The exponential map for time-varying vector fields[1]

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Abstract

The exponential map that characterises the flows of vector fields is the key in understanding the basic structural attributes of control systems in geometric control theory. However, this map does not exist due to the lack of completeness of flows for general vector fields.

An appropriate substitute is devised for the exponential map, not by trying to force flows to be globally defined by any compactness assumptions on the manifold, but by a categorical development of spaces of vector fields and flows, thus allowing for systematic localisation of such spaces. That is to say, we give a presheaf construction of the exponential map for vector fields with measurable time-dependence and continuous parameter-dependence in the category of general topological spaces. Moreover, all regularities in state are considered, including the cases of continuous, finitely differentiable, smooth and holomophic. Using geometric descriptions of suitable topologies for vector fields and for local diffeomorphisms, the homeomorphism of the exponential map is derived by a uniform treatment for these regularities. Finally, a new sort of continuous dependence is proved, that of the fixed time local flow on the parameter which plays an important role in the establishment of the homeomorphism of the exponential map.

Contents

Chapter 1

Introduction

In geometric control theory, one can study linear and nonlinear control theory from the point of view of applications, or from a more fundamental point of view where the system structure is a key element. It is well understood that the language of systems such as I am interested in should be founded in the study of differential geometry and vector fields on manifolds (Agrachev and Sachkov, 2004; Bloch, 2003; Bullo and Lewis, 2005; Jurdjevic, 1997; Nijmeijer and Van der Schaft, 1990).

Understanding the basic structural attributes of control systems requires understanding how the system is connected to the trajectories of the system. For a control system

$$\xi'(t) = F(t, \xi(t), \mu(t))$$

with a control $\mu : \mathbb{T} \to \mathcal{C}$ and trajectory $\xi : \mathbb{T} \to M$, the controllability, reachability, and stabilisability properties depend solely on the understanding a family of flows

$$(t, t_0, x_0) \mapsto \Phi^F(t, t_0, x_0, \mu), \quad \mu \in \mathscr{U}$$

for some class of controls $\mathscr{U}$. This is typically thought of as the image of the family of time-varying vector fields $(t, x) \mapsto F_\mu(t, x) := F(t, \xi(t), \mu(t))$ under some "exponential map"

$$\begin{aligned} \exp : \{\text{vector fields}\} &\to \{\text{local flows}\} \\ F &\mapsto \Phi^F. \end{aligned}$$

The idea of the flow of a vector field seems so well understood that it barely merits any systematic explication. However, the accepted casual manner of this presentation has a deficiency, this being that there is no way to generally define the exponential map as a mapping from the Lie algebra of vector fields to the group of diffeomorphisms. Quite apart from any technical difficulties that arise from working with infinite-dimensional manifolds, the incompleteness of general vector fields causes any naïve definition to fail.

One might overcome this by working with compact manifolds or by working with only complete vector fields. Particularly, the assumption of completeness is one that is very often made in passing "for the sake of convenience." For manifolds that are compact, the problem of completeness can be overcome, and the desired exponential map, in fact, exists (Omori, 1970). For noncompact manifolds, one can work with vector fields with compact support. These compactness assumptions (for compact manifolds) or impositions (of compact support) are not satisfactory. For example, the compactness assumption fails for linear ordinary differential equations, and any theory not including these can hardly be said to be general. As another instance of the lacking of these compactness constructions, note that a real analytic vector field on a noncompact manifold can never have compact support. From our perspective, any theory not including analytic vector fields is not satisfactory.

Another route to the exponential map in the time-varying case involves coming up with some series representation for time-varying flows. The so-called Volterra series is an adaptation of the exponential series for time-varying vector fields, and rigorous versions of this work date back to (Agrachev and Gamkrelidze, 1979). A nice recent summary of this work can be found in the book of (Agrachev and Sachkov, 2004). The "inversion" of the Volterra series leads to the notion of a Baker–Campbell–Hausdorff formula in the time-varying case. An example of this can be found in the work of (Strichartz, 1987). These considerations have given rise to an area dedicated to using methods from the theory of free algebras. For a recent outline of these methods, we refer to (Kawski, 2021). While these techniques have proved to have significant value in geometric control theory, the problem of defining the exponential map for general vector fields still remains open.

The drawbacks of these approaches to this subject come in various forms. One such drawback concerns regularity of the vector fields, and definedness and convergence of the series. The series from this theory are comprised of differential operators that arise from iterating first-order differential operators, i.e., vector fields. In degenerate cases where some nilpotency can be assumed, only finite iterations are necessary. However, one cannot expect this to be the case generically, and so a general theory in this framework must allow for infinite iterations of first-order differential operators, i.e., at least infinite differentiability. Thus the theory simply does not apply to lower degrees of regularity. Moreover, even in the infinitely differentiable setting, the series do not converge in any meaningful way. This is not surprising since there are some aspects of Taylor series in these series representations, and so one expects, and it is indeed the case, that real analyticity is required for convergence. Again, lesser regularity is simply not represented by these series methods (and it is certainly not claimed to be represented in the literature on the subject).

Another limitation of the series representations of flows is that they are only made for a single time-varying vector field, whereas any sort of "exponential map" should give us some representation of a flow given any time-varying vector field. The problem here comes in two flavours. First, the completeness problem mentioned above appears in the series representations as well; the domain in space and time simply cannot be uniform over all vector fields. Second, if one is working with real analytic vector fields and asking for convergence of series, the region of convergence will depend on the specific vector field (Jafarpour and Lewis, 2014, Example 6.24). In any case, the lack of a fixed domain for flows and convergence creates a problem for series representation methods as a means for defining any general sort of exponential map.

This, then, leaves open the question of how one satisfactorily addresses the problem of defining the exponential map in any general way.

1.1. Contribution of the thesis

The general question about the existence of the exponential map is addressed in this thesis by considering, not vector fields and diffeomorphisms, but presheaves of vector fields and presheaves of local diffeomorphisms of various of regularities, which allow for systematic localisation of the components of what will become the exponential map, i.e.,

$$\exp : \{\text{presheaf of vector fields}\} \to \{\text{presheaf of local diffeomorphisms}\}.$$

Moreover, we present a methodology for working with vector fields with measurable time-dependence, continuous parameter-dependence with parameters in an arbitrary topological space, and for working with the resulting flows of such vector fields. The presheaf point of view provides a theory that integrates the fact that flows are only locally defined, even for vector fields that are globally defined. That is to say, we deal with the lack of completeness not by trying to force flows to be globally defined by some sort of assumption, but by making vector fields themselves locally defined, thus putting them on the same local footing as their flows. The categorical framework for talking about time-dependent vector fields and local flows herein allows one to infer the existence of a presheaf in the category of topological spaces, which plays an important role in the study of controllability of a control system.

Another attribute of the framework is that time-varying vector fields are considered with a variety of degrees of regularity with respect to state, namely, Lipschitz, finitely differentiable, smooth, real analytic, and holomorphic. In doing this, use is made of locally convex topologies for the spaces of vector fields with these degrees of regularity. The classes of time-varying and parameter-dependent vector fields are characterised by their continuity, measurability, and integrability with respect to these locally convex topologies. Within this framework, very general results are provided concerning existence, uniqueness, and regular dependence of flows on initial and final time, initial state, and parameters. These results include, and drastically extend, known results for properties of flows.

Moreover, the fact that this map is a homeomorphism is established upon the suitable topologies for sets of vector fields and flows using geometric decompositions of various jet bundles by various of connections. This framework is interesting in that it allows an elegant and uniform treatment of vector fields across various regularity classes.

1.2. An outline of the thesis

Roughly speaking, in Chapter 2 and 3 we develop the classes of vector fields and flows we use, and attributes of these. In the final Chapter, we prove properties of flows of vector fields, and define the exponential map and its properties.

In more details, Chapter 2 overviews the locally convex topologies for the space of sections of a vector bundle for various regularity classes presented in (Jafarpour and Lewis, 2014), including the newly developed topology in the real analytic case. Most importantly, we use these locally convex topologies to describe classes of time-dependent and parameter-dependent sections. The approach we take strictly extends the usual approach to parameter-dependence in the theory of ordinary differential equations, and allows, for example, vector fields that depend on a parameter in a general topological space.

In Chapter 3, we carefully and geometrically establish the basic results concerning existence and uniqueness of integral curves, and of the regular dependence of flows on initial time, final time, initial state, and parameter. We make a comment that the new type of continuity result for the "parameter to local flow" mapping provides a geometric toolbox for dealing with the exponential map which will be established in Chapter 4.

In Chapter 4, we carefully establish the presheaf exponential map in the category of topological spaces. We start by developing a categorical framework for talking about time-dependent vector fields and local flows, which allow one to infer the existence of a presheaf, here in the category of topological spaces. The presheaves we construct are put together by requiring that, in any product neighbourhood of a point in time, state, and parameter, the theory should agree with the "standard" theory of Section 3.1 and Section 3.2. Since the collection of product neighbourhoods are a basis for the open sets in the product, standard presheaf theory constructions then give a presheaf whose local sections over products agree with the prescribed ones. Essentially by taking inverse limits in these appropriate categories, we show that the theory of local flows is elegantly represented by the existence of an "exponential mapping" from the presheaf of vector fields to the presheaf of flows. Moreover, using the appropriate topologies we developed for time-dependent vector fields and the topology for local flows herein, this exponential map can be shown to be an homeomorphism onto its image by the universal property of the inverse limit.

1.3. Background and notation

We shall give a brief outline of the notations we use in the main body of the thesis. We shall mainly give definitions, establish the bare minimum of facts we require, and refer the reader to the references for details.

Manifolds, vector bundles, and jet bundles. We shall assume all manifolds to be Hausdorff, second countable, and connected. We shall work with manifolds and vector bundles coming from the different categories: smooth (i.e., infinitely differentiable), real analytic, and holomorphic (i.e., complex analytic). We shall use "class C^r" to denote these three cases, i.e., $r \in \{\infty, \omega, \mathrm{hol}\}$ for smooth, real analytic, and holomorphic, respectively. When $r = \mathrm{hol}$, we shall frequently ask that M be a Stein manifold; this means that there is a proper embedding of M in $\mathbb{C}^N$ for a suitable $N \in \mathbb{Z}_{>0}$ (Grauert and Remmert, 1955). A typical C^r-vector bundle we will denote by $\pi : E \to M$. The dual bundle we denote by E^*. The tangent bundle we denote by $\pi_{TM} : TM \to M$ and the cotangent bundle by $\pi_{T^*M} : T^*M \to M$.

While manifolds and vector bundles are smooth, real analytic, or holomorphic, we shall work with other sorts of geometric objects, e.g., functions, mappings, sections, that have various sorts of regularity. Let us introduce the terminology we shall use. Let $m \in \mathbb{Z}_{\geq 0}$ and let $m' \in \{0, \mathrm{lip}\}$. We shall work with objects with regularity $\nu \in \{m + m', \infty, \omega, \mathrm{hol}\}$. Thus $\nu = m$ means "m-times continuously differentiable," $\nu = m + \mathrm{lip}$ means "m-times continuously differentiable with locally Lipschitz top derivative," $\nu = \infty$ means "smooth", $\nu = \omega$ means "real analytic", and $\nu = \mathrm{hol}$ means "holomorphic." Given $\nu \in \{m + m', \infty, \omega, \mathrm{hol}\}$, we shall often say "let $r \in \{\infty, \omega, \mathrm{hol}\}$, as required." This has obvious meaning that $r = \mathrm{hol}$ when $\nu = \mathrm{hol}$, $r = \omega$ when $\nu = \omega$, and $r = \infty$ otherwise. We shall also use the terminology "let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, as appropriate." This means that $\mathbb{F} = \mathbb{C}$ when $r = \mathrm{hol}$ and $\mathbb{F} = \mathbb{R}$ otherwise.

If $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, and if $\nu \in \{m + m', \infty, \omega, \mathrm{hol}\}$, then the C^ν-sections of E are denoted by $\Gamma^\nu(E)$. By $C^\nu(M)$ we denote the set of C^ν-functions on M, noting that these are $\mathbb{F}$-valued, i.e., $\mathbb{C}$-valued when $\nu = \mathrm{hol}$. If M and N are C^r-manifolds, $C^\nu(M; N)$ denotes the set of C^ν-mappings from M to N. If $f \in C^{\nu+1}(M)$ and $X \in \Gamma^\nu(TM)$, we denote by $\mathscr{L}_X f$ or Xf the Lie derivative of f with respect to X. If $\Phi \in C^1(M; N)$, $T\Phi : TM \to TN$ denotes the derivative of Φ.

For a C^r-vector bundle $\pi : E \to M$, $r \in \{\infty, \omega, \mathrm{hol}\}$, we denote by $\pi_m : J_m E \to M$ the vector bundle of m-jets of sections of E; see (Kolář, Michor, and Slovák, 1993, §12.17) and (Saunders, 1989). For C^r-manifolds M and N, we denote by $\rho_0^m : J^m(M; N) \to M \times N$ the bundle of m-jets of mappings from M to N. We also have a fibre bundle

$$\rho_m \triangleq \mathrm{pr}_1 \circ \rho_0^m : J^m(M; N) \to M,$$

where pr_1 is the projection onto the first component. We signify the m-jet of a section, function, or mapping by use of the prefix j_m, i.e., $j_m\xi$, $j_m f$, or $j_m\Phi$. The set of jets of sections at x we denote by $J_x^m E$ and the set of jets of mappings at $(x, y) \in M \times N$ we denote by $J^m(M; N)_{(x,y)}$. As a special case, $J^m(M; \mathbb{R})$ denotes the bundle of m-jets of functions. We denote by $T_x^{*m} M = J^m(M; \mathbb{R})_{(x,0)}$ the jets of functions with value 0 at x, and $T^{*m} M = \cup_{x \in M} T_x^{*m} M$. The space $T_x^{*m} M$ has the structure of a $\mathbb{R}$-algebra specified by requiring that

$$\mathbf{m}_x^r \ni f \mapsto j_m f(x) \in T^* m_x M$$

be a $\mathbb{R}$-algebra homomorphism, with $\mathbf{m}_x^r \subseteq C^r(M)$ being the ideal of functions vanishing at x. We then note (Kolář, Michor, and Slovák, 1993, Proposition 12.9) that

$J^m(M;N)_{(x,y)}$ is identified with the set of $\mathbb{R}$-algebra homomorphisms from $T_y^{*m}N$ to $T_x^{*m}M$ according to

$$j_m\Phi(x)(j_m g(y)) = j_m(\Phi^* g)(x)$$

for Φ a smooth mapping defined in some neighbourhood of x and satisfying $\Phi(x) = y$. We note that $\Gamma^\nu(J^m E)$ can be thought of in the usual way since $\pi_m : J^m E \to M$ is a C^r-vector bundle. However, $J^m(M;N)$ is not, generally, a vector bundle; nonetheless, we shall denote by $\Gamma^\nu(J^m(M;N))$ the set of C^ν-sections of the bundle $\rho_m : J^m(M;N) \to M$.

Metrics and connections. We shall make use of Riemannian and fibre metrics for the same reason of convenience. Many definitions we make use a specific choice for such metrics, although none of the results depend on these choices. For $r \in \{\infty, \omega\}$, we let $\pi : E \to M$ be a C^r-vector bundle. We denote by $\mathbb{G}_M$ a C^r-Riemannian metric on M and by $\mathbb{G}_\pi$ a C^r-metric for the fibres of E. We make a note that the existence of these in the real analytic case is verified by (Jafarpour and Lewis, 2014, Lemma 2.4). The metrics $\mathbb{G}_M$ and $\mathbb{G}_\pi$ then induce metrics in all tensor products of TM and E and their duals. For simplicity, we just denote any such metric by $\mathbb{G}_{M,\pi}$.

We will frequently make use of the distance function on M associated with a Riemannian metric $\mathbb{G}$. In order to have constructions involving $\mathbb{G}$ make sense—in terms of not depending on the choice of Riemannian metric—we should verify that such constructions do not depend on the choice of this metric. Of course, this is not true for all manner of general assertions. However, Lemma A.1 captures what we need. This lemma will not surprise most readers, but we could not find a proof of this anywhere.

For convenience we shall make use of connections in representing certain objects that do not actually require a connection for their description. For $r \in \{\infty, \omega\}$, we let $\pi : E \to M$ be a C^r-vector bundle. We let ∇^M be a C^r-affine connection on M and we let ∇^π denote a C^r-linear connection in the vector bundle. The existence of these in the real analytic case is proved by (Jafarpour and Lewis, 2014, Lemma 2.4). Almost always we will not require ∇^M to be the Levi-Civita connection for the Riemannian metric $\mathbb{G}_M$, nor do we typically require there to be any metric relationship between ∇^π and $\mathbb{G}_\pi$. However, in our constructions for the Lipschitz topology, it is sometimes convenient to assume that ∇^M is the Levi-Civita connection for $\mathbb{G}_M$ and that ∇^π is $\mathbb{G}_\pi$-orthogonal, i.e., parallel transport consists of inner product preserving mappings. Thus, a safety-minded reader may wish to make these assumptions in all cases.

Chapter 2

Space of sections

We present a methodology for working with vector fields with measurable time-dependence and for working with the resulting flows of such vector fields. We begin in this section by characterizing time-varying vector fields on manifolds using locally convex topologies generated by a family of seminorms. This presentation of time-varying vector fields agrees with, and extends, the standard treatments. It closely follows from (Jafarpour and Lewis, 2014).

2.1. Measurable and integrable funtions with values in locally convex topological vector spaces

The topological characterisation relies on notions of measurability, integrability, and boundedness in the locally convex spaces $\Gamma^\nu(E)$. For an arbitrary locally convex space V, let us review some definitions.

(1) Let $(\mathcal{M}, \mathscr{A})$ be a measurable space. A function $\Psi : \mathcal{M} \to V$ is "measurable" if $\Phi^{-1}(\mathcal{B}) \in \mathscr{A}$ for every Borel set $\mathcal{B} \subseteq V$.

(2) It is possible to describe a notion of integral, called the "Bochner integral", for a function $\gamma : \mathbb{T} \to V$ that closely resembles the usual construction of the Lebesgue integral. A curve $\gamma : \mathbb{T} \to V$ is "Bochner integrable" if its Bochner integral exists and is "locally Bochner integrable" if the Bochner integral of $\gamma|\mathbb{T}'$ exists for any compact subinterval $\mathbb{T}' \subseteq \mathbb{T}$.

(3) Finally, a subset $\mathcal{B} \subseteq V$ is bounded if $p|\mathcal{B}$ is bounded for any continuous seminorm p on V. A curve $\gamma : \mathbb{T} \to V$ is "essentially von Neumann bounded" if there exists a bounded set $\mathcal{B}$ such that

$$\lambda(\{t \in \mathbb{T} \mid \gamma(t) \notin \mathcal{B}\}) = 0,$$

and is "locally essentially von Neumann bounded" if $\gamma|\mathbb{T}'$ is essentially von Neumann bounded for every compact subinterval $\mathbb{T}' \subseteq \mathbb{T}$.

2.2. Topologies on space of sections

In this section we will provide explicit seminorms that define the various topologies we use for local sections, corresponding to regularity classes $\nu \in \{m + m', \infty, \omega, \mathrm{hol}\}$. We shall not use much space to describe the nature of these topologies, but give a general sketch and refer the interested readers to (Jafarpour and Lewis, 2014) for more details.

2.2.1. Locally Lipschitz sections of vector bundles. As we are interested in ordinary differential equations with well-defined flows, we must, according to the usual theory, consider locally Lipschitz sections of vector bundles. In particular, we will find it essential to topologise the space of locally Lipschitz sections of $\pi : E \to M$. To define the seminorms for this topology, we make use of a "local least Lipschitz constant."

We let $\xi : M \to E$ be such that $\xi(x) \in E_x$ for every $x \in M$. For a piecewise differentiable curve $\gamma : [0, T] \to M$, we denote by $\tau_{\gamma,t} : E_{\gamma(0)} \to E_{\gamma(t)}$ the isomorphism of parallel translation along γ for each $t \in [0, T]$. We then define, for $K \subseteq M$ compact,

$$l_K(\xi) = \sup\left\{ \frac{\|\tau_{\gamma,1}^{-1}(\xi \circ \gamma(1)) - \xi \circ \gamma(0)\|_{\mathbb{G}_\pi}}{\ell_{\mathbb{G}_M}(\gamma)} \,\middle|\, \gamma : [0,1] \to M,\ \gamma(0), \gamma(1) \in K,\ \gamma(0) \neq \gamma(1) \right\},$$

which is the **K-sectional dilatation of** ξ. Here $\ell_{\mathbb{G}_M}$ is the length function on piecewise differentiable curves. We also define

$$\begin{aligned} \mathrm{dil}\ \xi : M &\to \mathbb{R}_{\geq 0} \\ x &\mapsto \inf\{l_{\mathrm{cl}}(\mathcal{U})(\xi) \,|\, \mathcal{U} \text{ is a relatively compact neighbourhood of } x\}, \end{aligned}$$

which is the **local sectional dilatation** of ξ. Note that, while the values taken by dil ξ will depend on the choice of a Riemannian metric $\mathbb{G}$, the property dil $\xi(x) < \infty$ for $x \in M$ is independent of $\mathbb{G}$, whence $\xi \in \Gamma^{\mathrm{lip}}(E)$ (Jafarpour and Lewis, 2014, Lemma 3.10).

