Integrable and absolutely continuous vector-valued functions

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Abstract

The theory of integration for functions with values in a topological vector space is a bit of a messy subject. There are many different choices for the integral, most distinct from one another in general, and with no compelling reason to always adopt one definition or another as the "right" one. No attempt is made here to resolve this question of which integral is the "right" integral, but we overview a few of the common notions and prove a couple of useful properties for the notion of integrability by seminorm. For this notion of integrability, the completeness of the space of integrable functions with values in a complete locally convex space is established. This space is then easily seen to be isomorphic to the completion of the projective tensor product of the usual L^1 space of scalar functions with the vector space. Absolute continuity for this notion of integrability is also considered. Here it is shown that the familiar properties of differentiation for absolutely continuous scalar functions holds in the vector-valued case.

Keywords. Locally convex spaces, integration, differentiation

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1. Introduction

We let $(\mathcal{X}, \mathscr{A}, \mu)$ be a complete σ-finite measure space, and we let $(\mathsf{V}, \mathscr{O})$ be an Hausdorff locally convex topological $\mathbb{F}$-vector space, $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. We consider the matter of integrability of functions $f \colon \mathcal{X} \to \mathsf{V}$. In the case that $(\mathsf{V}, \mathscr{O})$ is a Banach space, Bochner [1933] introduced a notion of integral that closely mirrors the usual construction of the integral for $\mathbb{R}$-valued functions. This theory has many of the nice properties of the usual theory of integration for scalar-valued functions, but it is not clear how it is to be best extended to the case of a general Hausdorff locally convex $(\mathsf{V}, \mathscr{O})$. In the general locally convex case, Pettis [1938] gives a "weak" definition of an integral which has many nice properties. However, it lacks a compelling existence theory. Thomas [1975] considers the case when $(\mathsf{V}, \mathscr{O})$ is a Suslin locally convex topological vector space, and uses properties of Suslin spaces in a substantial way to give useful existence results for the Pettis integral. The notion of integrability by seminorm seems to have first been developed by Garnir, De Wilde, and Schmets [1972]. It has been considered by a few authors since its introduction,

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including [Blondia 1981, Chakraborty and Ali 1993, Marraffa 2006]; however, compared to the Bochner and Pettis integrals, it is a comparatively not well-known approach. It is this notion of integrability by seminorm that we mainly consider in this paper. In order to give context, we shall also consider in brief the more common strong (Bochner) and weak (Pettis) integrals, alongside integrability by seminorm.

Our interest is in answering two related questions concerning the space of functions that are integrable by seminorm:

1. is this space of functions complete?

2. is this space of functions isomorphic to $L^1(\mathcal{X}; \mathbb{F}) \overline{\otimes}_\pi V$, the completion of the projective tensor product of the scalar L^1-space with V?

The connection of spaces of integrable vector-valued functions to the projective tensor product is firmly established at the level of book literature [Jarchow 1981, Köthe 1979, Schaefer and Wolff 1999]. In the case that $(V, \mathcal{O})$ is a Banach space, the answer to both questions we ask is, "Yes" [Jarchow 1981, Corollary 15.7.5], [Köthe 1979, page 199-200]. For general locally convex Hausdorff topological vector spaces, the questions are not typically addressed, at least not definitively. For instance, the constructions of Jarchow [1981, Corollary 15.7.2], Köthe [1979, page 199-200], and Schaefer and Wolff [1999, Theorem III.6.5] *define* the space of integrable functions to be the completion of the space of simple functions, and this is fairly easily shown to be isomorphic to $L^1(\mathcal{X}; \mathbb{F}) \overline{\otimes}_\pi V$. The matter of when this completion is identifiable as some collection of integrable functions is sidestepped in this approach. Even with the more concrete representation of functions integrable by seminorm, Blondia [1981, Theorem 3.1] only establishes that the completion of the space of functions integrable by seminorm is topologically isomorphic to $L^1(\mathcal{X}; \mathbb{F}) \overline{\otimes}_\pi V$.

The bottom line is that the two questions above seem to be unanswered, and as we believe them to be of some degree of importance in "closing the loop" on the theory of integrability by seminorm, and also of the meaning of $L^1(\mathcal{X}; \mathbb{F}) \overline{\otimes}_\pi V$, we answer both questions in the affirmative when $(V, \mathcal{O})$ is complete.

As well, we prove that, if $\mathbb{I} \subseteq \mathbb{R}$ is an interval, if $f : \mathbb{I} \to V$ is integrable by seminorm, and if

$$F(t) = \int_{t_0}^{t} f(\tau) \, d\lambda(\tau) \quad (\lambda \text{ is Lebesgue measure}),$$

i.e., F is locally absolutely continuous, then F is almost everywhere differentiable and its derivative is almost everywhere equal to f.

2. Measurable and integrable vector-valued functions

In this section we review the notion of integrability by seminorm. Since the existing presentations of the properties of this integral are quite fragmented, we feel that there is some benefit in proving the most fundamental of these properties in a self-contained manner, although these results can be cobbled together from the existing literature with some effort. We include in our definitions the notions of strong (i.e., Bochner) integrability and weak (i.e., Pettis) integrability for the pedagogical purpose of comparison. We shall not establish important results for these notions, but will refer to the literature when appropriate.

Throughout the paper, we use $\mathbb{F}$ to denote either $\mathbb{R}$ or $\mathbb{C}$. Unless indicated to the contrary, $(\mathcal{X}, \mathscr{A}, \mu)$ is a complete σ-finite measure space and $(V, \mathcal{O})$ is an Hausdorff locally

convex topological $\mathbb{F}$-vector space. We shall denote the collection of continuous seminorms for $(\mathsf{V}, \mathscr{O})$ by $\mathscr{P}$. We denote the topological dual of $(\mathsf{V}, \mathscr{O})$ by V'. We denote the dual pairing between V' and V by $\langle \cdot ; \cdot \rangle$. For $A \subseteq \mathsf{V}$,

$$A^\circ = \{\lambda \in \mathsf{V}' \mid |\langle \lambda ; v \rangle| \leq 1, \ v \in A\}$$

is the polar of A.

2.1. Measurability.
We first consider a few notions of measurability and how they are related.

A **μ-simple function** is a function $s \colon \mathfrak{X} \to \mathsf{V}$ of the form

$$s = \sum_{j=1}^{k} \chi_{A_j} v_j,$$

where $A_1, \ldots, A_k \in \mathscr{A}$ are pairwise disjoint and satisfy $\mu(A_j) < \infty$, $j \in \{1, \ldots, k\}$, and where $v_1, \ldots, v_k \in \mathsf{V}$. We use χ_A to denote the characteristic function of a set A.

The common notions of measurability are then the following.

2.1 Definition: (Measurability for vector-valued functions) A function $f \colon \mathfrak{X} \to \mathsf{V}$ is:

(i)   **measurable** if $f^{-1}(B) \in \mathscr{A}$ for every Borel set $B \subseteq \mathsf{V}$;

(ii) **strongly measurable** if there exists a sequence $(s_j)_{j \in \mathbb{Z}_{>0}}$ of simple functions and a subset $Z \subseteq \mathfrak{X}$ of measure zero such that $(s_j(x))_{j \in \mathbb{Z}_{>0}}$ converges to $f(x)$ for $x \in \mathfrak{X} \setminus Z$;

(iii) **measurable by seminorm** if, for each $p \in \mathscr{P}$, there exists a subset $Z_p \subseteq \mathfrak{X}$ of measure zero and a sequence $(s_{p,j})_{j \in \mathbb{Z}_{>0}}$ of simple functions such that $(s_{p,j}(x))_{j \in \mathbb{Z}_{>0}}$ converges to $f(x)$ in (V, p) for every $x \in \mathfrak{X} \setminus Z_p$;

(iv) **weakly measurable** if $\mathfrak{X} \ni x \mapsto \lambda \circ f(x)$ is measurable for every $\lambda \in \mathsf{V}'$.   •

Strong measurability implies measurability, which in turn implies measurability by seminorm. The former is proved by proving that

$$\mathscr{A}' = \{T \subseteq \mathsf{V} \mid f^{-1}(T) \in \mathscr{A}\}.$$

is a σ-algebra containing $\mathscr{O}$, and noting that, if the sequence $(s_j)_{j \in \mathbb{Z}_{>0}}$ converges pointwise almost everywhere to f, then, for $\mathfrak{O} \in \mathscr{O}$,

$$f^{-1}(\mathfrak{O}) = \bigcup_{N \in \mathbb{Z}_{>0}} \bigcup_{j=N}^{\infty} s_j^{-1}(\mathfrak{O}) \in \mathscr{A},$$

modulo some fiddling with sets of measure zero. The latter follows by continuity of p. To prove that measurability by seminorm implies weak measurability, we consider the usual notions of functions whose image is, in some sense, separable.

2.2 Definition: (Essentially separably-valued) A function $f \colon \mathfrak{X} \to \mathsf{V}$ is:

(i) ***essentially separably-valued*** if there exists a subset $Z \subseteq \mathfrak{X}$ of measure zero and a countable subset $C \subseteq \mathsf{V}$ such that $f(\mathfrak{X} \setminus Z) \subseteq \mathrm{cl}(C)$;

(ii) ***essentially separably-valued by seminorm*** if, for each $p \in \mathscr{P}$, there exists a subset $Z_p \subseteq \mathfrak{X}$ of measure zero and a countable subset $C_p \subseteq \mathsf{V}$ such that $f(\mathfrak{X} \setminus Z_p) \subseteq \mathrm{cl}_p(C_p)$ (here "cl_p" means closure in the seminormed vector space (V, p)). $\quad\bullet$

Obviously, if $(\mathsf{V}, \mathscr{O})$ is separable, then both of these conditions hold for *any* V-valued function. The notion of being essentially separably-valued plays an important rôle in the classical presentation of the Bochner integral for Banach spaces, via the so-called Pettis Measurability Theorem. Here we present this in the setting of measurability by seminorm. The idea of the proof is an exercise on page 247 of [Garnir, De Wilde, and Schmets 1972], the central ideas for which are to be found at various places in the text.

2.3 Theorem: (Pettis Measurability Theorem) *For $f \colon \mathfrak{X} \to \mathsf{V}$, the following statements are equivalent:*

(i) f is measurable by seminorm;

(ii) f is (a) weakly measurable and (b) essentially separably-valued by seminorm.

Proof: (i) $\implies$ (ii) Let $\lambda \in \mathsf{V}'$ and note that $p = |\lambda|$ is a continuous seminorm. By hypothesis, let $Z_p \subseteq \mathfrak{X}$ have measure zero and let $(s_{p,j})_{j \in \mathbb{Z}_{>0}}$ be a sequence of simple functions for which $(s_{p,j}(x))_{j \in \mathbb{Z}_{>0}}$ converges to $f(x)$ in (V, p) for every $x \in \mathfrak{X} \setminus Z_p$. Then, for $x \in \mathfrak{X} \setminus Z_p$,

$$\lim_{j \to \infty} |\langle \lambda; f(x) \rangle - \langle \lambda; s_{p,j}(x) \rangle| = \lim_{j \to \infty} |\langle \lambda; f(x) - s_{p,j}(x) \rangle|$$
$$= \lim_{j \to \infty} p(f(x) - s_{p,j}(x)) = 0.$$

Thus $(\langle \lambda; s_{p,j}(x) \rangle)_{j \in \mathbb{Z}_{>0}}$ converges to $\langle \lambda; f(x) \rangle$ for $x \in \mathfrak{X} \setminus Z_p$. Thus $\lambda \circ f$ is measurable, and so we have weak measurability of f.

