if $F \in \mathrm{Aff}^r(\mathsf{E})$, then $F(e) = \langle \lambda \circ \pi(e); e\rangle + f \circ \pi(e)$ for some $\lambda \in \Gamma^r(\mathsf{E}^*)$ and $f \in \mathrm{C}^r(\mathsf{M})$. Thus the splitting is obtained by either injecting $\mathrm{Lin}^r(\mathsf{E})$ into $\mathrm{Aff}^r(\mathsf{E})$ or projecting from $\mathrm{Aff}^r(\mathsf{E})$ to $\mathrm{C}^r(\mathsf{M})$. ■

The lemma ensures that

$$\mathrm{Aff}^r(\mathsf{B}) \simeq \Gamma^r(\mathsf{E}^*) \oplus \mathrm{C}^r(\mathsf{M}), \tag{2.8}$$

the direct sum being of $\mathrm{C}^r(\mathsf{M})$-modules, although this decomposition is not canonical, except in the case that B is a vector bundle. Moreover, an isomorphism (2.8) is determined by a choice of section of $\beta\colon \mathsf{B} \to \mathsf{M}$.

We close this section by considering functions induced on vector and jet bundles.

2.8 Definition: (Lifts and evaluations of one-forms and functions) Let $r \in \{\infty, \omega, \mathrm{hol}\}$, as required. Let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a C^r-vector bundle and let $\beta\colon \mathsf{B} \to \mathsf{M}$ be a C^r-affine bundle modelled on E.

(i) For $\alpha \in \Gamma^r(\mathsf{B}^{*,\mathrm{aff}})$, the ***vertical evaluation*** of α is $\alpha^{\mathrm{e}} \in \mathrm{Aff}^r(\mathsf{B})$ defined by $\alpha^{\mathrm{e}}(b_x) = \langle \alpha(x); b_x\rangle$.

(ii) For $\lambda \in \Gamma^r(\mathsf{E}^*)$, the ***vertical evaluation*** of λ is $\lambda^{\mathrm{e}} \in \mathrm{Lin}^r(\mathsf{E})$ defined by $\lambda^{\mathrm{e}}(e_x) = \langle \lambda(x); e_x\rangle$.

(iii) For $f \in \mathrm{C}^r(\mathsf{M})$, the ***horizontal lift*** of f is the function $f^{\mathrm{h}} \in \mathrm{C}^r(\mathsf{E})$ defined by $f^{\mathrm{h}} = \pi^* f$. •

2.4. Differential operators. In this section we first work with a general notion of a differential operator, one that applies to mappings between manifolds. We then specialise to affine and linear differential operators.

2.4.1. Differential operators on fibred manifolds. To motivate our constructions with differential operators, it is convenient to work in the more-or-less standard setting of fibred manifolds. In this setting, we have the following definition.

2.9 Definition: (Differential operator) Let $r \in \{\infty, \omega, \mathrm{hol}\}$, and let $\rho\colon \mathsf{X} \to \mathsf{M}$ and $\theta\colon \mathsf{Y} \to \mathsf{M}$ be C^r-fibred manifolds. Let $\beta\colon \mathsf{B} \to \mathsf{M}$ be a C^r-affine bundle modelled on a C^r-vector bundle $\pi\colon \mathsf{E} \to \mathsf{M}$.

(i) A ***C^r-differential operator of order m*** is a C^r-morphism of fibred manifolds $P\colon \mathsf{J}^m\mathsf{X} \to \mathsf{Y}$.

(ii) A C^r-differential operator of order m, $P\colon \mathsf{J}^m\mathsf{X} \to \mathsf{B}$ is ***fibre-affine*** if, for each $p \in \mathsf{X}$, $P|(\rho_0^m)^{-1}(p)$ is an affine mapping with values in $\mathsf{B}_{\rho(p)}$.

By $\mathrm{DO}_m^r(\mathsf{X};\mathsf{Y})$ and $\mathrm{FADO}_m^r(\mathsf{X};\mathsf{B})$ we denote the spaces of C^r-differential operators and fibre-affine differential operators of order m, respectively. •

In terms of usual notions of differential operators, suppose that $P \in \mathrm{DO}_m^r(\mathsf{X};\mathsf{Y})$ and that $\sigma \in \Gamma^r(\mathsf{X})$. Then we define $\widehat{P}(\sigma) \in \Gamma^r(\mathsf{Y})$ by asking that $\widehat{P}(\sigma)(x) = P(j_m\sigma(x))$.

We shall have a particular interest in the case of differential operators with values in the trivial line bundle $\mathbb{F}_{\mathsf{M}}$, which we regard as an affine bundle when we wish to think of fibre-affine differential operators. In this case, we abbreviate

$$\mathrm{DO}_m^r(\mathsf{X}) = \mathrm{DO}_m^r(\mathsf{X};\mathbb{F}_{\mathsf{M}}), \quad \mathrm{FADO}_m^r(\mathsf{X}) = \mathrm{FADO}_m^r(\mathsf{X};\mathbb{F}_{\mathsf{M}}).$$

We note that, if $P \in \mathrm{FADO}_m^r(\mathsf{X})$, then

$$P \circ j_m\sigma(x) = (x, P_0 \circ j_m\sigma(x))$$

for a fibre-affine function P_0 on $\rho_0^m\colon \mathsf{J}^m\mathsf{X} \to \mathsf{X}$. Thus the set of fibre-affine functions on $\mathsf{J}^m\mathsf{X}$, as a bundle over X, satisfies

$$\mathrm{FADO}_m^r(\mathsf{X}) \simeq \Gamma^r((\mathsf{J}^m\mathsf{X})^{*,\mathrm{aff}}).$$

We shall consistently use the symbol $\mathrm{FADO}_m^r(\mathsf{X})$ to denote the set of fibre-affine functions on $\rho_0^m\colon \mathsf{J}^m\mathsf{X} \to \mathsf{X}$. Since $\rho_0^m\colon \mathsf{J}^m\mathsf{X} \to \mathsf{X}$ is an affine bundle modelled on $\nu_m\colon \mathsf{J}^m\mathsf{V}\mathsf{X} \to \mathsf{X}$, we thus have the following diagram

$$0 \longrightarrow \mathrm{C}^r(\mathsf{X}) \longrightarrow \mathrm{FADO}_m^r(\mathsf{X}) \dashrightarrow \Gamma^r((\mathsf{J}^m\mathsf{V}\mathsf{X})^*) \longrightarrow 0$$

where the dashed arrow indicates the projection onto the linear part.

Let us indicate the setting in which we shall use the preceding development. We shall work with the setting where $\mathsf{X} = \mathsf{M} \times \mathsf{N}$ and where $\rho = \mathrm{pr}_1$. In this case, we denote by $\mathrm{FADO}_m^r(\mathsf{M} \times \mathsf{N})$ the C^r-fibre-affine differential operators of order m. In this case, since $\mathsf{J}^m\mathsf{V}(\mathsf{M};\mathsf{N})$ is a vector bundle over $\mathsf{J}^0(\mathsf{M};\mathsf{N})$, we can also consider the special case of fibre-affine differential operators that are indeed fibre-linear. In such a case, we denote by $\mathrm{FLDO}_m^r(\mathsf{M} \times \mathsf{N})$ the set of sections of the dual bundle of $\mathsf{J}^m(\mathsf{M};\mathsf{N})$ over $\mathsf{J}^0(\mathsf{M};\mathsf{N})$. We thus have the exact sequence

$$0 \longrightarrow \mathrm{C}^r(\mathsf{M} \times \mathsf{N}) \longrightarrow \mathrm{FADO}_m^r(\mathsf{M} \times \mathsf{N}) \longrightarrow \mathrm{FLDO}_m^r(\mathsf{M} \times \mathsf{N}) \longrightarrow 0$$

2.4.2. Affine and linear differential operators. The subject of linear differential operators in vector bundles is classical [Nicolaescu 1996, §10.1]. We shall need to generalise this to affine bundles, and we carry out the fairly straightforward generalisation here.

We begin with the definitions.

2.10 Definition: (Linear and affine differential operators) Let $r \in \{\infty, \omega, \mathrm{hol}\}$, and let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a C^r-vector bundle with $\beta\colon \mathsf{B} \to \mathsf{M}$ an affine bundle of class C^r modelled on E. Also let $\gamma\colon \mathsf{A} \to \mathsf{M}$ be a C^r-affine bundle modelled on the C^r-vector bundle $\theta\colon \mathsf{F} \to \mathsf{M}$. Let $m \in \mathbb{Z}_{\geq 0}$.

(i) A ***$\mathbf{C}^r$-linear differential operator of order m*** from E to F is a C^r-vector bundle mapping $P\colon \mathsf{J}^m\mathsf{E} \to \mathsf{F}$.

(ii) A ***$\mathbf{C}^r$-affine differential operator of order m*** from B to A is a C^r-affine bundle mapping $P\colon \mathsf{J}^m\mathsf{B} \to \mathsf{A}$.

By $\mathrm{LDO}^r_m(\mathsf{B};\mathsf{A})$ and $\mathrm{ADO}^r_m(\mathsf{B};\mathsf{A})$ we denote the spaces of C^r-linear differential operators and C^r-affine differential operators of order m, respectively. •

Let us illustrate how the sort of differential operators we define are operators that differentiate, in the usual sense. Let $P \in \mathrm{ADO}^r_m(\mathsf{B};\mathsf{A})$ and let $\sigma \in \Gamma^r(\mathsf{B})$. We define $\widehat{P}(\sigma) \in \Gamma^r(\mathsf{A})$ by asking that $\widehat{P}(\sigma)(x) = P(j_m\sigma(x))$. Similar characterisations are possible for linear differential operators, of course. We shall have a particular interest in the case when A is the trivial line bundle $\mathbb{F}_{\mathsf{M}}$, which we think of as an affine bundle or a vector bundle as we need. We shall abbreviate

$$\mathrm{ADO}^r_m(\mathsf{B}) = \mathrm{ADO}^r_m(\mathsf{B};\mathbb{F}_{\mathsf{M}}), \quad \mathrm{LDO}^r_m(\mathsf{E}) = \mathrm{LDO}^r_m(\mathsf{E};\mathbb{F}_{\mathsf{M}}).$$

Note that, if $P \in \mathrm{ADO}^r_m(\mathsf{B})$, then

$$P \circ j_m\sigma(x) = (x, P_0 \circ j_m\sigma(x))$$

for a fibre-affine function P_0 on $\beta_m\colon \mathsf{J}^m\mathsf{B} \to \mathsf{M}$. Thus the set of fibre-affine functions on $\mathsf{J}^m\mathsf{B}$ satisfies

$$\mathrm{ADO}^r_m(\mathsf{B}) \simeq \Gamma^r((\mathsf{J}^m\mathsf{B})^{*,\mathrm{aff}}).$$

We shall adhere to the convention of denoting these fibre-affine functions by $\mathrm{ADO}^r_m(\mathsf{B})$, so fixing one of the three possible pieces of notation. In a similar fashion, $\mathrm{LDO}^r_m(\mathsf{E})$ is to be thought of as the set of fibre-linear functions on $\mathsf{J}^m\mathsf{E}$, or equivalently as the set of sections of the dual bundle $\mathsf{J}^m\mathsf{E}^*$. Note that the above discussions, and Lemma 2.7, give the following exact short sequence:

$$0 \longrightarrow \mathrm{C}^r(\mathsf{M}) \longrightarrow \mathrm{ADO}^r_m(\mathsf{B}) \longrightarrow \mathrm{LDO}^r_m(\mathsf{E}) \longrightarrow 0$$

3. Gelfand duality for manifolds

In this section we overview and give a unified development of the topological embedding of a C^r-manifold into the weak dual of the space of C^r-functions for $r \in \{\infty, \omega, \mathrm{hol}\}$. We also consider the functorial aspects of this embedding, i.e., how it behaves relative to morphisms.

3.1. Algebras and ideals of functions. Let $r \in \{\infty, \omega, \mathrm{hol}\}$ and let $\mathbb{F} = \mathbb{C}$ in the case $r = \mathrm{hol}$ and let $\mathbb{F} = \mathbb{R}$ in the cases $r \in \{\infty, \omega\}$. Let M be a manifold of class C^r. Then $\mathrm{C}^r(\mathsf{M})$ has the structure of an $\mathbb{F}$-algebra, where the algebra structure is that inherited pointwise from the ring structure of $\mathbb{F}$:

$$(f+g)(x) = f(x) + g(x); \quad (af)(x) = a(f(x)); \quad (fg)(x) = f(x)g(x),$$

for $f, g \in \mathrm{C}^r(\mathsf{M})$ and $a \in \mathbb{F}$.

Motivated by the above, let us make some general comments about $\mathbb{F}$-algebras. Thus we let A be a commutative $\mathbb{F}$-algebra with unit 1_{A}. Let $\mathsf{I} \subseteq \mathsf{A}$ be an ideal, thinking of A as a mere ring. We note that A/I is an $\mathbb{F}$-algebra with the algebra operations

$$(r_1 + \mathsf{I}) + (r_2 + \mathsf{I}) = (r_1 + r_2) + \mathsf{I}, \quad (r_1 + \mathsf{I})(r_2 + \mathsf{I}) = r_1 r_2 + \mathsf{I}, \quad a \cdot (r + \mathsf{I}) = a \cdot r + \mathsf{I}.$$

Note that, given an $\mathbb{F}$-algebra A, we have a canonical injection $\nu_{\mathsf{A}} \colon \mathbb{F} \to \mathsf{A}$ given by $\nu_{\mathsf{A}}(a) = a \cdot 1_{\mathsf{A}}$. One easily verifies that ν_{A} is an homomorphism of $\mathbb{F}$-algebras. For $\mathbb{F}$-algebras we denote by $\mathrm{AHom}_{\mathbb{F}}(\mathsf{A}; \mathsf{B})$ the set of $\mathbb{F}$-algebra homomorphisms from the $\mathbb{F}$-algebra A to the $\mathbb{F}$-algebra B. We denote by $\mathrm{Aut}_{\mathbb{F}}(\mathsf{A})$ the set of $\mathbb{F}$-algebra isomorphisms of a $\mathbb{F}$-algebra A. An $\mathbb{F}$-algebra homomorphism $\phi \in \mathrm{AHom}_{\mathbb{F}}(\mathsf{A}; \mathsf{B})$ between $\mathbb{F}$-algebras is ***unital*** if $\phi(1_{\mathsf{A}}) = 1_{\mathsf{B}}$. Unital $\mathbb{F}$-valued homomorphisms have the following useful property.