The following characterisations of the local sectional dilatation are useful.

Lemma 2.1 (Local sectional dilatation using derivatives)**.** *For a C^∞-vector bundle $\pi : E \to M$ and for $\xi \in \Gamma^{\mathrm{lip}}(E)$, we have*

$$\mathrm{dil}\,\xi(x) = \inf\{\sup\{\|\nabla^{\pi_m}_{v_y}\xi\|_{\mathbb{G}_{M,\pi}} \mid y \in cl(\mathcal{U}),\ \|v_y\|_{\mathbb{G}_M} = 1,\ \xi \text{ differentiable at } y\} \mid \mathcal{U} \text{ is a relatively compact neighbourhood of } x\}.$$

Proof. (Jafarpour and Lewis, 2014, Lemma 3.12). □

Lemma 2.2 (Local sectional dilatation and sectional dilatation)**.** *Let $\pi : E \to M$ be a C^∞-vector bundle. Then, for each $x_0 \in M$, there exists a relatively compact neighbourhood $\mathcal{U}$ of x_0 such that*

$$l_{\mathrm{cl}(\mathcal{U})}(\xi) = \sup\{\mathrm{dil}\ \ \xi(x) \mid x \in \mathrm{cl}(\mathcal{U})\},\ \ \xi \in \Gamma^{\mathrm{lip}}(E).$$

Proof. We let $\mathcal{U}$ be a geodesically convex neighbourhood of x_0 so that

$$\operatorname{dil}\xi(x) = \sup\{\|\nabla^{\pi_m}_{v_y}\xi\|_{\mathbb{G}_{M,\pi}} \mid y \in \operatorname{cl}(\mathcal{U}),\ \|v_y\|_{\mathbb{G}_M} = 1,\ \xi \text{ differentiable at } y\}.$$

Thus $l_{\operatorname{cl}}(\mathcal{U})(\xi)$ is an upper bound for

$$\{\operatorname{dil}\xi(x) \mid x \in \operatorname{cl}(\mathcal{U})\}.$$

Next, let $\epsilon \in \mathbb{R}_{>0}$. Let $x \in \mathcal{U}$ and $v_x \in T_xM$ be such that (1) ξ is differentiable at x, (2) $\|v_x\|_{\mathbb{G}_M} = 1$, and (3) $l_{\operatorname{cl}}(\mathcal{U})(\xi) - \|\nabla^{\pi}_{v_x}\xi\|_{\mathbb{G}_{M,\pi}} < \frac{\epsilon}{2}$. Then let $\mathcal{V}$ be a geodesically convex neighbourhood of x such that $\operatorname{cl}(\mathcal{V}) \subseteq \mathcal{U}$ and such that

$$\sup\{\|\nabla^{\pi}_{v_y}\xi\|_{\mathbb{G}_{M,\pi}} \mid y \in \operatorname{cl}(\mathcal{V}),\ \|v_y\|_{\mathbb{G}_M} = 1,\ \xi \text{ differentiable at } y\} - \operatorname{dil}\xi(x) < \frac{\epsilon}{2}.$$

We have

$$\begin{aligned} l_{\operatorname{cl}(\mathcal{U})}(\xi) - \frac{\epsilon}{2} &< \sup\{\|\nabla^{\pi_m}_{v_y}\xi\|_{\mathbb{G}_{M,\pi}} \mid y \in \operatorname{cl}(\mathcal{V}),\ \|v_y\|_{\mathbb{G}_M} = 1,\ \xi \text{ differentiable at } y\} \\ &\leq l_{\operatorname{cl}(\mathcal{U})}(\xi). \end{aligned}$$

Therefore,

$$\begin{aligned} l_{\operatorname{cl}(\mathcal{U})}(\xi) - \epsilon &= l_{\operatorname{cl}(\mathcal{U})}(\xi) - \frac{\epsilon}{2} - \frac{\epsilon}{2} \\ &\leq \sup\{\|\nabla^{\pi_m}_{v_y}\xi\|_{\mathbb{G}_{M,\pi}} \mid y \in \operatorname{cl}(\mathcal{V}),\ \|v_y\|_{\mathbb{G}_M} = 1,\ \xi \text{ differentiable at } y\} + \operatorname{dil}\xi \\ &\quad - \sup\{\|\nabla^{\pi_m}_{v_y}\xi\|_{\mathbb{G}_{M,\pi}} \mid y \in \operatorname{cl}(\mathcal{V}),\ \|v_y\|_{\mathbb{G}_M} = 1,\ \xi \text{ differentiable at } y\} \\ &= \operatorname{dil}\xi(x). \end{aligned}$$

This shows that $l_{\operatorname{cl}}(\mathcal{U})(\xi)$ is the least upper bound for

$$\{\operatorname{dil}\xi(x) \mid x \in \operatorname{cl}(\mathcal{U})\},$$

as required. □

2.2.2. Fibre norms for jet bundles. Fibre norms for jet bundles of a vector bundle play an important role in our unified treatment of various classes of regularities. Our discussion begins with general constructions for the fibres of jet bundles. Let $r \in \{\infty, \omega\}$ and let M be a C^r-manifold. Let $\pi : E \to M$ be a C^r-vector bundle with $\pi_m : J^mE \to M$ its mth jet bundle. We shall suppose that we have a C^r-affine connection ∇^M on M and a C^r-vector bundle connection ∇^π in E. By additionally supposing that we have a C^r-Riemannian metric $\mathbb{G}_M$ on M and a C^r-fibre metric $\mathbb{G}_\pi$ on E, we shall give a C^r-fibre norm on J^mE.

Denote $T^m(T^*M)$ the m-fold tensor product of T^*M and $S^m(T^*M)$ the symmetric tensor bundle. The connection ∇^M induces a covariant derivative for tensor fields $A \in \Gamma^1(T^k_l(TM))$ on M, $k, l \in \mathbb{Z}_{\geq 0}$. This covariant derivative we denote by ∇^M, dropping the particular k and l. Similarly, the connection ∇^π induces a covariant derivative for sections $B \in \Gamma^1(T^k_l(E))$ of the tensor bundles associated with E, $k, l \in \mathbb{Z}_{\geq 0}$. This covariant derivative we denote by ∇^π, dropping the particular k and l. We will also consider differentiation of sections of $T^{k_1}_{l_1}(TM) \otimes T^{k_2}_{l_2}(E)$, and we denote the covariant derivative by $\nabla^{M,\pi}$. Note that

$$\nabla^{M,\pi,m}\xi \triangleq \underbrace{\nabla^{M,\pi} \cdots (\nabla^{M,\pi}}_{m-1 \text{ times}}(\nabla^\pi \xi)) \in \Gamma^\infty(T^m(T^*M) \otimes E).$$

For $\xi \in \Gamma^\infty(E)$ and $m \in \mathbb{Z}_{\geq 0}$, we define

$$D^m_{\nabla^M, \nabla^\pi}(\xi) = \mathrm{Sym}_m \otimes \mathrm{id}_E(\nabla^{M,\pi,m}\xi) \in \Gamma^\infty(S^m(T^*M) \otimes E).$$

where $\mathrm{Sym}_m : T^m(T^*M) \to S^m(T^*M)$ by

$$\mathrm{Sym}_m(v_1 \otimes \cdots \otimes v_m) = \frac{1}{m!} \sum_{\sigma \in \mathfrak{S}_m} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(m)}.$$

We take the convention that $D^0_{\nabla^M, \nabla^\pi}(\xi) = \xi$. We then have a map

$$\begin{aligned} S^m_{\nabla^M, \nabla^{M,\pi}} : J^m E &\to \bigoplus_{j=0}^m (S^j(T^*M) \otimes E) \\ j_m \xi(x) &\mapsto (\xi(x), D^1_{\nabla^M, \nabla^\pi}(\xi)(x), \ldots, D^m_{\nabla^M, \nabla^\pi}(\xi)(x)) \end{aligned}$$

that can be verified to be an isomorphism of vector bundles (Jafarpour and Lewis, 2013, Lemma 2.1). Note that inner products on the components of a tensor products induce an inner product on the tensor product in a natural way (Jafarpour and Lewis, 2013, Lemma 2.2). Then we have a fibre metric in all tensor spaces associated with TM and E and their tensor products. We shall denote by $G_{M,\pi}$ any of these various fibre metrics. In particular, we have a fibre metric $\mathbb{G}_{M,\pi}$ on $T^j(T^*M) \otimes E$ for each $j \in \mathbb{Z}_{\geq 0}$. This thus gives us a fibre metric $\mathbb{G}_{M,\pi,m}$ on $J^m E$ defined by

$$\mathbb{G}_{M,\pi,m}(j_m\xi(x), j_m\eta(x)) = \sum_{j=0}^m \mathbb{G}_{M,\pi}\left(\frac{1}{j!} D^j_{\nabla^M, \nabla^\pi}(\xi)(x), \frac{1}{j!} D^j_{\nabla^M, \nabla^\pi}(\eta)(x)\right). \tag{2.1}$$

Associated to this inner product on fibres is the norm on fibres, which we denote by $\|\cdot\|_{\mathbb{G}_{M,\pi,m}}$. We shall use these fibre norms continually in our descriptions of various topologies in the next few sections.

2.2.3. Seminorms for spaces of finitely differentiable sections. In this section we give a seminorm for sections of regularity $\nu = m \in \mathbb{Z}_{\geq 0}$. We again take $\pi : E \to M$ to be smooth vector bundle. For the space $\Gamma^m(E)$ of m-times continuously differentiable sections, we define seminorms p^m_K, $K \subseteq M$ compact, for $\Gamma^m(E)$ by

$$p^m_K(\xi) = \sup\{\|j_m\xi(x)\|_{G_{M,\pi,m}} \mid x \in K\}.$$

We call the locally convex topology on $\Gamma^m(E)$ defined by the family of seminorms p^m_K where $K \subseteq M$ compact, the **C$^\mathbf{m}$-topology**, and it is complete, Hausdorff, separable, and metrizable (Jafarpour and Lewis, 2013, §3.4).

2.2.4. Seminorms for spaces of Lipschitz sections. In this section we again work with a smooth vector bundle $\pi : E \to M$. Different from defining the fibre metrics from the last section, for the Lipschitz topologies the affine connection ∇^M is required to be the Levi-Civita connection for the Riemannian metric G_M and the linear connection $\nabla^\pi$is required to be G_π-orthogonal. Because we have the decomposition

$$J^m E \simeq \bigoplus_{j=0}^{m} (S^j(T^*M) \otimes E),$$

it follows that the vector bundle $J^m E$ has a C^r-connection ∇^{π_m}, defined by

$$\nabla^{\pi_m}_X j_m\xi = (S^m_{\nabla^M,\nabla^\pi})^{-1}(\nabla^\pi_X \xi, \nabla^{M,\pi}_X D^1_{\nabla^M,\nabla^\pi}(\xi), \dots, \nabla^{M,\pi}_X D^m_{\nabla^M,\nabla^\pi}(\xi)).$$

By Rademacher's Theorem (Federer, 1969, Theorem 3.1.6), if a section ξ is of class $C^{m+\mathrm{lip}}$, then its $(m+1)$-th derivative exists almost everywhere. Then by Lemma 2.1, we define

$$\begin{aligned} \mathrm{dil}\ j_m\xi(x) = \inf\{\sup\{\|\nabla^{\pi_m}_{v_y} j_m\xi\|_{\mathbb{G}_{M,\pi}} \mid y \in \mathrm{cl}(\mathcal{U}),\ \|v_y\|_{\mathbb{G}} = 1,\ j_m\xi \text{ differentiable at } y \\ \mid \mathcal{U} \text{ is a relatively compact neighbourhood of } x\}. \end{aligned}$$

which is the **local sectional dilatation** of ξ. Let $K \subseteq M$ be compact and define

$$\lambda^m_K(\xi) = \sup\{\mathrm{dil} j_m\xi(x) \mid x \in K\}$$

for $\xi \in \Gamma^{m+\mathrm{lip}}(E)$. We then can define a seminorm on $\xi \in \Gamma^{m+\mathrm{lip}}(E)$ by

$$p^{m+\mathrm{lip}}_K(\xi) = \max\{\lambda^m_K(\xi), p^m_K(\xi)\}.$$

We call the locally convex topology on $\Gamma^{m+\mathrm{lip}}(E)$ defined by the family of seminorms $p^{m+\mathrm{lip}}_K$, $K \subseteq M$ compact, the **$\mathbf{C^{m+lip}}$-topology**, and it is complete, Hausdorff, separable, and metrizable (Jafarpour and Lewis, 2014, §3.5).

2.2.5. Seminorms for spaces of smooth sections. Let $\pi : E \to M$ be a smooth vector bundle. Using the fibre norms from the preceding section, it is a straightforward matter to define appropriate seminorms that define the locally convex topology for $\Gamma^\infty(E)$. For $K \subseteq M$ compact and for $m \in \mathbb{Z}_{\geq 0}$, define a seminorm $p^\infty_{K,m}$ on $\Gamma^\infty(E)$ by

$$p^\infty_{K,m}(\xi) = \sup\{\|j_m\xi(x)\|_{\mathbb{G}_{M,\pi,m}} \mid x \in K\}.$$

We call the locally convex topology on $\Gamma^\infty(E)$ defined by the family of seminorms $p^\infty_{K,m}$, $K \subseteq M$ compact, $m \in \mathbb{Z}_{\geq 0}$, the **$\mathbf{C^\infty}$-topology**, and it is complete, Hausdorff, separable, and metrizable (Jafarpour and Lewis, 2014, §3.2).

2.2.6. Seminorms for spaces of holomorphic sections. For the topology of the holomorphic sections, we consider an holomorphic vector bundle $\pi : E \to M$ and denote by $\Gamma^{\mathrm{hol}}(E)$ the space of holomorphic sections of E. Let G_π be an Hermitian metric on the vector bundle and denote by $\|\cdot\|_{G_\pi}$ the associated fibre norm. For $K \subseteq M$ compact, denote by p_K^{hol} the seminorm on $\Gamma^{\mathrm{hol}}(E)$ defined by

$$p_K^{\mathrm{hol}}(\xi) = \sup\{\|\xi(z)\|_{\mathbb{G}_\pi} \mid z \in K\}.$$

The family of seminorms p_K^{hol} where $K \subseteq M$ compact, defines a locally convex topology for $\Gamma^{\mathrm{hol}}(E)$ we call the **$\mathrm{C}^{\mathrm{hol}}$-topology**, and it is complete, Hausdorff, separable, and metrizable (Jafarpour and Lewis, 2014, §4.2).

2.2.7. Seminorms for spaces of real analytic sections. The topology one considers for real analytic sections does not have the same attributes as smooth, finitely differentiable, Lipschitz, and holomorphic cases. There is a history to the characterisation of real analytic topologies, and we refer to [Jafarpour and Lewis 2014, §5.2] for four equivalent characterisations of the real analytic topology for the space of real analytic sections of a vector bundle. Here we will give the most elementary of these definitions to state, although it is probably not the most practical definition.

In this section we let $\pi : E \to M$ be a real analytic vector bundle and let $\Gamma^\omega(E)$ be the space of real analytic sections. Here we need all of the data used to define the seminorms in the finitely differentiable and smooth cases to topologise $\Gamma^\omega(E)$, only now we need this data to be real analytic. We refer to (Jafarpour and Lewis, 2014, Lemma 2.4) for the existence of this data. Therefore, we can define real analytic fibre metrics $\mathbb{G}_{M,\pi,m}$ on the jet bundles J^mE as in Section 2.2.2. To define seminorms for $\Gamma^\omega(E)$, we let $c_0(\mathbb{Z}_{\geq 0};\mathbb{R}_{>0})$ denote the space of sequences in $\mathbb{R}_{>0}$, indexed by $\mathbb{Z}_{\geq 0}$, and converging to zero. We shall denote a typical element of $c_0(\mathbb{Z}_{\geq 0};\mathbb{R}_{>0})$ by $\boldsymbol{a} = (a_j)_{j\in\mathbb{Z}_{\geq 0}}$. Now for $K \subseteq M$ compact, and $\boldsymbol{a} \in c_0(\mathbb{Z}_{\geq 0};\mathbb{R}_{>0})$, we define a seminorm $p^\omega_{K,\boldsymbol{a}}$ on $\Gamma^\omega(E)$ by

$$p^\omega_{K,\boldsymbol{a}}(\xi) = \sup\{a_0a_1...a_m\|j_m\xi(x)\|_{\mathbb{G}_{M,\pi,m}} \mid x \in K,\ m \in \mathbb{Z}_{\geq 0}\}.$$

The family of seminorms $p^\omega_{K,\boldsymbol{a}}$, $K \subseteq M$ compact, $\boldsymbol{a} \in c_0(\mathbb{Z}_{\geq 0};\mathbb{R}_{>0})$, defines a locally convex topology on $\Gamma^\omega(E)$ that we call the **C^ω-topology**, and it is complete, Hausdorff, separable, but not metrizable (Jafarpour and Lewis, 2014, §5.3).

2.2.8. Summary and notation. The preceding developments have been made for spaces of sections of a general vector bundle. We will primarily (but not solely) be interested in spaces of vector fields and functions. For vector fields, the seminorms above can be defined for an affine connection ∇^M on M, as this also serves as a vector bundle connection for $\pi_{TM} : TM \to M$. For vector fields, the decomposition of the jet bundle J^mTM is

$$S^m_{\nabla^M} : J^mTM \;\to\; \oplus_{j=0}^m(S^j(T^*M)\otimes TM)$$

$$j_m X(x) \quad \mapsto \quad (X(x), \mathrm{Sym}_1 \otimes \mathrm{id}_{TM}(\nabla^M X)(x), ..., \mathrm{Sym}_m \otimes \mathrm{id}_{TM}(\nabla^{M,m} X)(x)).$$

where

$$\nabla^{M,k} X \triangleq \underbrace{\nabla^M \cdots \nabla^M}_{k \text{ times}} X$$

For the Lipschitz topologies, one needs for ∇^M to be a metric connection, and so it may as well be the Levi-Civita connection associated with a Riemannian metric $\mathbb{G}$. With the Riemannian metric $\mathbb{G}$, this decomposition gives the various topologies for spaces of vector fields. Functions are particular instances of sections of a vector bundle, as we can identity a function with a section of the trivial line bundle $\mathbb{F}_M = M \times \mathbb{F}$. We denote by $C^\nu(M)$ the set of functions having the regularity ν coming from one of the classes of regularity we consider. Note that this bundle has a canonical flat connection ∇^π which, when translated to functions, amounts to the requirement that $\nabla^\pi_X f = \mathscr{L}_X f$. For $f \in C^\nu(M)$ when $\nu \geq m$, let us denote

$$\nabla^{M,0} f = f, \;\; \nabla^{M,1} f = df, \;\text{ and }\; \nabla^{M,m} f = \nabla^{M,m-1} df.$$

Then the decomposition of the jet bundle of $\mathbb{F}_M$ looks like

$$\begin{aligned} S^m_{\nabla^M} : J^m(M;\mathbb{R}) \quad &\to \quad \mathbb{R} \oplus (\oplus_{j=0}^m S^j(T^*M)) \\ j_m f(x) \quad &\mapsto \quad ((\nabla^{M,0} f)(x), (\nabla^{M,1} f)(x), ..., \mathrm{Sym}_m(\nabla^{M,m} f)(x)). \end{aligned}$$

This decomposition can be used to define the locally convex topologies for the various regularity classes of functions. Thus, if we suppose that we have a Riemannian metric G_M and affine connection ∇^M on M, there is induced a natural fibre metric G_m on $J^m(M;\mathbb{R})$ for each $m \in \mathbb{Z}_{\geq 0}$ by

$$G_{M,m}(j_m f(x), j_m g(x)) = \sum_{j=0}^{m} G_M\left(\frac{1}{j!}\mathrm{Sym}_j(\nabla^{M,j} f)(x), \frac{1}{j!}\mathrm{Sym}_j(\nabla^{M,j} g)(x)\right),$$

and the associated norm we denote by $\|\cdot\|_{G_{M,m}}$.

In the real case, the degrees of regularity are ordered according to

$$C^0 \supset C^{\mathrm{lip}} \supset C^1 \supset \cdots \supset C^m \supset C^{m+1} \supset \cdots \supset C^\infty \supset C^\omega, \tag{2.2}$$

and in the complex case the ordering is the same, of course, but with an extra C^{hol} on the right.

With all the topologies for our various cases of regularities, we will, for $K \subseteq M$ be compact, for $k \in \mathbb{Z}_{\geq 0}$, and for $\boldsymbol{a} \in c_0(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0})$, denote

$$p^\nu_K = \begin{cases} p^m_K, & \nu = m, \\ p^{m+\mathrm{lip}}_K, & \nu = m + \mathrm{lip}, \\ p^\infty_{K,m}, & \nu = \infty, \\ p^\omega_{K,\boldsymbol{a}}, & \nu = \omega, \\ p^{\mathrm{hol}}_K, & \nu = \mathrm{hol}. \end{cases} \tag{2.3}$$

The convenience and brevity more than make up for the slight loss of preciseness in this approach.

We comment that these seminorms make it clear that we have an ordering of the regularity classes as

$$m_1 < m_1 + \mathrm{lip} < \cdots < m_2 < m_2 + \mathrm{lip} < \cdots < \infty < \omega < \mathrm{hol}$$

from least regular (coarser topology) to more regular (finer topology), and where $m_1 < m_2$. There is also an obvious "arithmetic" of degrees of regularity that we will use without feeling the need to explain it.