Now let p be a continuous seminorm, let $Z_p \subseteq \mathfrak{X}$ have measure zero, and let $(s_{p,j})_{j \in \mathbb{Z}_{>0}}$ be a sequence of simple functions for which $(s_{p,j}(x))_{j \in \mathbb{Z}_{>0}}$ converges to $f(x)$ in (V, p) for every $x \in \mathfrak{X} \setminus Z_p$. Since $s_{p,j}$ takes values in a finite subset of V, the set $C_p = \cup_{j \in \mathbb{Z}_{>0}} \mathrm{image}(s_{p,j})$ is countable. The definition of measurability by seminorm ensures that $\mathrm{cl}_p(C_p) = f(\mathfrak{X} \setminus Z_p)$; thus f is essentially separably-valued by seminorm.

(ii) $\implies$ (i) Let p be a continuous seminorm and let $Z_p \subseteq \mathfrak{X}$ have measure zero and let $C_p \subseteq \mathsf{V}$ be countable and such that $f(\mathfrak{X} \setminus Z_p) \subseteq \mathrm{cl}_p(C_p)$. Let us write the points in C_p as $(v_j)_{j \in \mathbb{Z}_{>0}}$. Let $\mathcal{N} = p^{-1}([0,1])$ and note that, for $v \in \mathsf{V}$,

$$p(v) = \inf\{r \in \mathbb{R}_{>0} \mid v \in r\mathcal{N}\}$$
$$= \inf\{r \in \mathbb{R}_{>0} \mid r^{-1}v \in \mathcal{N}\}$$
$$= \inf\{r \in \mathbb{R}_{>0} \mid |\langle \lambda; r^{-1}v \rangle| \le 1, \ \lambda \in \mathcal{N}^\circ\}$$
$$= \inf\{r \in \mathbb{R}_{>0} \mid |\langle \lambda; v \rangle| \le r, \ \lambda \in \mathcal{N}^\circ\}$$
$$= \sup\{|\langle \lambda; v \rangle| \mid \lambda \in \mathcal{N}^\circ\}. \tag{2.1}$$

It is now convenient to prove a lemma.

1 Lemma: *With all of the preceding notation in place, there exists a countable subset $\Lambda \subseteq \mathcal{N}^\circ$ such that, for $x \in \mathcal{X} \setminus Z_p$, we have*

$$p \circ f(x) = \sup\{\langle \lambda; f(x) \rangle \mid \lambda \in \Lambda\}.$$

Proof: We are using the notation $C_p = \{v_j \mid j \in \mathbb{Z}_{>0}\}$. Consider a mapping

$$\begin{aligned} \Phi_k \colon \mathcal{N}^\circ &\to \mathbb{F}^k \\ \lambda &\mapsto (\langle \lambda; v_1 \rangle, \ldots, \langle \lambda; v_k \rangle). \end{aligned}$$

For each $k \in \mathbb{Z}_{>0}$, let $(\lambda_{k,l})_{l \in \mathbb{Z}_{>0}}$ be such that $\{\Phi_k(\lambda_{k,l}) \mid l \in \mathbb{Z}_{>0}\}$ is dense in image(Φ_k), this being possible since $\mathbb{F}^n$ is separable and since subsets of separable metric spaces are separable.

Now, for $\lambda \in \mathcal{N}^\circ$ and $k \in \mathbb{Z}_{>0}$, we can choose $l_k \in \mathbb{Z}_{>0}$ large enough that

$$|\langle \lambda_{k,l_k}; v_j \rangle - \langle \lambda; v_j \rangle| < \frac{1}{k}, \qquad j \in \{1, \ldots, k\}.$$

Let us take $\lambda_k = \lambda_{k,l_k}$, and verify that $\Lambda = \{\lambda_k \mid k \in \mathbb{Z}_{>0}\}$ meets the criteria of the lemma.

Let $x \in \mathcal{X} \setminus Z_p$ and let $\epsilon \in \mathbb{R}_{>0}$. Since $\mathrm{cl}_p(C_p) = f(\mathcal{X} \setminus Z_p)$, let $j \in \mathbb{Z}_{>0}$ be such that

$$|p \circ f(x) - p(v_j)| < \frac{\epsilon}{4}.$$

By (2.1), let $\lambda \in \mathcal{N}^\circ$ be such that

$$|p(v_j) - \langle \lambda; v_j \rangle| < \frac{\epsilon}{4}.$$

By the previous paragraph, let $k \in \mathbb{Z}_{>0}$ be such that

$$|\langle \lambda; v_j \rangle - \langle \lambda_k; v_j \rangle| < \frac{\epsilon}{4}.$$

Since $\lambda_k \in \mathcal{N}^\circ$, we have

$$|\langle \lambda_k; v_j \rangle - \langle \lambda_k; f(x) \rangle| = |\langle \lambda_k; f(x) - v_j \rangle| \le p(f(x) - v_j),$$

by (2.1). Then we compute

$$\begin{aligned} |p \circ f(x) - \langle \lambda_k; f(x) \rangle| &\le |p \circ f(x) - p(v_j)| + |p(v_j) - \langle \lambda; v_j \rangle| \\ &\quad + |\langle \lambda; v_j \rangle - |\langle \lambda_k; v_j \rangle|| + |\langle \lambda_k; v_j \rangle - \langle \lambda_k; f(x) \rangle| < \epsilon, \end{aligned}$$

which is as desired. ▼

By the lemma and by [Cohn 2013, Proposition 2.1.5(a)], we conclude that $p \circ f$ is measurable.

Now, for $j, k \in \mathbb{Z}_{>0}$, denote

$$A_{j,k} = \{x \in \mathcal{X} \setminus Z_p \mid f(x) \in v_j + k^{-1}\mathcal{N}\} = \{x \in \mathcal{X} \setminus Z_p \mid p(f(x) - v_j) \le k^{-1}\},$$

noting that measurability of $p \circ f$ implies that $A_{j,k} \in \mathscr{A}$, $j, k \in \mathbb{Z}_{>0}$. Density (with respect to the seminorm p) of C_p in $f(\mathcal{X} \setminus Z_p)$ implies that

$$f(\mathcal{X} \setminus Z_p) = \bigcup_{j \in \mathbb{Z}_{>0}} f(A_{j,k}), \qquad k \in \mathbb{Z}_{>0}.$$

Now define $A'_{1,k} = A_{1,k}$, $k \in \mathbb{Z}_{>0}$, and recursively take

$$A'_{j,k} = A_{j,k} \setminus \left(\bigcup_{l=1}^{k-1} A'_{j,l} \right), \qquad j \geq 2, \ k \in \mathbb{Z}_{>0},$$

and note that the sets $A'_{j,k}$, $j, k \in \mathbb{Z}_{>0}$, are themselves measurable, and moreover pairwise disjoint. We also still have

$$f(\mathcal{X} \setminus Z_p) = \bigcup_{j \in \mathbb{Z}_{>0}} f(A'_{j,k}), \qquad k \in \mathbb{Z}_{>0}.$$

Also, $v_j \in f(A'_{j,k})$, $j, k \in \mathbb{Z}_{>0}$. We then take

$$s_k \colon \mathcal{X} \to \mathsf{V}$$

$$x \mapsto \sum_{j=1}^{k} v_j \chi_{A'_{j,k}},$$

noting that s_k is a simple function by virtue of pairwise disjointness of the sets $A'_{j,k}$, $j, k \in \mathbb{Z}_{>0}$. Moreover, because the sets $A'_{j,k}$, $j, k \in \mathbb{Z}_{>0}$, are pairwise disjoint, if $x \in A'_{j,k}$ for some $k \in \mathbb{Z}_{>0}$ and some $j \in \{1, \dots, k\}$, then $s_k(x) = v_j$.

It remains to show that $(s_k(x))_{k \in \mathbb{Z}_{>0}}$ converges to $f(x)$ in (V, p) for every $x \in \mathcal{X} \setminus Z_p$. To see this, let $x \in \mathcal{X} \setminus Z_p$, let $\epsilon \in \mathbb{R}_{>0}$, and let $j \in \mathbb{Z}_{>0}$ be such that $p(f(x) - v_j) < \epsilon$. Let $N \geq j$ be such that $f(x) \in v_j + N^{-1}\mathcal{N}$. Then, for $k \geq N$,

$$x \in A'_{j,k} \implies s_k(x) = v_j \implies p(f(x) - s_k(x)) < \epsilon,$$

as desired. ∎

For Suslin spaces, Thomas [1975] shows that all four notions of measurability are equivalent and, furthermore, that there is a countable subset of V' for which it suffices to check weak measurability. When $(\mathsf{V}, \mathscr{O})$ is metrisable, then measurability by seminorm implies strong measurability.

2.4 Proposition: (Measurability by seminorm for metrisable spaces) *If $(\mathsf{V}, \mathscr{O})$ is metrisable, then $f \colon \mathcal{X} \to \mathsf{V}$ is measurable by seminorm if and only if it is strongly measurable.*

Proof: Let $(p_j)_{j \in \mathbb{Z}_{>0}}$ be a countable set of continuous seminorms generating the topology $\mathscr{O}$. Let $Z_{p_j} \subseteq \mathcal{X}$ be sets of measure zero and let $C_{p_j} \subseteq \mathsf{V}$ be countable sets such that $f(\mathcal{X} \setminus C_{p_j}) \subseteq \mathrm{cl}_{p_j}(C_{p_j})$, $j \in \mathbb{Z}_{>0}$. Let $Z = \cup_{j=1}^{k} Z_{p_j}$ and $C = \cup_{l=1}^{k} C_{p_j}$. Let p be a continuous seminorm for $(\mathsf{V}, \mathscr{O})$ and let $M \in \mathbb{R}_{>0}$ and $j_1, \dots, j_k \in \mathbb{Z}_{>0}$ be such that

$$p(v) \leq M \max\{p_{j_1}(v), \dots, p_{j_k}(v)\}.$$

Therefore, if $\mathcal{N}_{p_{j_l}} = p_{j_l}^{-1}([0,1))$ and $\mathcal{N}_p = p^{-1}([0,1))$ are the absolutely convex 0-neighbourhoods, we have

$$M^{-1}\mathcal{N}_{p_{j_l}} \subseteq \mathcal{N}_p \implies \mathrm{cl}_{p_{j_l}}(C_{p_{j_l}}) \subseteq \mathrm{cl}_p(C_{p_{j_l}}), \qquad l \in \{1, \ldots, k\}.$$

Now we have

$$x \in \mathcal{X} \setminus Z \implies x \in \mathcal{X} \setminus Z_{j_l} \implies f(x) \in \mathrm{cl}_{p_{j_l}}(C_{p_{j_l}}) \subseteq \mathrm{cl}_p(C_{p_{j_l}}) \subseteq \mathrm{cl}_p(C),$$

which gives the result. $\blacksquare$

2.2. Integrability. Next we consider notions of integrability for vector-valued functions. We again consider other notions that that of integrability by seminorm, just for pedagogical reasons.