3.1 Lemma: (Characterisation of kernels of unital algebra homomorphisms) *Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. For a commutative $\mathbb{F}$-algebra A with unit and an ideal $\mathsf{I} \subseteq \mathsf{A}$, the following statements are equivalent:*

(i) the map $\nu_{\mathsf{A}/\mathsf{I}} \colon \mathbb{F} \to \mathsf{A}/\mathsf{I}$ is an isomorphism;

(ii) there exists a unital homomorphism $\phi \in \mathrm{AHom}_{\mathbb{F}}(\mathsf{A}; \mathbb{F})$ of $\mathbb{F}$-algebras for which $\mathsf{I} = \ker(\phi)$.

Moreover, an ideal satisfying the two equivalent conditions is maximal.

Proof: (i) $\Longrightarrow$ (ii) Let us define $\phi \in \mathrm{AHom}_{\mathbb{F}}(\mathsf{A}; \mathbb{F})$ by $\phi(f) = \nu_{\mathsf{A}/\mathsf{I}}^{-1} \circ \pi_{\mathsf{I}}$, where $\pi_{\mathsf{I}} \colon \mathsf{A} \to \mathsf{A}/\mathsf{I}$ is the canonical projection. Note that $\nu_{\mathsf{A}/\mathsf{I}}(a) = a \cdot (1_{\mathsf{A}} + \mathsf{I})$ and so $\nu_{\mathsf{A}/\mathsf{I}}^{-1}(1_{\mathsf{A}} + \mathsf{I}) = 1$. Thus

$$\phi(1_{\mathsf{A}}) = \nu_{\mathsf{A}/\mathsf{I}}^{-1} \circ \pi_{\mathsf{I}}(1_{\mathsf{A}}) = \nu_{\mathsf{A}/\mathsf{I}}^{-1}(1_{\mathsf{A}} + \mathsf{I}) = 1,$$

and so ϕ is unital. Moreover, since $\nu_{\mathsf{A}/\mathsf{I}}$ is an isomorphism,

$$\ker(\phi) = \ker(\nu_{\mathsf{A}/\mathsf{I}}^{-1} \circ \pi_{\mathsf{I}}) = \ker(\pi_{\mathsf{I}}) = \mathsf{I}.$$

(ii) $\Longrightarrow$ (i) Suppose that $a \in \ker(\nu_{\mathsf{A}/\mathsf{I}})$. Then

$$0 + \mathsf{I} = \nu_{\mathsf{A}/\mathsf{I}}(a) = a \cdot 1_{\mathsf{A}} + \mathsf{I},$$

which implies that $a \cdot 1_{\mathsf{A}} \in \mathsf{I} = \ker(\phi)$. Thus

$$0 = \phi(a \cdot 1_{\mathsf{A}}) = a\phi(1_{\mathsf{A}}) = a.$$

Thus $\nu_{\mathsf{A}/\mathsf{I}}$ is injective. By the first isomorphism theorem [Hungerford 1980, Theorem IV.1.7] the map,

$$\mathsf{A}/\ker(\phi) \ni a + \ker(\phi) \mapsto \phi(a) \in \mathrm{image}(\phi) = \mathbb{F}$$

is an isomorphism. Thus $\mathsf{A}/\mathsf{I} = \mathsf{A}/\ker(\phi)$ is isomorphic to the one-dimensional $\mathbb{F}$-algebra $\mathbb{F}$. Moreover, $\nu_{\mathsf{A}/\mathsf{I}}$ is thus an injective mapping into a one-dimensional $\mathbb{F}$-algebra, and so is an isomorphism.

For the last assertion, let J be an ideal of A such that $\mathsf{I} \subseteq \mathsf{J}$. If $\mathsf{I} \subset \mathsf{J}$ then we must have $\mathsf{A}/\mathsf{J} \subset \mathsf{A}/\mathsf{I}$. Since $\mathsf{A}/\mathsf{I} \simeq \mathbb{F}$, this means that $\mathsf{A}/\mathsf{J} = \{0\}$, and so $\mathsf{J} = \mathsf{A}$, giving maximality of I. ■

For $x \in \mathsf{M}$, we have a unital $\mathbb{F}$-algebra homomorphism

$$\begin{aligned} \mathrm{ev}_x \colon \mathrm{C}^r(\mathsf{M}) &\to \mathbb{F} \\ f &\mapsto f(x) \end{aligned}$$

called the ***evaluation map***. The evaluation map has useful topological, as well as algebraic, structure. To describe this, first note that, as an $\mathbb{F}$-algebra homomorphism, ev_x is $\mathbb{F}$-linear. Thus $\mathrm{ev}_x \in \mathrm{C}^r(\mathsf{M})^*$. We shall also see that it is continuous, and so is an element of $\mathrm{C}^r(\mathsf{M})'$. We shall equip $\mathrm{C}^r(\mathsf{M})'$ with the weak-$*$ topology, that is the topology defined by the family of seminorms

$$\begin{aligned} p_f \colon \mathrm{C}^r(\mathsf{M})' &\to \mathbb{R}_{\geq 0} \\ \alpha &\mapsto |\alpha(f)|, \end{aligned}$$

for $f \in \mathrm{C}^r(\mathsf{M})$.

3.2. The embedding theorem for manifolds.

Our first main result is now the following.

3.2 Theorem: (Embedding of manifolds) *Let $r \in \{\infty, \omega, \mathrm{hol}\}$ and let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ as appropriate. Let M be a manifold of class C^r, Stein when $r = \mathrm{hol}$. Then the mapping*

$$\begin{aligned} \mathrm{ev}_{\mathsf{M}} \colon \mathsf{M} &\to \mathrm{C}^r(\mathsf{M})' \\ x &\mapsto \mathrm{ev}_x \end{aligned}$$

is an homeomorphism of M with the set of unital $\mathbb{F}$-algebra homomorphisms from $\mathrm{C}^r(\mathsf{M})$ to $\mathbb{F}$, where the latter has the topology induced by the weak-$*$ topology.

Proof: For the first part of the proof, we consider ev_{M} as taking values in $\mathrm{C}^r(\mathsf{M})^*$, i.e., taking values in the algebraic dual, which it obviously does.

First we show that ev_{M} is injective. Suppose that $x_1, x_2 \in \mathsf{M}$ are distinct. We claim that there exists $f \in \mathrm{C}^r(\mathsf{M})$ such that $f(x_1) = 0$ and $f(x_2) = 1$. For $r = \infty$, one proves this using bump functions. In case $r \in \{\omega, \mathrm{hol}\}$ this is still true, although a partition of unity argument no longer works. Instead, we use the fact that the hypotheses of the theorem ensure that there exists a proper C^r-embedding $\iota_{\mathsf{M}} \colon \mathsf{M} \to \mathbb{F}^N$ for sufficiently large N; Let $\Psi \colon \mathbb{F}^N \to \mathbb{F}^N$ be an affine (and so C^r) isomorphism such that $\Psi \circ \iota_{\mathsf{M}}(x_1) = \mathbf{0}$ and $\Psi \circ \iota_{\mathsf{M}}(x_2) = \boldsymbol{e}_1$, where $\boldsymbol{e}_j$, $j \in \{1, \dots, N\}$, are the standard basis vectors. For $j \in \{1, \dots, N\}$, let $\mathrm{pr}_j \colon \mathbb{F}^N \to \mathbb{F}$ be the projection onto the jth component, noting that pr_j is of class C^r. Then the function $f = \mathrm{pr}_1 \circ \Psi \circ \iota_{\mathsf{M}}$ has the desired property that $f(x_1) = 0$ and $f(x_2) = 1$. Then we have that $\mathrm{ev}_{x_1}(f) \neq \mathrm{ev}_{x_2}(f)$, i.e., $\mathrm{ev}_{x_1} \neq \mathrm{ev}_{x_2}$. Thus ev_{M} is injective.

Next we show that ev_{M} is surjective. Let $\psi \colon \mathrm{C}^r(\mathsf{M}) \to \mathbb{F}$ be a unital $\mathbb{F}$-algebra homomorphism. For $f \in \mathrm{C}^r(\mathsf{M})$, denote

$$Z(f) = \{x \in M \mid f(x) = 0\}.$$

We then have the following useful lemma.

1 Lemma: *If* $f^1, \dots, f^k \in \ker(\psi)$*, then* $\cap_{j=1}^k Z(f^j) \neq \varnothing$*.*

Proof: Suppose that $\cap_{j=1}^k Z(f^j) = \varnothing$. Then, for each $x \in \mathsf{M}$, there exists $j \in \{1, \dots, k\}$ such that $f^j(x) \neq 0$. Then, if $[g]_x \in \mathscr{C}^r_{x,\mathsf{M}}$, we can write $[g]_x = ([g]_x[1/f^j]_x)[f^j]_x$. Thus we conclude that $[f^1]_x, \dots, [f^k]_x$ generate $\mathscr{C}^r_{x,\mathsf{M}}$ for each $x \in \mathsf{M}$. Define the surjective sheaf morphism $\Psi\colon (\mathscr{C}^r_{\mathsf{M}})^k \to \mathscr{C}^r_{\mathsf{M}}$ by requiring that

$$\Psi([g^1]_x, \dots, [g^k]_x) = [g^1]_x[f^1]_x + \dots + [g^k]_x[f^k]_x.$$

Since the sheaves serving as the domain and codomain of Ψ are coherent in the real analytic and holomorphic cases (by [Grauert and Remmert 1984, Consequence A.4.2.1]), we can then use (1) the vanishing of sheaf cohomology for sheaves over $\mathscr{C}^\infty_{\mathsf{M}}$ ([Wells Jr. 2008, Proposition 2.3.11], along with [Wells Jr. 2008, Examples 2.3.4(d, e)] and [Wells Jr. 2008, Proposition 2.3.5]), in the case $r = \infty$ or (2) Cartan's Theorem B in the cases $r \in \{\omega, \mathrm{hol}\}$ to conclude that the map

$$\begin{aligned} \Psi_{\mathsf{M}}\colon \mathrm{C}^r(\mathsf{M})^k &\to \mathrm{C}^r(\mathsf{M}) \\ (g^1, \dots, g^k) &\mapsto g^1 f^1 + \dots + g^k f^k \end{aligned}$$

is surjective. In particular, there exists $g^1, \dots, g^k \in \mathrm{C}^r(\mathsf{M})$ such that

$$g^1(x)f^1(x) + \dots + g^k(x)f^k(x) = 1, \qquad x \in \mathsf{M}.$$

Since $\ker(\psi)$ is an ideal, we conclude that $1_{\mathsf{M}} \in \ker(\psi)$, contradicting the assumption that ψ is unital. ▼

We now can complete the proof of surjectivity of ev_{M}. Let $\iota_{\mathsf{M}}\colon \mathsf{M} \to \mathbb{F}^N$ be a proper C^r-embedding. Let $\chi^1, \dots, \chi^N \in \mathrm{C}^r(\mathsf{M})$ be defined by

$$\iota_{\mathsf{M}}(x) = (\chi^1(x), \dots, \chi^N(x)).$$

Define $f^j \in \mathrm{C}^r(\mathsf{M})$ by $f^j = \chi^j - \psi(\chi^j)1_{\mathsf{M}}$, $j \in \{1, \dots, N\}$. Clearly $f^j \in \ker(\psi)$. By the lemma, $\cap_{j=1}^N Z(f^j) \neq \varnothing$. If $x \in Z(f^j)$, then the definition of f^j gives $\chi^j(x) = \psi(\chi^j)$. Thus, if $x \in \cap_{j=1}^N Z(f^j)$, then

$$\iota_{\mathsf{M}}(x) = (\psi(\chi^1), \dots, \psi(\chi^N)).$$

Since ι_{M} is injective, $\cap_{j=1}^N Z(f^j)$ is a singleton, say $\cap_{j=1}^N Z(f^j) = \{x\}$.

Now let $f \in \ker(\psi)$. Then

$$Z(f) \cap \{x\} = Z(f) \cap \left(\bigcap_{j=1}^N Z(f^j) \right).$$

By the lemma, $Z(f) \cap \{x\} \neq \varnothing$, and so we must have $x \in Z(f)$. As this argument is valid for every $f \in \ker(\psi)$, we conclude that, if $f \in \ker(\psi)$, then $f(x) = 0$. In other words, $\ker(\psi) \subseteq \ker(\mathrm{ev}_x)$. Since $\ker(\psi)$ and $\ker(\mathrm{ev}_x)$ are both maximal ideals, $\ker(\psi) = \ker(\mathrm{ev}_x)$. Let $f \in \mathrm{C}^r(\mathsf{M})$ and define $g = f - f(x)1_{\mathsf{M}}$. Then $\mathrm{ev}_x(g) = 0$ and so $g \in \ker(\mathrm{ev}_x) = \ker(\psi)$. Thus

$$0 = \psi(g) = \psi(f) - f(x) \quad \Longrightarrow \quad \psi(f) = f(x),$$

i.e., $\psi = \mathrm{ev}_x$.

Note that this establishes a bijection between M and the unital $\mathbb{F}$-algebra homomorphisms from $\mathrm{C}^r(\mathsf{M})$ to $\mathbb{F}$. It remains to prove the topological assertions of the theorem.

To this end, let us first show that ev_{M} is well-defined, in that ev_x is continuous for each $x \in \mathsf{M}$. Note that the C^r-topology is finer than the C^0-topology, so it suffices to show that ev_x is continuous in the C^0-topology on $\mathrm{C}^r(\mathsf{M})$. The C^0-topology (by definition) is the locally convex topology with the seminorms

$$p^0_{\mathcal{K}}(f) = \sup\{|f(x)| \mid x \in \mathcal{K}\}, \qquad \mathcal{K} \subseteq \mathsf{M} \text{ compact.}$$

Let $\mathcal{K} \subseteq \mathsf{M}$ be compact such that $x \in \mathcal{K}$. Then

$$|\mathrm{ev}_x(f)| = |f(x)| \leq p^0_{\mathcal{K}}(f)$$

for $f \in \mathrm{C}^r(\mathsf{M})$, which gives continuity of ev_x in the C^0-topology.

Now we prove that ev_{M} is continuous. Let $x_0 \in \mathsf{M}$ and let $\mathcal{O} \subseteq \mathrm{C}^r(\mathsf{M})'$ be a neighbourhood of ev_{x_0}. Let $f^1, \dots, f^k \in \mathrm{C}^r(\mathsf{M})$ and $r_1, \dots, r_k \in \mathbb{R}_{>0}$ be such that

$$\bigcap_{j=1}^k \{\alpha \in \mathrm{C}^r(\mathsf{M})' \mid |\alpha(f^j) - f^j(x_0)| < r_j\} \subseteq \mathcal{O}.$$

Let $\mathcal{U}$ be a neighbourhood of x_0 such that $|f^j(x) - f^j(x_0)| < r_j$, $j \in \{1, \dots, k\}$, $x \in \mathcal{U}$. Then, if $x \in \mathcal{U}$, $\mathrm{ev}_{\mathsf{M}}(x) \in \mathcal{O}$ and this gives continuity of ev_{M}.