2.3. Time- and parameter-dependent sections and functions

In this section we introduce the classes of vector fields, depending on both time and parameter, that we work with. In our presentation, we shall make use of measurable and integrable functions with values in a locally convex topological vector space. This is classical in the case of Banach spaces, but is not as fleshed out in the general case.

2.3.1. Time-dependent sections. We carefully introduce in this section the class of time-dependent vector fields we consider, and which were quickly introduced in Section 1.2. First of all, since all topological vector spaces we consider are Suslin spaces, all standard notions of measurability coincide (Thomas, 1975, Theorem 1). Thus, for example, one can take as one's notion of measurability the naive one that preimages of Borel sets are measurable. The notion of integrability we use is "integrability by seminorm," and seems to originate in (Garnir, De Wilde, and Schmets, 1972). We refer to (Lewis, 2021) for details and further references. For complete Suslin spaces, such as we are working with, integrability by seminorm amounts to the requirement that the application of any continuous seminorm to the vector-valued functions yields a function in the usual scalar L^1-space.

With these technicalities stowed away for safety and convenience, we now make our definitions. Let $m \in \mathbb{Z}_{\geq 0}$, let $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega, \mathrm{hol}\}$, and let $r \in \{\infty, \omega, \mathrm{hol}\}$, as required. Let $\pi : E \to M$ be a C^r-vector bundle and let $\mathbb{T} \subseteq \mathbb{R}$ be an interval. We say that $\xi : \mathbb{T} \to \Gamma^\nu(E)$ is **locally integrally bounded** if, for any continuous seminorm p for $\Gamma^\nu(E)$ and for $\mathbb{S} \subseteq \mathbb{T}$ compact, $p \circ (\xi|\mathbb{S}) \in L^1(\mathbb{T}; \mathbb{R})$. We give a locally convex topology for the set of locally integrally bounded sections by the seminorms

$$p^\nu_{K,\mathbb{S}}(\xi) = \int_{\mathbb{S}} p^\nu_K(\xi(t))\, \mathrm{d}t, \quad K \subseteq M, \mathbb{S} \subseteq \mathbb{T} \text{ compact},$$

where p^ν_K is the seminorm defined by (2.3). Thus, colloquially, the set of locally integrally bounded mappings is just $L^1_{\mathrm{loc}}(\mathbb{T}; \Gamma^\nu(E))$. We abbreviate this space by $\Gamma^\nu_{\mathrm{LI}}(\mathbb{T}; E)$. Explicit characterisations of these spaces are given by (Jafarpour and Lewis, 2014).

Definition 2.3 (Classes of time-varying sections)**.** For a vector bundle $\pi : E \to M$ of class C^r and an interval $\mathbb{T} \subseteq \mathbb{R}$, let $\xi : \mathbb{T} \times M \to E$ satisfy $\xi(t,x) \in E_x$ for each $(t,x) \in \mathbb{T} \times M$. Denote by ξ_t $(t \in \mathbb{T})$, the map $x \mapsto \xi(t,x)$ and suppose that $\xi_t \in \Gamma^\nu(E)$ for every $t \in \mathbb{T}$. Then ξ is:

(i) a *Carathéodory section of class* C^ν if the curve $\mathbb{T} \ni t \mapsto \xi_t \in \Gamma^\nu(E)$ is measurable;

(ii) *(locally) integrally* C^ν*-bounded* if the curve $\mathbb{T} \ni t \mapsto \xi_t \in \Gamma^\nu(E)$ is (locally) Bochner integrable;

(iii) *(locally) essentially* C^ν*-bounded* if the curve $\mathbb{T} \ni t \mapsto \xi_t \in \Gamma^\nu(E)$ is (locally) essentially von Neumann bounded.

We denote:

(iv) the set of Carathéodory sections of class C^ν by $\Gamma^\nu_{\mathrm{CF}}(\mathbb{T}; E)$;

(v) the set of (locally) integrally C^ν-bounded sections by $(\Gamma^\nu_{\mathrm{LI}}(\mathbb{T}; E))$ $\Gamma^\nu_{\mathrm{I}}(\mathbb{T}; E)$;

(vi) the set of (locally) essentially C^ν-bounded sections by $(\Gamma^\nu_{\mathrm{LB}}(\mathbb{T}; E))$ $\Gamma^\nu_{\mathrm{B}}(\mathbb{T}; E)$;

We shall also find some alternative, more expansive, notation profitable, notation that is the "standard" adaptation from the usual notation for functions. For a time-domain $\mathbb{T}$ we denote:

1. $\mathrm{CF}(\mathbb{T}; \Gamma^\nu(E))$: Carathéodory sections of class C^ν;
2. $\mathrm{L}^1(\mathbb{T}; \Gamma^\nu(E))$: integrally C^ν-bounded sections;
3. $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{T}; \Gamma^\nu(E))$: integrally C^ν-bounded sections;
4. $\mathrm{L}^\infty(\mathbb{T}; \Gamma^\nu(E))$: essentially C^ν-bounded sections;
5. $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{T}; \Gamma^\nu(E))$: locally essentially C^ν-bounded sections.

The spaces of integrable or essentially bounded sections have natural topologies, which we now describe.

1. The space $\mathrm{L}^1(\mathbb{T}; \Gamma^\nu(E))$ has a locally convex topology defined by seminorms $p^\nu_{K,1}$, $K \subseteq M$ compact, given by

$$p^\nu_{K,1}(\xi) = \int_{\mathbb{T}} p^\nu_K \circ \xi_t \, \mathrm{d}t,$$

where p^ν_K is a seminorm for the topology on $\Gamma^\nu(E)$. Here we make the abuse of notation by possibly omitting extra bits of notation from the seminorm p^ν_K, cf. (2.3). We recall that

$$\mathrm{L}^1(\mathbb{T}; \Gamma^\nu(E)) \simeq \mathrm{L}^1(\mathbb{T}; \mathbb{R}) \widehat{\otimes}_\pi \Gamma^\nu(E),$$

where $\widehat{\otimes}_\pi$ denotes the completed projective tensor product (Jarchow, 1981, Corrolary 15.7.2).

2. The space $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{T};\Gamma^\nu(E))$ has a locally convex topology defined by seminorms $p^\nu_{K,\mathbb{S},1}$, $K \subseteq M$ compact, $\mathbb{S} \subseteq \mathbb{T}$ compact, given by
$$p^\nu_{K,\mathbb{S},1}(\xi) = \int_{\mathbb{S}} p^\nu_K \circ \xi_t \,\mathrm{d}t, \tag{2.1}$$
where seminorms p^ν_K are as above. We note that $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{T};\Gamma^\nu(E))$ is the inverse limit of the inverse system $\{\mathrm{L}^1_{\mathrm{loc}}(\mathbb{S};\Gamma^\nu(E))\}_{\mathscr{K}_\mathbb{T}}$, where $\mathscr{K}_\mathbb{T}$ is the directed set of compact subintervals of $\mathbb{T}$ ordered by $\mathbb{S}_1 \preceq \mathbb{S}_2$ if $\mathbb{S}_1 \subseteq \mathbb{S}_2$.

3. The space $\mathrm{L}^\infty(\mathbb{T};\Gamma^\nu(E))$ has a locally convex topology defined by seminorms $p^\nu_{K,\infty}$, $K \subseteq M$ compact, given by
$$p^\nu_{K,\infty}(\xi) = \operatorname{ess\,sup}\{p^\nu_K \circ \xi_t \mid t \in \mathbb{T}\},$$
where seminorms p^ν_K are as above. We have
$$\mathrm{L}^\infty(\mathbb{T};\Gamma^\nu(E)) \simeq \mathrm{L}^\infty(\mathbb{T};\mathbb{R})\widehat{\otimes}_\pi\Gamma^\nu(E).$$

4. The space $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{T};\Gamma^\nu(E))$ has a locally convex topology defined by seminorms $p^\nu_{K,\mathbb{S},\infty}$ $K \subseteq M$ compact, $\mathbb{S} \subseteq \mathbb{T}$ compact, given by
$$p^\nu_{K,\mathbb{S},\infty}(\xi) = \operatorname{ess\,sup}\{p^\nu_K \circ \xi_t \mid t \in \mathbb{S}\},$$
where seminorms p^ν_K are as above. We note that $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{T};\Gamma^\nu(E))$ is the inverse limit of the inverse system $\{\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{S};\Gamma^\nu(E))\}_{\mathscr{K}_\mathbb{T}}$.

Our method of working with vector fields and their flows is to use general globally defined functions to replace local coordinates. As such, functions assume an important role in our presentation. Bearing in mind that functions are sections of the trivial line bundle, the above general definitions for sections of vector bundles apply specifically to functions, and yield the spaces $C^\nu_{\mathrm{LI}}(\mathbb{T};M)$ of time-dependent functions $f:\mathbb{T}\to C^\nu(M)$.

The following lemma indicates how we will convert sections and vector fields into functions.

Lemma 2.4 (Time-dependent functions from time-dependent sections and vector fields)**.** *Let $m \in \mathbb{Z}_{\geq 0}$, let $m' \in \{0, lip\}$, let $\nu \in \{m+m', \infty, \omega, hol\}$, and let $r \in \{\infty, \omega, hol\}$, as required. Let $\beta : B \to M$ be a C^r-affine bundle modelled on the C^r- vector bundle $\pi : E \to M$, and let $\mathbb{T} \subseteq \mathbb{R}$ be an interval. When $\nu = hol$, assume that M is a Stein manifold. Then $X \in \Gamma^\nu_{LI}(\mathbb{T};TM)$ if and only if $Xf \in C^\nu_{LI}(\mathbb{T};B)$ for any $f \in C^r(M)$.*

Proof. We note that the seminorms on $\Gamma^\nu(TM)$ provided by (2.3) give the initial topology associated with the mappings $X \mapsto \mathscr{L}_X f$, $f \in C^r(M)$ (Jafarpour and Lewis, 2014, Section 3). We observe that if $\pi_E : E \to M$ and $\pi_F : F \to N$ are C^r-vector bundles, $\nu_1, \nu_2 \in \{m+m', \infty, \omega, \mathrm{hol}\}$ are two regularity classes, and if $\phi : \Gamma^{\nu_1}(E) \to \Gamma^{\nu_2}(F)$ is a continuous linear map, then
$$\phi \circ \xi \in \Gamma^{\nu_1}_{\mathrm{LI}}(\mathbb{T};F), \quad \xi \in \Gamma^{\nu_2}_{\mathrm{LI}}(\mathbb{T};E),$$
since continuous linear maps preserve continuity of seminorms. The converse is true by universal property of the initial topology. Then the desired conclusion follows. □

The following result gives a more familiar characterisation of locally integrally bounded functions of class C^{lip}, bearing in mind our policy of introducing a Riemannian metric whenever it is convenient.

Lemma 2.5 (Property of time-varying locally Lipschitz functions)**.** *Let M be a smooth manifold, let $\mathbb{T}$ be a time-domain, and let $f \in C^{lip}_{LI}(\mathbb{T}; M)$. If $K \subseteq M$ is compact, then there exists $l \in L^1_{\mathrm{loc}}(\mathbb{T}; \mathbb{R}_{\geq 0})$ such that*

$$|f(t,x_1) - f(t,x_2)| \leq l(t)\, \mathrm{d}_{\mathbb{G}}(x_1,x_2), \quad t \in \mathbb{T},\ x_1,x_2 \in K.$$

Proof. Since functions are to be thought of as sections of the trivial line bundle $\mathbb{R}_M = M \times \mathbb{R}$, and since we use the flat connection on this bundle, we have, for any compact set $K \subseteq M$ and for $g \in C^{\mathrm{lip}}(M)$,

$$\begin{aligned} l_K(g) &= \sup\left\{ \frac{|g\circ\gamma(1) - g\circ\gamma(0)|}{l_{\mathbb{G}}(\gamma)} \ \middle|\ \gamma : [0,1] \to M, \gamma(0),\ \gamma(1) \in K,\ \gamma(0) \neq \gamma(1) \right\} \\ &= \sup\left\{ \frac{|g(x_1) - g(x_2)|}{\mathrm{d}_{\mathbb{G}}(x_1,x_2)} \ \middle|\ x_1,x_2 \in K,\ x_1 \neq x_2 \right\}. \end{aligned}$$

Let $K \subseteq M$ be compact. Let $x \in K$ and let $\mathcal{U}_x$ be a neighbourhood of x such that, by Lemma 2.2, for $g \in C^{\mathrm{lip}}(M)$, we have $\lambda^0_{\mathrm{cl}(\mathcal{U}_x)}(g) = l_{\mathrm{cl}(\mathcal{U}_x)}(g)$. Since $f \in C^{\mathrm{lip}}_{\mathrm{LI}}(\mathbb{T}; M)$, there exists $l_x \in L^1_{\mathrm{loc}}(\mathbb{T}; \mathbb{R}_{\geq 0})$ such that

$$\mathrm{dil} f(t,y) \leq l_x(t), \quad (t,y) \in \mathbb{T} \times \mathrm{cl}(\mathcal{U}_x).$$

Thus, for $x_1, x_2 \in \mathcal{U}_x$, we have

$$|f(t,x_1) - f(t,x_2)| \leq l_x(t)\, \mathrm{d}_{\mathbb{G}}(x_1,x_2), \quad t \in \mathbb{T}.$$

By compactness of K, there exist $x_1, \dots, x_m \in K$ such that $K \subseteq \cup_{j=1}^m \mathcal{U}_{x_j}$. By the Lebesgue Number Lemma (D. Burago, Y. Burago, and Ivanov, 2001, Theorem 1.6.11), there exists $r \in \mathbb{R}_{>0}$ with the property that, if $x_1, x_2 \in K$ satisfy $\mathrm{d}_{\mathbb{G}}(x_1,x_2) < r$, then there exists $j \in \{1, \dots, m\}$ such that $x_1, x_2 \in \mathcal{U}_{x_j}$. Since $C^{\mathrm{lip}}_{\mathrm{LI}}(\mathbb{T}; M) \subseteq C^0_{\mathrm{LI}}(\mathbb{T}; M)$, there exists $\kappa \in L^1_{\mathrm{loc}}(\mathbb{T}; \mathbb{R}_{\geq 0})$ such that $|f(t,x)| \leq \kappa(t)$ for $(t,x) \in \mathbb{T} \times K$. Let

$$l(t) = \max\left\{ l_{x_1}(t), \dots, l_{x_m}(t), \frac{2\kappa(t)}{r}, \quad t \in \mathbb{T} \right\},$$

noting that $l \in L^1_{\mathrm{loc}}(\mathbb{T}; \mathbb{R}_{\geq 0})$. Let $x_1, x2 \in K$. If $\mathrm{d}_{\mathbb{G}}(x_1,x_2) < r$, then let $j \in \{1, \dots, m\}$ be such that $x_1, x_2 \in \mathcal{U}_{x_j}$, and then we have

$$|f(t,x_1) - f(t,x_2)| \leq l_{x_j}(t)\, \mathrm{d}_{\mathbb{G}}(x_1,x_2) \leq l(t)\, \mathrm{d}_{\mathbb{G}}(x_1,x_2), \quad t \in \mathbb{T}.$$

If $\mathrm{d}_{\mathbb{G}}(x_1,x_2) \geq r$, then

$$|f(t,x_1) - f(t,x_2)| \leq |f(t,x_1)| + |f(t,x_2)| \leq 2\kappa(t) \leq \frac{2\kappa(t)}{r}\, \mathrm{d}_{\mathbb{G}}(x_1,x_2) \leq l(t)\, \mathrm{d}_{\mathbb{G}}(x_1,x_2),$$

which gives the result. □

2.3.2. Integrable sections along a curve.. We shall require the notion of a section of a vector bundle over a curve. Thus we let $r \in \{\infty, \omega, \mathrm{hol}\}$ and let $\pi : E \to M$ be a C^r-vector bundle. Let $\mathbb{T} \subseteq \mathbb{R}$ be an interval. We denote by $\mathrm{Aff}^r(E) \subseteq C^r(E)$ the set of C^r-functions G on the manifold E for which $G|E_x$ is affine for each $x \in M$. We denote by $L^1_{\mathrm{loc}}(\mathbb{T}; E)$ the mappings $\Gamma : \mathbb{T} \to E$ for which $G \circ \Gamma \in L^1_{\mathrm{loc}}(\mathbb{T}; \mathbb{F})$ for every $G \in \mathrm{Aff}^r(E)$. Note that, if $\Gamma \in L^1_{\mathrm{loc}}(\mathbb{T}; E)$, then there is a mapping $\gamma : \mathbb{T} \to M$ specified by the requirement that the diagram

$$\begin{tikzcd} \mathbb{T} \arrow[r, "\Gamma"] \arrow[dr, "\gamma"'] & E \arrow[d, "\beta"] \\ & M \end{tikzcd}$$

commute. We can think of Γ as being a section of E over γ. We topologise $L^1_{\mathrm{loc}}(\mathbb{T}; E)$ by giving it the initial topology associated with the mappings

$$\begin{aligned} \alpha_F : L^1_{\mathrm{loc}}(\mathbb{T}; E) &\to L^1(\mathbb{S}; \mathbb{F}) \\ \Gamma &\mapsto G \circ \Gamma|\mathbb{S}, \end{aligned}$$

where $G \in \mathrm{Aff}^r(E)$ and $\mathbb{S} \subseteq \mathbb{T}$ a compact subinterval.

Associated to these spaces of integrable sections over a curve, we have a few constructions and technical results whose importance will be made apparent at various points during the subsequent presentation. We consider the space $C^0(\mathbb{T}; M)$ with the topology (indeed, uniformity) defined by the family of semimetrics

$$d_{\mathbb{S},M}(\gamma_1, \gamma_2) = \sup\{\mathrm{d}_{\mathbb{G}}(\gamma_1(t), \gamma_2(t)) \mid t \in \mathbb{S}\}, \quad \mathbb{S} \subseteq \mathbb{T} \text{ a compact interval.} \tag{2.2}$$

We consider this in the following context. We consider C^r-vector bundles $\pi_E : E \to M$ and $\pi_F : F \to N$. We abbreviate

$$E^* \otimes F = \mathrm{pr}_1^* E^* \otimes \mathrm{pr}_2^* F,$$

where $\mathrm{pr}_1 : M \times N \to M$ and $\mathrm{pr}_2 : M \times N \to N$ are the projections. We regard $E^* \otimes F$ as a vector bundle over $M \times N$. The fibre over (x, y) we regard as

$$E_x^* \otimes F_y \simeq \mathrm{Hom}_{\mathbb{F}}(E_x; F_y).$$

Note that the total spaces E and F of these vector bundles, and so also $E^* \otimes F$, inherit a Riemannian metric from a Riemannian metrics on their base spaces and fibre metrics (Lewis, 2020, §4.1). Thus $E^* \otimes F$ possess the associated distance function, and we shall make use of this to define, as in (2.2), a topology on the space $C^0(\mathbb{T}; E^* \otimes F)$. If $\Gamma \in C^0(\mathbb{T}; E^* \otimes F)$, then we have induced mappings

$$\gamma_M \in C^0(\mathbb{T}; M), \quad \gamma_N \in C^0(\mathbb{T}; N)$$

obtained by first projecting to $M \times N$, and then projecting onto the components of the product. Note that $\Gamma(t) \in \mathrm{Hom}_{\mathbb{F}}(E_{\pi_E \circ \Gamma(t)}; F_{\pi_F \circ \Gamma(t)})$, $t \in \mathbb{T}$.

If $\xi : \mathbb{T} \times M \to E$ satisfies $\xi(t,x) \in E_x$ and if $\Gamma : \mathbb{T} \to E^* \otimes F$, then we can define

$$\begin{aligned} \xi_\Gamma : \mathbb{T} &\to F \\ t &\mapsto \Gamma(t)(\xi(t,\gamma_M(t))). \end{aligned}$$

We call ξ_Γ the composite section associated with ξ and Γ. The following lemma shows that this mapping is integrable, under suitable hypotheses on ξ and Γ.