For a simple function $s = \sum_{j=1}^k \chi_{A_j} v_j$, the *integral* of s is

$$\int_{\mathcal{X}} s \, \mathrm{d}\mu = \sum_{j=1}^k \mu(A_j) v_j \in \mathsf{V}.$$

It is routine to show that this definition of integral is independent of the particular representation of a simple function.

The notions of integrability that we write down are then the following.

2.5 Definition: (Integrability for vector-valued functions) Let $f \colon \mathcal{X} \to \mathsf{V}$.

(i) The function f is *strongly integrable* if there exist a sequence $(s_j)_{j \in \mathbb{Z}_{>0}}$ of simple functions and a subset $Z \subseteq \mathcal{X}$ of measure zero such that:

(a) $(s_j(x))_{j \in \mathbb{Z}_{>0}}$ converges to $f(x)$ for $x \in \mathcal{X} \setminus Z$;

(b) for each $p \in \mathscr{P}$ and for each $j \in \mathbb{Z}_{>0}$, the function $p \circ (f - s_j)$ is integrable and

$$\lim_{j \to \infty} \int_{\mathcal{X}} p \circ (f - s_j) \, \mathrm{d}\mu = 0;$$

(c) for $A \in \mathscr{A}$, the limit $\lim_{j \to \infty} \int_A s_j \, \mathrm{d}\mu$ exists.

We denote the *strong integral* of f by

$$(S)\int_{\mathcal{X}} f \, \mathrm{d}\mu = \lim_{j \to \infty} \int_{\mathcal{X}} s_j \, \mathrm{d}\mu.$$

(ii) The function f is *integrable by seminorm* if, for each $p \in \mathscr{P}$ and for each $A \in \mathscr{A}$, there exists a sequence $(s_{p,j})_{j \in \mathbb{Z}_{>0}}$ of simple functions, a subset $Z_p \subseteq \mathcal{X}$ of measure zero, and $I_A(f) \in \mathsf{V}$ such that:

(a) $(s_{p,j}(x))_{j \in \mathbb{Z}_{>0}}$ converges to $f(x)$ in (V, p) for $x \in \mathcal{X} \setminus Z_p$;

(b) $x \mapsto p(f(x) - s_{p,j}(x))$ is integrable for $j \in \mathbb{Z}_{>0}$ and

$$\lim_{j \to \infty} \int_{\mathcal{X}} p \circ (f - s_{p,j}) \, \mathrm{d}\mu = 0;$$

(c) we have

$$\lim_{j \to \infty} p\left(\int_A s_{p,j} \, \mathrm{d}\mu - I_A(f)\right) = 0.$$

We denote the **integral** of f by

$$\int_{\mathfrak{X}} f \, \mathrm{d}\mu = I_{\mathfrak{X}}(f).$$

(iii) The function f is **weakly integrable** if, for each $A \in \mathscr{A}$, there exists $I_A(f) \in \mathsf{V}$ such that, for each $\lambda \in \mathsf{V}'$:

   (a) $\lambda \circ f$ is measurable;

   (b) $\lambda \circ f$ is integrable;

   (c) we have

$$\langle \lambda; I_A(f) \rangle = \int_A \lambda \circ f \, \mathrm{d}\mu.$$

We denote the **weak integral** of f by

$$(W) \int_{\mathfrak{X}} f \, \mathrm{d}\mu = I_{\mathfrak{X}}(f). \qquad\qquad \bullet$$

For our purposes, as we are mainly interested in integrability by seminorm we simply refer to the "integral" as that version of the integral arising in this case, just for the sake of brevity. The weak integral is the classical Pettis integral. The strong integral gives the classical Bochner integral when $(\mathsf{V}, \mathscr{O})$ is a Banach space. In the general case of a locally convex space, there seems to not be much known about the strong integral. An approach using approximation by nets of simple functions is presented by Beckmann and Deitmar [2011]. A difficulty with this approach concerns measurability, where nets interact poorly with measurability. This is overcome by Beckmann and Deitmar by making a blanket assumption of measurability. This, however, makes things like the Dominated Convergence Theorem problematic.

One readily verifies, using the triangle inequality, that if f is integrable by seminorm (or, indeed, strongly integrable), then $p \circ f$ is integrable and

$$p\left(\int_{\mathfrak{X}} f \, \mathrm{d}\mu\right) \leq \int_{\mathfrak{X}} p \circ f \, \mathrm{d}\mu, \qquad p \in \mathscr{P}. \tag{2.2}$$

It is also relatively easy to establish that, if f is integrable by seminorm (or, indeed, strongly or weakly integrable), and if $\phi \colon \mathsf{V} \to \mathsf{U}$ is a continuous linear map, then

$$\int_{\mathfrak{X}} \phi \circ f \, \mathrm{d}\mu = \phi\left(\int_{\mathfrak{X}} f \, \mathrm{d}\mu\right). \tag{2.3}$$

In terms of comparing these integrals, it is obvious that strong integrability implies integrability by seminorm, and that the values for the two integrals agree. The converse assertion also clearly holds for normed vector spaces. In the following result we give the relationship between integrability by seminorm and weak integrability. As with the Pettis Measurability Theorem, this result can be pieced together from material in [Garnir, De Wilde, and Schmets 1972], entering the presentation around page 253.

2.6 Theorem: (Comparison of integrals) *If $f\colon \mathfrak{X} \to \mathsf{V}$ is measurable by seminorm, then the following are equivalent:*

  (i) f is integrable by seminorm;

  (ii) f is weakly integrable and $p \circ f$ is integrable for every continuous seminorm p for $(\mathsf{V}, \mathscr{O})$.

Moreover, if either of the two conditions is satisfied, then

$$\int_{\mathfrak{X}} f \, \mathrm{d}\mu = (W) \int_{\mathfrak{X}} f \, \mathrm{d}\mu.$$

Proof: (i) $\implies$ (ii) By (2.2), we know that $p \circ f$ is integrable for every $p \in \mathscr{P}$. By (2.3), $\lambda \circ f$ is integrable and, for each $\lambda \in \mathsf{V}'$,

$$\int_{\mathfrak{X}} \lambda \circ f \, \mathrm{d}\mu = \lambda \left(\int_{\mathfrak{X}} f \, \mathrm{d}\mu \right).$$

The definition of the weak integral then ensures that f is weakly integrable with weak integral equal to the integral.

(ii) $\implies$ (i) First let us prove a lemma.

1 Lemma: *If $f\colon \mathfrak{X} \to \mathsf{V}$ is measurable by seminorm and is such that $p \circ f$ is integrable for every $p \in \mathscr{P}$, then, for every $p \in \mathscr{P}$, there exists a sequence $(s_{p,j})_{j \in \mathbb{Z}_{>0}}$ of simple functions and a subset $Z_p \subseteq \mathfrak{X}$ of measure zero such that*

$$\lim_{j \to \infty} p(f(x) - s_{p,j}(x)) = 0, \qquad x \in \mathfrak{X} \setminus Z_p,$$

and

$$\lim_{j \to \infty} \int_{\mathfrak{X}} p \circ (f - s_{p,j}) \, \mathrm{d}\mu = 0.$$

Proof: We use σ-finiteness of the measure to define $F\colon \mathfrak{X} \to \mathbb{R}_{>0}$ by

$$F = \sum_{j=1}^{\infty} \frac{1}{1 + 2^j \mu(A_j)} \chi_{A_j},$$

where $\mathfrak{X} = \cup_{j \in \mathbb{Z}_{>0}} A_j$, $\mu(A_j) < \infty$, and the family A_j, $j \in \mathbb{Z}_{>0}$, is pairwise disjoint. Then F is measurable and, since

$$\int_{\mathfrak{X}} \sum_{j=1}^{N} \frac{1}{1 + 2^j \mu(A_j)} \chi_{A_j} \, \mathrm{d}\mu < \sum_{j=1}^{N} \frac{1}{2^j} \leq 1,$$

by the Beppo–Levi Theorem [Cohn 2013, Corollary 2.4.2], we can conclude that F is integrable. One then defines a finite measure μ_F on $(\mathfrak{X}, \mathscr{A})$ by

$$\mu_F(A) = \int_{\mathfrak{X}} F \chi_A \, \mathrm{d}\mu,$$

and one can easily see that each of the measures μ and μ_F is absolutely continuous with respect to the other.

Measurability of f by seminorm means that, for each $p \in \mathscr{P}$, there exists a sequence $(t_{p,j})_{j \in \mathbb{Z}_{>0}}$ of simple functions and a subset $Z_p \subseteq \mathcal{X}$ of measure zero such that

$$\lim_{j \to \infty} p(f(x) - t_{p,j}(x)) = 0, \qquad x \in \mathcal{X} \setminus Z_p.$$

Let

$$A_{p,j} = \{x \in \mathcal{X} \mid p \circ t_{p,j}(x) \le p \circ f(x) + F(x)\}$$

and let $s_{p,j} = \chi_{A_{p,j}} t_{p,j}$, and note that $s_{p,j}$ is a simple function for each $j \in \mathbb{Z}_{>0}$. By [Cohn 2013, Proposition 2.1.3], $A_{p,j} \in \mathscr{A}$, $j \in \mathbb{Z}_{>0}$. We then have

$$\lim_{j \to \infty} |p \circ f(x) - p \circ t_{p,j}(x)| \le \lim_{j \to \infty} p(f(x) - t_{p,j}(x)) = 0, \qquad x \in \mathcal{X} \setminus Z_p,$$

and so $\mathcal{X} \setminus Z_p \subseteq \cup_{j \in \mathbb{Z}_{>0}} A_{p,j}$. Therefore,

$$\lim_{j \to \infty} p(f(x) - s_{p,j}(x)) = 0, \qquad x \in \mathcal{X} \setminus Z_p.$$

Since

$$p(f(x) - s_{p,j}(x)) \le p \circ f(x) + F(x), \qquad x \in \mathcal{X} \setminus Z_p, \ j \in \mathbb{Z}_{>0},$$

and since the right-hand side is integrable by hypothesis, the Dominated Convergence Theorem [Cohn 2013, Theorem 2.4.5] then gives

$$\lim_{j \to \infty} \int_{\mathcal{X}} p \circ (f - s_{p,j}) \, \mathrm{d}\mu = 0,$$

which is the result.   ▼

Continuing with the proof, let $(s_{p,j})_{j \in \mathbb{Z}_{>0}}$ be a sequence of simple functions satisfying the conclusions of the lemma. This already gives the first of the conditions for integrability by seminorm. As for the second,

$$p(f(x) - s_{p,j}(x)) \le p \circ f(x) + p \circ s_{p,j}(x), \qquad x \in \mathcal{X} \setminus Z_p, \ j \in \mathbb{Z}_{>0},$$

which gives integrability of the left-hand side, giving the second of the conditions of integrability by seminorm by the lemma.