Finally, we show that ev_{M} is an homeomorphism onto its image. As above, we let $\iota_{\mathsf{M}} \colon \mathsf{M} \to \mathbb{F}^N$ be a proper C^r-embedding of M in N-dimensional Euclidean space and denote by $\chi^1, \dots, \chi^N$ the coordinate functions restricted to M, which are of class C^r. Let $x_0 \in \mathsf{M}$ and let $\mathcal{U}$ be a neighbourhood of x_0 in the standard topology of M. Let $r \in \mathbb{R}_{>0}$ be sufficiently small that

$$\left\{x \in \mathsf{M} \;\middle|\; \sum_{j=1}^N |\chi^j(x) - \chi^j(x_0)| < r\right\} \subseteq \mathcal{U}.$$

Note that

$$\begin{aligned}
\iota_{\mathsf{M}}(\mathsf{M}) \cap & \left(\bigcap_{j=1}^N \{\alpha \in \mathrm{C}^r(\mathsf{M})' \mid |\alpha(\chi^j) - \chi^j(x_0)| < r/N\}\right) \\
&= \{\iota_{\mathsf{M}}(x) \in \iota_{\mathsf{M}}(\mathsf{M}) \mid |\chi^j(x) - \chi^j(x_0)| < r/N,\ j \in \{1, \dots, N\}\} \\
&\subseteq \left\{\iota_{\mathsf{M}}(x) \in \iota_{\mathsf{M}}(\mathsf{M}) \;\middle|\; \sum_{j=1}^N |\chi^j(x) - \chi^j(x_0)| < r\right\} \subseteq \iota_{\mathsf{M}}(\mathcal{U}).
\end{aligned}$$

This shows that $\mathcal{U}$ is open in the topology induced by ev_{M}. That is to say, the topology on M induced by ev_{M} is finer than the standard topology, which shows that ev_{M} is open onto its image. ■

We note that the 1–1 correspondence of M with the unital $\mathbb{F}$-algebra homomorphisms does not require any topology. That is to say, we do not require that $\mathrm{ev}_{\mathsf{M}}(x) = \mathrm{ev}_x$ be continuous, and the fact that ev_{M} is an homeomorphism onto its image is additional to the 1–1 correspondence.

We claim that the assignment to an object M in the category of C^r-manifolds of the object $\mathrm{C}^r(\mathsf{M})$ in the category of $\mathbb{F}$-algebras is injective. Indeed, suppose that $\mathrm{C}^r(\mathsf{M}_1) = \mathrm{C}^r(\mathsf{M}_2)$, equality being as $\mathbb{F}$-algebras. Then certainly these algebras have the same collection of unital $\mathbb{F}$-algebra homomorphisms. But then this implies that $\mathsf{M}_1 = \mathsf{M}_2$ by the theorem.

3.3. Mappings of manifolds and homomorphisms of algebras. An important facet of Gelfand duality is that it assigns not only algebraic objects to manifolds, but homomorphisms of these algebraic objects to mappings of manifolds.

The result we state along these lines is the following.

3.3 Theorem: (Mappings as homomorphisms, diffeomorphisms as automorphisms) *Let $r \in \{\infty, \omega, \mathrm{hol}\}$ and let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, as appropriate. Let M and N be C^r-manifolds, Stein if $r = \mathrm{hol}$. Then the following statements hold:*

(i) if $\Phi \in \mathrm{C}^r(\mathsf{M}; \mathsf{N})$, then $\Phi^* \colon \mathrm{C}^r(\mathsf{N}) \to \mathrm{C}^r(\mathsf{M})$ is a continuous $\mathbb{F}$-algebra homomorphism;

(ii) the mapping $\Phi \mapsto \Phi^*$ is a bijection from $\mathrm{C}^r(\mathsf{M}; \mathsf{N})$ to $\mathrm{AHom}_{\mathbb{F}}(\mathrm{C}^r(\mathsf{N}); \mathrm{C}^r(\mathsf{M}))$;

(iii) if $\Phi \in \mathrm{Diff}^r(\mathsf{M})$, then Φ^* is a continuous automorphism of $\mathrm{C}^r(\mathsf{M})$;

(iv) the mapping $\Phi \mapsto \Phi^*$ is a bijection from $\mathrm{Diff}^r(\mathsf{M})$ to $\mathrm{Aut}_{\mathbb{F}}(\mathrm{C}^r(\mathsf{M}))$.

Proof: (i) Since Φ^* is clearly $\mathbb{F}$-linear and since

$$\Phi^*(fg)(x) = (fg) \circ \Phi(x) = (f \circ \Phi(x))(g \circ \Phi(x)) = \Phi^* f(x) \Phi^* g(x) = (\Phi^* f \Phi^* g)(x),$$

for $f, g \in \mathrm{C}^r(\mathsf{N})$ and $x \in \mathsf{M}$, we conclude that Φ^* is an $\mathbb{F}$-algebra homomorphism. Continuity of Φ^* is proved in [Lewis 2023, Theorem 5.26] in the real analytic case. The smooth and holomorphic cases follow along similar lines, but are easier, cf. [Lewis 2023, Remark 5.27].

(ii) First we show that $\Phi \mapsto \Phi^*$ is an injective mapping. Suppose that $\Phi_1^* = \Phi_2^*$. Let $\iota_{\mathsf{N}} \colon \mathsf{N} \to \mathbb{F}^N$ be a C^r-embedding with coordinate functions $\chi^1, \dots, \chi^N \in \mathrm{C}^r(\mathsf{N})$. Then we have

$$\begin{aligned}
& \Phi_1^* \chi^j(x) = \Phi_2^* \chi^j(x), \qquad j \in \{1, \dots, N\},\ x \in \mathsf{M}, \\
\Longrightarrow\quad & \chi^j \circ \Phi_1(x) = \chi^j \circ \Phi_2(x), \qquad j \in \{1, \dots, N\},\ x \in \mathsf{M}, \\
\Longrightarrow\quad & \iota_{\mathsf{M}} \circ \Phi_1(x) = \iota_{\mathsf{M}} \circ \Phi_2(x), \qquad x \in \mathsf{M}, \\
\Longrightarrow\quad & \Phi_1(x) = \Phi_2(x), \qquad x \in \mathsf{M},
\end{aligned}$$

as desired.

To show that $\Phi \mapsto \Phi^*$ is surjective, we shall construct a right inverse of this mapping. Thus let $\gamma \in \mathrm{AHom}_{\mathbb{F}}(\mathrm{C}^r(\mathsf{N}); \mathrm{C}^r(\mathsf{M}))$ and define $\Phi_\gamma \colon \mathsf{M} \to \mathsf{N}$ as follows. Let $x \in \mathsf{M}$ and $f, g \in \mathrm{C}^r(\mathsf{N})$, and note that

$$\mathrm{ev}_x \circ \gamma(1_{\mathsf{N}}) = \mathrm{ev}_x(1_{\mathsf{M}}) = 1$$

and

$$\mathrm{ev}_x \circ \gamma(fg) = \mathrm{ev}_x(\gamma(f)\gamma(g)) = \gamma(f)(x)\gamma(g)(x) = (\mathrm{ev}_x \circ \gamma(f))(\mathrm{ev}_x \circ \gamma(g)),$$

from which we conclude that $\mathrm{ev}_x \circ \gamma\colon \mathrm{C}^r(\mathsf{N}) \to \mathbb{F}$ is a unital $\mathbb{F}$-algebra homomorphism. Thus, by Theorem 3.2, there exists $y_x \in \mathsf{N}$ such that $\mathrm{ev}_{y_x} = \mathrm{ev}_x \circ \gamma$. We define $\Phi_\gamma(x) = y_x$.

We claim that $\Phi_\gamma \in \mathrm{C}^r(\mathsf{M};\mathsf{N})$. Indeed, let $f \in \mathrm{C}^r(\mathsf{N})$ and note that

$$\Phi_\gamma^* f(x) = f(y_x) = \mathrm{ev}_{y_x}(f) = \mathrm{ev}_x \circ \gamma(f) = \gamma(f)(x),$$

i.e., $\Phi_\gamma^* f = \gamma(f) \in \mathrm{C}^r(\mathsf{M})$. We claim that this implies that Φ_γ is of class C^r. Indeed, let $x_0 \in \mathsf{M}$ and denote $y_0 = \Phi_\gamma(x_0)$. Let $(\mathcal{U}, \phi)$ be a coordinate chart for M about x_0 whose coordinate functions we denote by $\chi^1, \dots, \chi^n$. Let $(\mathcal{V}, \psi)$ be a coordinate chart for N about y_0 whose coordinate functions $\eta^1, \dots, \eta^k$ are restrictions of globally defined functions of class C^r. This is possible by Lemma 1.1. The mapping

$$\begin{aligned}\boldsymbol{\eta}\colon \mathsf{N} &\to \mathbb{F}^k\\ y &\mapsto (\eta^1(y), \dots, \eta^k(y))\end{aligned}$$

is a diffeomorphism from a neighbourhood $\mathcal{V}' \subseteq \mathcal{V}$ of y_0 to a neighbourhood $\mathcal{W}$ of $\boldsymbol{\eta}(y_0) \in \mathbb{F}^k$. Since $\Phi_\gamma^* \boldsymbol{\eta}$ is continuous by hypothesis, there is a neighbourhood $\mathcal{U}'$ of x such that $\Phi_\gamma^* \boldsymbol{\eta}(\mathcal{U}') \subseteq \mathcal{W}$. Thus $\Phi_\gamma(\mathcal{U}') \subseteq \mathcal{V}'$. Therefore, we can assume without loss of generality that $\Phi_\gamma(\mathcal{U}) \subseteq \mathcal{V}$. We denote

$$\begin{aligned}\boldsymbol{\chi}\colon \mathcal{U} &\to \mathbb{F}^n\\ x &\mapsto (\chi^1(x), \dots, \chi^n(x)).\end{aligned}$$

Note that the local representative of Φ_γ in the charts $(\mathcal{U}, \phi)$ and $(\mathcal{V}, \psi)$ is

$$\begin{aligned}\boldsymbol{\Phi}_\gamma\colon \phi(\mathcal{U}) &\to \psi(\mathcal{V})\\ \boldsymbol{x} &\mapsto \boldsymbol{\eta} \circ \Phi_\gamma \circ \boldsymbol{\chi}^{-1}.\end{aligned}$$

Since $\boldsymbol{\eta} \circ \Phi_\gamma$ is of class C^r (by hypothesis) and $\boldsymbol{\chi}^{-1}$ is of class C^r, the local representative of Φ_γ is of class C^r, and this shows that Φ_γ is of class C^r.

Moreover, the equality $\Phi_\gamma^* = \gamma$ proved above is exactly the statement that the mapping $\gamma \mapsto \Phi_\gamma$ is a right inverse of the mapping $\Phi \mapsto \Phi^*$, and this completes the proof of this part of the theorem.

(iii) This follows from part (i) since the inverse of Φ^* is $\Phi_* = (\Phi^{-1})^*$ in the case that Φ is a diffeomorphism.

(iv) This follows from part (ii), just as part (iii) follows from (i). ■

3.4 Corollary: (Gelfand duality for manifolds) *Let $r \in \{\infty, \omega, \mathrm{hol}\}$ and let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, as appropriate. The category of C^r-manifolds is a full subcategory of the opposite category of the category of $\mathbb{F}$-algebras via the functor given by $\mathsf{M} \mapsto \mathrm{C}^r(\mathsf{M})$ on objects and by $\Phi \mapsto \Phi^*$ on morphisms.*

4. Gelfand duality for vector and affine bundles

Of course, Theorem 3.2 applies just as well to the total space of a vector or affine bundle, at least when one realises that the total space of a affine bundle over a Stein manifold is a Stein manifold (which we shall do in Proposition 6.1). Since we already know that there is a 1–1 correspondence between C^r-manifolds and their $\mathbb{F}$-algebras of C^r-functions, when specialising to manifolds with additional structure (such as the structure of a vector or affine bundle), we imagine that we should restrict ourselves to consideration of subsets of the algebras of C^r-functions. Specifically, we work with a particular class of $\mathbb{F}$-linear mappings $\phi\colon \mathrm{Aff}^r(\mathsf{B}) \rightarrow \mathbb{F}$. We require these to be compatible with (1) the $\mathbb{F}$-algebra structure on $\mathrm{C}^r(\mathsf{M})$ and (2) the affine space structure of the fibres of B. We must also devise suitable morphisms for this algebraic structure that capture the essential algebraic features of mappings between affine bundles.

4.1. Semialgebras. Let us first contrive a general setting, i.e., a category, for the class of functions in which we are interested.

4.1 Definition: (Semialgebra) Let R be a commutative ring with unit. An ***R-semialgebra*** is a triple $(\mathsf{M}, \mathsf{A}, \kappa)$ for which

(i) A is an R-algebra,

(ii) M is an A-module, and

(iii) $\kappa \in \mathrm{Hom}_{\mathsf{A}}(\mathsf{A}; \mathsf{M})$.

We call A the ***nonlinear part*** of the semialgebra and the A-module $\mathsf{M}/\operatorname{image}(\kappa)$ the ***linear part*** of the semialgebra, denoted by $L(\mathsf{M}, \mathsf{A}, \kappa)$. •

4.2 Definition: (Morphism of semialgebras) A ***morphism*** of R-semialgebras $(\mathsf{M}_1, \mathsf{A}_1, \kappa_1)$ and $(\mathsf{M}_2, \mathsf{A}_2, \kappa_2)$ is a pair (ϕ, ϕ_0) such that $\phi \in \mathrm{Hom}_{\mathsf{R}}(\mathsf{M}_1; \mathsf{M}_2)$ and $\phi_0 \in \mathrm{AHom}_{\mathsf{R}}(\mathsf{A}_1; \mathsf{A}_2)$ are such that the diagram of R-modules

$$\begin{tikzcd} \mathsf{M}_1 \arrow[r, "\phi"] & \mathsf{M}_2 \\ \mathsf{A}_1 \arrow[u, "\kappa_1"] \arrow[r, "\phi_0"'] & \mathsf{A}_2 \arrow[u, "\kappa_2"'] \end{tikzcd}$$

commutes and such that

$$\phi(a_1 x_1) = \phi_0(a_1)\phi(x_1), \qquad a_1 \in \mathsf{A}_1,\ x_1 \in \mathsf{M}_1. \tag{4.1}$$

We denote by

$$\mathrm{Hom}_{\mathsf{R}}((\mathsf{M}_1, \mathsf{A}_1, \kappa_1); (\mathsf{M}_2, \mathsf{A}_2, \kappa_2))$$

the set of R-semialgebra morphisms. •

We leave to the reader the simple exercise of checking that, if $(\mathsf{M}_1, \mathsf{A}_1, \kappa_1)$, $(\mathsf{M}_2, \mathsf{A}_2, \kappa_2)$, and $(\mathsf{M}_3, \mathsf{A}_3, \kappa_3)$ are R-semialgebras, and if

$$(\phi, \phi_0) \in \mathrm{Hom}_{\mathsf{R}}((\mathsf{M}_1, \mathsf{A}_1, \kappa_1); (\mathsf{M}_2, \mathsf{A}_2, \kappa_2)), \quad (\psi, \psi_0) \in \mathrm{Hom}_{\mathsf{R}}((\mathsf{M}_2, \mathsf{A}_2, \kappa_2); (\mathsf{M}_3, \mathsf{A}_3, \kappa_3)),$$

then

$$(\psi \circ \phi, \psi_0 \circ \phi_0) \in \mathrm{Hom}_{\mathsf{R}}((\mathsf{M}_1, \mathsf{A}_1, \kappa_1); (\mathsf{M}_3, \mathsf{A}_3, \kappa_3)).$$

Obviously $(\mathrm{id}_{\mathsf{M}}, \mathrm{id}_{\mathsf{A}})$ is an R-semialgebra morphism from $(\mathsf{M}, \mathsf{A}, \kappa)$ to itself, and it has the usual attributes of an identity morphism. In short, we have a category of R-semialgebras. We denote by $\mathrm{Aut}_{\mathsf{R}}(\mathsf{M}, \mathsf{A}, \kappa)$ the set of R-semialgebra isomorphisms of $(\mathsf{M}, \mathsf{A}, \kappa)$.