Lemma 2.6 (Integrability of composite section)**.** *Let $\pi_E : E \to M$ and $\pi_F : F \to M$ be C^∞-vector bundles and let $\mathbb{T} \subseteq \mathbb{R}$ be an interval. If $\xi \in \Gamma^0_{LI}(\mathbb{T};E)$ and if $\Gamma \in C^0(\mathbb{T};E^* \otimes F)$, then $\xi_\Gamma \in L^1_{\mathrm{loc}}(\mathbb{T};F)$.*

Proof. Let $G \in \mathrm{Aff}^\infty(F)$. We first show that $t \mapsto G \circ \xi_\Gamma$ is measurable on $\mathbb{T}$. Note that

$$t \mapsto G \circ \Gamma(s)(\xi(t,\gamma_M(s)))$$

is measurable for each $s \in \mathbb{T}$ and that

$$s \mapsto G \circ \Gamma(s)(\xi(t,\gamma_M(s))) \tag{2.3}$$

is continuous for each $t \in \mathbb{T}$. Let $[a,b] \subseteq \mathbb{T}$ be compact, let $k \in \mathbb{Z}_{>0}$, and denote

$$t_{k,j} = a + \frac{j-1}{k}(b-a), \quad j \in \{1,\dots,k+1\}.$$

Also denote

$$\mathbb{T}_{k,j} = [t_{k,j}, t_{k,j+1}), \quad j \in \{1,\dots,k-1\},$$

and $\mathbb{T}_{k,k} = [t_{k,k}, t_{k,k+1}]$. Then define $g_k : \mathbb{T} \to \mathbb{R}$ by

$$g_k(t) = \sum_{j=1}^{k} G \circ \Gamma(t_{k,j})(\xi(t,\gamma_M(t_{k,j})))\chi_{t_{k,j}}.$$

Note that g_k is measurable, being a sum of products of measurable functions (Cohn, 2013, Proposition 2.1.7). By continuity of (2.3) for each $t \in \mathbb{T}$, we have

$$\lim_{k\to\infty} g_k(t) = G \circ \Gamma(t)(\xi(t,\gamma_M(t))), \quad t \in [a,b],$$

showing that $t \mapsto G \circ \Gamma(t)(\xi(t,\gamma_M(t)))$ is measurable on $[a,b]$, as pointwise limits of measurable functions are measurable (Cohn, 2013, Proposition 2.1.5). Since the compact interval $[a,b] \subseteq \mathbb{T}$ is arbitrary, we conclude that $t \mapsto G \circ \Gamma(t)(\xi(t,\gamma_M(t)))$ is measurable on $\mathbb{T}$.

Let $\mathbb{S} \subseteq \mathbb{T}$ be compact and let $K \subseteq M$ be a compact set for which $\gamma_M(\mathbb{S}) \subseteq K$. Since $\xi \in \Gamma^0_{\mathrm{LI}}(\mathbb{T};M)$ and since Γ is continuous, there exists $h \in L^1(\mathbb{S};\mathbb{R}_{\geq 0})$ be such that

$$|G \circ \Gamma(t)(\xi(t,x))| \leq h(t) \qquad (t,x) \in \mathbb{S} \times K,$$

In particular, this shows that $t \mapsto G \circ \Gamma(t)(\xi(t,\gamma_M(t)))$ is integrable on $\mathbb{S}$ and so locally integrable on $\mathbb{T}$. ∎

The following simplified version of the lemma will be useful.

Corollary 2.7 (Integrability of composite section). *Let M be a C^∞-manifold and let $\mathbb{T} \subseteq \mathbb{R}$ be an interval. If $f \in C^0_{LI}(\mathbb{T}; M)$, if $\gamma \in C^0(\mathbb{T}; M)$, and if we define $f_\gamma : \mathbb{T} \to \mathbb{R}$ by $f_\gamma(t) = f(t, \gamma(t))$, then $f_\gamma \in L^1_{\mathrm{loc}}(\mathbb{T}; \mathbb{R})$.*

Proof. Apply the lemma with $E = F = M \times \mathbb{R}$ (so that sections are identified with functions) and $\Gamma(t) = ((\gamma(t), \gamma(t)), \mathrm{id}_{\mathbb{R}})$. □

We also have the mapping

$$\begin{aligned} \Psi_{\mathbb{T},E,\xi} : C^0(\mathbb{T}; E^* \otimes F) &\to L^1_{\mathrm{loc}}(\mathbb{T}; F) \\ \Gamma &\mapsto \xi_\Gamma, \end{aligned}$$

which is well-defined by Lemma 2.6. The following lemma gives the continuity of this mapping.

Lemma 2.8 (Continuity of curve to composite section map). *Let $\pi_E : E \to M$ and $\pi_F : F \to M$ be C^∞-vector bundles, let $\mathbb{T} \subseteq \mathbb{R}$ be an interval, and let $\xi \in \Gamma^0_{LI}(\mathbb{T}; E)$. Then $\Psi_{\mathbb{T},E,\xi}$ is continuous.*

Proof. Let $G \in \mathrm{Aff}^\infty(F)$ and let $\mathbb{S} \subseteq \mathbb{T}$ be a compact interval. Let $\Gamma_j \in C^0(\mathbb{T}; E^* \otimes F)$, $j \in \mathbb{Z}_{>0}$, be a sequence of curves converging to $\Gamma \in C^0(\mathbb{T}; E^* \otimes F)$. Since $\Gamma(\mathbb{S})$ is compact and $E^* \otimes F$ is locally compact, we can find a precompact neighbourhood $\mathcal{W}$ of $\Gamma(\mathbb{S})$. Then, for $N \in \mathbb{Z}_{>0}$ sufficiently large, we have $\Gamma_j(\mathbb{S}) \subseteq \mathcal{W}$, $j \geq N$ by uniform convergence. Therefore, we can find a compact set $L \subseteq E^* \otimes F$ such that $\Gamma^j(\mathbb{S}) \subseteq L$, $j \in \mathbb{Z}_{>0}$, and $\Gamma(\mathbb{S}) \subseteq L$. Let $g \in L^1(\mathbb{S}; \mathbb{R}_{\geq 0})$ be such that

$$|G \circ \Gamma(t)(\xi(t, x))| \leq h(t) \quad (t, x) \in \mathbb{S} \times K,$$

this since $\xi \in \Gamma^0_{\mathrm{LI}}(\mathbb{T}; M)$ and since Γ is continuous. Then, for fixed $t \in \mathbb{S}$, continuity of $x \mapsto G \circ \Gamma(t)(\xi(t, x))$ ensures that

$$\lim_{j\to\infty} G \circ \Gamma(t)(\xi(t, \gamma_{M,j}(t))) = G \circ \Gamma(t)(\xi(t, \gamma_M(t))).$$

We also have

$$|G \circ \Gamma(t)(\xi(t, \gamma_{M,j}(t)))| \leq g(t), \quad t \in \mathbb{S}.$$

Therefore, by the Dominated Convergence Theorem (Cohn, 2013, Theorem 2.4.5)

$$\lim_{j\to\infty} \int_{\mathbb{S}} G \circ \Gamma(t)(\xi(t, \gamma_{M,j}(t)))\, \mathrm{d}t = \int_{\mathbb{S}} G \circ \Gamma(t)(\xi(t, \gamma_M(t)))\, \mathrm{d}t,$$

which gives the desired continuity. □

2.3.3. Time- and parameter-dependent sections.. Now we turn our attention to vector fields depending on both parameter and time, as we outlined in Section 1.2.

Definition 2.9. Let $m \in \mathbb{Z}_{\geq 0}$, let $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega, \mathrm{hol}\}$, and let $r \in \{\infty, \omega, \mathrm{hol}\}$, as required. Let $\pi : E \to M$ be a vector bundle of class C^r, let $\mathbb{T} \subseteq \mathbb{R}$ be a time interval, and let $\mathcal{P}$ be a topological space. Consider a map $\xi : \mathbb{T} \times M \times \mathcal{P} \to E$ with the property that $\xi(t, x, p) \in E_x$ for each $(t, x, p) \in \mathbb{T} \times M \times \mathcal{P}$. Denote by $\xi^p : \mathbb{T} \times M \to E$ the map $\xi^p(t, x) = \xi(t, x, p)$. Then ξ is a:

(i) ***separately continuous parameter-dependent, (locally) integrally bounded section of class*** $\mathbf{C}^\nu$ if $\xi^p \in \Gamma^\nu_{\mathrm{I}}(\mathbb{T}; E)$ $(\Gamma^\nu_{\mathrm{LI}}(\mathbb{T}; E))$ for each p and $p \to \xi^p(t, x)$ is continuous for each $(t, x) \in \mathbb{T} \times M$;

(ii) ***parameter-dependent, (locally) integrally bounded section of class*** $\mathbf{C}^\nu$ if it is a separately continuous parameter-dependent, (locally) integrally bounded section of class $\mathbf{C}^\nu$ and if the map $\mathcal{P} \ni p \to \xi^p \in \Gamma^\nu_{\mathrm{I}}(\mathbb{T}; E)$ $(\Gamma^\nu_{\mathrm{LI}}(\mathbb{T}; E))$ is continuous;

(iii) ***separately continuous parameter-dependent, (locally) essentially bounded section of class*** $\mathbf{C}^\nu$ if $\xi^p \in \Gamma^\nu_{\mathrm{B}}(\mathbb{T}; E)$ $(\Gamma^\nu_{\mathrm{LB}}(\mathbb{T}; E))$ for each $p \in \mathcal{P}$ and if $p \to \xi^p(t, x)$ is continuous for each $(t, x) \in \mathbb{T} \times M$;

(iv) ***parameter-dependent, (locally) essentially bounded section of class*** $\mathbf{C}^\nu$ if it is a separately continuous parameter-dependent, (locally) essentially bounded section of class $\mathbf{C}^\nu$ and if the map $\mathcal{P} \ni p \to \xi^p \in \Gamma^\nu_{\mathrm{B}}(\mathbb{T}; E)$ $(\Gamma^\nu_{\mathrm{LB}}(\mathbb{T}; E))$ is continuous.

We denote:

(v) the set of parameter-dependent, (locally) integrally bounded section of class C^ν by $\Gamma^\nu_{\mathrm{PLI}}(\mathbb{T}; E; \mathcal{P})$;

(vi) the set of parameter-dependent, (locally) essentially bounded section of class C^ν by $(\Gamma^\nu_{\mathrm{PLB}}(\mathbb{T}; E; \mathcal{P}))$ $\Gamma^\nu_{\mathrm{PB}}(\mathbb{T}; E; \mathcal{P})$.

As with solely parameter-dependent and solely time-varying sections, we have more fulsome notation for time-varying, parameter-dependent sections that is sometimes useful. To see this, we first observe that the spaces

$$\mathrm{L}^1(\mathbb{T}; \Gamma^\nu(E)), \quad \mathrm{L}^1_{\mathrm{loc}}(\mathbb{T}; \Gamma^\nu(E)), \quad \mathrm{L}^\infty(\mathbb{T}; \Gamma^\nu(E)), \quad \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{T}; \Gamma^\nu(E))$$

have topologies, as described above. We then have the spaces

1. $C^0(\mathcal{P}; \mathrm{L}^1(\mathbb{T}; \Gamma^\nu(E)))$,
2. $C^0(\mathcal{P}; \mathrm{L}^1_{\mathrm{loc}}(\mathbb{T}; \Gamma^\nu(E)))$,
3. $C^0(\mathcal{P}; \mathrm{L}^\infty(\mathbb{T}; \Gamma^\nu(E)))$, and

4. $C^0(\mathcal{P};\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{T};\Gamma^\nu(E)))$.

This notation is sufficiently obvious that it does not warrant explanation. As in the solely time-varying case, we shall not always state result in all four cases of "integrable," "locally integrable," "essentially bounded," and "locally essentially bounded," although all of the obvious results hold, and we shall use them.

To give a slightly explicit characterisation of membership in $\Gamma^\nu_{\mathrm{PLI}}(\mathbb{T};E;\mathcal{P})$, we note that the conditions for such membership on ξ are, just by definition: for each $p_0\in\mathcal{P}$, for each compact $K\subseteq M$ and $\mathbb{S}\subseteq\mathbb{T}$, and for each $\epsilon\in\mathbb{R}_{>0}$, there exists a neighbourhood $\mathcal{O}\subseteq\mathcal{P}$ of p_0 such that

$$\int_{\mathbb{S}} p^\nu_K(\xi^p_t-\xi^{p_0}_t)\,\mathrm{d}t<\epsilon,\quad p\in\mathcal{O},\tag{2.4}$$

where seminorms p^ν_K are as above. We can, moreover, topologise these spaces. The ***compact open topology*** on $C^0(\mathcal{P};\mathrm{L}^1(\mathbb{T};\Gamma^\nu(E)))$ is the locally convex topology defined by the family of seminorms $p^\nu_{K,\mathbb{T},1,L}$, $L\subseteq\mathcal{P}$ compact and $K\subseteq M$ compact, given by

$$p^\nu_{K,\mathbb{T},1,L}(\xi)=\sup\{p^\nu_{K,\mathbb{T},1}\circ\xi^p\mid p\in L\},$$

where $p^\nu_{K,\mathbb{T},1}$ is a seminorm for the topology on $\mathrm{L}^1(\mathbb{T};\Gamma^\nu(E))$. Similarly, the ***compact open topology*** on $C^0(\mathcal{P};\mathrm{L}^1_{\mathrm{loc}}(\mathbb{T};\Gamma^\nu(E)))$ is the locally convex topology defined by the family of seminorms $p^\nu_{K,\mathbb{S},1,L}$, $L\subseteq\mathcal{P}$ compact, $K\subseteq M$ compact, and $\mathbb{S}\subseteq\mathbb{T}$ compact given by

$$p^\nu_{K,\mathbb{S},1,L}(\xi)=\sup\{p^\nu_{K,\mathbb{S},1}\circ\xi^p\mid p\in L\},$$

where $p^\nu_{K,\mathbb{S},1}$ is a seminorm for the topology on $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{T};\Gamma^\nu(E))$.

The following result then characterises time-and parameter-dependent Lipschitz functions, just as in the time-dependent case.

Lemma 2.10 (Property of time- and parameter-dependent locally Lipschitz functions)**.** *Let M be a smooth manifold, let $\mathbb{T}$ be a time-domain, let $\mathcal{P}$ be a topological space, and let $f\in C^{\mathrm{lip}}_{PLI}(\mathbb{T};M;\mathcal{P})$. If $K\subseteq M$ is compact, if $\mathbb{S}\subseteq\mathbb{T}$ is a compact interval, and if $p_0\in\mathcal{P}$, then there exists $C\in\mathbb{R}_{>0}$ and a neighbourhood $\mathcal{O}$ of p_0 such that*

$$\int_{\mathbb{S}}|f(t,x_1,p)-f(t,x_2,p)|\,\mathrm{d}t\le C\,\mathrm{d}_{\mathbb{G}}(x_1,x_2),\quad x_1,x_2\in K,\ p\in\mathcal{O}.$$

Proof. Let $K\subseteq M$ and $\mathbb{S}\subseteq\mathbb{T}$ be compact, and let $p_0\in\mathcal{P}$. Let $x\in K$ and, as in the proof of Lemma 2.5, let $\mathcal{U}_x$ be a neighbourhood of x and let $\ell_x\in L^1(\mathbb{S};\mathbb{R}_{\ge0})$ be such that

$$\mathrm{dil}\ f(t,y,p_0)\le\ell_x(t)\,\mathrm{d}_{\mathbb{G}}(x_1,x_2),\quad (t,y)\in\mathbb{S}\times\mathrm{cl}(\mathcal{U}_x).$$

According to (2.4), there exists a neighbourhood $\mathcal{O}_x$ of p_0 such that

$$\int_{\mathbb{S}}\mathrm{dil}\ (f^p-f^{p_0})(t,y)\,\mathrm{d}t<1,\quad (t,y,p)\in\mathbb{S}\times\mathcal{U}_x\times\mathcal{O}_x.$$

Therefore, by the triangle inequality,

$$\int_{\mathbb{S}} \operatorname{dil} f^p(t,y)\,\mathrm{d}t \le \int_{\mathbb{S}} \operatorname{dil}\,(f^p - f^{p_0})(t,y)\,\mathrm{d}t + \int_{\mathbb{S}} \operatorname{dil} f^{p_0}(t,y)\,\mathrm{d}t < \underbrace{1 + \int_{\mathbb{S}} \ell_x(t)\,\mathrm{d}t}_{C_x}$$

for all $(t,y,p) \in \mathbb{S} \times \mathcal{U}_x \times \mathcal{O}_x$. Thus, by Lemma 2.2, there exists $C_x \in \mathbb{R}_{>0}$ such that

$$\int_{\mathbb{S}} \frac{f(t,x_1,p) - f(t,x_2,p)}{\mathrm{d}_{\mathbb{G}}(x_1,x_2)}\,\mathrm{d}t \le \int_{\mathbb{S}} \lambda^0_{\mathrm{cl}(\mathcal{U}_x)}(f^p_t)\,\mathrm{d}t \le C_x$$

for $x_1, x_2 \in \mathcal{U}_x$ distinct and for $p \in \mathcal{O}_x$. By compactness of K, there exists $x_1,\dots,x_m \in K$ such that $K \subseteq \cup_{j=1}^m \mathcal{U}_{x_j}$. By the Lebesgue Number Lemma (D. Burago, Y. Burago, and Ivanov, 2001, Theorem 1.6.11), there exists $r \in \mathbb{R}_{>0}$ with the property that, if $x_1, x_2 \in K$ satisfies $\mathrm{d}_{\mathbb{G}}(x_1,x_2) < r$, then there exists $j \in \{1,\dots,m\}$ such that $x_1, x_2 \in \mathcal{U}_{x_j}$. Since $C^0_{\mathrm{LI}}(\mathbb{T};M;\mathcal{P}) \subseteq C^{\mathrm{lip}}_{\mathrm{LI}}(\mathbb{T};M;\mathcal{P})$, by (2.4) there exists a neighbourhood $\mathcal{O}'$ of p_0 such that

$$\int_{\mathbb{S}} |f(t,x,p)|\,\mathrm{d}t < 1, \quad (t,y,p) \in \mathbb{S} \times K \times \mathcal{O}'.$$

Let

$$C = \max\left\{C_{x_1},\dots,C_{x_m},\frac{r}{2}\right\}$$

and let $\mathcal{O} = \mathcal{O}' \cap (\cap_{j=1}^m \mathcal{O}_{x_j})$. Let $x_1, x_2 \in K$ and $p \in \mathcal{O}$. If $\mathrm{d}_{\mathbb{G}}(x_1,x_2) < r$, then let $j \in \{1,\dots,m\}$ be such that $x_1, x_2 \in \mathcal{U}_{x_j}$, and then we have

$$\int_{\mathbb{S}} |f(t,x_1,p) - f(t,x_2,p)|\,\mathrm{d}t \le C_{x_j}\,\mathrm{d}_{\mathbb{G}}(x_1,x_2) \le C\,\mathrm{d}_{\mathbb{G}}(x_1,x_2).$$

If $\mathrm{d}_{\mathbb{G}}(x_1,x_2) \ge r$, then

$$\begin{aligned}\int_{\mathbb{S}} |f(t,x_1,p) - f(t,x_2,p)|\,\mathrm{d}t &\le \int_{\mathbb{S}} |f(t,x_1,p)|\,\mathrm{d}t + \int_{\mathbb{S}} |f(t,x_2,p)|\,\mathrm{d}t \\ &< 2 \le \frac{2}{r}\,\mathrm{d}_{\mathbb{G}}(x_1,x_2) \le C\,\mathrm{d}_{\mathbb{G}}(x_1,x_2),\end{aligned}$$

which gives the result. $\square$

Chapter 3

Vector fields and flows

For the class of time- and parameter-dependent vector fields introduced in the previous section, we give a geometric characterisation of their integral curves and flows, and prove the more standard results on the manner in which the flow depends on its arguments. While the arguments used in the proofs bear an unsurprising similarity to the standard proofs, we highlight four important points of departure:

1. in the time-dependent setting, we are using a class of vector fields that is new, so the proofs necessarily reflect this by being different than standard proofs;

2. as throughout the chapter, we eschew the use of coordinates in favour of globally defined functions, as this is an important ingredient in our approach.

3. give geometric proofs for standard results, globally expressed, concerning continuous (and more regular, when appropriate) dependence of terminal state on initial state, initial and final time.

4. prove a new type of continuity result for the "parameter to local flow" mapping.

3.1. Integral curves for vector fields

In this section, we will give geometric definitions and characterisations of integral curves and flows. We begin by defining and characterising integral curves in our global framework.

Definition 3.1 (Integral curve)**.** Let $m \in \mathbb{Z}_{\geq 0}$, let $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega, \mathrm{hol}\}$, and let $r \in \{\infty, \omega, \mathrm{hol}\}$, as required. Let M be a C^r-manifold, let $\mathbb{T} \subseteq \mathbb{R}$ be an interval, let $X \in \Gamma^{\nu}_{\mathrm{LI}}(\mathbb{T}; TM)$. An **integral curve** for X is a locally absolutely continuous curve $\xi : \mathbb{T}' \to M$ such that

(i) $\mathbb{T}' \subseteq \mathbb{T}$ and

(ii) $\xi'(t) = X(t, \xi(t))$ for almost every $t \in \mathbb{T}'$.

An integral curve $\xi : \mathbb{T}' \to M$ is ***maximal*** if, given any other integral curve $\eta : \mathbb{T}'' \to M$ for which $\eta(t) = \xi(t)$ for some $t \in \mathbb{T}' \cap \mathbb{T}''$, we have $\mathbb{T}'' \subseteq \mathbb{T}'$.

The following result, while admittedly simple, characterises integral curves in a way that will be of essential use to our approach.