It remains to verify the third of the conditions for integrability by seminorm, and that the value of the integral agrees with the weak integral. Let $I_{\mathcal{X}}(f)$ be the weak integral of f. Weak integrability implies that

$$\lambda(I_{\mathcal{X}}(f)) = \int_{\mathcal{X}} \lambda \circ f \, \mathrm{d}\mu,$$

which gives

$$\lambda(I_{\mathcal{X}}(f) - I_{\mathcal{X}}(s_{p,j})) = \int_{\mathcal{X}} \lambda \circ (f - s_{p,j}) \, \mathrm{d}\mu.$$

Using (2.1), for a continuous seminorm p with associated 0-filter neighbourhood $\mathcal{N} = p^{-1}([0,1])$, we have

$$\begin{aligned}
p(I_{\mathcal{X}}(f) - I_{\mathcal{X}}(s_{p,j})) &= \sup\{|\langle \lambda; I_{\mathcal{X}}(f) - I_{\mathcal{X}}(s_{p,j})\rangle| \mid \lambda \in \mathcal{N}^{\circ}\} \\
&\le \sup\left\{ \int_{\mathcal{X}} |\langle \lambda; f - s_{p,j}\rangle| \, \mathrm{d}\mu \ \middle| \ \lambda \in \mathcal{N}^{\circ} \right\} \\
&= \int_{\mathcal{X}} p \circ (f - s_{p,j}) \, \mathrm{d}\mu,
\end{aligned}$$

By the lemma, letting $j \to \infty$ we get conclusions of the theorem.   ■

A function $f\colon \mathfrak{X} \to \mathsf{V}$ is **integrally bounded** if $p \circ f$ is integrable for every continuous seminorm p for $(\mathsf{V}, \mathscr{O})$. The following theorem shows the importance of this notion for integrability by seminorm for functions taking values in complete spaces. Our proof here follows [Blondia 1981, Theorem 2.10].

2.7 Theorem: (Integrability for complete locally convex topological vector spaces) *If $(\mathsf{V}, \mathscr{O})$ is complete, the following are equivalent for a function $f\colon \mathfrak{X} \to \mathsf{V}$:*

   (i) f is integrable by seminorm;

   (ii) f is measurable by seminorm and integrally bounded.

Proof: (i) $\Longrightarrow$ (ii) If f is integrable (by seminorm), then it is measurable by seminorm, by definition. By (2.2), f is also integrally bounded.

   (ii) $\Longrightarrow$ (i) By the Lemma from the proof of Theorem 2.6, for $p \in \mathscr{P}$, there exists a subset $Z_p \subseteq \mathfrak{X}$ of measure zero and a sequence $(s_{p,j})_{j \in \mathbb{Z}_{>0}}$ of simple functions such that

$$\lim_{j \to \infty} p(f(x) - s_{p,j}(x)) = 0, \qquad x \in \mathfrak{X} \setminus Z_p,$$

and

$$\lim_{j \to \infty} \int_{\mathfrak{X}} p \circ (f - s_{p,j}) \, \mathrm{d}\mu = 0. \tag{2.4}$$

Let us define a directed set

$$I = \{(P,k) \mid P \subseteq \mathscr{P} \text{ is finite and } k \in \mathbb{Z}_{>0}\},$$

with the partial order

$$(P_1, k_1) \preceq (P_2, k_2) \iff P_1 \subseteq P_2, \; k_1 \leq k_2.$$

Let $P \subseteq \mathscr{P}$ be finite and let $p_P \in \mathscr{P}$ be such that

$$p(v) \leq p_P(v), \qquad v \in \mathsf{V}, \; p \in P.$$

By (2.4), for $k \in \mathbb{Z}_{>0}$, let $s_{P,k}\colon \mathfrak{X} \to \mathsf{V}$ be a simple function for which

$$\int_{\mathfrak{X}} p_P \circ (f - s_{P,k}) \, \mathrm{d}\mu < \frac{1}{k}.$$

We claim that, for $A \in \mathscr{A}$, $(\int_A s_{P,k} \, \mathrm{d}\mu)_{(P,k) \in I}$ is a Cauchy net in V. Indeed, let q be a continuous seminorm and let $\epsilon \in \mathbb{R}_{>0}$. Let $(P_0, k_0) \in I$ be such that $q \in P_0$ and such that $k_0^{-1} < \frac{\epsilon}{2}$. Then, for $(P_0, k_0) \preceq (P_1, k_1), (P_2, k_2)$, we have

$$q\left(\int_A (s_{P_1,k_1} - s_{P_2,k_2}) \, \mathrm{d}\mu\right) \leq \int_A q \circ (f - s_{P_1,k_1}) \, \mathrm{d}\mu + \int_A q \circ (f - s_{p_2,k_2}) \, \mathrm{d}\mu$$

$$\leq \int_{\mathfrak{X}} p_{P_1} (f - s_{P_1,k_1}) \, \mathrm{d}\mu + \int_{\mathfrak{X}} p_{P_2} (f - s_{P_2,k_2}) \, \mathrm{d}\mu < \epsilon,$$

showing that, indeed, the net $(\int_A s_{P,k} \, \mathrm{d}\mu)_{(P,k) \in I}$ is Cauchy. We conclude, then, from completeness of $(\mathsf{V}, \mathscr{O})$ that there exists $v_A \in \mathsf{V}$ such that

$$\lim_{(P,k) \in I} \int_A s_{P,k} \, \mathrm{d}\mu = v_A.$$

Moreover, let q be a continuous seminorm and let $\epsilon \in \mathbb{R}_{>0}$. Let $(P, k) \in I$ be such that $q \in P$, $k^{-1} < \frac{\epsilon}{3}$, and

$$\int_{\mathcal{X}} p_P \circ (f - s_{P,k}) \, \mathrm{d}\mu < \frac{\epsilon}{3}, \quad q\left(\int_A s_{P,k} \, \mathrm{d}\mu - v_A\right) < \frac{\epsilon}{3}.$$

Also let $s_{q,j} = s_{\{q\},j}$, $j \in \mathbb{Z}_{>0}$. Then, for $j \geq k$,

$$q\left(\int_A s_{q,j} \, \mathrm{d}\mu - v_A\right) \leq \int_A q \circ (f - s_{q,j}) \, \mathrm{d}\mu + \int_A q \circ (f - s_{P,k}) \, \mathrm{d}\mu + q\left(\int_A s_{P,k} \, \mathrm{d}\mu - v_A\right)$$

$$\leq \int_{\mathcal{X}} p_P \circ (f - s_{P,k}) \, \mathrm{d}\mu + \int_{\mathcal{X}} p_P \circ (f - s_{P,k}) \, \mathrm{d}\mu + q\left(\int_A s_{P,k} \, \mathrm{d}\mu - v_A\right) < \epsilon,$$

giving integrability by seminorm. ∎

A useful version of the Dominated Convergence Theorem holds for integrability by seminorm.

2.8 Theorem: (Dominated Convergence Theorem) *For a sequence $f_j \colon \mathcal{X} \to \mathsf{V}$, $j \in \mathbb{Z}_{>0}$, assume the following:*

(i) f_j, $j \in \mathbb{Z}_{>0}$, is integrable by seminorm;

(ii) there exists $f \colon \mathcal{X} \to \mathsf{V}$ and a subset $Z \subseteq \mathsf{V}$ of measure zero such that $\lim_{j \to \infty} f_j(x) = f(x)$ for $x \in \mathcal{X} \setminus Z$;

(iii) there exists $g \in \mathrm{L}^1(\mathcal{X}; \mathbb{R}_{\geq 0})$ and, for each $p \in \mathscr{P}$, subsets $Z_{p,j} \subseteq \mathcal{X}$, $j \in \mathbb{Z}_{>0}$, such that $p \circ f_j(x) \leq g(x)$ for $x \in \mathcal{X} \setminus Z_{p,j}$, $j \in \mathbb{Z}_{>0}$.

Then f is integrable by seminorm and

$$\lim_{j \to \infty} \int_{\mathcal{X}} f_j \, \mathrm{d}\mu = \int_{\mathcal{X}} f \, \mathrm{d}\mu.$$

Proof: Let us first show that f is measurable by seminorm. Since f_j, $j \in \mathbb{Z}_{>0}$, is measurable by seminorm, by Theorem 2.3, f_j is essentially separably-valued by seminorm. Thus, for each $j \in \mathbb{Z}_{>0}$ and each continuous seminorm p, there is a subset $Z'_{p,j} \subseteq \mathcal{X}$ of measure zero and a countable subset $C_{p,j} \subseteq \mathsf{V}$ such that $f_j(\mathcal{X} \setminus Z'_{p,j}) \subseteq \mathrm{cl}_p(C_{p,j})$. Let $C_p = \cup_{j \in \mathbb{Z}_{>0}} C_{p,j}$ and $Z'_p = Z \cup_{j \in \mathbb{Z}_{>0}} Z'_{p,j}$ so that, $f(\mathcal{X} \setminus Z'_p) \subseteq \mathrm{cl}(C_p)$. Thus f is essentially separably-valued by seminorm. Again by Theorem 2.3, f_j is weakly measurable for each $j \in \mathbb{Z}_{>0}$. Thus, for each $\lambda \in \mathsf{V}'$, $\lambda \circ f_j$ is measurable. Therefore, for $x \in \mathcal{X} \setminus Z$,

$$\lambda(f(x)) = \lambda\left(\lim_{j \to \infty} f_j(x)\right) = \lim_{j \to \infty} \lambda(f_j(x)),$$

and so $\lambda \circ f$ is the almost everywhere limit of a sequence of measurable functions, whence it is measurable by [Cohn 2013, Proposition 2.1.5]. Thus f is weakly measurable, and hence measurable by seminorm according to Theorem 2.3.

Now, given that $p \circ f_j(x) \leq g(x)$ for $x \in \mathcal{X} \setminus Z_{p,j}$ and $j \in \mathbb{Z}_{>0}$, and given that $\lim_{j \to \infty} f_j(x) = f(x)$ for $x \in \mathcal{X} \setminus Z$, we have

$$p \circ f(x) = \lim_{j \to \infty} p \circ f_j(z) \leq g(x), \qquad x \in \mathcal{X} \setminus (Z \cup_{j \in \mathbb{Z}_{>0}} Z_{p,j}).$$

As we proved during the course of the proof of Theorem 2.3, $p \circ f$ is measurable, whence $p \circ f$ is integrable. Therefore, by the Lemma from the proof of Theorem 2.6, there exists a sequence $(s_{p,j})_{j \in \mathbb{Z}_{>0}}$ of simple functions and a subset $Z_p \subseteq \mathfrak{X}$ of measure zero such that

$$\lim_{j \to \infty} p(f(x) - s_{p,j}(x)) = 0, \qquad x \in \mathfrak{X} \setminus Z_p,$$

and

$$\lim_{j \to \infty} \int_{\mathfrak{X}} p \circ (f - s_{p,j}) \, \mathrm{d}\mu = 0.$$

This is gives the second of the conditions for integrability by seminorm.

Finally, by (2.2), we have

$$p \left(\int_{\mathfrak{X}} f \, \mathrm{d}\mu - \int_{\mathfrak{X}} f_j \, \mathrm{d}\mu \right) \leq \int_{\mathfrak{X}} p \circ (f - f_j) \, \mathrm{d}\mu.$$

By the usual Dominated Convergence Theorem [Cohn 2013, Theorem 2.4.5], which holds since

$$p(f(x) - f_j(x)) \leq 2g(x), \qquad x \in \mathfrak{X} \setminus (Z \cup \cup_{j \in \mathbb{Z}_{>0}} Z_{p,j}), \ j \in \mathbb{Z}_{>0},$$

we have

$$\lim_{j \to \infty} \int_{\mathfrak{X}} p \circ (f - f_j) \, \mathrm{d}\mu = 0.$$

As this hold for every continuous seminorm p, we obtain

$$\int_{\mathfrak{X}} f \, \mathrm{d}\mu = \lim_{j \to \infty} \int_{\mathfrak{X}} f_j \, \mathrm{d}\mu$$

as desired. ∎

3. The space $\mathsf{L}^1(\mathfrak{X}; \mathsf{V})$

In this section we answer the two questions posed in the introduction. First let us recall the standard constructions, a good presentation of which is that of Jarchow [1981, §15.7].