Morphisms of semialgebras induce morphisms on their linear parts.

4.3 Lemma: (Morphism induced on linear part of semialgebras) *Let* R *be a ring, and let* $(\mathsf{M}_1, \mathsf{A}_1, \kappa_1)$ *and* $(\mathsf{M}_2, \mathsf{A}_2, \kappa_2)$ *be* R*-semialgebras. If*

$$(\phi, \phi_0) \in \mathrm{Hom}_{\mathsf{R}}((\mathsf{M}_1, \mathsf{A}_1, \kappa_1); (\mathsf{M}_2, \mathsf{A}_2, \kappa_2)),$$

then there is an induced R*-module homomorphism*

$$L(\phi, \phi_0) \in \mathrm{Hom}_{\mathsf{R}}(L(\mathsf{M}_1, \mathsf{A}_1, \kappa_1); L(\mathsf{M}_2, \mathsf{A}_2, \kappa_2))$$

for which the diagram

$$\begin{tikzcd} \mathsf{A}_1 \arrow[r, "\kappa_1"] \arrow[d, "\phi_0"'] & \mathsf{M}_1 \arrow[r] \arrow[d, "\phi"'] & L(\mathsf{M}_1, \mathsf{A}_1, \kappa_1) \arrow[d, "L(\phi,\phi_0)"] \\ \mathsf{A}_2 \arrow[r, "\kappa_2"] & \mathsf{M}_2 \arrow[r] & L(\mathsf{M}_2, \mathsf{A}_2, \kappa_2) \end{tikzcd}$$

of R*-modules commutes and for which*

$$L(\phi, \phi_0)(a_1(x_1 + \mathrm{image}(\kappa_1))) = \phi_0(a_1)L(\phi, \phi_0)(x_1 + \mathrm{image}(\kappa_1))$$

for $a_1 \in \mathsf{A}_1$*,* $x_1 + \mathrm{image}(\kappa_1) \in L(\mathsf{M}_1, \mathsf{A}_1, \kappa_1)$*.*

Proof: Define

$$L(\phi, \phi_0)(x_1 + \mathrm{image}(\kappa_1)) = \phi(x_1) + \mathrm{image}(\kappa_2).$$

Let us show that $L(\phi, \phi_0)$ is well-defined. This is standard. Indeed, suppose that $x_1' - x_1 = \kappa_1(a_1)$ so that $x_1 + \mathrm{image}(\kappa_1) = x_1' + \mathrm{image}(\kappa_1)$. Then

$$\begin{aligned} \phi(x_1') = \phi(x_1 + \kappa_1(a_1)) &= \phi(x_1) + \phi \circ \kappa_1(a_1) \\ &= \phi(x_1) + \kappa_2 \circ \phi_0(a_1), \end{aligned}$$

showing that $L(\phi, \phi_0)$ is indeed well-defined. We also have

$$\begin{aligned} L(\phi, \phi_0)(a_1(x_1 + \mathrm{image}(\kappa_1))) &= L(\phi, \phi_0)(a_1x_1 + \mathrm{image}(\kappa_1)) \\ &= \phi(a_1x_1) + \mathrm{image}(\kappa_2) \\ &= \phi_0(a_1)\phi(x_1) + \mathrm{image}(\kappa_2) \\ &= \phi_0(a_1)(\phi(x_1) + \mathrm{image}(\kappa_2)) \\ &= \phi_0(a_1)L(\phi, \phi_0)(x_1 + \mathrm{image}(\kappa_1)), \end{aligned}$$

as claimed. ■

Let us consider the examples of semialgebras.

4.4 Examples: (Semialgebras)

1. If R is a ring and A is an R-algebra, we can identify A it in a natural way with the R-semialgebra $(\mathsf{A}, \mathsf{A}, \mathrm{id}_{\mathsf{A}})$. The linear part of such semialgebras is the zero module. One can see that morphisms of semialgebras of this form are essentially homomorphisms of algebras since they arise from diagrams like the following:

$$\begin{tikzcd} \mathsf{A}_1 \arrow[r, "\phi_0"] & \mathsf{A}_2 \\ \mathsf{A}_1 \arrow[r, "\phi_0"'] \arrow[u, "\mathrm{id}_{\mathsf{A}_1}"] & \mathsf{A}_2 \arrow[u, "\mathrm{id}_{\mathsf{A}_2}"'] \end{tikzcd}$$

The intertwining condition (4.1) then reads

$$\phi_0(a_1 b_1) = \phi_0(a_1)\phi_0(b_1), \qquad a_1, b_1 \in \mathsf{A}_1,$$

i.e., it expresses that the semialgebra morphisms are algebra morphisms.

2. Let $r \in \{\infty, \omega, \mathrm{hol}\}$ and let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, as appropriate. If $\beta\colon \mathsf{B} \to \mathsf{M}$ is a C^r-affine bundle modelled on the C^r-vector bundle $\pi\colon \mathsf{E} \to \mathsf{M}$, then $(\mathrm{Aff}^r(\mathsf{B}), \mathrm{C}^r(\mathsf{M}), \beta^*)$ is an $\mathbb{F}$-semialgebra. The nonlinear part of the semialgebra is the algebra $\mathrm{C}^r(\mathsf{M})$ of functions and the linear part is the $\mathrm{C}^r(\mathsf{M})$-module $\mathrm{Aff}^r(\mathsf{B})/\mathrm{C}^r(\mathsf{M}) \simeq \mathrm{Lin}^r(\mathsf{E})$.

   If $\beta_1\colon \mathsf{B}_1 \to \mathsf{M}_1$ and $\beta_2\colon \mathsf{B}_2 \to \mathsf{M}_2$ are C^r-affine bundles and if $\Phi\colon \mathsf{B}_1 \to \mathsf{B}_2$ is a C^r-affine bundle map over $\Phi_0\colon \mathsf{M}_1 \to \mathsf{M}_2$, then we claim that (Φ^*, Φ_0^*) is an $\mathbb{F}$-semialgebra morphism from $(\mathrm{Aff}^r(\mathsf{B}_2), \mathrm{C}^r(\mathsf{M}_2), \beta_2^*)$ to $(\mathrm{Aff}^r(\mathsf{B}_1), \mathrm{C}^r(\mathsf{M}_1), \beta_1^*)$. The commutativity of the diagram

$$\begin{tikzcd} \mathsf{B}_1 \arrow[r, "\Phi"] \arrow[d, "\beta_1"'] & \mathsf{B}_2 \arrow[d, "\beta_2"] \\ \mathsf{M}_1 \arrow[r, "\Phi_0"'] & \mathsf{M}_2 \end{tikzcd}$$

   gives the commuting of the diagram

$$\begin{tikzcd} \mathrm{Aff}^r(\mathsf{B}_2) \arrow[r, "\Phi^*"] & \mathrm{Aff}^r(\mathsf{B}_1) \\ \mathrm{C}^r(\mathsf{M}_1) \arrow[u, "\beta_1^*"] \arrow[r, "\Phi_0^*"'] & \mathrm{C}^r(\mathsf{M}_2) \arrow[u, "\beta_2^*"'] \end{tikzcd}$$

   Let us show that $\Phi^* F$ is fibre-affine if $F \in \mathrm{Aff}^r(\mathsf{B}_2)$. Let $x_1 \in \mathsf{M}_1$ and note that

$$\Phi^* F|\mathsf{B}_{1,x_1} = (F|\mathsf{B}_{2,\Phi_0(x_1)}) \circ \Phi|\mathsf{B}_{1,x_1}.$$

   Since F is fibre-affine and Φ is affine, we see that, indeed, $\Phi^* F$ is fibre-affine. Finally, if $f_2 \in \mathrm{C}^r(\mathsf{M}_2)$ and $F_2 \in \mathrm{Aff}^r(\mathsf{B}_2)$, then

$$\Phi^*(f_2 F_2)(b_1) = (f_2 F_2)(\Phi(b_1)) = f_2(\Phi_0(\beta(b_1))) F_2 \circ \Phi(b_1),$$

   which gives the intertwining condition (4.1). •

The last example makes it evident what we will be going for here; we will arrive at the category of affine bundles being a full subcategory of the category of $\mathbb{F}$-semialgebras. Analogously to our presentation in Section 3, for $b \in \mathsf{B}$ we have the $\mathbb{F}$-linear mapping

$$\begin{aligned} \mathrm{Ev}_b\colon \mathrm{Aff}^r(\mathsf{B}) &\to \mathbb{F} \\ F &\mapsto F(b). \end{aligned}$$

Clearly Ev_b is unital, i.e., $\mathrm{Ev}_b(1_{\mathsf{B}}) = 1$. We claim that

$$(\mathrm{Ev}_b, \mathrm{ev}_{\beta(b)}) \in \mathrm{Hom}_{\mathbb{F}}((\mathrm{Aff}^r(\mathsf{B}), \mathrm{C}^r(\mathsf{M}), \beta^*); (\mathbb{F}, \mathbb{F}; \mathrm{id}_{\mathbb{F}})).$$

Indeed, the diagram

$$\begin{tikzcd} \mathrm{Aff}^r(\mathsf{M}) \arrow[r, "\mathrm{Ev}_b"] & \mathbb{F} \arrow[d, equal] \\ \mathrm{C}^r(\mathsf{M}) \arrow[u, "\beta^*"] \arrow[r, "\mathrm{ev}_{\beta(b)}"'] & \mathbb{F} \end{tikzcd}$$

clearly commutes. Also,

$$\mathrm{Ev}_b(fF) = f(\beta(b))F(b) = \mathrm{ev}_{\beta(b)}(f)\mathrm{Ev}_b(F), \qquad f \in \mathrm{C}^r(\mathsf{M}),\ F \in \mathrm{Aff}^r(\mathsf{B}),$$

and so the intertwining condition (4.1) also holds. We shall often grammatically identify $(\mathrm{Ev}_b, \mathrm{ev}_{\beta(b)})$ with Ev_b for convenience. Thus, for example, we may say that $(\mathrm{Ev}_b, \mathrm{ev}_{\beta(b)})$ is an element of $\mathrm{Aff}^r(\mathsf{B})^*$, when we mean to apply this assertion only to Ev_b.

We note that, as a particular facet of its definition, we have $\mathrm{Ev}_b \in \mathrm{Aff}^r(\mathsf{B})^*$, i.e., Ev_b is a member of the algebraic dual. We shall see, moreover, that it is a member of the topological dual $\mathrm{Aff}^r(\mathsf{B})'$. We equip $\mathrm{Aff}^r(\mathsf{B})'$ with the weak-$*$ topology defined by the family of seminorms

$$\begin{aligned} P_F \colon \mathrm{Aff}^r(\mathsf{B})' &\to \mathbb{R}_{\geq 0} \\ \nu &\mapsto |\nu(F)|, \end{aligned}$$

for $F \in \mathrm{Aff}^r(\mathsf{B})$. We similarly use the weak-$*$ topology for $\mathrm{Lin}^r(\mathsf{E})'$.

4.2. The embedding theorem for affine bundles. The following result now gives us the desired topological embedding for affine bundles.

4.5 Theorem: (Embedding of affine bundles) *Let $r \in \{\infty, \omega, \mathrm{hol}\}$ and let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ as appropriate. Let $\pi \colon \mathsf{E} \to \mathsf{M}$ be a vector bundle of class C^r, assuming that M is Stein in the case $r = \mathrm{hol}$. Let $\beta \colon \mathsf{B} \to \mathsf{M}$ be an affine bundle modelled on E. Then the mapping*

$$\begin{aligned} \mathrm{Ev}_{\mathsf{B}} \colon \mathsf{B} &\to \mathrm{Aff}^r(\mathsf{B})' \\ b &\mapsto \mathrm{Ev}_b \end{aligned}$$

is an homeomorphism of B with the set of unital $\mathbb{F}$-semialgebra morphisms from $(\mathrm{Aff}^r(\mathsf{B}), \mathrm{C}^r(\mathsf{M}), \beta^*)$ to $(\mathbb{F}, \mathbb{F}, \mathrm{id}_{\mathbb{F}})$, where the latter has the topology induced by the weak-$*$ topology. Moreover, $\mathrm{Ev}_{\mathsf{B}}|\mathsf{B}_x$ is an affine map for each $x \in \mathsf{M}$.

Proof: Let us show that Ev_{B} is injective. Let $b_1, b_2 \in \mathsf{B}$ be distinct. As in Corollary 6.3, let $\iota_{\mathsf{B}} \colon \mathsf{B} \to \mathbb{F}^N \times \mathbb{F}^N$ be an injective C^r-affine bundle mapping over a proper C^r-embedding $\iota_{\mathsf{M}} \colon \mathsf{M} \to \mathbb{F}^N$.

Suppose first that $\beta(b_1) \neq \beta(b_2)$. Then, as in the proof of Theorem 3.2, there exists $f \in \mathrm{C}^r(\mathsf{M})$ such that $f \circ \beta(b_1) = 0$ and $f \circ \beta(b_2) = 1$. Then $\beta^* f \in \mathrm{Aff}^r(\mathsf{B})$ is such that $\beta^* f(b_1) = 0$ and $\beta^* f(b_2) = 1$. Thus $\mathrm{Ev}_{\mathsf{E}}(b_1) \neq \mathrm{Ev}_{\mathsf{E}}(b_2)$ in this case.