Lemma 3.2 ("Weak" characterisation of integral curves)**.** *Let $m \in \mathbb{Z}_{\geq 0}$, let $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m+m', \infty, \omega, hol\}$, and let $r \in \{\infty, \omega, hol\}$, as required. Let M be a C^r manifold, Stein when $\nu = hol$, let $\mathbb{T} \subseteq \mathbb{R}$ be an interval, and let $X \in \Gamma^{\nu}_{LI}(\mathbb{T}; TM)$. For a curve $\xi : \mathbb{T}' \to M$, the following statements are equivalent:*

(i) ξ is a integral curve for X;

(ii) for any $t_0 \in \mathbb{T}'$ and any $f \in C^\infty(M)$,

$$f \circ \xi(t) = f \circ \xi(t_0) + \int_{t_0}^{t} Xf(s, \xi(s))\, \mathrm{d}s;$$

(iii) for any $t_0 \in \mathbb{T}'$ and any $f \in C^r(M)$,

$$f \circ \xi(t) = f \circ \xi(t_0) + \int_{t_0}^{t} Xf(s, \xi(s))\, \mathrm{d}s.$$

Proof. (i) $\Longrightarrow$ (ii). Let $t_0 \in \mathbb{T}'$ and $f \in C^\infty(M)$. Then we have

$$\begin{aligned} f \circ \xi(t_0) + \int_{t_0}^{t} Xf(s, \xi(s))\, \mathrm{d}s &= f \circ \xi(t_0) + \int_{t_0}^{t} \langle df(\xi(s)); \xi'(s)\rangle\, \mathrm{d}s \\ &= f \circ \xi(t_0) + \int_{t_0}^{t} \frac{\mathrm{d}}{\mathrm{d}s} f \circ \xi(s)\, \mathrm{d}s = f \circ \xi(t), \end{aligned}$$

as claimed.

(ii) $\Longrightarrow$ (iii). This follows since real analytic and holomorphic functions are smooth.

(iii) $\Longrightarrow$ (i). Let $t_0 \in \mathbb{T}'$ and let $\chi^1, \dots, \chi^n \in C^r(M)$ be such that

$$(\mathrm{d}\chi^1(\xi(t_0)), \dots, \mathrm{d}\chi^1(\xi(t_0)))$$

is a basis for $T^*_{\xi(t_0)}M$. The existence of such functions follows from the existence of globally defined C^r-coordinate functions about any point in M for smooth, real analytic, and Stein manifolds. One then argues, just as in the proof of the first implication above, that

$$\chi^j \circ \xi(t) = \chi^j \circ \xi(t_0) + \int_{t_0}^{t} X\chi^j(s, \xi(s))\, \mathrm{d}s$$

for each $j \in \{1, \dots, n\}$. This gives

$$(\chi^j \circ \xi)'(t) = X\xi^j(t, \xi(t)), \qquad j \in \{1, \dots, n\}, \text{ a.e. } t \in \mathbb{T}'.$$

The linear independence of $\mathrm{d}\chi^1, \dots, \mathrm{d}\chi^n$ in a neighbourhood of $\xi(t_0)$ gives $\xi'(t) = X(t, \xi(t))$ for almost every t in some neighbourhood of t_0. As this holds for every $t_0 \in \mathbb{T}'$, we conclude that ξ is an integral curve for X. $\square$

The next lemma is an adaptation of the preceding lemma for curves that are not necessarily integral curves of vector fields. In the statement of the result, we make use of the notation

$$\mathbb{T}_{t_0,\alpha} = \mathbb{T} \cap [t_0 - \alpha, t_0 + \alpha].$$

Lemma 3.3 (Curves determined by functions)**.** *Let M be a manifold of class C^∞, let $\mathbb{T} \subseteq \mathbb{R}$ be an interval, and let $\eta^j \in C^0(\mathbb{T};\mathbb{R})$. Let $x_0 \in M$ and suppose that $\chi^1,\dots,\chi^n \in C^\infty(M)$ are such that $(\mathrm{d}\chi^1(x_0),\dots,\mathrm{d}\chi^n(x_0))$ is a basis for $T^*_{x_0}M$. If $\eta^j(t_0) = \chi^j(x_0)$, then the following statements hold:*

(i) there exists $\alpha \in \mathbb{R}_{>0}$ and $\gamma \in C^0(\mathbb{T}_{t_0,\alpha};M)$ such that

$$\chi^j \circ \gamma(t) = \eta^j(t), \quad j \in \{1,\dots,n\},\ t \in \mathbb{T}_{t_0,\alpha};$$

(ii) with $\chi^1,\dots,\chi^n$ and γ as in part (i), α can be chosen so that the curve γ is unique in that any two such curves agree on the intersection of their domains.

Proof. Consider the map

$$\begin{aligned} f : M &\to \mathbb{R}^n \\ x &\mapsto (\chi^1(x),\dots,\chi^n(x)). \end{aligned}$$

By the Inverse Function Theorem, f is a diffeomorphism from a neighbourhood $\mathcal{U}$ of x_0 onto a neighbourhood $\mathcal{V}$ of $f(x_0)$. Let $\alpha \in \mathbb{R}_{>0}$ be sufficiently small that

$$(\eta^1(t),\dots,\eta^n(t)) \in \mathcal{V}, \quad j \in \{1,\dots,n\},\ t \in \mathbb{T}_{t_0,\alpha}.$$

Thus there is a unique continuous curve $\gamma : \mathbb{T}_{t_0,\alpha} \to \mathcal{U}$ satisfying $\chi^j \circ \gamma(t) = \eta^j(t)$ for $j \in \{1,\dots,n\}$ and $t \in \mathbb{T}_{t_0,\alpha}$. □

The next step should be that of existence and uniqueness of integral curves for time-dependent vector fields (or time- and parameter-dependent vector fields with the parameter fixed). This, however, requires no special measures since the condition that $\Gamma^{\mathrm{lip}}_{\mathrm{LI}}(\mathbb{T};TM)$ returns exactly the condition required for existence and uniqueness of local integral curves, and, hence, also maximal integral curves, i.e., those defined on the largest possible time interval. Thus we shall not give an independent proof here. However, we will give a a precise formulation for this in the presence of parameters, with our particular form of parameter-dependence, and we shall take for granted the usual existence and uniqueness results for maximal integral curves.

3.2. Flows for vector fields

With the notion of an integral curve at hand, we can define flows of time-varying vector fields.

Definition 3.4 (Domain of a vector field, flow of a vector field)**.** Let $m \in \mathbb{Z}_{\geq 0}$, let $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega, \mathrm{hol}\}$, and let $r \in \{\infty, \omega, \mathrm{hol}\}$, as required. Let M be a C^r manifold, let $\mathbb{T} \subseteq \mathbb{R}$ be a time-domain, let $\mathcal{P}$ be a topological space, and let $X \in \Gamma^{\nu}_{\mathrm{PLI}}(\mathbb{T}; TM; \mathcal{P})$.

(i) For $(t_0, x_0, p_0) \in \mathbb{T} \times M \times \mathcal{P}$, denote

$$J_X(t_0, x_0, p_0) = \bigcup \{J \subseteq \mathbb{T} \mid J \text{ is an interval and there exists an integral curve } \xi : J \to M \text{ for } X^{p_0} \text{ satisfying } \xi(t_0) = x_0\}.$$

The interval $j_X(t_0, x_0, p_0)$ is the interval of existence for X^{p_0} for the initial condition (t_0, x_0).

(ii) For $(t_1, x_0, p_0) \in \mathbb{T} \times M \times \mathcal{P}$, denote

$$I_X(t_1, x_0) = \{t_0 \in \mathbb{T} \mid t_1 \in J_X(t_0, x_0, p_0)\}.$$

(iii) For For $(t_1, x_0, p_0) \in \mathbb{T} \times M \times \mathcal{P}$, denote

$$D_X(t_1, t_0, p_0) = \{x \in M \mid t_1 \in J_X(t_0, x, p_0)\}.$$

(iv) Denote

$$D_X = \{(t_0, t_0, x_0, p_0) \in \mathbb{T} \times \mathbb{T} \times M \times \mathcal{P} \mid t_1 \in J_X(t_0, x_0, p_0)\}.$$

(v) The ***flow*** for X is the mapping

$$\begin{aligned} \Phi^X : D_X &\to M \\ (t_1, t_0, x_0, p_0) &\mapsto \xi(t_1), \end{aligned}$$

where ξ is the integral curve for X^{p_0} satisfying $\xi(t_0) = x_0$.

We will work with time-varying vector fields both with and without parameter dependence. When we work with vector fields that are time-dependent but not parameter-dependent, we will simply omit the argument corresponding to the parameter without further mention. We shall also use the notation

$$\Phi^{X^p}_{t_1, t_0} : D_X(t_1, t_0, p) \to M$$

when convenient.

The flow has the following elementary properties that follow from the definitions and by the uniqueness of integral curves:

(i) for each $(t_0, x_0) \in \mathbb{T} \times M$, $(t_0, t_0, x_0) \in D_X$ and $\Phi^X(t_0, t_0, x_0) = x_0$;

(ii) if $(t_2, t_1, x) \in D_X$, then $(t_3, t_2, \Phi^X(t_2, t_1, x)) \in D_X$ if and only if $(t_3, t_1, x) \in D_X$ and, if this holds, then

$$\Phi^X(t_3, t_1, x) = \Phi^X(t_3, t_2, \Phi^X(t_2, t_1, x));$$

(iii) if $(t_2, t_1, x) \in D_X$, then $(t_1, t_2, \Phi^X(t_2, t_1, x)) \in D_X$ and $\Phi^X(t_1, t_2, \Phi^X(t_2, t_1, x)) = x$.

We can now state a "standard" theorem in our nonstandard framework. In the statement and proof of the result, we will find the notation

$$|t_0, t_1| = \begin{cases} [t_0, t_1], & t_1 \geq t_0, \\ [t_1, t_0], & t_1 \leq t_0. \end{cases}$$

useful, for $t_0, t_1 \in \mathbb{T}$. We also denote by $\mathrm{LocFlow}^\nu(\mathbb{S}'; \mathbb{S}; \mathcal{U})$ the set of local flows defined on $\mathbb{S}' \times \mathbb{S} \times \mathcal{U} \subseteq \mathbb{T} \times \mathbb{T} \times M$, and

$$\mathcal{V}^\nu_{\mathbb{S}' \times \mathbb{S} \times \mathcal{U}} = \{X \in \Gamma^\nu_{\mathrm{LI}}(\mathbb{T}; TM) \mid \mathbb{S}' \times \mathbb{S} \times \mathcal{U} \subseteq D_X, \ \ \mathbb{S} \subseteq \mathbb{S}'\},$$

the space of time-verying sections whose flows are defined on $\mathbb{S}' \times \mathbb{S} \times \mathcal{U}$.

Theorem 3.5 (Continuous dependence of flows)**.** *Let M be a C^∞-manifold, let $\mathbb{T} \subseteq \mathbb{R}$ be an interval, let $\mathcal{P}$ be a topological space, and let $X \in \Gamma^{\mathrm{lip}}_{\mathrm{PLI}}(\mathbb{T}; TM; \mathcal{P})$. Then the following statements hold:*

(i) for $(t_0, x_0, p_0) \in \mathbb{T} \times M \times \mathcal{P}$, $J_X(t_0, x_0, p_0)$ is a relatively open subinterval of $\mathbb{T}$;

(ii) for $(t_0, x_0, p_0) \in \mathbb{T} \times M \times \mathcal{P}$, the curve

$$\begin{aligned} \gamma_{(t_0, x_0, p_0)} : J_X(t_0, x_0, p_0) &\to M \\ t &\mapsto \Phi^X(t, t_0, x_0, p_0) \end{aligned}$$

is well-defined and locally absolutely continuous;

(iii) for $(t_1, x_0, p_0) \in \mathbb{T} \times M \times \mathcal{P}$, $I_X(t_1, x_0, p_0)$ is a relatively open subinterval of $\mathbb{T}$;

(iv) for $(t_1, x_0, p_0) \in \mathbb{T} \times M \times \mathcal{P}$, the curve

$$\begin{aligned} \iota_{(t_1, x_0, p_0)} : I_X(t_1, x_0, p_0) &\to M \\ t &\mapsto \Phi^X(t_1, t, x_0, p_0) \end{aligned}$$

is well-defined and locally absolutely continuous;

(v) for $t_1, t_0 \in \mathbb{T}$ and $p_0 \in \mathcal{P}$, $D_X(t_1, t_0, p_0)$ is open in M;

(vi) *for* $t_1, t_0 \in \mathbb{T}$ *and* $p_0 \in \mathcal{P}$ *for which* $D_X(t_1, t_0, p_0) \neq \varnothing$, $\Phi^{X p_0}_{t_1,t_0}$ *is a locally bi-Lipschitz homeomorphism onto its image;*

(vii) D_x *is relatively open in* $\mathbb{T} \times \mathbb{T} \times M \times \mathcal{P}$*;*

(viii) *the map*

$$\Phi^X : D_X \to M$$

is continuous;

(ix) *for* $(t_0, x_0, p_0) \in \mathbb{T} \times M \times \mathcal{P}$*, and for* $\epsilon \in \mathbb{R}_{>0}$*, there exists* $\alpha \in \mathbb{R}_{>0}$*, a neighbourhood* $\mathcal{U}$ *of* x_0*, and a neighbourhood* $\mathcal{O}$ *of* p_0 *such that*

$$\sup J_X(t, x, p) > \sup J_X(t_0, x_0, p_0) - \epsilon, \qquad \inf J_X(t, x, p) < \inf J_X(t_0, x_0, p_0) + \epsilon$$

for all $(t, x, p) \in \operatorname{int}(\mathbb{T}_{t_0,\alpha}) \times \mathcal{U} \times \mathcal{O}$*;*

(x) *for* $(t_1, t_0, x_0, p_0) \in D_X$*, the curves*

$$|t_0, t_1| \ni t \mapsto \Phi^X(t, t_0, x, p) \in M$$

converge uniformly to

$$|t_0, t_1| \ni t \mapsto \Phi^X(t, t_0, x_0, p_0) \in M$$

as $(x, p) \to (x_0, p_0)$*.*

Proof. (i) Since $J_X(t_0, x_0, p_0)$ is a union of intervals, each of which contains t_0, it follows that it is itself an interval. To show that it is an open subset of $\mathbb{T}$, we show that, if $t \in J_X(t_0, x_0, p_0)$, there exists $\epsilon \in \mathbb{R} > 0$ such that $\mathbb{T}_{t,\epsilon} \subseteq J_X(t_0, x_0, p_0)$.

First let us consider the case when t is not an endpoint of $\mathbb{T}$, in the event that $\mathbb{T}$ contains one or both of its endpoints. In this case, by definition of $J_X(t_0, x_0, p_0)$, there is an open interval $J \subseteq \mathbb{T}$ containing t_0 and t, and an integral curve $\xi : J \to M$ for X_{p_0} satisfying $\xi(t_0) = x_0$. In particular, there exists $\epsilon \in \mathbb{R}_{>0}$ such that $(t - \epsilon, t + \epsilon) \subseteq J \subseteq J_X(t_0, x_0, p_0)$, which gives the desired conclusion in this case.

Next suppose that t is the right endpoint of $\mathbb{T}$, which we assume is contained in $\mathbb{T}$, of course. In this case, by definition of $J_X(t_0, x_0, p_0)$, there is an interval $J \subseteq \mathbb{T}$ containing t_0 and t, and an integral curve $\xi : J \to M$ for X_{p_0} satisfying $\xi(t_0) = x_0$. In this case, there exists $\epsilon \in \mathbb{R}_{>0}$ such that

$$\mathbb{T}_{t,\epsilon} = (t - \epsilon, t] \subseteq J_X(t_0, x_0, p_0),$$

which gives the desired conclusion in this case. A similar argument gives the desired conclusion when t is the left endpoint of $\mathbb{T}$.

(ii) That $\gamma_{(t_0,x_0,p_0)}$ is defined in $J_X(t_0, x_0, p_0)$ was proved as part of the preceding part of the proof. The assertion that $\gamma_{(t_0,x_0,p_0)}$ is locally absolutely continuous follows from the usual existence and uniqueness theorem.

(iii) This follows similarly to part (ii).

We will defer the proof of part (iv) to the end of the proof.

We shall prove the assertions (v) and (vi) of the theorem together, first by proving that these conditions hold locally, and then giving an extension argument to give the global version of the statement. Let us first prove a few technical lemmata that will be useful to us. First we give the initial part of the local version of the theorem.

Lemma 3.6. *Let M be a C^∞-manifold, let $\mathbb{T} \subseteq \mathbb{R}$ be a time domain, let $\mathcal{P}$ be a topological space, and let $X \in \Gamma^{\mathrm{lip}}_{LI}(\mathbb{T}, TM; \mathcal{P})$. For each $(t_0, x_0, p_0) \in \mathbb{T} \times M \times \mathcal{P}$, there exist $\alpha \in \mathbb{R}_{>0}$, a neighborhood $\mathcal{U} \subseteq M$ of x_0 and a neighborhood $\mathcal{O} \subseteq \mathcal{P}$ of p_0 such that, $(t, t_0, x, p) \in D_X$ for each $t \in \mathbb{T}_{t_0,\alpha}$, $x \in \mathcal{U}$ and $p \in \mathcal{O}$. Moreover, α, $\mathcal{U}$ and $\mathcal{O}$ can be chosen such that:*

(i) the map
$$\mathcal{U} \ni x \mapsto \Phi^{X^p}_{t_1,t_0}(x)$$
is Lipschitz for every $p \in \mathcal{O}$ and every $t_1 \in \mathbb{T}_{t_0,\alpha}$;

(ii) the map
$$\mathbb{T}_{t_0,\alpha} \times \mathcal{U} \times \mathcal{O} \ni (t, x, p) \mapsto \Phi^X(t, t_0, x, p)$$
is continuous.

Proof. We first essentially prove the local existence and uniqueness result, including the role of parameters. We make use of an arbitrarily selected Riemannian metric $\mathbb{G}$. Let $\chi^1, \dots, \chi^n \in C^\infty(M)$ be such that $\{\mathrm{d}\chi^1(x_0), \dots, \mathrm{d}\chi^n(x_0)\}$ is a basis for $T^*_{x_0}M$. Let $R \in \mathbb{R}_{>0}$ be such that
$$\mathcal{U} \coloneqq \{x \in M \mid \mathrm{d}_{\mathbb{G}}(x, x_0) < R\}$$
is geodesically convex (Kobayashi and Nomizu, 1963, Proposition IV.3.4). We choose R sufficiently small that $\mathrm{d}\chi^1, \dots, \mathrm{d}\chi^n$ are linearly independent at points in $\mathcal{U}$. By Lemma A.1, there exists $C \in \mathbb{R}_{>0}$, such that
$$\begin{aligned} &C^{-1} \sup\{|\chi^j(x_1) - \chi^j(x_2)| \mid j \in \{1, \dots, n\}\} \\ &\qquad \le \mathrm{d}_{\mathbb{G}}(x_1, x_2) \le C \sup\{|\chi^j(x_1) - \chi^j(x_2)| : j \in \{1, \dots, n\}\}, \quad x_1, x_2 \in \mathcal{U}. \end{aligned} \tag{3.1}$$

For $x \in M$ and $a \in \mathbb{R}_{>0}$, we denote by
$$\mathcal{U}(a, x) \subseteq \cap_{j=1}^n (\chi^j)^{-1}(\chi^j(x) - a, \chi^j(x) + a)$$
the connected component of the set on the right containing x. With $\mathcal{U}$ chosen above, note that $\mathcal{U}(a, x)$ is a neighborhood, homeomorphic to an n-dimensional ball, of x for $x \in \mathcal{U}$ and for a sufficiently small that $\mathcal{U}(a, x) \subseteq \mathcal{U}$.

Following from (Jafarpour and Lewis, 2014, Theorem 6.4:(v) and (vi)) and making use of the universal property of the initial topology, one can easily see that $X\chi^j \in$

$C^{\mathrm{lip}}_{\mathrm{PLI}}(\mathbb{T};M;\mathcal{P})$ for $j \in \{1,\dots,n\}$. Let $r' \in \mathbb{R}_{>0}$ be such that $\mathcal{U}(r',x_0) \subseteq \mathcal{U}$. Let $r = \frac{r'}{2}$, and let $\lambda \in (0,1)$. There exists $\alpha \in \mathbb{R}_{>0}$ s.t.

$$\int_{\mathbb{T}_{t_0,\alpha}} |X\chi^j(s,x,p_0)| ds < \frac{r}{2} \tag{3.2}$$

and

$$\int_{\mathbb{T}_{t_0,\alpha}} \mathrm{dil}(X\chi^j)(s,x,p_0) ds < \frac{\lambda}{2C} \tag{3.3}$$

for $x \in \mathcal{U}(r',x_0)$ and $j \in \{1,\dots,n\}$. Since $X\chi^j \in C^{\mathrm{lip}}_{\mathrm{PLI}}(\mathbb{T};M;\mathcal{P})$, $j \in \{1,\dots,n\}$, there exists a neighbourhood $\mathcal{O}$ of p_0 such that

$$\int_{\mathbb{T}_{t_0,\alpha}} |X\chi^j(s,x,p) - X\chi^j(s,x,p_0)| ds < \frac{r}{2} \tag{3.4}$$

and

$$\int_{\mathbb{T}_{t_0,\alpha}} \mathrm{dil}((X\chi^j)^p - (X\chi^j)^{p_0})(s,x) ds < \frac{\lambda}{2C} \tag{3.5}$$

for $x \in \mathcal{U}(r',x_0)$, $p \in \mathcal{O}$, and $j \in \{1,\dots,n\}$.

Applying triangle inequality to (3.2)-(3.5), we have

$$\int_{\mathbb{T}_{t_0,\alpha}} |X\chi^j(s,x,p)| ds < r \tag{3.6}$$

and

$$\int_{\mathbb{T}_{t_0,\alpha}} \mathrm{dil}(X\chi^j)(s,x,p) ds < \frac{\lambda}{C} \tag{3.7}$$

for all $x \in \mathcal{U}(r',x_0)$, $p \in \mathcal{O}$, and $j \in \{1,\dots,n\}$.