The set of simple functions is denoted by $\widehat{\mathsf{S}}(\mathfrak{X}; \mathsf{V})$. Denote

$$\mathsf{Z}(\mathfrak{X}; \mathsf{V}) = \{ s \in \widehat{\mathsf{S}}(\mathfrak{X}; \mathsf{V}) \mid \mu(\{ x \in \mathfrak{X} \mid s(x) \neq 0 \}) = 0 \}$$

and

$$\mathsf{S}(\mathfrak{X}; \mathsf{V}) = \widehat{\mathsf{S}}(\mathfrak{X}; \mathsf{V}) / \mathsf{Z}(\mathfrak{X}; \mathsf{V}),$$

i.e., $\mathsf{S}(\mathfrak{X}; \mathsf{V})$ is the set of equivalence classes of simple functions agreeing almost everywhere. For a continuous seminorm p for $(\mathsf{V}, \mathscr{O})$, define a seminorm

$$p_{\mathfrak{X}, \mathsf{V}} \colon \mathsf{S}(\mathfrak{X}; \mathsf{V}) \to \mathbb{R}$$

$$[s] \mapsto \int_{\mathfrak{X}} p \circ s \, \mathrm{d}\mu,$$

noting that the definition is obviously independent of the choice of representative from $[s]$. We then take the locally convex topology on $\mathsf{S}(\mathfrak{X}; \mathsf{V})$ defined by the seminorms $p_{\mathfrak{X}, \mathsf{V}}$ as p runs

over the collection of continuous seminorms for $(\mathsf{V}, \mathscr{O})$. In the scalar-valued case, i.e., when $\mathsf{V} = \mathbb{F}$, then we have the norm

$$p_1([s]) = \int_{\mathcal{X}} |s| \, \mathrm{d}\mu, \qquad [s] \in \mathrm{S}(\mathcal{X}; \mathbb{F}),$$

giving the usual locally convex topology for the space of equivalence classes of scalar-valued simple functions. Jarchow [1981, Theorem 15.7.1] shows that the bilinear mapping

$$(s, v) \mapsto (t \mapsto s(t)v)$$

defines a topological isomorphism

$$\mathrm{S}(\mathcal{X}; \mathsf{V}) \simeq \mathrm{S}(\mathcal{X}; \mathbb{F}) \otimes_\pi \mathsf{V},$$

$\otimes_\pi$ being the projective tensor product [Jarchow 1981, Chapter 15].

Now we extend these constructions to the space of integrable functions.

3.1 Definition: (Spaces of integrable functions) Let $(\mathcal{X}, \mathscr{A}, \mu)$ be a σ-finite measure space and let $(\mathsf{V}, \mathscr{O})$ be an Hausdorff locally convex topological vector space.

(i) We shall refer to a function $f \colon \mathcal{X} \to \mathsf{V}$ as being **integrable** if it is integrable by seminorm.

(ii) We denote by $\widehat{\mathrm{L}}^1(\mathcal{X}; \mathsf{V})$ the set of integrable functions.

(iii) Denote

$$\mathsf{Z}_{\mathscr{P}}(\mathcal{X}; \mathsf{V}) = \{ f \in \widehat{\mathrm{L}}^1(\mathcal{X}; \mathsf{V}) \mid \mu(\{x \in \mathcal{X} \mid p \circ f(x)\}) = 0 \text{ for all } p \in \mathscr{P} \}.$$

(iv) Denote $\mathrm{L}^1(\mathcal{X}; \mathsf{V}) = \widehat{\mathrm{L}}^1(\mathcal{X}; \mathsf{V}) / \mathsf{Z}_{\mathscr{P}}(\mathcal{X}; \mathsf{V})$. •

The seminorms above for simple functions obviously extend to integrable functions. Thus, for $p \in \mathscr{P}$, define a seminorm

$$p_{\mathcal{X}, \mathsf{V}} \colon \mathrm{L}^1(\mathcal{X}; \mathsf{V}) \to \mathbb{R}$$
$$[f] \mapsto \int_{\mathcal{X}} p \circ f \, \mathrm{d}\mu.$$

We then take the locally convex topology on $\mathrm{L}^1(\mathcal{X}; \mathsf{V})$ defined by the seminorms $p_{\mathcal{X}, \mathsf{V}}$. In the scalar-valued case we have the norm

$$p_1([f]) = \int_{\mathcal{X}} |f| \, \mathrm{d}\mu, \qquad [f] \in \mathrm{L}^1(\mathcal{X}; \mathbb{F}).$$

We note that $\mathrm{S}(\mathcal{X}; \mathsf{V})$ is dense in $\mathrm{L}^1(\mathcal{X}; \mathsf{V})$ by definition, and so we have topological isomorphisms

$$\overline{\mathrm{L}^1(\mathcal{X}; \mathsf{V})} \simeq \overline{\mathrm{S}(\mathcal{X}; \mathsf{V})} \simeq \mathrm{L}^1(\mathcal{X}; \mathbb{F}) \overline{\otimes}_\pi \mathsf{V} \simeq \mathrm{L}^1(\mathcal{X}; \mathbb{F}) \overline{\otimes}_\pi \overline{\mathsf{V}}, \tag{3.1}$$

by virtue of [Jarchow 1981, Proposition 15.2.3(a)], and noting that taking the completion of a projective tensor product is the same as taking the completion of the projective tensor product of the completions [Jarchow 1981, Corollary 15.2.4]. Here we use $\overline{\cdot}$ to denote the completion, and $\overline{\otimes}_\pi$ denotes that we are taking the completion of the projective tensor

product. As we commented upon in the introduction, many authors *define* the L^1-space of vector-valued functions to be equal to the completion of $\mathrm{S}(\mathfrak{X};\mathsf{V})$. This is, of course, unsatisfactory because one does not know very much about what objects in the completion look like. Other authors merely note (3.1), and do not consider the question of when one can append "$\mathrm{L}^1(\mathfrak{X};\mathsf{V}) \simeq$" to the left of this equation.

The following theorem addresses this question.

3.2 Theorem: (Spaces of integrable functions and the projective tensor product)
If $(\mathsf{V},\mathscr{O})$ is complete, then $\mathrm{L}^1(\mathfrak{X};\mathsf{V})$ is complete, and in consequence we have a topological isomorphism

$$\mathrm{L}^1(\mathfrak{X};\mathsf{V}) \simeq \mathrm{L}^1(\mathfrak{X};\mathbb{F})\overline{\otimes}_\pi\mathsf{V}.$$

Proof: Note that the "in consequence" assertion follows from (3.1), and so we must prove the completeness assertion.

First we prove the result in the important special case of Banach spaces. Usually this is proved by means of, "it's just like the usual case," but we elect to give the complete proof.

1 Lemma: *If $(\mathsf{V},\mathscr{O})$ is a Banach space, then $\mathrm{L}^1(\mathfrak{X};\mathsf{V})$ is a Banach space.*

Proof: We let p be the norm. We need only prove sequential completeness. Let $([f_j])_{j\in\mathbb{Z}_{>0}}$ be a Cauchy sequence in $\mathrm{L}^1(\mathfrak{X};\mathsf{V})$. For $k \in \mathbb{Z}_{>0}$, let $j_k \in \mathbb{Z}_{>0}$ be such that $p_{\mathfrak{X},\mathsf{V}}(f_j - f_{j_k}) < 2^{-k}$ for $k \in \mathbb{Z}_{>0}$. Without loss of generality, we can arrange that $j_k < j_{k+1}$, $k \in \mathbb{Z}_{>0}$. Then

$$C \triangleq p_{\mathfrak{X},\mathsf{V}}(f_{j_1}) + \sum_{k=1}^{\infty} p_{\mathfrak{X},\mathsf{V}}(f_{j_{k+1}} - f_{j_k}) < \infty.$$

Define

$$g_m \colon \mathfrak{X} \to \mathbb{R}_{\geq 0}$$

$$x \mapsto p(f_{j_1}(x)) + \sum_{k=1}^{m} p(f_{j_{k+1}}(x) - f_{j_k}(x))$$

for $m \in \mathbb{Z}_{>0}$. Let $g(x) = \lim_{m\to\infty} g_m(x)$, the limit existing (possibly infinite) as it is a limit of a monotonically increasing sequence. We have

$$p_1(g_m) = p_1\left(p \circ f_{j_1} + \sum_{k=1}^{m} p \circ (f_{j_{k+1}} - f_{j_k})\right)$$

$$\leq p_1(p \circ f_{j_1}) + \sum_{k=1}^{m} p_1(p \circ (f_{j_{k+1}} - f_{j_k})) < C.$$

By the Beppo Levi Theorem [Cohn 2013, Corollary 2.4.2],

$$p_1(g) = \lim_{m\to\infty} p_1(g_m) < \infty, \implies g = p \circ f_{j_1} + \sum_{k=1}^{\infty} p \circ (f_{j_{k+1}} - f_{j_k}) \in \mathrm{L}^1(\mathfrak{X},\mathbb{R}_{\geq 0}).$$

Thus there exists a subset $Z \subseteq \mathfrak{X}$ of measure zero such that the series

$$p(f_{j_1}(x)) + \sum_{k=1}^{\infty} p(f_{j_{k+1}}(x) - f_{j_k}(x))$$

with positive terms converges for all $x \in \mathfrak{X} \setminus Z$. Since an absolutely convergent series in a complete normed vector space converges, the series

$$f_{j_1}(x) + \sum_{k=1}^{\infty} (f_{j_{k+1}}(x) - f_{j_k}(x))$$

converges for every $x \in \mathfrak{X} \setminus Z$. Since the mth partial sum for this series is $f_{j_{m+1}}(x)$, we can define

$$f(x) = \begin{cases} \lim_{m \to \infty} f_{j_m}(x), & x \in \mathfrak{X} \setminus Z, \\ 0, & x \in Z. \end{cases}$$

We claim that f is measurable by seminorm. By Theorem 2.3, this is equivalent to assuming that f_{j_m}, $m \in \mathbb{Z}_{>0}$, is essentially separably-valued by seminorm and weakly measurable, and proving that f has these same attributes. For $m \in \mathbb{Z}_{>0}$, let $(s_{m,l})_{l \in \mathbb{Z}_{>0}}$ be a sequence of simple functions and let $Z_m \subseteq \mathfrak{X}$ be a set of measure zero such that $\lim_{l \to \infty} s_{m,l}(x) = f_{j_m}(x)$ for $x \in \mathfrak{X} \setminus Z_m$. Let $C_{m,l} \subseteq \mathsf{V}$ be the finite set of values taken by $s_{m,l}$ and let $C = \cup_{m,l \in \mathbb{Z}_{>0}} C_{m,l}$. Then $f(x) \in \operatorname{cl}(C)$ for $x \in \mathfrak{X} \setminus (Z \cup \cup_{m \in \mathbb{Z}_{>0}} Z_m)$, and so f is essentially separably-valued by seminorm. Moreover, for $\lambda \in \mathsf{V}'$, $\lambda \circ f(x) = \lim_{m \to \infty} \lambda \circ f_{j_m}(x)$ for $x \in \mathfrak{X} \setminus Z$, and so $\lambda \circ f$ is measurable being the almost everywhere pointwise limit of measurable functions [Cohn 2013, Proposition 2.1.5(c)]. Thus f is weakly measurable.