Now suppose that $\beta(b_1) = \beta(b_2)$. Let $\Psi \colon \mathbb{F}^N \to \mathbb{F}^N$ be a linear isomorphism (thus of class C^r) such that $\Psi \circ \mathrm{pr}_2 \circ \iota_{\mathsf{B}}(b_1) = \boldsymbol{e}_1$ and $\Psi \circ \mathrm{pr}_2 \circ \iota_{\mathsf{B}}(b_2) = \boldsymbol{e}_2$, where $\boldsymbol{e}_1, \dots, \boldsymbol{e}_N$ are the standard basis vectors. Denote

$$\begin{aligned} \overline{\Psi} \colon \mathbb{F}^N \times \mathbb{F}^N &\to \mathbb{F}^N \times \mathbb{F}^N \\ (\boldsymbol{x}, \boldsymbol{v}) &\mapsto (\boldsymbol{x}, \Psi(\boldsymbol{v})). \end{aligned}$$

Note that $\overline{\Psi}$ is a C^r-affine bundle isomorphism over $\mathrm{id}_{\mathbb{F}^N}$. For $j \in \{1, \dots, N\}$, denote

$$\begin{aligned}\mathrm{Pr}_j \colon \mathbb{F}^N \times \mathbb{F}^N &\to \mathbb{F}\\ ((x_1, \dots, x_N), (v_1, \dots, v_N)) &\mapsto v_j.\end{aligned}$$

Note that Pr_j is of class C^r. Define $F \in \mathrm{C}^r(\mathsf{B})$ by $F = \mathrm{Pr}_1 \circ \overline{\Psi} \circ \iota_{\mathsf{B}}$. Note that F is fibre-affine since ι_{B} and $\overline{\Psi}$ are injective affine bundle mappings and since Pr_1 is fibre-affine. Moreover, $F(b_1) = 1$ and $F(b_2) = 0$. Thus we again conclude that $\mathrm{Ev}_{\mathsf{E}}(b_1) \neq \mathrm{Ev}_{\mathsf{E}}(b_2)$.

Next we prove that Ev_{B} is surjective. Let (ψ, ψ_0) be a unital $\mathbb{F}$-semialgebra morphism from $(\mathrm{Aff}^r(\mathsf{B}), \mathrm{C}^r(\mathsf{M}), \beta^*)$ to $(\mathbb{F}, \mathbb{F}, \mathrm{id}_{\mathbb{F}})$. We then have the following useful lemma.

1 Lemma: *If $F^1, \dots, F^k \in \ker(\psi)$, then $\cap_{j=1}^k Z(F^j) \neq \varnothing$.*

Proof: Suppose that $\cap_{j=1}^k Z(F^j) = \varnothing$. Let us write $F^j = (\lambda^j)^{\mathrm{e}}$ for $\lambda^j \in \Gamma^r(\mathsf{B}^{*,\mathrm{aff}})$. Then, for each $x \in \mathsf{M}$ and for each $b \in \mathsf{B}_x$, there exists $j \in \{1, \dots, k\}$ such that $F^j(b) \neq 0$. Thus $\lambda^1(x), \dots, \lambda^k(x)$ span $\mathsf{B}_x^{*,\mathrm{aff}}$ and so $[\lambda^1]_x, \dots, [\lambda^k]_x$ generate $\mathscr{G}^r_{x,\mathsf{B}^{*,\mathrm{aff}}}$. Thus we have a surjective sheaf morphism $\Psi \colon (\mathscr{C}^r_{\mathsf{M}})^k \to \mathscr{G}^r_{\mathsf{B}^{*,\mathrm{aff}}}$ defined by requiring that

$$\Psi([g^1]_x, \dots, [g^k]_x) = [g^1]_x[\lambda^1]_x + \dots + [g^k]_x[\lambda^k]_x.$$

Since the sheaves serving as the domain and codomain of Ψ are coherent in the real analytic and holomorphic cases (by [Grauert and Remmert 1984, Consequence A.4.2.1]), we can then use (1) the vanishing of sheaf cohomology for sheaves over $\mathscr{C}^\infty_{\mathsf{M}}$ ([Wells Jr. 2008, Proposition 2.3.11], along with [Wells Jr. 2008, Examples 2.3.4(d, e)] and [Wells Jr. 2008, Proposition 2.3.5]), in the case $r = \infty$ or (2) Cartan's Theorem B in the cases $r \in \{\omega, \mathrm{hol}\}$ to conclude that the map

$$\begin{aligned}\Psi_{\mathsf{B}} \colon \mathrm{C}^r(\mathsf{M})^k &\to \mathrm{Aff}^r(\mathsf{B})\\ (g^1, \dots, g^k) &\mapsto \beta^* g^1 F^1 + \dots + \beta^* g^k F^k\end{aligned}$$

is surjective. In particular, there exists $g^1, \dots, g^k \in \mathrm{C}^r(\mathsf{M})$ such that

$$\beta^* g^1 F^1 + \dots + \beta^* g^k F^k = 1_{\mathsf{B}} = \beta^* 1_{\mathsf{M}}.$$

Thus

$$\begin{aligned}\psi(\beta^* g^1 F^1 + \dots + \beta^* g^k F^k) &= \psi(\beta^* g^1 F^1) + \dots + \psi(\beta^* g^k F^k)\\ &= \psi_0(g^1)\psi(F^1) + \dots + \psi_0(g^k)\psi(F^k) = 0.\end{aligned}$$

This is in contradiction with ψ being unital. ▼

We now can complete the proof of surjectivity of Ev_{B}. Let $\iota_{\mathsf{B}} \colon \mathsf{B} \to \mathbb{F}^N \times \mathbb{F}^N$ be an injective C^r-affine bundle mapping over a proper C^r-embedding $\iota_{\mathsf{M}} \colon \mathsf{M} \to \mathbb{F}^N$. Let

$$((\chi^1, \dots, \chi^N), (\mu^1, \dots, \mu^N))$$

be the coordinate functions for this embedding with $\chi^1, \dots, \chi^N \in \mathrm{C}^r(\mathsf{M})$ and $\mu^1, \dots, \mu^N \in \mathrm{Aff}^r(\mathsf{B})$. Define $f^j \in \mathrm{C}^r(\mathsf{M})$ by $f^j = \chi^j - \psi_0(\chi^j)1_{\mathsf{M}}$, $j \in \{1, \dots, N\}$. Define $F^j \in \mathrm{Aff}^r(\mathsf{B})$

by $F^j = \mu^j - \psi(\mu^j)1_{\mathsf{B}}$. Clearly $f^j \in \ker(\psi_0)$ and $F^j \in \ker(\psi)$. Note that $\beta^* f^j \in \ker(\psi)$ thanks to the diagram

$$\begin{tikzcd} \mathrm{C}^r(\mathsf{M}) \arrow[r, "\beta^*"] \arrow[dr, "\psi_0"'] & \mathrm{Aff}^r(\mathsf{B}) \arrow[d, "\psi"] \\ & \mathbb{F} \end{tikzcd} \tag{4.2}$$

By the lemma,

$$\left(\bigcap_{j=1}^N Z(\beta^* f^j)\right) \cap \left(\bigcap_{j=1}^N Z(F^j)\right) \neq \varnothing.$$

If $b \in Z(\beta^* f^j)$, then the definition of f^j gives $\chi^j(\beta(b))) = \psi_0(\chi^j)$. If $b \in Z(F^j)$, then the definition of F^j gives $\mu^j(b) = \psi(\mu^j)$. Thus, if

$$b \in \left(\bigcap_{j=1}^N Z(\beta^* f^j)\right) \cap \left(\bigcap_{j=1}^N Z(F^j)\right),$$

then

$$\iota_{\mathsf{B}}(b) = ((\psi_0(\chi^1), \dots, \psi_0(\chi^N)), (\psi(\mu^1), \dots, \psi(\mu^N))).$$

Since ι_{B} is injective, there exists $b \in \mathsf{B}$ such that

$$\left(\bigcap_{j=1}^N Z(\beta^* f^j)\right) \cap \left(\bigcap_{j=1}^N Z(F^j)\right) = \{b\}.$$

Now let $F \in \ker(\psi)$. Then

$$Z(F) \cap \{b\} = Z(F) \cap \left(\bigcap_{j=1}^N Z(\beta^* f^j)\right) \cap \left(\bigcap_{j=1}^N Z(\mu^j)\right).$$

By the lemma, $Z(F) \cap \{b\} \neq \varnothing$, and so we must have $b \in Z(F)$. As this argument is valid for every $F \in \ker(\psi)$, we conclude that, if $F \in \ker(\psi)$, then $F(b) = 0$. In other words, $\ker(\psi) \subseteq \ker(\mathrm{Ev}_b)$. In particular, if $\beta^* f \in \ker(\psi)$, then

$$0 = \psi(\beta^* f) = \psi_0(f).$$

Thus, $\ker(\psi_0) \subseteq \ker(\mathrm{ev}_{\beta(b)})$. We then argue as in the proof of Theorem 3.2 that $\psi_0 = \mathrm{ev}_{\beta(b)}$. Let us next show that $\ker(\mathrm{Ev}_b) \subseteq \ker(\psi)$. By Corollary 2.5, let $\sigma \in \Gamma^r(\mathsf{B})$ be such that $\sigma(\beta(b)) = b$. Then, for $F \in \ker(\mathrm{Ev}_b)$,

$$0 = F(b) = F \circ \sigma(\beta(b)).$$

Thus $F \circ \sigma \in \ker(\mathrm{ev}_{\beta(b)}) = \ker(\psi_0)$, and so $F \circ \sigma \in \ker(\psi \circ \beta^*)$ by (4.2). Thus

$$0 = \psi \circ \beta^*(F \circ \sigma) = \psi(F \circ \sigma \circ \beta) = \psi(F),$$

i.e., if $F \in \ker(\mathrm{Ev}_b)$, then $F \in \ker(\psi)$.

Finally, let $F \in \mathrm{Aff}^r(\mathsf{B})$ and define $G = F - F(b)1_{\mathsf{B}}$. Then $\mathrm{Ev}_b(G) = 0$ and so $G \in \ker(\mathrm{Ev}_b) = \ker(\psi)$. Thus

$$0 = \psi(G) = \psi(F) - F(b) \quad \Longrightarrow \quad \psi(F) = F(b),$$

i.e., $\psi = \mathrm{Ev}_b$.

Next we show that the restriction of Ev_{B} to fibres is affine. Let $x \in \mathsf{M}$ and define

$$A_x \colon \mathsf{B}_x \to \mathrm{Aff}^r(\mathsf{B})^*$$

by

$$\langle A_x(b); F\rangle = \langle \mathrm{Ev}_{\mathsf{B}}(b); F\rangle = F(b).$$

Similarly, define

$$L_x \colon \mathsf{E}_x \to \mathrm{Lin}^r(\mathsf{E})^*$$

by

$$\langle L_x(e); F\rangle = \langle \mathrm{Ev}_{\mathsf{E}}(e); F\rangle = F(e).$$

To show that A_x is affine, we will show that, for $b \in \mathsf{B}_x$ and $e \in \mathsf{E}_x$, $e \mapsto A_x(b+e) - A_x(b)$ is (1) independent of b and (2) linear in e. Thus, let $b \in \mathsf{B}_x$ and let $e \in \mathsf{E}_x$, and let $F \in \mathrm{Aff}^r(\mathsf{B})$. We have

$$\begin{aligned}\langle A_x(b+e); F\rangle &= \langle \mathrm{Ev}_{\mathsf{B}}(b+e); F\rangle = F(b+e)\\ &= F(b) + L(F)(e) = \langle A_x(b); F\rangle + \langle L_x(e); L(F)\rangle,\end{aligned}$$

where $L(F) \in \mathrm{Lin}^r(\mathsf{E})$ is the linear part of F. Thus

$$\langle A_x(b+e); F\rangle - \langle A_x(b); F\rangle = \langle L_x(e); L(F)\rangle.$$

This verifies that $\mathrm{Ev}_{\mathsf{B}}|\mathsf{B}_x$ is affine, as asserted.

The above shows that Ev_{B} is a bijection between B and the unital $\mathbb{F}$-semialgebra morphisms from $(\mathrm{Aff}^r(\mathsf{B}), \mathrm{C}^r(\mathsf{M}), \beta^*)$ to $(\mathbb{F}, \mathbb{F}, \mathrm{id}_{\mathbb{F}})$. It remains to prove the topological assertions of the theorem.

To show that Ev_{E} is well-defined, one must show that $\mathrm{Ev}_b \in \mathrm{Aff}^r(\mathsf{B})'$. Note that the C^r-topology is finer than the C^0-topology, so it suffices to show that Ev_b is continuous in the C^0-topology on $\mathrm{Aff}^r(\mathsf{B})$. Let $\mathcal{K} \subseteq \mathsf{B}$ be compact such that $b \in \mathcal{K}$. Then

$$|\mathrm{Ev}_b(F)| = |F(b)| \le p^0_{\mathcal{K}}(F)$$

for $F \in \mathrm{Aff}^r(\mathsf{B})$, giving the desired continuity. (Here $p^0_{\mathcal{K}}$ is a seminorm for the C^0-topology, as described in the proof of Theorem 3.2.)

Next we show that Ev_{B} is continuous. Let $b_0 \in \mathsf{B}$ and let $\mathcal{O} \subseteq \mathrm{Aff}^r(\mathsf{B})'$ be a neighbourhood of Ev_{b_0}. Let $F^1, \dots, F^k \in \mathrm{Aff}^r(\mathsf{B})$ and $r_1, \dots, r_k \in \mathbb{R}_{>0}$ be such that

$$\{\alpha \in \mathrm{Aff}^r(\mathsf{B})' \mid |\alpha(F^j) - F^j(b_0)| < r_j,\ j \in \{1, \dots, k\}\} \subseteq \mathcal{O}.$$

Let $\mathcal{V}$ be a neighbourhood of b_0 such that $|F^j(b) - F^j(b_0)| < r_j$, $j \in \{1, \dots, k\}$, $b \in \mathcal{V}$. Then, if $b \in \mathcal{V}$, $\mathrm{Ev}_{\mathsf{B}}(b) \in \mathcal{O}$ and this gives continuity of Ev_{B}.