The inequality (3.7), with the aid of Lemma 2.10, give,

$$\int_{|t_0,t|} |X\chi^j(s,x_1,p) - X\chi^j(s,x_2,p)| ds < \frac{\lambda}{C} \mathrm{d}_{\mathbb{G}}(x_1,x_2) \tag{3.8}$$

for $x_1, x_2 \in \mathcal{U}(r',x_0)$, $j \in \{1,\dots,n\}$, and $p \in \mathcal{O}$.

If $y \in \mathcal{U}(r,x_0)$, then $\mathcal{U}(r,y) \subseteq \mathcal{U}(r',x_0)$. Therefore, (3.6) and (3.8) hold for all $x, x_1, x_2 \in \mathcal{U}(r,y)$, $t \in \mathbb{T}_{t_0,\alpha}$, $j \in \{1,\dots,n\}$ and $p \in \mathcal{O}$.

Denote $\mathcal{U} \coloneqq \mathcal{U}(r,x_0)$, and let $x \in \mathcal{U}$. For each $j \in \{1,\dots,n\}$, we denote by $\phi_0^j \in C^0(\mathbb{T}_{t_0,\alpha},\mathbb{R})$ the constant mapping $\phi_0^j = \chi^j(x)$, and denote by $\overline{B}(r,\phi_0^j)$ the closed ball of radius r about ϕ_0^j. For each $p \in \mathcal{O}$, we have the mapping

$$\begin{aligned} F^j_{X^p}: \quad & \overline{B}(r,\phi_0^j) \to C^0(\mathbb{T}_{t_0,\alpha},\mathbb{R}) \\ & \phi^j \mapsto \chi^j(x) + \int_{|t_0,t|} X\chi^j(s,\gamma_\phi(s),p)\, \mathrm{d}s, \end{aligned}$$

for $j \in \{1,\dots,n\}$, and where $\gamma_\phi \in C^0(\mathbb{T}_{t_0,\alpha},M)$ is the unique curve satisfying

$$\chi^j \circ \gamma_\phi = \phi^j(t), \qquad t \in \mathbb{T}_{t_0,\alpha},\ j \in \{1,\dots,n\},$$

cf. Lemma 3.3. Note that the definition of r ensures that γ_ϕ so defined takes values in $\mathcal{U}$.

Now we claim that $\operatorname{Im}(\gamma_\phi) \subseteq \mathcal{U}(r', x_0)$. Indeed, since $x \in \mathcal{U} = \mathcal{U}(r, x_0)$, then

$$\mathcal{U}(r,x) \subseteq \cap_{j=1}^n (\chi^j)^{-1}(\chi^j(x) - r, \chi^j(x) + r) \subseteq \mathcal{U}(r', x_0).$$

Since $\phi^j \in \overline{B}(r, \phi_0^j)$ and $\gamma_\phi(t) = (\chi^j)^{-1} \circ \phi^j(t)$ for all $j \in \{1, ..., n\}$, then

$$\gamma_\phi(t) \in \cap_{j=1}^n (\chi^j)^{-1}(\phi_0^j(t) - r, \phi_0^j(t) + r) = \cap_{j=1}^n (\chi^j)^{-1}(\chi^j(x) - r, \chi^j(x) + r) \subseteq \mathcal{U}(r', x_0).$$

We then claim that $F_{X^p}^j(\overline{B}(r, \phi_0^j)) \subseteq \overline{B}(r, \phi_0^j)$, $j \in \{1, ..., n\}, p \in \mathcal{O}$. Indeed, if $\phi^j \in \overline{B}(r, \phi_0^j)$, $j \in \{1, ..., n\}$, we have

$$|F_{X^p}^j \circ \phi^j(t) - \phi_0^j(t)| \le \int_{|t_0,t|} |X\chi^j(s, \gamma_\phi(s), p)| \, \mathrm{d}s < r.$$

We also claim that the mapping

$$\prod_{j=1}^n \overline{B}(r, \phi_0^j) \ni (\phi^1, ..., \phi^n) \mapsto (F_{X^p}^1 \circ \phi^1, ..., F_{X^p}^n \circ \phi^n) \in \prod_{j=1}^n \overline{B}(r, \phi_0^j) \tag{3.9}$$

is a contraction mapping for each $p \in \mathcal{O}$, where $\prod_{j=1}^n \overline{B}(r, \phi_0^j)$ is given the product metric. Indeed, let $\phi_1^j, \phi_2^j \in \overline{B}(r, \phi_0^j)$, $j \in \{1, ..., n\}$. Let $\gamma_1, \gamma_2 \in C^0(\mathbb{T}_{t_0,\alpha}, M)$ be the corresponding curves satisfying

$$\chi^j \circ \gamma_a(t) = \phi_a^j(t), \quad t \in \mathbb{T}_{t_0,\alpha}, \ j \in \{1, ..., n\}, \ a \in \{1, 2\},$$

cf. Lemma 3.3. Then we have, for each $j \in \{1, ..., n\}$,

$$\begin{aligned} |F_{X^p}^j \circ \phi_1^j(t) - F_{X^p}^j \circ \phi_2^j(t)| &\le \int_{|t_0,t|} |X\chi^j(s, \gamma_1(s), p) - X\chi^j(s, \gamma_2(s), p)| \, \mathrm{d}s \\ &\le \frac{\lambda}{C} \sup\{\mathrm{d}_{\mathbb{G}}(\gamma_1(s), \gamma_2(s)) \mid s \in |t_0, t|\} \\ &\le \lambda \sup\{|\phi_1^k(s) - \phi_2^k(s)| \mid k \in \{1, ..., n\}\}, \ s \in \mathbb{T}_{t_0,\alpha}, \end{aligned}$$

from which the desired conclusion follows.

By the Contraction Mapping Theorem (Abraham, Marsden, and Ratiu, 1988, Theorem 1.2.6) there exists a unique fixed point for the mapping (3.9) in $\prod_{j=1}^n \overline{B}(r, \phi_0^j)$; let us denote the components of this unique fixed point by ϕ^j, $j \in \{1, ..., n\}$. Let us also denote by $\xi \in C^0(\mathbb{T}_{t_0,\alpha}; M)$ the corresponding curve in M, noting that

$$\chi^j \circ \xi(t) = \chi^j(x) + \int_{|t_0,t|} X\chi^j(s, \xi(s), p) \, \mathrm{d}s$$

for all $p \in \mathcal{O}$, cf. Lemma 3.3. It remains to show that ξ is an integral curve for X^p satisfying $\xi(t_0) = x$. Observe that $\xi(t_0) = x$ is obvious. We can, then, follow the proof of part (iii) of Lemma 3.2 to see that ξ is an integral curve for X^p.

We next prove uniqueness of this integral curve on $\mathbb{T}_{t_0,\alpha}$. Suppose that $\eta : \mathbb{T}_{t_0,\alpha} \to M$ is another integral curve satisfying $\eta(t_0) = x_0$. By Lemma 3.2 we have

$$f \circ \eta(t) = f(x) + \int_{|t_0,t|} Xf(s, \eta(s))\, \mathrm{d}s$$

for every $f \in C^\infty(M)$. It then follows that, if we define

$$\begin{aligned} \phi^j : \mathbb{T}_{t_0,\alpha} &\to \mathbb{R} \\ t &\mapsto \chi^j \circ \eta(t), \end{aligned}$$

then $(\phi^1, \ldots, \phi^n)$ is a fixed point of the mapping (3.9). Since this fixed point is unique, we must have

$$\chi^j \circ \eta(t) = \chi^j \circ \xi(t), \quad j \in \{1, \ldots, n\}, t \in \mathbb{T}_{t_0,\alpha}.$$

By Lemma 3.3 we conclude that $\eta = \xi$. One can also prove global uniqueness of integral curves using the standard arguments.

From the above, we conclude that

$$\mathbb{T}_{t_0,\alpha} \times \{t_0\} \times \mathcal{U} \times \mathcal{O} \subseteq D_X$$

and that

$$f \circ \Phi^X(t, t_0, x, p) = f(x) + \int_{|t,t_0|} Xf(s, \Phi^X(s, t_0, x, p), p)\, \mathrm{d}s$$

for $(t, x, p) \in \mathbb{T}_{t_0,\alpha} \times \mathcal{U} \times \mathcal{O}$ and $f \in C^\infty(M)$. This proves the existential part of the lemma.

(i) Fix $p \in \mathcal{O}$. By (3.1) and (3.8), we have

$$\begin{aligned} &\int_{|t_0,t|} |X^p\chi^j(s, x_1) - X^p\chi^j(s, x_2)|\, \mathrm{d}t \\ &\qquad \leq \lambda \max\{|\chi^l(x_1) - \chi^l(x_2)| \mid l \in \{1, \ldots, k\}\}, \ t \in \mathbb{T}_{t_0,\alpha}, \ x_1, x_2 \in \mathcal{U}. \end{aligned}$$

Let $t \in \mathbb{T}_{t_0,\alpha}$ be such that $t \geq t_0$ and let $x_1, x_2 \in \mathcal{U}$. We then have

$$f \circ \Phi^{X^p}(t, t_0, x_a) = f(x_a) + \int_{|t,t_0|} Xf(s, \Phi^{X^p}(s, t_0, x_a))\, \mathrm{d}s, \quad a \in \{1, 2\}, \ f \in C^\infty(M).$$

Thus, for $j \in \{1, \ldots, n\}$,

$$\begin{aligned} &|\chi^j \circ \Phi^{X^p}(t, t_0, x_1) - \chi^j \circ \Phi^{X^p}(t, t_0, x_2)| \\ &\quad \leq |\chi^j(x_1) - \chi^j(x_2)| + \int_{|t_0,t|} |X^p\chi^j(s, \Phi^{X^p}(s, t_0, x_1)) - X^p\chi^j(s, \Phi^{X^p}(s, t_0, x_2))|\, \mathrm{d}s \\ &\quad \leq |\chi^j(x_1) - \chi^j(x_2)| \\ &\qquad + \lambda \sup\left\{|\chi^l \circ \Phi^{X^p}(s, t_0, x_1) - \chi^l \circ \Phi^{X^p}(s, t_0, x_2)| \;\middle|\; s \in [t_0, t], \ l \in \{1, \ldots, n\}\right\} \end{aligned}$$

Abbreviate

$$\xi^p(s) = \max\left\{|\chi^l \circ \Phi^{X^p}(s, t_0, x_1) - \chi^l \circ \Phi^{X^p}(s, t_0, x_2)| \;\middle|\; l \in \{1, \ldots, n\}\right\}$$

and

$$\delta^p = \sup\{\xi^p(s) \;\mid\; s \in \mathbb{T}_{t_0,\alpha}\}$$

The definitions then give

$$\delta^p \leq \xi^p(t_0) + \lambda\delta^p \implies \delta^p \leq (1-\lambda)^{-1}\xi^p(t_0).$$

Since

$$\xi^p(t_0) = \max\left\{|\chi^j(x_1) - \chi^j(x_2)| \;\middle|\; j \in \{1,...,n\}\right\},$$

together with (3.1), we have

$$\mathrm{d}_{\mathbb{G}}(\Phi^{X^p}(t,t_0,x_1),\Phi^{X^p}(t,t_0,x_2)) \leq C\xi^p(t) \leq C\delta^p \leq (1-\lambda)^{-1}\,\mathrm{d}_{\mathbb{G}}(x_1,x_2),$$

which shows that $\Phi^{X^p}_{t,t_0}|\mathcal{U}$ is Lipschitz. Incidentally, the Lipschitz constant is independent of $t \in \mathbb{T}_{t_0,\alpha}$ and $p \in \mathcal{O}$.

(ii) Let $t_1, t_2 \in \mathbb{T}_{t_0,\alpha}$ be such that $t_0 \leq t_1 \leq t_2$, let $x_1, x_2 \in \mathcal{U}$, and let $p_1, p_2 \in \mathcal{O}$. We first have, for $t \in [t_0, t_1]$,

$$\begin{aligned}
&|\chi^j \circ \Phi^X(t,t_0,x_1,p_1) - \chi^j \circ \Phi^X(t,t_0,x_2,p_2)| \\
&\quad\leq |\chi^j(x_1) - \chi^j(x_2)| \\
&\qquad + \int_{|t_0,t|} |X\chi^j(s,\Phi^X(s,t_0,x_1,p_1),p_1) - X\chi^j(s,\Phi^X(s,t_0,x_2,p_2),p_2)|\,\mathrm{d}s \\
&\quad\leq |\chi^j(x_1) - \chi^j(x_2)| \\
&\qquad + \int_{|t_0,t|} |X\chi^j(s,\Phi^X(s,t_0,x_1,p_1),p_1) - X\chi^j(s,\Phi^X(s,t_0,x_2,p_2),p_1)|\,\mathrm{d}s \\
&\qquad + \int_{|t_0,t|} |X\chi^j(s,\Phi^X(s,t_0,x_2,p_2),p_1) - X\chi^j(s,\Phi^X(s,t_0,x_2,p_2),p_2)|\,\mathrm{d}s
\end{aligned}$$

for $j \in \{1,...,n\}$. Hence, for $j \in \{1,...,n\}$ and $t \in [t_0,t_1]$, we use (3.1) and (3.8) to give

$$\begin{aligned}
&\int_{|t_0,t|} |X\chi^j(s,\Phi^X(s,t_0,x_1,p_1),p_1) - X\chi^j(s,\Phi^X(s,t_0,x_2,p_2),p_1)|\,\mathrm{d}s \\
&\qquad\qquad \leq \lambda \max\left\{|\chi^l \circ \Phi^X(s,t_0,x_1,p_1) - \chi^l \circ \Phi^X(s,t_0,x_2,p_2)| \;\middle|\; l \in \{1,...,n\}\right\}.
\end{aligned}$$

We also clearly have

$$\begin{aligned}
&\int_{|t_0,t|} |X\chi^j(s,\Phi^X(s,t_0,x_2,p_2),p_1) - X\chi^j(s,\Phi^X(s,t_0,x_2,p_2),p_2)|\,\mathrm{d}s \leq \rho \triangleq \\
&\quad \max\left\{\int_{\mathbb{T}_{t_0,\alpha}} |X\chi^j(s,\Phi^X(s,t_0,x_2,p_2),p_1) - X\chi^j(s,\Phi^X(s,t_0,x_2,p_2),p_2)|\,\mathrm{d}s \;\middle|\; j \in \{1,...,n\}\right\}.
\end{aligned}$$

Let us denote

$$\xi(s) = \max\left\{|\chi^l \circ \Phi^{X^p}(s,t_0,x_1,p_1) - \chi^l \circ \Phi^X(s,t_0,x_2,p_2)| \;\middle|\; l \in \{1,...,n\}\right\}$$

and
$$\delta = \sup\{\xi(s) \mid s \in \mathbb{T}_{t_0,\alpha}\}$$
so that
$$\delta \le \xi(t_0) + \lambda\delta^p \implies \delta \le (1-\lambda)^{-1}\xi(t_0).$$
As per (3.1), let $C \in \mathbb{R}_{>0}$ be such that
$$\xi(t_0) \le C^{-1}\,\mathrm{d}_{\mathbb{G}}(x_1, x_2).$$
Then we have
$$\begin{aligned}
&|\chi^j \circ \Phi^X(t_1, t_0, x_1, p_1) - \chi^j \circ \Phi^X(t_2, t_0, x_2, p_2)| \\
&\quad \le |\chi^j \circ \Phi^X(t_1, t_0, x_1, p_1) - \chi^j \circ \Phi^X(t_1, t_0, x_2, p_2)| \\
&\qquad + |\chi^j \circ \Phi^X(t_2, t_0, x_2, p_2) - \chi^j \circ \Phi^X(t_1, t_0, x_2, p_2)| \\
&\quad \le \xi(t_0) + \lambda\delta + \rho + \int_{|t_1,t_2|} |X\chi^j(s, \Phi^X(s, t_0, x_2, p_2), p_2)|\,\mathrm{d}s \\
&\quad \le \frac{C^{-1}}{1-\lambda}\,\mathrm{d}_{\mathbb{G}}(x_1, x_2) + \rho + \int_{|t_1,t_2|} |X\chi^j(s, \Phi^X(s, t_0, x_2, p_2), p_1)|\,\mathrm{d}s \\
&\qquad + \int_{|t_1,t_2|} |X\chi^j(s, \Phi^X(s, t_0, x_2, p_2), p_1) - X\chi^j(s, \Phi^X(s, t_0, x_2, p_2), p_2)|\,\mathrm{d}s
\end{aligned}$$
Now we choose a neighbourhood $\mathcal{V}$ of x_1, $\sigma \in \mathbb{R}_{>0}$, and and a neighbourhood $\mathcal{O}' \subseteq \mathcal{O}$ of p_1 such that
$$\frac{C^{-1}}{1-\lambda}\,\mathrm{d}_{\mathbb{G}}(x_1, x_2) < \frac{\epsilon}{4}, \quad x_2 \in \mathcal{V};$$
$$\int_{|t_1,t_2|} |X\chi^j(s, \Phi^X(s, t_0, x_2, p_2), p_1)|\,\mathrm{d}s < \frac{\epsilon}{4}, \quad |t_1 - t_2| < \sigma;$$
$$\int_{|t_1,t_2|} |X\chi^j(s, \Phi^X(s, t_0, x_2, p_2), p_1) - X\chi^j(s, \Phi^X(s, t_0, x_2, p_2), p_2)|\,\mathrm{d}s < \frac{\epsilon}{4}, \quad p_2 \in \mathcal{O}';$$
$$\int_{\mathbb{T}_{t_0,\alpha}} |X\chi^j(s, \Phi^X(s, t_0, x_2, p_2), p_1) - X\chi^j(s, \Phi^X(s, t_0, x_2, p_2), p_2)|\,\mathrm{d}s < \frac{\epsilon}{4}, \quad p_2 \in \mathcal{O}'.$$
The last inequality implies that $\rho < \frac{\epsilon}{4}$. Thus we have
$$|\chi^j \circ \Phi^X(t_1, t_0, x_1, p_1) - \chi^j \circ \Phi^X(t_2, t_0, x_2, p_2)| < \epsilon, \quad x_2 \in \mathcal{V},\ p_2 \in \mathcal{O}',\ |t_1 - t_2| < \sigma,$$
which gives the continuity of $(t, x, p) \mapsto \chi^j \circ \Phi^X(t, t_0, x, p)$, and so the continuity of $(t, x, p) \mapsto \Phi^X(t, t_0, x, p)$. ▽

The next lemma is a refinement of the preceding one, giving the local version of the theorem statement.

Lemma 3.7. *Let M be a C^∞-manifold, let $\mathbb{T} \subseteq \mathbb{R}$ be a time domain, let $\mathcal{P}$ be a topological space, and let $X \in \Gamma^{\mathrm{lip}}_{LI}(\mathbb{T}, TM; \mathcal{P})$. For each $(t_0, x_0, p_0) \in \mathbb{T} \times M \times \mathcal{P}$, there exist $\alpha \in \mathbb{R}_{>0}$, a neighborhood $\mathcal{U} \subseteq M$ of x_0, and a neighborhood $\mathcal{O} \subseteq \mathcal{P}$ of p_0 such that* *(i)* $\mathbb{T}_{t_0,\alpha} \times \{t_0\} \times \mathcal{U} \times \mathcal{O} \subseteq D_X$,

(ii) *the map*

$$\mathbb{T}_{t_0,\alpha} \times \mathcal{U} \times \mathcal{O} \ni (t,x,p) \mapsto \Phi^X(t,t_0,x,p)$$

is continuous, and

(iii) *the map*

$$\mathcal{U} \ni x \mapsto \Phi^{X^p}_{t,t_0}(x) \in M$$

is s bi-Lipschitz homeomorphism onto its image for every $p \in \mathcal{O}$ *and every* $t \in \mathbb{T}_{t_0,\alpha}$.

Proof. (i) and (ii) can be proven by Lemma 3.6.