Next we claim that $([f_j])_{j \in \mathbb{Z}_{>0}}$ converges to $[f]$ and that $[f] \in \mathrm{L}^1(\mathfrak{X}; \mathsf{V})$. Let $\epsilon \in \mathbb{R}_{>0}$ and let $N \in \mathbb{Z}_{>0}$ be such that

$$p_{\mathfrak{X}, \mathsf{V}}(f_j - f_{j'}) < \epsilon, \qquad j, j' \geq j_N.$$

For $m \geq N$ and $j \geq j_N$, $p_{\mathfrak{X}, \mathsf{V}}(f_j - f_{j_m}) < \epsilon$. Then, by Fatou's Lemma [Cohn 2013, Theorem 2.4.4],

$$p_1(p \circ (f - f_j)) = p_1 \left(\liminf_{m \to \infty} p \circ (f_{j_m} - f_j) \right)$$
$$\leq \liminf_{m \to \infty} p_1(p \circ (f_{j_m} - f_j)) \leq \epsilon.$$

From this we conclude that $p_{\mathfrak{X}, \mathsf{V}}(f - f_j) < \infty$ and so $f - f_j \in \mathrm{L}^1(\mathfrak{X}; \mathsf{V})$ for $j \geq j_N$ by Theorem 2.7. Therefore, $p_{\mathfrak{X}, \mathsf{V}}(f) \leq p_{\mathfrak{X}, \mathsf{V}}(f - f_j) + p_{\mathfrak{X}, \mathsf{V}}(f_j) < \infty$, and so also $f \in \mathrm{L}^1(\mathfrak{X}; \mathsf{V})$. We also conclude that $\lim_{j \to \infty} f_j = f$. ▼

Now we consider the general case. For $p \in \mathscr{P}$, we have the subspace

$$\mathsf{N}_p = \{v \in \mathsf{V} \mid p(v) = 0\},$$

and, on the quotient space $\mathsf{V}_p = \mathsf{V}/\mathsf{N}_p$ we have the norm

$$p_{/\mathsf{N}_p}(v + \mathsf{N}_p) = \inf\{p(v + u) \mid u \in \mathsf{N}_p\}.$$

Since $(\mathsf{V}, \mathscr{O})$ is complete, the seminormed space (V, p) is complete and so the quotient $(\mathsf{V}_p, p_{/\mathsf{N}_p})$ is a complete normed vector space by [Jarchow 1981, Proposition 4.4.1] (we use a notion of completeness that holds for non-Hausdorff spaces, but this is not problematic). We use the obvious partial order for $\mathscr{P}$:

$$p \preceq p' \iff p(v) \leq p'(v), \qquad v \in \mathsf{V}.$$

For $p \preceq p'$ we have continuous linear mappings

$$\phi_{p,p'} \colon \mathsf{V}_{p'} \to \mathsf{V}_p$$
$$v + \mathsf{N}_{p'} \mapsto v + \mathsf{N}_p$$

with dense image [cf. Jarchow 1981, §2.9]. Moreover, by [Jarchow 1981, Theorem 2.9.2], $(\mathsf{V}, \mathscr{O})$ is topologically isomorphic to the inverse limit of the inverse system $(((\mathsf{V}_p, p))_{p \in \mathscr{P}}, (\phi_{p,p'})_{p \preceq p'})$ of Banach spaces, and the mappings induced by the inverse limit are the quotient mappings

$$\pi_{/\mathsf{N}_p} \colon \mathsf{V} \to \mathsf{V}_p, \qquad p \in \mathscr{P}.$$

For $p \in \mathscr{P}$, define

$$\hat{\pi}_{/\mathsf{N}_p} \colon \mathrm{L}^1(\mathcal{X}; \mathsf{V}) \to \mathrm{L}^1(\mathcal{X}; \mathsf{V}_p)$$
$$[f] \mapsto [\pi_{/\mathsf{N}_p} \circ f].$$

Continuity of the quotient mappings $\pi_{/\mathsf{N}_p}$ and (2.3) ensure that $[\pi_{\mathsf{N}_p} \circ f] \in \mathrm{L}^1(\mathcal{X}; \mathsf{V}_p)$. In like manner, we have mappings

$$\hat{\phi}_{p,p'} \colon \mathrm{L}^1(\mathcal{X}; \mathsf{V}_{p'}) \to \mathrm{L}^1(\mathcal{X}; \mathsf{V}_p) \qquad p, p' \in \mathscr{P}, \ p \preceq p'.$$
$$[f] \mapsto [\phi_{p,p'} \circ f],$$

Now let $([f_i])_{i \in I}$ be a Cauchy net in $\mathrm{L}^1(\mathcal{X}; \mathsf{V})$. As Cauchy nets are preserved by continuous linear maps, $(\hat{\pi}_{/\mathsf{N}_p}([f_i]))_{i \in I}$ is a Cauchy net in $(\mathsf{V}_p, p_{/\mathsf{N}_p})$, and so converges by the lemma above. We denote the limit by $[f_p]$.

We claim that $[f_p] = \hat{\phi}_{p,p'}([f_{p'}])$ for $p, p' \in \mathscr{P}$, $p \preceq p'$. Indeed,

$$[f_p] = \lim_{i \in I} \hat{\pi}_{/\mathsf{N}_p}([f_i]) = \lim_{i \in I} [\pi_{/\mathsf{N}_p} \circ f_i]$$
$$= \lim_{i \in I} [\phi_{p,p'} \circ \pi_{/\mathsf{N}_{p'}} \circ f_i] = \left[\phi_{p,p'} \left(\lim_{i \in I} \pi_{/\mathsf{N}_{p'}} \circ f_i \right) \right]$$
$$= \hat{\phi}_{p,p'}([f_{p'}]),$$

as claimed.

Note that, for each $p \in \mathscr{P}$, we have the diagram

$$\mathsf{V} \xrightarrow{\pi_{/\mathsf{N}_p}} \mathsf{V}_p$$

with f from $\mathcal{X}$ to V and f_p from $\mathcal{X}$ to V_p.

This defines, using the universal property of the inverse limit in the category of sets, the mapping f assigned to the dashed vertical arrow.

We need to show that $[f]$ is independent of the choice of representative $[f_p]$, $p \in \mathscr{P}$. For each $p \in \mathscr{P}$, let $f_p, f'_p \colon \mathcal{X} \to \mathsf{V}$ be such that $[f_p] = [f'_p]$, meaning that there exists a subset $Z_p \subseteq \mathcal{X}$ of measure zero such that $f_p(x) = f'_p(x)$ for $x \in \mathcal{X} \setminus Z_p$. Let $f, f' \colon \mathcal{X} \to \mathsf{V}$ be the associated mappings, as in the preceding paragraph. By [Jarchow 1981, Proposition 4.2.1], $\pi_{/\mathsf{N}_p}$ is open, and so there exists $C_p \in \mathbb{R}_{>0}$ such that

$$p(v) \leq C_p p_{/\mathsf{N}_p}(\pi_{/\mathsf{N}_p}(v)), \qquad v \in \mathsf{V}.$$

Thus, for $x \in \mathcal{X}$, we have

$$p \circ f(x) \le C_p p_{/\mathsf{N}_p} \circ f_p(x), \quad p \circ f'(x) \le C_p p_{/\mathsf{N}_p} \circ f'_p(x).$$

In particular, for $x \in \mathcal{X} \setminus Z_p$, $p \circ f(x) = p \circ f'(x)$. As this holds for every $p \in \mathscr{P}$, $[f] = [f']$.

By integrability of f_p, take a subset $Z_p \subseteq \mathcal{X}$ of measure zero, a sequence $(s_{p,j})_{j \in \mathbb{Z}_{>0}}$ of simple functions, and, for $A \in \mathscr{A}$, a vector $v_{p,A} \in \mathsf{V}_p$ such that

1. $\lim_{j \to \infty} s_{p,j}(x) = f_p(x)$ for $x \in \mathcal{X} \setminus Z_p$,

2. $\lim_{j \to \infty} \int_{\mathcal{X}} p_{/\mathsf{N}_p} \circ (f_p - s_{p,j}) \, \mathrm{d}\mu = 0$, and

3. $\lim_{j \to \infty} p_{/\mathsf{N}_p} (\int_A s_{p,j} \, \mathrm{d}\mu - v_{p,A}) = 0$.

Let $s'_{p,j} \in \widehat{\mathsf{S}}(\mathcal{X}; \mathsf{V})$ be such that $\pi_{/\mathsf{N}_p} \circ s'_{p,j} = s_{p,j}$ (this is easily done as simple functions take finitely many values).

Then, for $x \in \mathcal{X} \setminus Z_p$,

$$\lim_{j \to \infty} p(f(x) - s'_{p,j}(x)) \le \lim_{j \to \infty} C_p p_{/\mathsf{N}_p}(f_p(x) - s_{p,j}(x)) = 0.$$

This gives the first condition for integrability of f by seminorm.

Next we have

$$\int_{\mathcal{X}} p \circ (f - s'_{p,j}) \, \mathrm{d}\mu \le C_p \int_{\mathcal{X}} p_{/\mathsf{N}_p} \circ (f - s_{p,j}),$$

from which we conclude that $x \mapsto p(f(x) - s'_{p,j}(x))$ is integrable and that

$$\lim_{j \to \infty} \int_{\mathcal{X}} p \circ (f - s'_{p,j}) \, \mathrm{d}\mu = 0,$$

giving the second condition for integrability of f by seminorm.

For the final of the conditions for integrability of f by seminorm, consider the diagram

$$
\begin{array}{ccc}
\mathsf{V} & \xrightarrow{\;\;\pi_{/\mathsf{N}_p}\;\;} & \mathsf{V}_p \\[4pt]
{\scriptstyle I_A} \uparrow \vdots & & \uparrow {\scriptstyle \int_A} \\[4pt]
\mathrm{L}^1(\mathcal{X}; \mathsf{V}) & \xrightarrow[\widehat{\pi}_{/\mathsf{N}_p}]{} & \mathrm{L}^1(\mathcal{X}; \mathsf{V}_p)
\end{array}
$$

By the universal property of inverse limits in the category of topological vector spaces, there is a unique continuous linear mapping indicated by the vertical arrow. We then have

$$p \left(\int_A s'_{p,j} \, \mathrm{d}\mu - I_A(f) \right) \le C_p p_{/\mathsf{N}_p} \left(\int_A s_{p,j} \, \mathrm{d}\mu - v_{p,A} \right),$$

which, upon taking the limit as $j \to \infty$, gives the theorem. ∎

We can also consider in our setting the notion of local integrability. It is convenient to do this in the setting of Borel measures on Hausdorff topological spaces [Cohn 2013, §7.2], an example of which is the Lebesgue measure on Euclidean space. Typically, for a coherent theory, one restricts to locally compact topological spaces, but for the purposes of the definition this is not necessary. Thus, if $(\mathcal{S}, \mathscr{O}_{\mathcal{S}})$ is an Hausdorff topological space, we

consider a σ-algebra $\mathscr{A}$ containing the Borel σ-algebra, and we let μ be a measure for the measurable space $(\mathcal{S}, \mathscr{A})$. If $T \subseteq \mathcal{S}$ has the subspace topology, then we denote

$$\mathscr{A}_T = \{ A \cap T \mid A \in \mathscr{A} \}.$$

We note that $\mathscr{A}_T$ contains the Borel σ-algebra of T with the subspace topology by virtue of [Cohn 2013, Lemma 7.2.2]. We also denote $\mu_T = \mu | \mathscr{A}_T$.