Finally, we show that Ev_{B} is an homeomorphism onto its image. As above, we let $\iota_{\mathsf{B}} \colon \mathsf{B} \to \mathbb{F}^N \times \mathbb{F}^N$ be an injective C^r-affine bundle mapping over a proper C^r-embedding

$\iota_{\mathsf{M}}\colon \mathsf{M} \to \mathbb{F}^N$. Define functions $\chi^j \in \mathrm{C}^r(\mathsf{M})$ and $\mu^j \in \mathrm{C}^r(\mathsf{B})$, $j \in \{1,\dots,N\}$, by the requirement that

$$\iota_{\mathsf{B}}(b) = ((\chi^1 \circ \beta(b),\dots,\chi^N \circ \beta(b)),(\mu^1(b),\dots,\mu^N(b))).$$

Let $b_0 \in \mathsf{B}$ and let $\mathcal{V}$ be a neighbourhood of b_0 in the standard topology of B. Let $r \in \mathbb{R}_{>0}$ be sufficiently small that

$$\left\{b \in \mathsf{B} \;\middle|\; \sum_{j=1}^N (|\chi^j \circ \beta(b) - \chi^j \circ \beta(b_0)| + |\mu^j(b) - \mu^j(b_0)|) < r\right\} \subseteq \mathcal{V}.$$

Note that

$$\begin{aligned}
\iota_{\mathsf{B}}(\mathsf{B}) \cap &\left(\bigcap_{j=1}^N \left\{\alpha \in \mathrm{Aff}^r(\mathsf{B})' \;\middle|\; |\alpha(\beta^*\chi^j) - \chi^j \circ \beta(b_0)|, |\alpha(\mu^j) - \mu^j(b_0)|, < r/2N\right\}\right)\\
&= \left\{\iota_{\mathsf{B}}(b) \in \iota_{\mathsf{B}}(\mathsf{B}) \;\middle|\; |\chi^j \circ \beta(b) - \chi^j \circ \beta(b_0)|, |\mu^j(b) - \mu^j(b_0)| < r/2N,\ j \in \{1,\dots,N\}\right\}\\
&\subseteq \left\{\iota_{\mathsf{B}}(b) \in \iota_{\mathsf{B}}(\mathsf{B}) \;\middle|\; \sum_{j=1}^N (|\chi^j \circ \beta(b) - \chi^j \circ \beta(b_0)| + |\mu^j(b) - \mu^j(b_0)|) < r\right\} \subseteq \iota_{\mathsf{B}}(\mathcal{V}).
\end{aligned}$$

This shows that $\mathcal{V}$ is open in the topology induced by Ev_{B}. That is to say, the topology on B induced by Ev_{B} is finer than the standard topology, which shows that Ev_{B} is open onto its image. ■

The theorem applies, obviously, to the special case of vector bundles. Note that, when working with vector bundles, one *cannot* take the class of functions to be the fibre-linear functions $\mathrm{Lin}^r(\mathsf{E})$. The theorem holds in this case with

$$\mathrm{Ev}_{\mathsf{E}}\colon \mathsf{E} \to \mathrm{Aff}^r(\mathsf{E})',$$

i.e., the theorem is "the same" for vector bundles as for affine bundles. The reason is that fibre-linear functions are unable to encode any information about the base manifold M, whereas the affine functions contain functions on M as a submodule. The only distinguishing feature of vector bundles as compared to affine bundles in this setup is that, for vector bundles, one has a canonical decomposition

$$\mathrm{Aff}^r(\mathsf{E}) = \mathrm{C}^r(\mathsf{M}) \oplus \mathrm{Lin}^r(\mathsf{E}).$$

We note that the 1–1 correspondence of B with the unital, $\mathbb{F}$-semialgebra morphisms does not require any topology. That is to say, we do not require that $\mathrm{Ev}_{\mathsf{B}}(b) = \mathrm{Ev}_b$ be continuous, and the fact that Ev_{B} is an homeomorphism onto its image is additional to the 1–1 correspondence.

We claim that the assignment to an object B in the category of C^r-affine bundles of the object $(\mathrm{Aff}^r(\mathsf{B}), \mathrm{C}^r(\mathsf{M}), \beta^*)$ in the category of $\mathbb{F}$-semialgebras is injective. Indeed, suppose that

$$(\mathrm{Aff}^r(\mathsf{B}_1), \mathrm{C}^r(\mathsf{M}_1), \beta_1^*) = (\mathrm{Aff}^r(\mathsf{B}_2), \mathrm{C}^r(\mathsf{M}_2), \beta_2^*),$$

with equality being as $\mathbb{F}$-semialgebras. Then $\mathrm{C}^r(\mathsf{M}_1) = \mathrm{C}^r(\mathsf{M}_2)$ as $\mathbb{F}$-algebras, and so $\mathsf{M}_1 = \mathsf{M}_2$, as we argued following Theorem 3.2. Also, the set of unital $\mathbb{F}$-linear mappings of $\mathrm{Aff}^r(\mathsf{B}_1)$ must agree with those of $\mathrm{Aff}^r(\mathsf{B}_2)$, whence $\mathsf{B}_1 = \mathsf{B}_2$ by virtue of the theorem. Since $\beta_1^* = \beta_2^*$, $\beta_1 = \beta_2$. Also since $\beta_1^* = \beta_2^*$, the linear parts of the semialgebras must agree, but the linear parts are $\mathrm{Lin}^r(\mathsf{E}_1)$ and $\mathrm{Lin}^r(\mathsf{E}_2)$, where E_1 and E_2 are the model vector bundles. This, however, means that $\mathsf{E}_1^* = \mathsf{E}_2^*$, whence $\mathsf{E}_1 = \mathsf{E}_2$.

4.3. Mappings of vector and affine bundles and homomorphisms of algebras of affine functions. To complete the story of Gelfand duality for affine bundles, we need to ensure that morphisms in the category of affine bundles and in the category of $\mathbb{F}$-semialgebras behave as do those for the category of manifolds. To this end, we have the following analogue of Theorem 3.3.

4.6 Theorem: (Affine bundle mappings as homomorphisms, affine bundle isomorphisms as automorphisms) *Let $r \in \{\infty, \omega, \mathrm{hol}\}$ and let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, as appropriate. Let $\alpha\colon \mathsf{A} \to \mathsf{M}$ and $\beta\colon \mathsf{B} \to \mathsf{N}$ be C^r-affine bundles. When $r = \mathrm{hol}$, we assume the base manifolds M and N are Stein. Then the following statements hold:*

(i) if $(\Phi, \Phi_0) \in \mathrm{AB}^r(\mathsf{A}; \mathsf{B})$, then

$$(\Phi^*, \Phi_0^*) \in \mathrm{Hom}_{\mathbb{F}}((\mathrm{Aff}^r(\mathsf{A}), \mathrm{C}^r(\mathsf{M}), \alpha^*); (\mathrm{Aff}^r(\mathsf{B}), \mathrm{C}^r(\mathsf{N}), \beta^*)),$$

and Φ and Φ_0 are continuous;

(ii) the mapping $(\Phi, \Phi_0) \mapsto (\Phi^*, \Phi_0^*)$ is a bijection from $\mathrm{AB}^r(\mathsf{A}; \mathsf{B})$ to

$$\mathrm{Hom}_{\mathbb{F}}((\mathrm{Aff}^r(\mathsf{A}), \mathrm{C}^r(\mathsf{M}), \alpha^*); (\mathrm{Aff}^r(\mathsf{B}), \mathrm{C}^r(\mathsf{N}), \beta^*));$$

(iii) taking $\mathsf{N} = \mathsf{M}$, if $(\Phi, \mathrm{id}_{\mathsf{M}})$ is an isomorphism of the affine bundles A and B, then

$$(\Phi^*, \mathrm{id}_{\mathsf{M}}^*) \in \mathrm{Hom}_{\mathbb{F}}((\mathrm{Aff}^r(\mathsf{A}), \mathrm{C}^r(\mathsf{M}), \alpha^*); (\mathrm{Aff}^r(\mathsf{B}), \mathrm{C}^r(\mathsf{M}), \beta^*))$$

is an $\mathbb{F}$-semialgebra isomorphism and Φ^* is continuous;

(iv) the mapping $(\Phi, \mathrm{id}_{\mathsf{M}}) \mapsto (\Phi^*, \mathrm{id}_{\mathsf{M}}^*)$ is a bijection from the set of affine bundle isomorphisms of A to the set $\mathrm{Aut}_{\mathbb{F}}(\mathrm{Aff}^r(\mathsf{B}), \mathrm{C}^r(\mathsf{M}), \beta^*)$.

Proof: (i) We verified in Example 4.4–2 that (Φ^*, Φ_0^*) is a morphism of $\mathbb{F}$-semialgebras. Continuity of Φ^* and Φ_0^* follows from [Lewis 2023, Theorem 5.26] in the manner explained in the corresponding part of the proof of Theorem 3.3.

(ii) First we show that $(\Phi, \Phi_0) \mapsto (\Phi^*, \Phi_0^*)$ is injective. Suppose that $(\Phi_1^*, \Phi_{1,0}^*) = (\Phi_2^*, \Phi_{2,0}^*)$. Let $\iota_{\mathsf{B}}\colon \mathsf{B} \to \mathbb{F}^N \times \mathbb{F}^N$ be an injective C^r-affine bundle mapping with coordinate functions

$$((\chi^1, \dots, \chi^N), (\mu^1, \dots, \mu^N))$$

satisfying $\chi^1, \dots, \chi^N \in \mathrm{C}^r(\mathsf{N})$ and $\mu^1, \dots, \mu^N \in \mathrm{Aff}^r(\mathsf{B})$. As in the proof of Theorem 3.3, we have $\Phi_{1,0} = \Phi_{2,0}$. We also have

$$\begin{aligned} & \Phi_1^* \mu^j(b) = \Phi_2^* \mu^j(b), \qquad j \in \{1, \dots, N\},\ b \in \mathsf{B}, \\ \Longrightarrow\quad & \mu^j \circ \Phi_1(b) = \mu^j \circ \Phi_2(b), \qquad j \in \{1, \dots, N\},\ b \in \mathsf{B}, \\ \Longrightarrow\quad & \iota_{\mathsf{B}} \circ \Phi_1(b) = \iota_{\mathsf{B}} \circ \Phi_2(b), \qquad b \in \mathsf{B}, \\ \Longrightarrow\quad & \Phi_1(b) = \Phi_2(b), \qquad b \in \mathsf{B}, \end{aligned}$$

which, combined with the fact that $\Phi_{1,0} = \Phi_{2,0}$, gives the desired conclusion.

To show that $(\Phi, \Phi_0) \mapsto (\Phi^*, \Phi_0^*)$ is surjective, we shall construct a right inverse of this mapping. Thus let

$$(\gamma, \gamma_0) \in \mathrm{Hom}_{\mathbb{F}}((\mathrm{Aff}^r(\mathsf{B}), \mathrm{C}^r(\mathsf{N}), \beta^*); (\mathrm{Aff}^r(\mathsf{A}), \mathrm{C}^r(\mathsf{M}), \mu^*)),$$

and let $\Phi_{\gamma_0} \in \mathrm{C}^r(\mathsf{M}; \mathsf{N})$ be as in the proof of Theorem 3.3; thus $\gamma_0 = \Phi_{\gamma_0}^*$. Now define $\Phi_\gamma \colon \mathsf{A} \to \mathsf{B}$ as follows. Let $a \in \mathsf{A}_x$ and $F \in \mathrm{Aff}^r(\mathsf{B})$, and note that

$$\mathrm{Ev}_a \circ \gamma(1_{\mathsf{B}}) = \mathrm{ev}_x \circ \gamma_0(1_{\mathsf{N}}) = \mathrm{ev}_x(1_{\mathsf{M}}) = 1.$$

Also, for $g \in \mathrm{C}^r(\mathsf{N})$,

$$\mathrm{Ev}_a \circ \gamma(\beta^* g) = \mathrm{Ev}_a \circ \gamma_0(g) = \mathrm{ev}_x \circ \Phi_{\gamma_0}^*(g),$$

giving the commuting of the diagram

$$\begin{tikzcd} \mathrm{C}^r(\mathsf{N}) \arrow[r, "\beta^*"] \arrow[dr, "\mathrm{ev}_x \circ \Phi_{\gamma_0}^*"'] & \mathrm{Aff}^r(\mathsf{B}) \arrow[d, "\mathrm{Ev}_a \circ \gamma"] \\ & \mathbb{F} \end{tikzcd}$$

For $g \in \mathrm{C}^r(\mathsf{N})$ and $G \in \mathrm{Aff}^r(\mathsf{B})$, we have

$$\mathrm{Ev}_a \circ \gamma(gG) = \mathrm{Ev}_a(\gamma_0(g)\gamma(G)) = (\mathrm{ev}_x \circ \Phi_{\gamma_0}^*(g))(\mathrm{Ev}_a \circ \gamma(G)),$$

which is a verification of the intertwining condition (4.1). From all of this, we conclude that $(\mathrm{Ev}_a \circ \gamma, \mathrm{ev}_x \circ \Phi_{\gamma_0}^*)$ is a unital $\mathbb{F}$-semialgebra morphism from $(\mathrm{Aff}^r(\mathsf{B}), \mathrm{C}^r(\mathsf{N}), \beta^*)$ to $(\mathbb{F}, \mathbb{F}, \mathrm{id}_{\mathbb{F}})$. Thus, by Theorem 4.5, there exists $b_a \in \mathsf{N}$ such that $\mathrm{Ev}_{b_a} = \mathrm{Ev}_a \circ \gamma$. We define $\Phi_\gamma(a) = b_a$.