(iii) Let α', $\mathcal{U}'$ and $\mathcal{O}'$ be as in Lemma 3.6, and let $\alpha \in (0,\alpha']$, $\mathcal{U} \subseteq \mathcal{U}'$, and $\mathcal{O} \subseteq \mathcal{O}'$ be such that

$$\Phi^X(t,t_0,x,p) \in \mathcal{U}', \quad (t,x,p) \in \mathbb{T}_{t_0,\beta} \times \mathcal{U} \times \mathcal{O},$$

this is possible by Lemma 3.6 (i). Let $t \in \mathbb{T}_{t_0,\alpha}$, $x \in \mathcal{U}$, $p \in \mathcal{O}$, and denote

$$\mathcal{V} = \Phi^{X^p}_{t,t_0}(\mathcal{U}) \subseteq \mathcal{U}'.$$

Since $y \triangleq \Phi^X(t,t_0,x,p) \in \mathcal{U}'$ and $t \in \mathbb{T}_{t_0,\alpha}$, there exists a neighborhood $\mathcal{V}'$ of y such that, if $y' \in \mathcal{V}'$, then $(t,t_0,y',p) \in D_X$. Moreover, since $\Phi^{X^p}_{t_0,t}$ is continuous and Lipschitz, we can choose $\mathcal{V}'$ sufficiently small that $\Phi^{X^p}_{t_0,t}(y') \in \mathcal{U}$ if $y' \in \mathcal{V}'$. By Lemma 3.6, $\Phi^{X^p}_{t_0,t}|\mathcal{V}'$ is Lipschitz. Therefore, there is a neighborhood of x on which the restriction of $\Phi^{X^p}_{t_0,t}$ is invertible, Lipschitz, and with a Lipschitz inverse. ∇

We now need to show that parts (v) and (vi) of the theorem hold globally. Let $(t_0,x_0,p_0) \in \mathbb{T} \times M \times \mathcal{P}$, and denote by $J_+(t_0,x_0,p_0) \subseteq \mathbb{T}$ the set of $b > t_0$ such that, for each $b' \in [t_0,b)$, there exists a relatively open interval $J \subseteq \mathbb{T}$, a neighborhood $\mathcal{U}$ of x_0, and a neighborhood $\mathcal{O}$ of p_0 such that

1. $b' \in J$,
2. $J \times \{t_0\} \times \mathcal{U} \times \mathcal{O} \subseteq D_X$,
3. $J \times \mathcal{U} \times \mathcal{O} \ni (t,x,p) \mapsto \Phi^X(t,t_0,x,p) \in M$ is continuous, and
4. the map $\mathcal{U} \ni x \mapsto \Phi^X(t,t_0,x,p)$ is locally bi-Lipschitz homeomorphism onto its image for every $p \in \mathcal{O}$ and every $t \in J$.

Proof. By Lemma 3.7, $J_+(t_0,x_0,p_0) \neq \varnothing$. Then we consider two cases.

The first case is $J_+(t_0,x_0,p_0) \cap [t_0,\infty) = \mathbb{T} \cap [t_0,\infty)$. In this case, for each $t \in \mathbb{T}$ with $t \geq t_0$, there exists a relatively open interval $J \subseteq \mathbb{T}$, a neighborhood $\mathcal{U}$ of x_0, and a neighborhood $\mathcal{O}$ of p_0 such that

1. $t \in J$,

2. $J \times \{t_0\} \times \mathcal{U} \times \mathcal{O} \subseteq D_X$,

3. $J \times \mathcal{U} \times \mathcal{O} \ni (\tau, x, p) \mapsto \Phi^X(\tau, t_0, x, p) \in M$ is continuous, and

4. the map $\mathcal{U} \ni x \mapsto \Phi^X(\tau, t_0, x, p)$ is locally bi-Lipschitz homeomorphism onto its image for every $p \in \mathcal{O}$ and every $\tau \in J$.

The second case is $J_+(t_0, x_0, p_0) \cap [t_0, \infty) \subsetneq \mathbb{T} \cap [t_0, \infty)$. Now we show this is impossible. In this case we let $t_1 = \sup J_+(t_0, x_0, p_0)$ and note that $t_1 \neq \sup \mathbb{T}$. We claim that $t_1 \in J_Y(t_0, x_0, p_0)$. If this were not the case, then we must have $b \triangleq \sup J_X(t_0, x_0, p_0) < t_1$. Since $b \in J_+(t_0, x_0, p_0)$, there must be a relatively open interval $J \subseteq \mathbb{T}$ containing b such that $t \in J_X(t_0, x_0, p_0)$ for all $t \in J$. But, since there are $t's$ in J larger than b, this contradicts the definition of b, and so we conclude that $t_1 \in J_X(t_0, x_0, p_0)$.

We denote $x_1 = \Phi^X(t_1, t_0, x_0, p_0)$. By Lemma 3.7, there exists $\alpha_1 \in \mathbb{R}_{>0}$, a neighborhood $\mathcal{V}_1$ of x_1 such that $(t, t_1, x, p) \in D_X$ for every $t \in \mathbb{T}_{t_1,\alpha_1}$, $x \in \mathcal{V}_1$, and $p \in \mathcal{O}_1$, and such that the map

$$\mathbb{T}_{t_1,\alpha_1} \times \mathcal{V}_1 \times \mathcal{O}_1 \ni (t, x, p) \mapsto \Phi^X(t, t_1, x, p)$$

is continuous, and the map

$$\mathcal{V}_1 \ni x \mapsto \Phi^X(t, t_1, x, p)$$

is a locally bi-Lipschitz homeomorphism onto its image for every $t \in \mathbb{T}_{t_1,\alpha_1}$ and $p \in \mathcal{O}_1$. Let $\mathcal{V}'_1 \subseteq \mathcal{V}_1$ be such that $\mathrm{cl}(\mathcal{V}') \subseteq \mathcal{V}_1$. Since $t \mapsto \Phi^X(t, t_0, x_0, p_0)$ is continuous and $\Phi^X(t_1, t_0, x_0, p_0) = x_1$, let $\delta \in \mathbb{R}_{>0}$ be such that $\delta < \frac{\alpha}{2}$ and $\Phi^X(t, t_0, x_0, p_0) \in \mathcal{V}'_1$ for $t \in (t_1 - \delta, t_1)$. Now let $\tau_1 \in (t_1 - \delta, t_1)$ and, by our hypotheses on t_1, there exists an open interval J, a neighborhood $\mathcal{U}'_1$ of x_0, and a neighborhood $\mathcal{O}'_1$ of p_0 such that

1. $\tau_1 \in J$,

2. $J \times \{t_0\} \times \mathcal{U}'_1 \times \mathcal{O}'_1 \subseteq D_X$,

3. $J \times \mathcal{U}'_1 \times \mathcal{O}'_1 \ni (\tau, x, p) \mapsto \Phi^X(\tau, t_0, x, p) \in M$ is continuous, and

4. the map $\mathcal{U}'_1 \ni x \mapsto \Phi^X(\tau, t_0, x, p)$ is a locally bi-Lipschitz homeomorphism onto its image for every $\tau \in J$ and $p \in \mathcal{O}'_1$.

We choose J, $\mathcal{U}'_1$ and $\mathcal{O}'_1$ sufficiently small that

$$\{\Phi^X(t, t_0, x, p) \mid t \in J,\ x \in \mathcal{U}'_1,\ p \in \mathcal{O}'_1\} \subseteq \mathcal{V}'_1$$

This is possible by the continuity of the flow. We, moreover, assume that $\mathcal{O}'_1$ is sufficiently small as to be a subset of $\mathcal{O}_1$.

We claim that

$$\mathbb{T}_{\tau_1,\alpha_1} \times \{t_0\} \times \mathcal{U}'_1 \times \mathcal{O}'_1 \subseteq D_X.$$

We first show that

$$[\tau_1, \tau_1 + \alpha_1) \times \{t_0\} \times \mathcal{U}'_1 \times \mathcal{O}'_1 \subseteq D_X. \tag{3.10}$$

For $x \in \mathcal{U}'_1$ and $p' \in \mathcal{O}'_1$, we have $(\tau_1, t_0, x, p) \in D_X$ since $\tau_1 \in J$. By definition of J, $\Phi^X(\tau_1, t_0, x, p) \in \mathcal{V}'_1$. By definition of $\tau_1, t_1 - \tau_1 < \delta < \frac{\alpha_1}{2}$. Then, by definition of α_1 and $\mathcal{V}_1$,

$$(t_1, \tau_1, \Phi^X(\tau_1, t_0, x, p'), p) \in D_X$$

for every $x \in \mathcal{U}'_1$, $p' \in \mathcal{O}'_1$, and $p \in \mathcal{O}_1$. From this we conclude that $(t_1, t_0, x, p) \in D_X$ for every $x \in \mathcal{U}'_1$ and $p \in \mathcal{O}'_1$. Now, since

$$t \in [\tau_1, \tau_1 + \alpha_1) \implies t \in \mathbb{T}_{t_1, \alpha_1},$$

we have $(t, t_1, \Phi^X(t_1, t_0, x, p), p) \in D_X$ for every $t \in \mathbb{T}_{\tau_1, \alpha_1}$, $x \in \mathcal{U}'_1$, and $p \in \mathcal{O}'_1$. Since

$$\Phi^X(t, t_1, \Phi^X(t_1, t_0, x, p), p) = \Phi^X(t, t_0, x, p),$$

we conclude (3.10). A similar argument gives

$$\mathbb{T}_{\tau_1, \alpha_1} \times \{t_0\} \times \mathcal{U}'_1 \times \mathcal{O}'_1 \subseteq D_X.$$

Now we claim the map

$$\mathcal{T}_{\tau_1, \alpha_1} \times \mathcal{U}'_1 \times \mathcal{O}'_1 \ni (t, x, p) \mapsto \Phi^X(t, t_0, x, p)$$

is continuous. This map is continuous at

$$(t, x, p) \in (\tau_1 - \alpha_1, \tau_1] \times \mathcal{U}'_1 \times \mathcal{O}'_1$$

by definition of τ_1. For $t \in (\tau_1, \tau_1 + \alpha_1)$ we have

$$\Phi^X(t, t_0, x, p) = \Phi^X(t, \tau_1, \Phi^X(\tau_1, t_0, x, p), p)$$

and continuity follows since the composition of continuous maps is continuous.

Next we claim the map

$$\mathcal{U}'_1 \ni x \mapsto \Phi^Y(t, t_0, x, p)$$

is a locally bi-Lipschitz homeomorphism onto its image for every $t \in \mathcal{T}_{\tau_1, \alpha_1}$ and $p \in \mathcal{O}'_1$. By definition of τ_1, the map

$$\Phi^{X^p}_{t, t_0} : \mathcal{U}'_1 \mapsto \mathcal{V}'_1$$

is locally bi-Lipschitz onto its image for $t \in (\tau_1 - \alpha_1, \tau_1]$ and $p \in \mathcal{O}'_1$. We also have that

$$\Phi^Y_{t, \tau_1} : \mathcal{V}_1 \mapsto M$$

is a locally bi-Lipschitz homeomorphism onto its image for $t \in (\tau_1, \tau_1 + \alpha_1)$. Since the composition of locally bi-Lipschitz homeomorphisms onto their image is a locally bi-Lipschitz homeomorphism onto its image, our assertion follows.

By our arguments, we have an open interval J', a neighborhood $\mathcal{U}'$ of x_0, and a neighbourhood $\mathcal{O}'_1$ of p_0 such that

1. $t_1 \in J'$,
2. $J' \times \{t_0\} \times \mathcal{U}'_1 \times \mathcal{O}'_1 \subseteq D_X$,
3. $J' \times \mathcal{U}'_1 \times \mathcal{O}'_1 \ni (t, x, p) \mapsto \Phi^X(t, t_0, x, p) \in M$ is continuous, and
4. the map $\mathcal{U}'_1 \ni x \mapsto \Phi^X(t, t_0, x, p)$ is locally bi-Lipschitz homeomorphism onto its image for every $t \in J'$ and $p \in \mathcal{O}'_1$.

This contradicts the fact that $t_1 = \sup J_+(t_0, x_0, p_0)$ and so the condition

$$J_+(t_0, x_0, p_0) \cap [t_0, \infty) \subsetneq \mathbb{T} \cap [t_0, \infty)$$

cannot obtain.

One similar shows that it must be the case that $J_-(t_0, x_0, p_0) \cap (-\infty, t_0] = \mathbb{T} \cap (-\infty, t_0]$, where $J_-(t_0, x_0)$ has the obvious definition. ∇

Finally, we note that Φ^X_{t,t_0} is injective by uniqueness of solutions for X. Now, assertions (v) and (vi) of the theorem follow, since the notions of "locally bi-Lipschitz homeomorphism" can be tested locally, i.e., in a neighbourhood of any point. We have shown something more, in fact, namely that, along with parts (v) and (vi) of the theorem, the set

$$\{(t, x, p) \in \mathbb{T} \times M \times \mathcal{P} \mid (t, t_0, x, p) \in D_X\}$$

is open for each t_0, and that the mapping $(t, x, p) \mapsto \Phi^X(t, t_0, x, p)$ is continuous. We shall now use this fact and a lemma, to prove assertions (vii) and (viii) together.

Lemma 3.8. *Let $(t_1, t_0, x_0, p_0) \in D_X$. As Lemma 3.7, there exist $\alpha_1 \in \mathbb{R}_{>0}$, a neighborhood $\mathcal{U}_1$ of x_0, and a neighborhood $\mathcal{O}_1$ of p_0 such that*

$$\mathbb{T}_{t_1,\alpha_1} \times \{t_0\} \times \mathcal{U}_1 \times \mathcal{O}_1 \subseteq D_X,$$

and the map $(t, x, p) \mapsto \Phi^X(t, x_0, x, p)$ is continuous on this domain. Then the map

$$(t, x, p) \mapsto \Phi^X(t_0, t, x, p)$$

is continuous for (t, x, p) nearby (t_0, x_0, p_0).

Proof. To see this, we proceed rather as in the proof of Lemma 3.6. Let U, $\chi^1, \dots, \chi^n$, and C be just as in the initial part of the proof of Lemma 3.6. We also choose $\alpha, r \in R_{>0}$ and a neighbourhood $\mathcal{O}$ of p_0 such that $\mathcal{U}(r, x_0) \in \mathcal{U} \cap \mathcal{U}_1$ and such that

$$\int_{|t,t_0|} |X\chi^j(s, x, p)| \, \mathrm{d}s < \frac{r}{2} \tag{3.11}$$

and

$$\int_{|t,t_0|} |X\chi^j(s, x_1, p) - X\chi^j(s, x_2, p)| \, \mathrm{d}s < \frac{\lambda}{C} \mathrm{d}_{\mathbb{G}}(x_1, x_2), \tag{3.12}$$

for all $t \in \mathbb{T}_{t_0,\alpha}$, $x, x_1, x_2 \in \mathcal{U}$, $j \in \{1, \dots, n\}$ and $p \in \mathcal{O}$.

Let $x \in \mathcal{U}(r/2, x_0)$. For each $j \in \{1, ..., n\}$, we denote by $\phi_0^j \in C^0(\mathbb{T}_{t_0,\alpha}, \mathbb{R})$ the constant mapping $\phi_0^j = \chi^j(x_0)$, and denote by $\overline{B}(r, \phi_0^j)$ the closed ball of radius r about ϕ_0^j. For each $p \in \mathcal{O}$, we have the mapping

$$\begin{aligned} F_{X^p}^j : \quad & \overline{B}(r, \phi_0^j) \to C^0(\mathbb{T}_{t_0,\alpha}, \mathbb{R}) \\ & \phi^j \mapsto \chi^j(x) + \int_{|t,t_0|} X\chi^j(s, \gamma_\phi(s), p) ds, \end{aligned}$$

for $j \in \{1, ..., n\}$, and where $\gamma_\phi \in C^0(\mathbb{T}_{t_0,\alpha}, M)$ is the unique curve satisfying

$$\chi^j \circ \gamma_\phi = \phi^j(t), \qquad t \in \mathbb{T}_{t_0,\alpha},\ j \in \{1, ..., n\},$$

cf. Lemma 3.3. Note that similar to the proof in Lemma 3.6, the definition of r ensures that γ_ϕ so defined takes values in $\mathcal{U}'$.

We then claim that $F_{X^p}^j(\overline{B}(r, \phi_0^j)) \subseteq \overline{B}(r, \phi_0^j)$, $j \in \{1, ..., n\}$, $p \in \mathcal{O}$. Indeed, if $\phi^j \in \overline{B}(r, \phi_0^j)$, $j \in \{1, ..., n\}$, we have

$$|F_{X^p}^j \circ \phi^j(t) - \phi_0^j(t)| \le |\chi^j(x) - \chi^j(x_0)| + \int_{|t,t_0|} |X\chi^j(s, \gamma_\phi(s))| \, \mathrm{d}s < r.$$

We also claim that the mapping

$$\prod_{j=1}^n \overline{B}(r, \phi_0^j) \ni (\phi^1, ..., \phi^n) \mapsto (F_{X^p}^1 \circ \phi^1, ..., F_{X^p}^n \circ \phi^n) \in \prod_{j=1}^n \overline{B}(r, \phi_0^j) \tag{3.13}$$

is a contraction mapping for each $p \in \mathcal{O}$, where $\prod_{j=1}^n \overline{B}(r, \phi_0^j)$ is given the product metric. Indeed, let $\phi_1^j, \phi_2^j \in \overline{B}(r, \phi_0^j)$, $j \in \{1, ..., n\}$. Let $\gamma_1, \gamma_2 \in C^0(\mathbb{T}_{t_0,\alpha}, \mathbb{R})$ be the corresponding curves satisfying

$$\chi^j \circ \gamma_a(t) = \phi_a^j(t), \qquad t \in \mathbb{T}_{t_0,\alpha},\ j \in \{1, ..., n\},\ a \in \{1, 2\},$$

using Lemma 3.3. Then we have, for each $j \in \{1, ..., n\}$,

$$\begin{aligned} |F_{X^p}^j \circ \phi_1^j(t) - F_{X^p}^j \circ \phi_2^j(t)| &\le \int_{|t_0,t|} |X\chi^j(s, \gamma_1(s), p) - X\chi^j(s, \gamma_2(s), p)| \, \mathrm{d}s \\ &\le \frac{\lambda}{C} \mathrm{d}_{\mathbb{G}}(\gamma_1(s), \gamma_2(s)) \\ &\le \lambda \sup\{|\phi_1^k(s) - \phi_2^k(s)| \mid k \in \{1, ..., n\}\},\ s \in \mathbb{T}_{t_0,\alpha} \end{aligned}$$

from which the desired conclusion follows.

By the Contraction Mapping Theorem, there exists a unique fixed point for the mapping (3.13), which gives rise to a curve $\xi : \mathbb{T}_{t_0,\alpha} \mapsto \mathcal{U}$ satisfying

$$f \circ \xi(t) = f(x) + \int_{|t,t_0|} Xf(s, \xi(s), p) \, \mathrm{d}s.$$

Differentiating with respect to t shows that ξ is an integral curve for X and $\xi(t_0) = x$. This shows that, if $t \in \mathbb{T}_{t_0,\alpha}$, $x \in \mathcal{U}(r/2, x_0)$, and $p \in \mathcal{O}$, then we have $\Phi^X(t_0, t, x, p) \in \mathcal{U}(r, x_0)$ and

$$f \circ \Phi^X(t_0, t, x, p) = f(x) + \int_{|t,t_0|} Xf(s, \Phi^Y(t_0, s, x, p), p)\, \mathrm{d}s$$

for $f \in C^\infty(M)$. We conclude that there exists $\alpha_0, r_0 \in \mathbb{R}_{>0}$ and a neighbourhood $\mathcal{O}$ of p_0 such that

$$\Phi^X(t_0, t, x, p) \in \mathcal{U}_1, \qquad (t, x, p) \in \mathbb{T}_{t_0,\alpha_0} \times \mathcal{U}(r_0, x_0) \times \mathcal{O}.$$

We then show that the map $(t, x, p) \mapsto \Phi^X(t_0, t, x, p)$ is continuous on $\mathbb{T}_{t_0,\alpha_0} \times \mathcal{U}(r_0, x_0) \times \mathcal{O}$, just as in the proof of Lemma 3.6.

Finally, if

$$(t', t_0', x, p) \in \mathbb{T}_{t,\alpha} \times \mathbb{T}_{t_0,\alpha_0} \times \mathcal{U}(r_0, x_0) \times \mathcal{O}',$$

then

$$\Phi^X(t', t_0', \Phi^X(t_0, t_0', x, p), p) = \Phi^X(t', t_0', x, p),$$

which shows both that D_X is open and that Φ^X is continuous, since that composition of continuous mappings is continuous. ∇

(ix) Let $T_+ = \sup J_X(t_0, x_0, p_0)$. Then $(T_+ - \epsilon, t_0, x_0, p_0) \in D_X$. Since D_X is open, there exists a neighbourhood $\mathcal{U}$ of x_0 and a neighbourhood $\mathcal{O}$ of p_0 such that

$$\{T_+ - \frac{\epsilon}{2}\} \times \mathbb{T}_{t_0,\alpha} \times \mathcal{U} \times \mathcal{O} \subseteq D_X.$$

In other words, $[t_0, T_+ - \frac{\epsilon}{2}] \subseteq J_X(t, x, p)$ for every $(t, x, p) \in \mathbb{T}_{t_0,\alpha} \times \mathcal{U} \times \mathcal{O}$. Thus, for such (t, x, p),

$$\sup J_X(t, x, p) \geq T_+ - \frac{\epsilon}{2} > T_+ - \epsilon = \sup J_X(t_0, x_0, p_0) - \epsilon,$$

as claimed. A similar argument holds for the left endpoint of intervals of existence.