With this notation, we can make the following definition, denoting by $\mathscr{K}(\mathcal{S})$ the collection of compact subsets of $\mathcal{S}$.

3.3 Definition: (Locally integrable vector-valued function) Let $(\mathsf{V}, \mathscr{O})$ be an Hausdorff locally convex topological vector space, let $(\mathcal{S}, \mathscr{O}_\mathcal{S})$ be an Hausdorff topological space, let $\mathscr{A}$ be a σ-algebra on $\mathcal{S}$ containing the Borel σ-algebra, and let $\mu \colon \mathscr{A} \to \overline{\mathbb{R}}_{\geq 0}$ be a measure. A function $f \colon \mathcal{S} \to \mathsf{V}$ is **locally integrable** if, for each $\mathcal{K} \in \mathscr{K}(\mathcal{S})$, $f | \mathcal{K} \in \mathrm{L}^1(\mathcal{K}; \mathsf{V})$. $\bullet$

Let us consider how to topologise the collection of all locally integrable functions. We note that $\mathscr{K}(\mathcal{S})$ is a directed set by $\mathcal{K}_1 \preceq \mathcal{K}_2$ if $\mathcal{K}_1 \subseteq \mathcal{K}_2$. Thus we can take

$$\mathrm{L}^1_{\mathrm{loc}}(\mathcal{S}; \mathsf{V}) = \varprojlim_{\mathcal{K} \in \mathscr{K}(\mathcal{S})} \mathrm{L}^1(\mathcal{K}; \mathsf{V}).$$

We note that, if $f \colon \mathcal{S} \to \mathsf{V}$ is locally integrable, then $[f | \mathcal{K}] \in \mathrm{L}^1(\mathcal{K}; \mathsf{V})$, and so $\mathrm{L}^1_{\mathrm{loc}}(\mathcal{S}; \mathsf{V})$ does indeed contain all locally integrable functions, or more properly, their equivalence classes.

4. Absolutely continuous vector-valued functions on the line

For $\mathbb{R}$-valued functions there is an ϵ-δ definition of absolute continuity [Cohn 2013, page 135], which is then shown to be equivalent to the function being almost everywhere differentiable with a locally integrable derivative [Cohn 2013, Corollary 6.3.8]. We adopt the latter point of view with our definition.

4.1 Definition: (Absolutely continuous vector-valued function) Let $\mathbb{I} \subseteq \mathbb{R}$ be an interval. A function $F \colon \mathbb{I} \to \mathsf{V}$ is **locally absolutely continuous** if there exists $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{I}; \mathsf{V})$ such that

$$F(t) = F(t_0) + \int_{t_0}^t f(\tau) \, \mathrm{d}\lambda(\tau). \qquad\qquad \bullet$$

We can now show that, for this notion of absolute continuity and for the notion of integrability by seminorm, one has the following useful property of locally absolutely continuous functions.

4.2 Theorem: (Almost everywhere differentiability of locally absolutely continuous functions) *Let $\mathbb{I} \subseteq \mathbb{R}$ be an interval, and let $F \colon \mathbb{I} \to \mathsf{V}$ be absolutely continuous and given by*

$$F(t) = F(t_0) + \int_{t_0}^t f(\tau) \, \mathrm{d}\lambda(\tau)$$

for $t_0 \in \mathbb{I}$ and $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{I}; \mathsf{V})$. Then F is differentiable almost everywhere and

$$\lim_{s \to 0} \frac{F(t+s) - F(t)}{s} = f(t), \qquad a.e. \ t \in \mathbb{I}.$$

Proof: We first prove a technical lemma. In the statement of the lemma, we denote by

$$D_0 = [-\tfrac{1}{2}, \tfrac{1}{2}]^n \subseteq \mathbb{R}^n$$

the disk centred at 0 with side length 1 and we denote by λ_n the Lebesgue measure for $\mathbb{R}^n$.

1 Lemma: *Let* $\mathcal{U} \subseteq \mathbb{R}^n$ *be open and let* $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathcal{U}; \mathbb{R}_{\geq 0})$, *i.e.,* f *is locally integrable in the sense of Lebesgue. Then*

$$\lim_{s \to 0} \int_{D_0} f(\boldsymbol{x} + s\boldsymbol{\xi}) \, \mathrm{d}\lambda_n(\boldsymbol{\xi}) = f(\boldsymbol{x})$$

for almost every $\boldsymbol{x} \in \mathcal{U}$.

Proof: Let

$$f_s(\boldsymbol{x}) = \int_{D_0} f(\boldsymbol{x} + s\boldsymbol{\xi}) \, \mathrm{d}\lambda_n(\boldsymbol{\xi})$$

and

$$m(\boldsymbol{x}) = \liminf_{s \to 0} f_s(\boldsymbol{x}), \qquad M(\boldsymbol{x}) = \limsup_{s \to 0} f_s(\boldsymbol{x}).$$

We first claim that

$$\lambda_n(\{\boldsymbol{x} \in \mathcal{U} \mid M(\boldsymbol{x}) = \infty\}) = 0.$$

Suppose that this is not so, and let $A \subseteq \mathcal{U}$ be the set of positive measure on which M is infinite. Since $\mathcal{U}$ is covered by a countable collection of bounded open disks, there must exist a bounded open disk D with $\mathrm{cl}(D) \subseteq \mathcal{U}$ such that $A' \triangleq A \cap D$ has positive measure. For $s \in \mathbb{R}_{>0}$ and for $\boldsymbol{x} \in \mathcal{U}$, we denote by $sD_0 + \boldsymbol{x}$ the disk with side length $|s|$ centred at $\boldsymbol{x}$. For $a \in \mathbb{R}_{\geq 0}$, denote

$$\mathscr{C}_a = \{sD_0 + \boldsymbol{x} \mid f_s(\boldsymbol{x}) > a\}.$$

We claim that $\mathscr{C}_a$ is a Vitali cover of A', as in [Cohn 2013, page 164]. Indeed, let $\boldsymbol{x} \in A'$ and let $\delta \in \mathbb{R}_{>0}$. Since $M(\boldsymbol{x}) = \infty$, there exists $s \in \mathbb{R}$ with $|s| < \delta$ such that $f_s(\boldsymbol{x}) > a$, and we conclude that, indeed, $\mathscr{C}_a$ is a Vitali cover. Thus, by the Vitali Covering Lemma [Cohn 2013, Theorem 6.2.1], there exists a sequence $D_j \triangleq s_j D_0 + \boldsymbol{x}_j$, $j \in \mathbb{Z}_{>0}$, of disjoint disks such that $D_j \subseteq D$, $j \in \mathbb{Z}_{>0}$, and such that the set of points from A' not covered by $\cup_{j \in \mathbb{Z}_{>0}} D_j$ has measure zero. Then, by a change of variable,

$$\int_{D_j} f(\boldsymbol{x}) \, \mathrm{d}\lambda_n(\boldsymbol{x}) = |s_j|^n \int_{D_0} f(\boldsymbol{x}_j + s_j \boldsymbol{\xi}) \, \mathrm{d}\lambda_n(\boldsymbol{\xi}) = |s_j|^n f_{s_j}(\boldsymbol{x}_j).$$

Since $|s_j|^n = \lambda_n(D_j)$,

$$\int_{D_j} f(\boldsymbol{x}) \, \mathrm{d}\lambda_n(\boldsymbol{x}) > a\lambda_n(D_j).$$

Therefore, if $E = \cup_{j \in \mathbb{Z}_{>0}} D_j$, we have

$$\int_D f(\boldsymbol{x}) \, \mathrm{d}\lambda_n(\boldsymbol{x}) \geq \int_E f(\boldsymbol{x}) \, \mathrm{d}\lambda_n(\boldsymbol{x}) \geq a\lambda_n(E) \geq a\lambda_n(A') > 0.$$

Therefore, since $a \in \mathbb{R}_{>0}$ is arbitrary, we conclude that f is not locally integrable. This contradiction gives our claim that M is infinite on a set of zero measure.

Now we claim that

$$\lambda_n(\{\boldsymbol{x} \in \mathcal{U} \mid m(\boldsymbol{x}) < M(\boldsymbol{x})\}) = 0.$$

As in the preceding step, we do this by contradiction, so that

$$A = \{\boldsymbol{x} \in \mathcal{U} \mid m(\boldsymbol{x}) < M(\boldsymbol{x})\}$$

has positive measure. Just as in the preceding step, this means that there is a bounded open disk D with $\mathrm{cl}(D) \subseteq \mathcal{U}$ such that $A' \triangleq A \cap D$ has positive measure. For $\boldsymbol{x} \in A'$, there exist $q, q' \in \mathbb{Q}$ such that

$$m(\boldsymbol{x}) < q < q' < M(\boldsymbol{x}).$$

For $q, q' \in \mathbb{Q}$ satisfying $q < q'$, we can denote

$$A'_{q,q'} = \{\boldsymbol{x} \in A' \mid m(\boldsymbol{x}) < q < q' < M(\boldsymbol{x})\}.$$

Note that

$$A' = \cup\{A'_{q,q'} \mid q, q' \in \mathbb{Q}, \ q < q'\},$$

and this union is a countable union. Therefore, there must be some $q, q' \in \mathbb{Q}$ with $q < q'$ and with $\lambda_n(A'_{q,q'}) > 0$. We fix such a q and q'. By definition of Lebesgue measure (or by [Cohn 2013, Proposition 1.4.1]), let $\mathcal{O} \subseteq D$ be an open set such that $A'_{q,q'} \subseteq \mathcal{O}$ and such that

$$\lambda_n(\mathcal{O}) < \sqrt{q'/q}\,\lambda_n(A'_{q,q'}).$$

Using the notation as above, let

$$\mathscr{C}_q = \{sD_0 + \boldsymbol{x} \mid f_s(\boldsymbol{x}) < q\}.$$

Similarly to the preceding part of the proof, $\mathscr{C}_q$ is a Vitali cover of $A_{q,q'}$, and so we can find a sequence $D_j \triangleq s_j D_0 + \boldsymbol{x}_j \subseteq \mathcal{O}$, $j \in \mathbb{Z}_{>0}$, of disjoint disks from $\mathscr{C}_q$ that covers $A'_{q,q'}$ except for a set of measure zero. By a computation with a change of variable, just as above, we have

$$\int_{D_j} f(\boldsymbol{x})\,\mathrm{d}\lambda_n(\boldsymbol{x}) < q\lambda_n(D_j).$$

Let $E = \cup_{j \in \mathbb{Z}_{>0}} D_j$ and calculate

$$\int_E f(\boldsymbol{x})\,\mathrm{d}\lambda_n(\boldsymbol{x}) < q\lambda_n(E) \leq q\lambda_n(\mathcal{O}) \leq \sqrt{qq'}\,\lambda_n(A'_{q,q'}). \tag{4.1}$$