We claim that $(\Phi_\gamma, \Phi_{\gamma_0}) \in \mathrm{AB}^r(\mathsf{A}; \mathsf{B})$. As we have already seen in the proof of Theorem 3.3, $\gamma = \Phi_{\gamma_0}^*$ and $\Phi_{\gamma_0} \in \mathrm{C}^r(\mathsf{M}; \mathsf{N})$. Now let $F \in \mathrm{Aff}^r(\mathsf{B})$ and note that

$$\Phi_\gamma^* F(a) = F(b_a) = \mathrm{Ev}_{b_a}(F) = \mathrm{Ev}_a \circ \gamma(F) = \gamma(F)(a),$$

i.e., $\Phi_\gamma^* F = \gamma(F) \in \mathrm{Aff}^r(\mathsf{B})$. We claim that this implies that Φ_γ is of class C^r. Indeed, let $a_0 \in \mathsf{A}$ and denote $b_0 = \Phi_\gamma(a_0)$. Let $(\mathcal{U}, \phi)$ be an affine bundle chart for B about $\alpha(a_0)$ whose coordinate functions we denote by

$$((\chi^1, \dots, \chi^n), (\mu^1, \dots, \mu^k)).$$

Let $(\mathcal{V}, \psi)$ be an affine bundle chart for B about $\beta(b_0)$ whose coordinate functions

$$((\eta^1, \dots, \eta^m), (\nu^1, \dots, \nu^l))$$

are restrictions of globally defined functions of class C^r (fibre-affine functions, in the case of $\nu^1, \dots, \nu^l$). This is possible by Corollary 6.4. The mapping

$$\begin{aligned} \boldsymbol{\eta} \times \boldsymbol{\nu} \colon \mathsf{B} &\to \mathbb{F}^m \times \mathbb{F}^l \\ b &\mapsto ((\eta^1(\beta(b)), \dots, \eta^m(\beta(b))), (\nu^1(b), \dots, \nu^k(b))) \end{aligned}$$

is a fibre-affine isomorphism onto its image from an affine bundle coordinate neighbourhood $\mathcal{V}' \subseteq \mathcal{V}$ of b_0 to a neighbourhood $\mathcal{W} \times \mathbb{F}^l$ of $\boldsymbol{\eta}(b_0) \times \{\mathbf{0}\} \in \mathbb{F}^m \times \mathbb{F}^l$. Since $\Phi^*_{\gamma_0}\boldsymbol{\eta}$ and $\Phi^*_\gamma\boldsymbol{\nu}$ are continuous by hypothesis, there is an affine bundle coordinate neighbourhood $\mathcal{U}'$ of a_0 such that

$$\Phi_\gamma(\boldsymbol{\eta} \times \boldsymbol{\nu})(\mathcal{U}') \subseteq \mathcal{W} \times \mathbb{F}^l.$$

Thus $\Phi_\gamma(\mathcal{U}') \subseteq \mathcal{V}'$. Therefore, we can assume without loss of generality that $\Phi_\gamma(\mathcal{U}) \subseteq \mathcal{V}$. We denote

$$\begin{aligned} \boldsymbol{\chi} \times \boldsymbol{\alpha} \colon \mathcal{U} &\to \mathbb{F}^n \times \mathbb{F}^k \\ x &\mapsto ((\chi^1(\alpha(a)), \dots, \chi^n(\alpha(a))), (\mu^1(a), \dots, \mu^k(a))). \end{aligned}$$

Note that the local representative of Φ_γ in the charts $(\mathcal{U}, \phi)$ and $(\mathcal{V}, \psi)$ is

$$\begin{aligned} \boldsymbol{\Phi}_\gamma \colon \phi(\mathcal{U}) &\to \psi(\mathcal{V}) \\ \boldsymbol{x} &\mapsto (\boldsymbol{\eta} \times \boldsymbol{\nu}) \circ \Phi_\gamma \circ (\boldsymbol{\chi} \times \boldsymbol{\alpha})^{-1}. \end{aligned}$$

Since $(\boldsymbol{\eta} \times \boldsymbol{\nu}) \circ \Phi_\gamma$ is of class C^r (by hypothesis) and $(\boldsymbol{\chi} \times \boldsymbol{\alpha})^{-1}$ is of class C^r, the local representative of Φ_γ is of class C^r, and this shows that Φ_γ is of class C^r.

Moreover, the equality $\Phi^*_\gamma = \gamma$ proved above is exactly the statement that the mapping $\gamma \mapsto \Phi_\gamma$ is a right inverse of the mapping $\Phi \mapsto \Phi^*$, and this completes the proof of this part of the theorem.

(iii) This follows from part (i) since the inverse of Φ^* is $\Phi_* = (\Phi^{-1})^*$ in the case that Φ is an affine bundle isomorphism.

(iv) This follows from part (ii), just as part (iii) follows from (i). ■

4.7 Corollary: (Gelfand duality for affine bundles) *Let $r \in \{\infty, \omega, \mathrm{hol}\}$ and let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, as appropriate. The category of C^r-affine bundles is a full subcategory of the opposite category of the category of $\mathbb{F}$-semialgebras via the functor given by*

$$(\beta \colon \mathsf{B} \to \mathsf{M}) \mapsto (\mathrm{Aff}^r(\mathsf{B}), \mathrm{C}^r(\mathsf{M}), \beta^*)$$

on objects and by $(\Phi, \Phi_0) \mapsto (\Phi^*, \Phi_0^*)$ on morphisms.

5. Gelfand duality for jet bundles

As an application of the results of Section 4, we shall embed jet bundles into duals of function spaces in a way that respects the structure of jet bundles.

5.1. Embedding jet bundles of sections of affine bundles. In this section we refine the development in the preceding section for affine bundles to jet bundles of affine bundles. Thus we suppose that $r \in \{\infty, \omega, \mathrm{hol}\}$ and that $\beta \colon \mathsf{B} \to \mathsf{M}$ is a C^r-affine bundle modelled on the C^r-vector bundle $\pi \colon \mathsf{E} \to \mathsf{M}$. For $m \in \mathbb{Z}_{\geq 0}$, we have the C^r-affine bundle $\beta_m \colon \mathsf{J}^m\mathsf{B} \to \mathsf{M}$. As we saw in Section 2.4.2, the set of fibre-affine functions on this latter affine bundle is naturally identified with the set $\mathrm{ADO}^r_m(\mathsf{B})$ of C^r-affine differential operators with values in $\mathbb{F}$. Thus we have the following short exact sequence of $\mathrm{C}^r(\mathsf{M})$-modules:

$$0 \longrightarrow \mathrm{C}^r(\mathsf{M}) \xrightarrow{\beta_m^*} \mathrm{ADO}^r_m(\mathsf{B}) \longrightarrow \mathrm{LDO}^r_m(\mathsf{E}) \longrightarrow 0$$

This puts us squarely in the setting of Section 4. That is to say, we can consider unital $\mathbb{F}$-semialgebra morphisms

$$(\psi, \psi_0) \in \mathrm{Hom}_{\mathbb{F}}((\mathrm{ADO}^r_m(\mathsf{B}), \mathrm{C}^r(\mathsf{M}), \beta^*_m), (\mathbb{F}, \mathbb{F}, \mathrm{id}_{\mathbb{F}}))$$

as a subset of the topological dual $\mathrm{ADO}^r_m(\mathsf{B})'$.

Thus we have the following result.

5.1 Theorem: (Embedding jet bundle of sections into the space of differential operators) *Let $r \in \{\infty, \omega, \mathrm{hol}\}$ and let $\beta\colon \mathsf{B} \to \mathsf{M}$ be a C^r-affine bundle modelled on the C^r-vector bundle $\pi\colon \mathsf{E} \to \mathsf{M}$, assuming that M is Stein in case $r = \mathrm{hol}$. Let $m \in \mathbb{Z}_{\geq 0}$. Then the mapping*

$$\begin{aligned} \mathrm{Ev}^m_{\mathsf{B}}\colon \mathsf{J}^m\mathsf{B} &\to \mathrm{ADO}^r_m(\mathsf{B})' \\ j_m\sigma(x) &\mapsto \mathrm{ev}_{j_m\sigma(x)} \end{aligned}$$

is an homeomorphism of $\mathsf{J}^m\mathsf{B}$ with the set of unital $\mathbb{F}$-semialgebra morphisms from $(\mathrm{ADO}^r_m(\mathsf{B}), \mathrm{C}^r(\mathsf{M}), \beta^*_m)$ to $(\mathbb{F}, \mathbb{F}, \mathrm{id}_{\mathbb{F}})$, where the latter has the topology induced by the weak-$*$-topology. Moreover, $\mathrm{Ev}^m_{\mathsf{B}}|\mathsf{J}^m\mathsf{B}_x$ is an affine map for each $x \in \mathsf{M}$.

We note that there are many other affine bundles in this setting arising from (2.3) and (2.4). This will give rise to corresponding embeddings, and we leave to the reader the chore of developing the notation required to state the results. As a hint, we note that the fibre-affine functions in this setting will be homogeneous differential operators.

5.2. Embedding jet bundles of mappings. Next we consider a suitable embedding of the jet bundle of mappings between manifolds. Here we let $r \in \{\infty, \omega, \mathrm{hol}\}$ and let M and N be C^r-manifolds. Recall that $\mathsf{J}^0(\mathsf{M};\mathsf{N}) = \mathsf{M} \times \mathsf{N}$. For $m \in \mathbb{Z}_{\geq 0}$, we have the affine bundle $\rho^m_0\colon \mathsf{J}^m(\mathsf{M};\mathsf{N}) \to \mathsf{J}^0(\mathsf{M};\mathsf{N})$ modelled on the vector bundle $\mathsf{J}^m(\mathsf{V}\mathsf{J}^0(\mathsf{M};\mathsf{N}))$. As we saw in Section 2.4.1, the set of C^r-fibre-affine functions on this affine bundle is identified with the set $\mathrm{FADO}^r_m(\mathsf{M} \times \mathsf{N})$ of fibre-affine differential operators of order m with values in $\mathbb{F}$. Therefore, we have the following short exact sequence of $\mathrm{C}^r(\mathsf{M})$-modules:

$$0 \longrightarrow \mathrm{C}^r(\mathsf{J}^0(\mathsf{M};\mathsf{N})) \xrightarrow{(\rho^m_0)^*} \mathrm{FADO}^r_m(\mathsf{M} \times \mathsf{N}) \longrightarrow \Gamma^r((\mathsf{J}^m\mathsf{V}\mathsf{J}^0(\mathsf{M};\mathsf{N}))^*) \longrightarrow 0$$

Again, we are in the setting of Section 4, and so we can consider unital $\mathbb{F}$-semialgebra morphisms

$$(\psi, \psi_0) \in \mathrm{Hom}_{\mathbb{F}}((\mathrm{FADO}^r_m(\mathsf{M} \times \mathsf{N}), \mathrm{C}^r(\mathsf{J}^0(\mathsf{M};\mathsf{N})), (\rho^m_0)^*), (\mathbb{F}, \mathbb{F}, \mathrm{id}_{\mathbb{F}}))$$

as a subset of the topological dual $\mathrm{FADO}^r_m(\mathsf{M} \times \mathsf{N})'$. The embedding result one then has is the following.

5.2 Theorem: (Embedding jet bundle of mappings into the space of differential operators) *Let $r \in \{\infty, \omega, \mathrm{hol}\}$, and let M and N be C^r-manifolds, assuming them to be Stein when $r = \mathrm{hol}$. Let $m \in \mathbb{Z}_{\geq 0}$. Then the mapping*

$$\begin{aligned} \mathrm{Ev}^m_{\mathsf{M}\times\mathsf{N}}\colon \mathsf{J}^m(\mathsf{M};\mathsf{N}) &\to \mathrm{FADO}^r_m(\mathsf{M} \times \mathsf{N})' \\ j_m\Phi(x) &\mapsto \mathrm{ev}_{j_m\Phi(x)} \end{aligned}$$

is an homeomorphism of $\mathsf{J}^m(\mathsf{M};\mathsf{N})$ *with the set of unital* $\mathbb{F}$*-semialgebra morphisms from* $(\mathrm{FADO}^r_m(\mathsf{M}\times\mathsf{N}), \mathrm{C}^r(\mathsf{M}\times\mathsf{N}), (\rho_0^m)^*)$ *to* $(\mathbb{F},\mathbb{F},\mathrm{id}_{\mathbb{F}})$*, where the latter has the topology induced by the weak-$*$ topology. Moreover,* $\mathrm{Ev}^m_{\mathsf{M}\times\mathsf{N}}|(\rho_0^m)^{-1}(x,y)$ *is an affine map for each* $x,y\in\mathsf{M}\times\mathsf{N}$*.*

We note that there are many other affine bundles in this setting arising from (2.1). This will give rise to corresponding embeddings, and we leave to the reader the pleasure of developing the notation required to state the results. As with our hint above for vector bundles, we comment that the fibre-affine functions arising in this case will be homogeneous differential operators.

6. Smooth, real analytic, and holomorphic versions of the Serre–Swan Theorem

In this section, to wrap up our collection of interconnected results, we prove a version of the Serre–Swan Theorem for vector bundles in the three regularity categories with which we are working in this paper.

Our proof relies on embedding the total space of a vector bundle in a suitable Euclidean space. In the smooth and real analytic cases, this follows without problem from the embedding theorems in these cases (see, for example, the proof of Lemma 1.1 for references). In the case of vector bundles over Stein manifolds, that the total space is itself is Stein is required to use the corresponding embedding theorem. This is well-known to be true, and is typically attributed to Serre, and without reference as near as we can tell. Related problems are discussed in [Forstnerič 2011, §4.21]. In any case, let us prove here the result we need, since we make substantial use of the corollaries that follow it.

6.1 Proposition: (The total space of a vector bundle over a Stein manifold is a Stein manifold) *If* $\pi\colon\mathsf{E}\to\mathsf{M}$ *is an holomorphic vector bundle over a Stein manifold, then* E *is a Stein manifold.*

Proof: We will show the following three things:

1. for each $e\in\mathsf{E}$, there exists an holomorphic chart for E about e whose coordinate functions are globally defined holomorphic functions;
2. holomorphic functions on E separate points, i.e., if $e_1,e_2\in\mathsf{E}$ are distinct, then there exists $f\in\mathrm{C}^{\mathrm{hol}}(\mathsf{E})$ such that $f(e_1)\neq f(e_2)$;
3. E is holomorphically convex, i.e., if $L\subseteq\mathsf{E}$ is compact, then the set
$$\mathrm{hconv}(L)\triangleq\{e\in\mathsf{E}\mid\ |f(e)|\leq p^0_L(f),\ f\in\mathrm{C}^{\mathrm{hol}}(\mathsf{E})\}$$
is compact.

These suffice to show that E is a Stein manifold by any of the various definitions.

Let us prove these in order.

Let $e\in\mathsf{E}$. Let $z=\pi(e)$ and let $(\mathcal{U},\phi)$ be an holomorphic chart for M about z whose coordinate functions are globally defined holomorphic functions; this is possible since M is Stein. Let $(\alpha^1,\dots,\alpha^m)$ be a basis for E^*_z. By Cartan's Theorem A, let $\sigma^1,\dots,\sigma^m\in\Gamma^{\mathrm{hol}}(\mathsf{E}^*)$

be such that $\sigma^j(z) = \alpha^j$, $j \in \{1, \dots, m\}$. Shrink $\mathcal{U}$ so that $(\sigma^1(z'), \dots, \sigma^m(z'))$ is a basis for $\mathsf{E}^*_{z'}$ for $z' \in \mathcal{U}$. Then, if n is the dimension of M, define a chart map for $\pi^{-1}(\mathcal{U})$ by

$$\begin{aligned}\Phi\colon \pi^{-1}(\mathcal{U}) &\to \mathbb{C}^n \times \mathbb{C}^m \\ e_{z'} &\mapsto (\phi(z'), (\langle \sigma^1(z'); e_{z'} \rangle, \dots, \langle \sigma^m(z'); e_{z'} \rangle)).\end{aligned}$$

This is the required holomorphic chart for E in a neighbourhood of e whose coordinate functions are globally defined holomorphic functions.