Again by definition of Lebesgue measure, let $\mathcal{O}' \subseteq D$ be an open set such that $E \subseteq \mathcal{O}'$ and such that

$$\int_{\mathcal{O}'} f(\boldsymbol{x})\,\mathrm{d}\lambda_n(\boldsymbol{x}) \leq \sqrt{q'/q} \int_E f(\boldsymbol{x})\,\mathrm{d}\lambda_n(\boldsymbol{x}). \tag{4.2}$$

We now take

$$\mathscr{C}_{q'} = \{sD_0 + \boldsymbol{x} \mid f_t(\boldsymbol{x}) > q'\},$$

which we verify to be a Vitali cover of $\mathcal{O}' \cap A'_{q,q'}$. Let $D'_j \triangleq s'_j D_0 + \boldsymbol{x}'_j \subseteq \mathcal{O}'$, $j \in \mathbb{Z}_{>0}$, be a sequence of disjoint disks that covers $\mathcal{O}' \cap A'_{q,q'}$ except for a set of measure zero. Since

$E \subseteq \mathcal{O}'$ and since E covers $A'_{q,q'}$ except for a set of measure zero, the disks D'_j cover $A'_{q,q'}$ except for a set of measure zero. As we have done already twice, we arrive at

$$\int_{D'_j} f(\boldsymbol{x}) \, \mathrm{d}\lambda_n(\boldsymbol{x}) > q'\lambda_n(D'_j).$$

Taking $E' = \cup_{j\in\mathbb{Z}_{>0}} D'_j$, we have

$$\int_{\mathcal{O}'} f(\boldsymbol{x}) \, \mathrm{d}\lambda_n(\boldsymbol{x}) \geq \int_{E'} f(\boldsymbol{x}) \, \mathrm{d}\lambda_n(\boldsymbol{x}) > q'\lambda_n(E') \geq q'\lambda_n(A'_{q,q'}).$$

Combining this with (4.2) gives

$$\int_{E} f(\boldsymbol{x}) \, \mathrm{d}\lambda_n(\boldsymbol{x}) > \sqrt{qq'}\lambda_n(A'_{q,q'}),$$

in contradiction with (4.1). We conclude from this contradiction that the set of points at which $m(\boldsymbol{x}) < M(\boldsymbol{x})$ must have measure zero.

We conclude from the above that $\lim_{s\to 0} f_s(\boldsymbol{x})$ exists almost everywhere; let us denote the limit as $g(\boldsymbol{x})$ when it exists. We now show that, at almost all such points where the limit $g(\boldsymbol{x})$ exists, we, in fact, have $g(\boldsymbol{x}) = f(\boldsymbol{x})$. Let D be a bounded disk with $\mathrm{cl}(D) \subseteq \mathcal{U}$. We have, noting that $\lambda_n(D_0) = 1$,

$$\begin{aligned}
\int_{D} |f_s(\boldsymbol{x}) - f(\boldsymbol{x})| \, \mathrm{d}\lambda_n(\boldsymbol{x}) &= \int_{D} \left| \int_{D_0} (f(\boldsymbol{x} + s\boldsymbol{\xi}) - f(\boldsymbol{x})) \, \mathrm{d}\lambda_n(\boldsymbol{\xi}) \right| \, \mathrm{d}\lambda_n(\boldsymbol{x}) \\
&\leq \int_{D} \int_{D_0} |f(\boldsymbol{x} + s\boldsymbol{\xi}) - f(\boldsymbol{x})| \, \mathrm{d}\lambda_n(\boldsymbol{\xi}) \, \mathrm{d}\lambda_n(\boldsymbol{x}) \\
&= \int_{D_0} \int_{D} |f(\boldsymbol{x} + s\boldsymbol{\xi}) - f(\boldsymbol{x})| \, \mathrm{d}\lambda_n(\boldsymbol{x}) \, \mathrm{d}\lambda_n(\boldsymbol{\xi}),
\end{aligned}$$

using Fubini's Theorem [Cohn 2013, Theorem 5.2.2] and for s small enough that $D + sD_0 \subseteq \mathcal{U}$. By continuity of translation in L^1, we then have

$$\lim_{s\to 0} \int_{D} |f_s(\boldsymbol{x}) - f(\boldsymbol{x})| \, \mathrm{d}\lambda_n(\boldsymbol{x}) = 0.$$

By [Cohn 2013, Proposition 3.1.5], it follows that f_s converges to f pointwise almost everywhere as $s \to 0$ on D. Now let $(s_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence converging to 0. Define $g_j = f_{s_j}$. Let $(D_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence of bounded disks that cover $\mathcal{U}$. Choose a subsequence $(g_{1,j})_{j\in\mathbb{Z}_{>0}}$ of $(g_j)_{j\in\mathbb{Z}_{>0}}$ and a subset $Z_1 \subseteq D_1$ of measure zero such that $\lim_{j\to\infty} g_{1,j}(\boldsymbol{x}) = f(\boldsymbol{x})$ for $\boldsymbol{x} \in D_1 \setminus Z_1$. Next, choose a subsequence $(g_{2,j})_{j\in\mathbb{Z}_{>0}}$ of $(g_{1,j})_{j\in\mathbb{Z}_{>0}}$ and a subset $Z_2 \subseteq D_1 \cup D_2$ of measure zero such that $\lim_{j\to\infty} g_{2,j}(\boldsymbol{x}) = f(\boldsymbol{x})$ for $\boldsymbol{x} \in (D_1 \cup D_2) \setminus Z_2$. Carrying on this way, we get subsequences $(g_{k,j})_{j\in\mathbb{Z}_{>0}}$ and subsets $Z_k \subseteq D_1 \cup \cdots \cup D_k$ of measure zero such that

$$\lim_{j\to\infty} g_{k,j}(\boldsymbol{x}) = f(\boldsymbol{x}), \qquad \boldsymbol{x} \in (D_1 \cup \cdots \cup D_k) \setminus Z_k, \ k \in \mathbb{Z}_{>0}.$$

Now let $\boldsymbol{x} \in \mathcal{U} \setminus (\cup_{j\in\mathbb{Z}_{>0}} Z_j)$ and let $\epsilon \in \mathbb{R}_{>0}$. Let $k \in \mathbb{Z}_{>0}$ be such that $\boldsymbol{x} \in D_1 \cup \cdots \cup D_k$. Let $N \geq k$ be such that $g_{j,j}(\boldsymbol{x}) < \epsilon$ for $j \geq N$. This shows that the subsequence $(g_{j,j})_{j\in\mathbb{Z}_{>0}}$ of $(g_j)_{j\in\mathbb{Z}_{>0}}$ converges pointwise almost everywhere to f in $\mathcal{U}$. Since $(g_j)_{j\in\mathbb{Z}_{>0}}$ converges pointwise almost everywhere to g, we conclude that $(g_j)_{j\in\mathbb{Z}_{>0}}$ converges pointwise almost everywhere to f. Since the sequence $(s_j)_{j\in\mathbb{Z}_{>0}}$ is arbitrary, we conclude that $\lim_{s\to 0} f_s(\boldsymbol{x}) = f(\boldsymbol{x})$ for almost every $\boldsymbol{x}$, as is asserted by the lemma.   ▼

The next, much simpler, lemma is a consequence of preceding technical lemma.

2 Lemma: *Let* $\mathbb{I} \subseteq \mathbb{R}$ *be an interval and let* $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{I}; \mathsf{V})$. *Then, for each* $p \in \mathscr{P}$,

$$\lim_{s \to 0} \int_0^1 p(f(t + s\tau) - f(t)) \, \mathrm{d}\lambda(\tau) = 0$$

for almost every $t \in \mathbb{I}$.

Proof: We take $p \in \mathscr{P}$. By Theorem 2.3, let $C_p \subseteq \mathsf{V}$ be a countable subset and let $Z'_p \subseteq \mathbb{I}$ have measure zero and such that $f(\mathbb{I} \setminus Z'_p) \subseteq \mathrm{cl}_p(C_p)$. Write $C_p = \{v_{p,j} \mid j \in \mathbb{Z}_{>0}\}$. By Theorem 2.6, $p \circ f$ is locally integrable. Therefore, for each $j \in \mathbb{Z}_{>0}$, by the previous lemma, there exists a subset $Z_{p,j} \subseteq \mathbb{I}$ of measure zero such that we have

$$\lim_{s \to 0} \int_0^1 p(f(t + s\tau) - v_{p,j}) \, \mathrm{d}\lambda(\tau) = p(f(t) - v_{p,j})$$

for every $t \in \mathbb{I} \setminus Z_{p,j}$. Taking $Z_p = \cup_{j \in \mathbb{Z}_{>0}} Z_{p,j}$, we have

$$\lim_{s \to 0} \int_0^1 p(f(t + s\tau) - v_{p,j}) \, \mathrm{d}\lambda(\tau) = p(f(t) - v_{p,j})$$

for every $t \in \mathbb{I} \setminus Z_p$ and every $j \in \mathbb{Z}_{>0}$. Now, for $t \in \mathbb{I} \setminus (Z_p \cup Z'_p)$, let $j \in \mathbb{Z}_{>0}$ be such that $p(f(t) - v_{p,j}) < \frac{\epsilon}{3}$. Then there exists s sufficiently small that

$$\left| \int_0^1 p(f(t + s\tau) - v_{p,j}) \, \mathrm{d}\lambda(\tau) - p(f(t) - v_{p,j}) \right| < \frac{\epsilon}{3}.$$

Therefore,

$$\int_0^1 p(f(t + s\tau) - v_{p,j}) \, \mathrm{d}\lambda(\tau) < \frac{2\epsilon}{3}.$$

As a result, for the chosen s and j,

$$\int_0^1 p(f(t + s\tau) - f(t)) \, \mathrm{d}\lambda(\tau) \leq \int_0^1 p(f(t + s\tau) - v_{p,j}) \, \mathrm{d}\lambda(\tau) + \int_0^1 p(f(t) - v_{p,j}) \, \mathrm{d}\lambda(\tau) < \epsilon,$$

giving the desired conclusion.   ▼

By the lemma just preceding, for $p \in \mathscr{P}$, we have

$$\lim_{s \to 0} p \left(\int_0^1 f(t + s\tau) \, \mathrm{d}\lambda(\tau) - \int_0^1 f(t) \, \mathrm{d}\lambda(\tau) \right) \leq \lim_{s \to 0} \int_0^1 p(f(t + s\tau) - f(t)) \, \mathrm{d}\lambda(\tau) = 0$$

for almost every $t \in \mathbb{I}$. Therefore,

$$\lim_{s \to 0} \int_0^1 f(t + s\tau) \, \mathrm{d}\lambda(\tau) = \int_0^1 f(t) \, \mathrm{d}\lambda(\tau) = f(t)$$

for almost every $t \in \mathbb{I}$. By a change of variable,

$$\lim_{s \to 0} \frac{1}{s} \int_t^{t+s} f(\tau) \, \mathrm{d}\lambda(\tau) = f(t)$$

for almost every $t \in \mathbb{I}$. Thus, for almost every $t \in \mathbb{I}$,

$$\lim_{s \to 0} \frac{F(t + s) - F(t)}{s} = f(t),$$

which is the theorem.   ■

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