Let $e_1, e_2 \in \mathsf{E}$ be distinct. If $\pi(e_1) \neq \pi(e_2)$, then let $f \in \mathrm{C}^{\mathrm{hol}}(\mathsf{E})$ be such that $f \circ \pi(e_1) \neq f \circ \pi(e_2)$, this being possible since M is Stein. Then $\pi^* f(e_1) \neq \pi^* f(e_2)$ and so $\pi^* f$ separates e_1 and e_2. Now suppose that $\pi(e_1) = \pi(e_2) = z$. Suppose that $e_1 \neq 0$, without loss of generality. Let $\alpha \in \mathsf{E}^*_z$ be such that $\alpha(e_1) = 1$ and $\alpha(e_2) = 0$. By Cartan's Theorem A, let $\sigma \in \Gamma^{\mathrm{hol}}(\mathsf{E}^*)$ be such that $\sigma(z) = \alpha$. Define $f \in \mathrm{C}^{\mathrm{hol}}(\mathsf{E})$ by $f(e) = \langle \sigma \circ \pi(e); e \rangle$. Since $f(e_1) = 1$ and $f(e_2) = 0$, f separates e_1 and e_2.

We shall show that, if $(e_j)_{j \in \mathbb{Z}_{>0}}$ is a sequence in E with no accumulation point, there exists $F \in \mathrm{C}^{\mathrm{hol}}(\mathsf{E})$ such that $\limsup_{j \to \infty} |F(e_j)| = \infty$. First of all, if the sequence $(\pi(e_j))_{j \in \mathbb{Z}_{>0}}$ has no accumulation point, then, since M is holomorphically convex, there exists $f \in \mathrm{C}^{\mathrm{hol}}(\mathsf{M})$ such that $\limsup_{j \to \infty} |f \circ \pi(e_j)| = \infty$, and since $\pi^* f \in \mathrm{C}^{\mathrm{hol}}(\mathsf{E})$ this gives the desired conclusion in this case. So suppose that $(\pi(e_j))_{j \in \mathbb{Z}_{>0}}$ has an accumulation point, and let us pass to a subsequence in order to obtain the assumption that $(e_j)_{j \in \mathbb{Z}_{>0}}$ has no accumulation point and that $\lim_{j \to \infty} \pi(e_j) = x \in \mathsf{M}$. Choose a local trivialisation $\Phi\colon \mathsf{E}|\mathcal{U} \to \mathcal{U} \times \mathbb{C}^m$, where $\mathcal{U}$ is a neighbourhood of x. Let us write

$$\Phi(e) = (\pi(e), (g^1(e), \dots, g^m(e))),$$

where $g^1, \dots, g^m \in \mathrm{C}^{\mathrm{hol}}(\mathsf{E}|\mathcal{U})$ are linear on fibres. Thus, if we define

$$\alpha^j(e) = (\pi(e), g^j(e)), \qquad j \in \{1, \dots, \},$$

then $\alpha^1, \dots, \alpha^m \in \Gamma^{\mathrm{hol}}(\mathsf{E}^*|\mathcal{U})$. By Cartan's Theorem A, there exists $\sigma_1, \dots, \sigma_k \in \Gamma^{\mathrm{hol}}(\mathsf{E}^*)$ and $f^1, \dots, f^k \in \mathrm{C}^{\mathrm{hol}}(\mathcal{U})$ such that

$$\alpha^j = f^1 \cdot \sigma_1 + \dots + f^k \sigma_k, \qquad j \in \{1, \dots, m\},$$

possibly after shrinking $\mathcal{U}$. By hypothesis, the sequence $(\mathrm{pr}_2 \circ \Phi(e_j))_{j \in \mathbb{Z}_{>0}}$ in $\mathbb{C}^m$ does not have an accumulation point. Therefore, we must have that, for some $a \in \{1, \dots, m\}$, the sequence $(g^a(e_j))_{j \in \mathbb{Z}_{>0}}$ has no accumulation point. Since the function g^a is linear on fibres, $\limsup_{j \to \infty} |g^a(e_j)| = \infty$. Therefore, for some $b \in \{1, \dots, k\}$, we must have $\limsup_{j \to \infty} |\mathrm{pr}_2 \circ \sigma_a(e_j)| = \infty$, furnishing us with the desired conclusion. ■

Using the preceding, we can now prove a vector bundle version of the various embedding theorems.

6.2 Proposition: (Embedding of vector bundles) *If $r \in \{\infty, \omega, \mathrm{hol}\}$ and if $\pi\colon \mathsf{E} \to \mathsf{M}$ is a C^r-vector bundle, with M Stein when $r = \mathrm{hol}$, then there exist $N \in \mathbb{Z}_{\geq 0}$ and an injective C^r-vector bundle mapping $\iota_{\mathsf{E}}\colon \mathsf{E} \to \mathbb{F}^N \times \mathbb{F}^N$ over a proper C^r-embedding $\iota_{\mathsf{M}}\colon \mathsf{M} \to \mathbb{F}^N$.*

Proof: By [Whitney 1936, Lemma 19] in the case of $r = \infty$, [Grauert 1958, Theorem 3] in the case of $r = \omega$, and [Remmert 1954] and Proposition 6.1 in the holomorphic case, there exists a proper C^r-embedding Ψ of E in $\mathbb{F}^N$ for some $N \in \mathbb{Z}_{>0}$. There is then an induced proper C^r-embedding ι_{M} of M in $\mathbb{F}^N$ by restricting Ψ to the zero section of E. Let us take the subbundle $\hat{\mathsf{E}}$ of $\mathsf{T}\mathbb{F}^N|\iota_{\mathsf{M}}(\mathsf{M})$ whose fibre at $\iota_{\mathsf{M}}(x) \in \iota_{\mathsf{M}}(\mathsf{M})$ is

$$\hat{\mathsf{E}}_{\iota_{\mathsf{M}}(x)} = T_{0_x}\Psi(\mathsf{V}_{0_x}\mathsf{E}).$$

Now recall that $\mathsf{E} \simeq \zeta^*\mathsf{VE}$, where $\zeta\colon \mathsf{M} \to \mathsf{E}$ is the zero section [cf. Kolář, Michor, and Slovák 1993, page 55]. Let us abbreviate $\iota_{\mathsf{E}} = T\Psi|\zeta^*\mathsf{VE}$. We then have the following diagram

$$\begin{tikzcd} \mathsf{E} \simeq \zeta^*\mathsf{VE} \arrow[r, "\iota_{\mathsf{E}}"] \arrow[d, "\pi"'] & \mathbb{F}^N \times \mathbb{F}^N \arrow[d, "\mathrm{pr}_2"] \\ \mathsf{M} \arrow[r, "\iota_{\mathsf{M}}"'] & \mathbb{F}^N \end{tikzcd}$$

describing an injective mapping ι_{E} of C^r-vector bundles over the proper C^r-embedding ι_{M}, with the image of ι_{E} being $\hat{\mathsf{E}}$. This is the assertion of the lemma. ■

Combining this result with Corollary 2.4 we have the following.

6.3 Corollary: (Embedding of affine bundles) *Let $r \in \{\infty, \omega, \mathrm{hol}\}$, let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a C^r-vector bundle, with M Stein when $r = \mathrm{hol}$, and let $\beta\colon \mathsf{B} \to \mathsf{M}$ be a C^r-affine bundle modelled on E. Then there exist $N \in \mathbb{Z}_{\geq 0}$ and an injective C^r-affine bundle mapping $\iota_{\mathsf{B}}\colon \mathsf{B} \to \mathbb{F}^N \times \mathbb{F}^N$ over a proper C^r-embedding $\iota_{\mathsf{M}}\colon \mathsf{M} \to \mathbb{F}^N$.*

We also get the following analogue of Lemma 1.1.

6.4 Corollary: (Existence of globally defined affine coordinates) *Let $r \in \{\infty, \omega, \mathrm{hol}\}$ and let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, as appropriate. Let $\beta\colon \mathsf{B} \to \mathsf{M}$ be a C^r-affine bundle modelled on the C^r-vector bundle $\pi\colon \mathsf{E} \to \mathsf{M}$, and suppose that M is Stein when $r = \mathrm{hol}$. Then, for any $x \in \mathsf{M}$, there exist an affine bundle chart $(\mathcal{V}, \psi)$ for B and a vector bundle chart $(\mathcal{U}, \phi)$ for E whose coordinate functions*

$$((\chi^1, \dots, \chi^n), (\alpha^1, \dots, \alpha^k))$$

and

$$((\chi^1, \dots, \chi^n), (\nu^1, \dots, \nu^k))$$

satisfy the following:

(i) $\chi^1, \dots, \chi^n$ are restrictions to $\beta(\mathcal{V})$ of globally defined C^r-functions;

(ii) $\alpha^1, \dots, \alpha^k$ are restrictions to $\mathcal{V}$ of globally defined C^r-fibre affine functions;

(iii) $\nu^1, \dots, \nu^k$ are restrictions to $\mathcal{U}$ of globally defined C^r-fibre linear functions.

Proof: The simple idea of the proof of Lemma 1.1 is easily adapted to this situation. ■

We can now state the Serre–Swan Theorem. We give a unified proof of this theorem for smooth vector bundles, real analytic vector bundles, and holomorphic vector bundles over a Stein base.

6.5 Theorem: (Serre–Swan Theorem for C^r-vector bundles) *Let $r \in \{\infty, \omega, \mathrm{hol}\}$ and let $\mathbb{F} = \mathbb{R}$ if $r \in \{\infty, \omega\}$ and let $\mathbb{F} = \mathbb{C}$ if $r = \mathrm{hol}$. Let M be a manifold of class C^r. If $r = \mathrm{hol}$ assume that M is Stein. The following statements hold:*

(i) if $\pi\colon \mathsf{E} \to \mathsf{M}$ is a vector bundle of class C^r, then $\Gamma^r(\mathsf{E})$ is a finitely generated projective module over $\mathrm{C}^r(\mathsf{M})$; that is to say, $\Gamma^r(\mathsf{E})$ is a direct summand of a finitely generated free module over $\mathrm{C}^r(\mathsf{M})$;

(ii) if $\mathscr{M}$ is a finitely generated projective module over $\mathrm{C}^r(\mathsf{M})$, then $\mathscr{M}$ is isomorphic to the module $\Gamma^r(\mathsf{E})$ of C^r-sections of a C^r-generalised subbundle of E.

Proof: (i) By Proposition 6.2, let $\iota_{\mathsf{E}}\colon \mathsf{E} \to \mathbb{F}^N \times \mathbb{F}^N$ be an injective C^r-vector bundle mapping over a proper C^r-embedding $\iota_{\mathsf{M}}\colon \mathsf{M} \to \mathbb{R}^N$. Thus we have E as isomorphic to a subbundle of the trivial bundle $\mathbb{F}^N_{\mathsf{M}} \triangleq \mathsf{M} \times \mathbb{F}^N$. Let $\langle\cdot,\cdot\rangle$ be the standard (Hermitian, if $\mathbb{F} = \mathbb{C}$) inner product on $\mathbb{F}^N$ which we think of as a vector bundle metric on $\mathbb{F}^N_{\mathsf{M}}$. Define G_x to be the orthogonal complement to E_x, noting that G is then a C^r-subbundle of $\mathbb{F}^N_{\mathsf{M}}$ and that $\mathbb{F}^N_{\mathsf{M}} = \mathsf{E} \oplus \mathsf{G}$. Let $\pi_1\colon \mathbb{F}^N_{\mathsf{M}} \to \mathsf{E}$ and $\pi_2\colon \mathbb{F}^N_{\mathsf{M}} \to \mathsf{G}$ be the projections, thought of as vector bundle morphisms. Note that $\Gamma^r(\mathbb{F}^N_{\mathsf{M}})$ is isomorphic, as a $\mathrm{C}^r(\mathsf{M})$-module, to $\mathrm{C}^r(\mathsf{M})^N$. Moreover, the map from $\Gamma^r(\mathbb{F}^N_{\mathsf{M}})$ to $\Gamma^r(\mathsf{E}) \oplus \Gamma^r(\mathsf{G})$ given by

$$\boldsymbol{\xi} \mapsto (\pi_1 \circ \boldsymbol{\xi}) \oplus (\pi_2 \circ \boldsymbol{\xi})$$

can be directly verified to be an isomorphism of $\mathrm{C}^r(\mathsf{M})$-modules. In particular, $\Gamma^r(\mathsf{E})$ is a summand of the free, finitely generated $\mathrm{C}^r(\mathsf{M})$-module $\Gamma^r(\mathbb{F}^N_{\mathsf{M}})$.

(ii) By definition, there exists a module $\mathscr{N}$ over $\mathrm{C}^r(\mathsf{M})$ such that

$$\mathscr{M} \oplus \mathscr{N} \simeq \underbrace{\mathrm{C}^r(\mathsf{M}) \oplus \cdots \oplus \mathrm{C}^r(\mathsf{M})}_{N \text{ factors}}.$$

The direct sum on the right is naturally isomorphic to the set of C^r-sections of the trivial vector bundle $\mathbb{F}^N_{\mathsf{M}} = \mathsf{M} \times \mathbb{F}^N$. Thus we can write $\mathscr{M} \oplus \mathscr{N} = \Gamma^r(\mathbb{F}^N_{\mathsf{M}})$. For $a \in \{1,2\}$, let $\Pi_a\colon \Gamma^r(\mathbb{F}^N_{\mathsf{M}}) \to \Gamma^r(\mathbb{F}^N_{\mathsf{M}})$ be the projection onto the ath factor. As per [Nelson 1967, §6] (essentially), associated with Π_a is a vector bundle map $\pi_a\colon \mathbb{F}^N_{\mathsf{M}} \to \mathbb{F}^N_{\mathsf{M}}$. Since $\Pi_a \circ \Pi_a = \Pi_a$ (by virtue of Π_a being a projection), $\pi_a \circ \pi_a = \pi_a$. To show that $\mathscr{M}$ is the set of sections of a vector subbundle of $\mathbb{F}^N_{\mathsf{M}}$ it suffices to show that π_1 has constant rank. One can easily show that $x \mapsto \operatorname{rank}(\pi_{a,x})$ is lower semicontinuous for $a \in \{1,2\}$. However, since $\operatorname{rank}(\pi_{1,x}) + \operatorname{rank}(\pi_{2,x}) = N$ for all $x \in \mathsf{M}$, if $x \mapsto \operatorname{rank}(\pi_{1,x})$ is lower semicontinuous at x_0, then $x \mapsto \operatorname{rank}(\pi_{2,x})$ is upper semicontinuous at x_0. Thus we conclude that both of these functions must be continuous at x_0. Since $x \mapsto \operatorname{rank}(\pi_{1,x})$ is integer-valued, it must therefore be constant. ■

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