(x) Let $t \in |t_0, t_1|$ and let $\epsilon \in \mathbb{R}_{>0}$. Following the argument and using notation inspired by the proof of part (ii) of Lemma 3.6, there is an interval $\mathbb{T}_t \subseteq \mathbb{T}$, a neighbourhood $\mathcal{V}_{t,\epsilon}$ of x_0, a neighbourhood $\mathcal{O}_{t,\epsilon} \subseteq \mathcal{P}$ of p_0, and $\chi_t^j \in C^\infty(M)$, $j \in \{1, \dots, n\}$, (n being the dimension of M) such that

$$\begin{aligned} |\chi_t^j \circ \Phi^X(t', t, \Phi^X(t, t_0, x, p), p) - \chi_t^j \circ \Phi^X(t', t, \Phi^X(t, t_0, x_0, p_0), p_0)| \leq C_t^{-1}\epsilon, \\ (t', x, p) \in \mathbb{T}_t \times \mathcal{V}_{t,\epsilon} \times \mathcal{O}_{t,\epsilon}, \quad j \in \{1, \dots, n\}, \end{aligned}$$

where $C_t \in \mathbb{R}_{>0}$ is such that

$$\mathrm{d}_{\mathbb{G}}(x_1, x_2) \leq C_t \max\left\{|\chi_t^j(x_1) - \chi_t^j(x_2)| \;\middle|\; j \in \{1, \dots, n\},\ x_1, x_2 \in \mathcal{V}_{t,\epsilon}\right\}.$$

Let $t_1,\dots,t_k \in |t_0,t_1|$ be such that $|t_0,t_1| \subseteq \cup_{j=1}^k \mathbb{T}_{t_j}$. For $t \in |t_0,t_1|$, let $j_t \in \{1,\dots,k\}$ be such that $t \in \mathbb{T}_{t_{j_t}}$, and let $x \in \cap_{j=1}^k \mathcal{V}_{t_j,\epsilon}$ and $p \in \cap_{j=1}^k \mathcal{O}_{t_j,\epsilon}$. Let $C = \max\{C_{t_1},\dots,C_{t_k}\}$. Then, if $t \in |t_0,t_1|$ and with $j \in \{1,\dots,k\}$ such that $t \in \mathbb{T}_{t_j}$,

$$\mathrm{d}_{\mathbb{G}}(\Phi^X(t,t_0,x,p),\Phi^X(t,t_0,x_0,p_0)) \le C|\chi_t^j \circ \Phi^X(t,t_0,x,p) - \chi_t^j \circ \Phi^X(t,t_0,x_0,p_0)| < \epsilon,$$

which gives the desired uniform convergence.

(ix) Since the parameter-dependence plays no role here, we suppose we are in the parameter-independent case to simplify the notation. By definition of $I_X(t_1,x_0)$, the curve $\iota_{(t_1,x_0)}$ is well-defined. We will show that it is locally absolutely continuous. We first prove this locally, and so work with a time-varying vector field X. Let $(t_0,x_0) \in \mathbb{T} \times M$ and let $\alpha \in \mathbb{R}_{>0}$ and $\mathcal{U}$ be a neighbourhood of x_0 such that

$$f \circ \iota_{(t_1,x_0)}(t) = f \circ \Phi^X(t_1,t,x_0) = f(x_0) + \int_{|t,t_1|} Xf(s,\Phi^Y(s,t,x_0))\,\mathrm{d}s,$$

for $t \in \mathbb{T}_{t_0,\alpha}$ and $x \in \mathcal{U}$. Following our constructions in the preceding parts of the proof, we work with functions $\chi^1,\dots,\chi^n \in C^\infty(M)$ whose differentials are linearly independent on $\mathcal{U}$. We also let $g, l \in L^1_{\mathrm{loc}}(\mathbb{T};\mathbb{R}_{\ge 0})$ be such that

$$|X\chi^j(t,x)| \le g(t)$$

and

$$|X\chi^j(t,x_1) - X\chi^j(t,x_2)| \le l(t)\max\{|\chi^m(x_1)-\chi^m(x_2)| \ \mid\ m \in \{1,\dots,n\}\}$$

for $t \in \mathbb{T}$, $x, x_1, x_2 \in \mathcal{U}$, and $j \in \{1,\dots,n\}$, this because $X\chi^j \in C^{\mathrm{lip}}_{\mathrm{LI}}(\mathbb{T};M)$ by Lemma 2.5, (3.1), and (Jafarpour and Lewis, 2014, Theorem 6.4: (v) and (vi)).

Let $\delta \in \mathbb{R}_{>0}$ be such that, if (a_l,b_l), $l \in \{1,\dots,k\}$, is a collection of pairwise disjoint subintervals of $\mathbb{T}_{t_0,\alpha}$ satisfying

$$\sum_{l=1}^k |b_l - a_l| < \delta,$$

then

$$\sum_{l=1}^k \int_{a_1}^{b_l} g(s)\,\mathrm{d}s < \epsilon \exp\left(-\int_{\mathbb{T}_{t_0,\alpha}} l(s)\,\mathrm{d}s\right).$$

Let $t_1 \in \mathbb{T}_{t_0,\alpha}$ and $j \in \{1,\dots,n\}$, and compute

$$\begin{aligned}
&\sum_{l=1}^k |\chi^j \circ \iota_{(t_1,x_0)}(b_l) - \chi^j \circ \iota_{(t_1,x_0)}(a_l)| \\
&= \sum_{l=1}^k |\chi^j \circ \Phi^X(t_1,b_l,x_0) - \chi^j \circ \Phi^X(t_1,a_l,x_0)| \\
&\le \sum_{l=1}^k \left| \int_{b_l}^{t_1} X\chi^j(s,\Phi^X(s,b_l,x_0))\,\mathrm{d}s - \int_{a_l}^{t_1} X\chi^j(s,\Phi^X(s,a_l,x_0))\,\mathrm{d}s \right|
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{l=1}^{k} \left| \int_{b_l}^{t_1} X\chi^j(s, \Phi^X(s, b_l, x_0))\,\mathrm{d}s - \int_{a_l}^{t_1} X\chi^j(s, \Phi^X(s, b_l, x_0))\,\mathrm{d}s \right| \\
&+ \sum_{l=1}^{k} \left| \int_{a_l}^{t_1} X\chi^j(s, \Phi^X(s, b_l, x_0))\,\mathrm{d}s - \int_{a_l}^{t_1} X\chi^j(s, \Phi^X(s, a_l, x_0))\,\mathrm{d}s \right| \\
&\leq \sum_{l=1}^{k} \int_{b_l}^{a_l} |X\chi^j(s, \Phi^X(s, b_l, x_0))|\,\mathrm{d}s \\
&+ \sum_{l=1}^{k} \int_{a_l}^{t_1} |X\chi^j(s, \Phi^X(s, b_l, x_0)) - X\chi^j(s, \Phi^X(s, a_l, x_0))|\,\mathrm{d}s \\
&\leq \sum_{l=1}^{k} \int_{b_l}^{a_l} g(s)\,\mathrm{d}s \\
&+ \sum_{l=1}^{k} \int_{a_l}^{t_1} l(s) \max\{|\chi^m \circ \Phi^X(s, b_l, x_0) - \chi^m \circ \Phi^X(s, a_l, x_0)| \mid m \in \{1, \dots, n\}\}\,\mathrm{d}s \\
&\leq \sum_{l=1}^{k} \int_{b_l}^{a_l} g(s)\,\mathrm{d}s \\
&+ \sum_{l=1}^{k} \int_{a_l}^{t_1} l(s) \max\{|\chi^m \circ \Phi^X(s, b_l, x_0) - \chi^m \circ \Phi^X(s, a_l, x_0)| \mid m \in \{1, \dots, n\}\}\,\mathrm{d}s \\
&+ \sum_{l=1}^{k} \int_{t_0-\alpha}^{a_l} l(s) \max\{|\chi^m \circ \Phi^X(s, b_l, x_0) - \chi^m \circ \Phi^X(s, a_l, x_0)| \mid m \in \{1, \dots, n\}\}\,\mathrm{d}s \\
&\leq \sum_{l=1}^{k} \int_{b_l}^{a_l} g(s)\,\mathrm{d}s \\
&+ \int_{t_0-\alpha}^{t_1} l(s) \sum_{l=1}^{k} \max\{|\chi^m \circ \Phi^X(s, b_l, x_0) - \chi^m \circ \Phi^X(s, a_l, x_0)| \mid m \in \{1, \dots, n\}\}\,\mathrm{d}s.
\end{aligned}
$$

For $t \in \mathbb{T}_{t_0,\alpha}$, let us denote

$$\kappa(t) = \sum_{l=1}^{k} \max |\chi^j \circ \iota_{(t,x_0)}(b_l) - \chi^j \circ \iota_{(t,x_0)}(a_l)| \mid j \in \{1, \dots, m\}.$$

Our computations above show that

$$\kappa(t) \leq \sum_{l=1}^{k} \int_{b_l}^{a_l} g(s)\,\mathrm{d}s + \int_{t_0-\alpha}^{t_1} l(s)\kappa(s)\,\mathrm{d}s.$$

Thus, by Gronwall's inequality (Sontag, 1998, Lemma C.3.1), we have

$$\kappa(t_1) \leq \exp\left(\int_{t_0-\alpha}^{t_1} l(s)\,\mathrm{d}s \right) \sum_{l=1}^{k} \int_{b_l}^{a_l} g(s)\,\mathrm{d}s.$$

This shows that, for every $t_1 \in \mathbb{T}_{t_0,\alpha}$,

$$\sum_{l=1}^{k} |\chi^j \circ \iota_{(t,x_0)}(b_l) - \chi^j \circ \iota_{(t,x_0)}(a_l)| < \epsilon.$$

Thus, in particular, $\iota(t_1, x_0)$ is absolutely continuous on $\mathbb{T}_{t_0,\alpha}$ for each $t_1 \in \mathbb{T}_{t_0,\alpha}$. Now we prove the suitable conclusion globally. Here we make use of another lemma.

Lemma 3.9. *Let $\mathcal{U} \subseteq \mathbb{R}^n$ and $\mathcal{U} \subseteq \mathbb{R}^m$ be open, let $\mathbb{T} \subseteq \mathbb{R}$ be a time-domain, let $\Phi : \mathcal{U} \to \mathcal{V}$ be locally Lipschitz, and let $\gamma : \mathbb{T} \to \mathcal{U}$ have locally absolutely continuous components. Then $\Phi \circ \gamma$ has locally absolutely continuous components.*

Proof. Let $[a, b] \subseteq \mathbb{T}$ be a compact subinterval. Since $\gamma([a, b])$ is compact and Φ is locally Lipschitz, there exists $L \in \mathbb{R}_{>0}$ such that

$$\|\Phi \circ \gamma(t_1) - \Phi \circ \gamma(t_2)\| \leq L\|\gamma(t_1) - \gamma(t2)\|, \quad t_1, t_2 \in [a, b].$$

Let $\epsilon \in \mathbb{R}_{>0}$ and let $\delta \in \mathbb{R}_{>0}$ be such that, if $((a_j, b_j))_{j\in\{1,\dots,k\}}$ is a family of disjoint intervals such that

$$\sum_{j=1}^{k} |b_j - a_j| < \delta,$$

then

$$\sum_{j=1}^{k} |\gamma^a(b_j) - \gamma^a(a_j)| < \frac{\epsilon}{L\sqrt{n}}, \quad a \in \{1, \dots, n\}.$$

Then, for any $a \in \{1, \dots, n\}$ and $\alpha \in \{1, \dots, m\}$, and using standard relationships between the 2-norm and the 1-norm for $\mathbb{R}^n$,

$$\begin{aligned} \sum_{j=1}^{k} |\Phi^\alpha \circ \gamma(b_j) - \Phi^\alpha \circ \gamma(a_j)| &\leq \sum_{j=1}^{k} \|\Phi \circ \gamma(b_j) - \Phi \circ \gamma(a_j)\| \leq \sum_{j=1}^{k} L\|\gamma(b_j) - \gamma(a_j)\| \\ &\leq L\sqrt{n} \max\left\{ \sum_{j=1}^{k} |\gamma^a(b_j) - \gamma^a(a_j)| \;\middle|\; j \in \{1, \dots, n\} \right\} < \epsilon \end{aligned}$$

whenever

$$\sum_{j=1}^{k} |b_j - a_j| < \delta.$$

∇

Now let $(t_1, t_0, x_0) \in D_X$ and let $\alpha \in \mathbb{R}_{>0}$ and $\mathcal{U}$ be as above. For $t \in \mathbb{T}_{t_0,\alpha}$ we have

$$f \circ \iota_{(t_1,x_0)}(t) = f \circ \Phi^X(t_1, t, x_0) = f \circ \Phi^X_{t_1,t_0} \circ \Phi^X_{t_0,t}(x_0).$$

By our computations above, the curve $t \mapsto \Phi^X_{t_0,t}(x_0)$ is absolutely continuous on $\mathbb{T}_{t_0,\alpha}$. Since $x \mapsto \Phi^X_{t_1,t_0}(x)$ is locally Lipschitz by part (vi), it follows from the previous lemma that $t \mapsto f \circ \iota_{(t_1,x_0)}(t)$ is absolutely continuous on $\mathbb{T}_{t_0,\alpha}$. Since local absolute continuity is a local property, i.e., it only needs to hold in any neighbourhood of any point, it follows that $t \mapsto f \circ \iota_{(t_1,x_0)}(t)$ is locally absolutely continuous. Thus, by definition, $\iota_{(t_1,x_0)}$ is locally absolutely continuous.

∇

The reader will note that a substantial portion of the proof is taken up with the extension of the usual local statements—such as one normally finds in presentations of ordinary differential equations—to global statements valid on the whole of the domain of the vector field. We feel as if it worth doing this carefully once, since it is not easy to find, and impossible to find in the generality we give here. A few useful consequences of the theorem follow.

Corollary 3.10 (Images of compact subsets of initial conditions are compact)**.** *Let M be a C^∞-manifold, let $\mathbb{T} \subset \mathbb{R}$ be an interval, and let $X \in \Gamma^{\mathrm{lip}}_{LI}(\mathbb{T}; TM)$. Let $K \subseteq M$ be compact and let $t_0, t_1 \in \mathbb{T}$ be such that*

$$|t_0, t_1| \times \{t_0\} \times K \subseteq D_X. \tag{3.14}$$

Then

$$\bigcup_{(t,x)\in|t_0,t_1|\times K} \Phi^X(t, t_0, x) \tag{3.15}$$

is compact.

Proof. This follows since the set (3.15) is the image of the compact set (3.14) under the continuous (by part (viii) of the theorem) map Φ^X. □

Corollary 3.11 (Robustness of compactness by variations of parameters)**.** *Let M be a C^∞-manifold, let $\mathbb{T} \subseteq \mathbb{R}$ be an interval, let $\mathcal{P}$ be a topological space, and let $X \in \Gamma^{\mathrm{lip}}_{PLI}(\mathbb{T}; TM; \mathcal{P})$. Let $K \subseteq M$ be compact, let $t_0, t_1 \in \mathbb{T}$, and let $p_0 \in \mathcal{P}$ be such that*

$$|t_0, t_1| \times \{t_0\} \times K \times \{p_0\} \subseteq D_X.$$

Then there exists a neighbourhood $\mathcal{O} \subseteq \mathcal{P}$ of p_0 such that

$$\bigcup_{(t,x,p)\in|t_0,t_1|\times K\times\mathcal{O}} \Phi^X(t, t_0, x, p)$$

is well-defined and precompact.

Proof. By the previous corollary,

$$K_0 \triangleq \bigcup_{(t,x)\in|t_0,t_1|\times K} \Phi^X(t, t_0, x)$$

is compact. Since M is locally compact, let $\mathcal{V}$ be a precompact neighbourhood of K_0. By part (x) of the theorem, for $x \in K$, let $\mathcal{U}_x$ be a neighbourhood of x and let $\mathcal{O}_x$ be a neighbourhood of p_0 such that

$$\bigcup_{(t,x,p)\in|t_0,t_1|\times(K\cap\mathcal{U}_x)\times\mathcal{O}_x} \Phi^X(t, t_0, x, p) \subseteq \mathcal{V}.$$

By compactness of K, let $x_1, \dots, x_m \in K$ be such that $K = \cup_{j=1}^m K \cap \mathcal{U}_{x_j}$ and let $\mathcal{O} = \cap_{j=1}^k \mathcal{O}_{x_j}$. Then

$$\Phi^X(t, t_0, x, p) \subseteq \mathcal{V}, \quad (t, x, p) \in |t_0, t_1| \times K \times \mathcal{O},$$

as desired. □

Corollary 3.12 (Uniform Lipschitz character of flows). *Let M be a C^∞-manifold, let $\mathbb{T} \subseteq \mathbb{R}$ be an interval, let $\mathcal{P}$ be a topological space, and let $X \in \Gamma^{\mathrm{lip}}_{PLI}(\mathbb{T}; TM; \mathcal{P})$. Let $K \subseteq M$ be compact, let $t_0, t_1 \in \mathbb{T}$, and let $p_0 \in \mathcal{P}$ be such that*

$$|t_0, t_1| \times \{t_0\} \times K \times \{p_0\} \subseteq D_X.$$

Then there exists a neighbourhood $\mathcal{O} \subseteq \mathcal{P}$ of p_0 and $C \in \mathbb{R}_{>0}$ such that

$$\mathrm{d}_{\mathbb{G}}(\Phi^X(t, t_0, x_1, p), \Phi^X(t, t_0, x_2, p)) \leq C\, \mathrm{d}_{\mathbb{G}}(x_1, x_2), \quad t \in |t_0, t_1|,\ x_1, x_2 \in K,\ p \in \mathcal{O}.$$

Proof. As in the proof of Lemma 3.6(i) from the theorem (see, especially, the last few lines of that part of the proof), for $(t, x) \in |t_0, t_1| \times K$, there exists an open interval $\mathbb{T}_{(t,x)} \subseteq |t_0, t_1|$ containing t, a neighbourhood $\mathcal{U}_{(t,x)} \subseteq M$ of x, and a neighbourhood $\mathcal{O}_{(t,x)}$ of p_0 such that

$$\mathbb{T}_{(t,x)} \times \{t_0\} \times \mathcal{U}_{(t,x)} \times \mathcal{O}_{(t,x)} \subseteq D_X,$$

and such that there exists $C_{(t,x)} \in \mathbb{R}_{>0}$ for which

$$\begin{aligned}\mathrm{d}_{\mathbb{G}}(\Phi^X(t', t_0, x_1, p), \Phi^X(t', t_0, x_2, p)) \leq C_{(t,x)}\, \mathrm{d}_{\mathbb{G}}(\Phi^X(t, t_0, x_1, p), \Phi^X(t, t_0, x_2, p)),\\ t \in |t_0, t_1|,\ x_1, x_2 \in K,\ p \in \mathcal{O}.\end{aligned} \tag{3.16}$$

Denote

$$\mathcal{N}_{(t,x)} = \{\Phi^X(t', t_0, x', p) \ \mid (t', x', p) \in \mathbb{T}_{(t,x)} \times \mathcal{U}_{(t,x)} \times \mathcal{O}_{(t,x)}\},$$

noting that $\mathcal{N}_{(t,x)}$ is a neighbourhood of $\Phi^X(t, t_0, x, p_0)$. By compactness of

$$K_x \triangleq \{\Phi^X(t, t_0, x, p_0) \ \mid \ t \in |t_0, t_1|\},$$

there exist $t_{x,1}, \ldots, t_{x,m_x} \in |t_0, t_1|$ such that

$$K_x \subseteq \bigcup_{j=1}^{m_x} \mathrm{int}(\mathcal{N}_{(t_{x,j}, x)}).$$

Let us also choose $t_{x,0} = t_0$ and so add to this finite cover the set $\mathcal{N}_{(t_{x,0},x)}$ associated with $t = t_0$. Let $\mathcal{O}_x = \cap_{j=0}^{m_x} \mathcal{O}_{(t_{x,j},x)}$ and $\mathcal{U}_x = \cap_{j=0}^{m_x} \mathcal{U}_{(t_{x,j},x)}$, and note that

$$|t_0, t_1| \times \{t_0\} \times K \cap \mathcal{U}_x \times \mathcal{O}_x \subseteq D_X.$$

Also note that

$$\mathcal{V}_x \triangleq \{\Phi^X(t, t_0, x', p) \ \mid (t, x', p) \in |t_0, t_1| \times \mathcal{U}_x \times \mathcal{O}_x\}$$

is a filter neighbourhood of K_x. Since the intervals $\mathbb{T}_{(t_{x,j},x)}$, $j \in \{0, \ldots, m_x\}$, cover $|t_0, t_1|$ and since $t_{x,0} = t_0$, for any $j \in \{1, \ldots, m\}$ we can write

$$\Phi^X(t_{x,j}, t_0, x, p) = \Phi^X_{t_{x,j_m}, t_{x,j_{m-1}}} \circ \Phi^X_{t_{x,j_1}, t_0} \tag{3.17}$$

for some $j_1, \dots, j_m \in \{1, \dots, m_x\}$ satisfying $t_{x,j_l} \in \mathbb{T}_{x,j_{l-1}}$, $l \in \{1, \dots, m\}$. Moreover, we can do this with at most m_x compositions.

Now let $x_1, x_2 \in \mathcal{U}_x$ and note that, for $t \in |t_0, t_1|$, $t \in \mathbb{T}_{(t_{x,j},x)}$ for some $j \in \{0, 1, \dots, m_x\}$. We also have $x_1, x_2 \in \mathcal{U}_{(t_{x,j},x)}$. Therefore, with this j chosen and for $p \in \mathcal{O}_x$,

$$\mathrm{d}_{\mathbb{G}}(\Phi^X(t, t_0, x_1, p), \Phi^X(t, t_0, x_2, p)) \le C_{t_j,x}\, \mathrm{d}_{\mathbb{G}}(\Phi^X(t_{x,j}, t_0, x_1, p), \Phi^X(t_{x,j}, t_0, x_2, p)).$$

Using the composition (3.17) and the bound (3.16) for each term in the composition, we have

$$\mathrm{d}_{\mathbb{G}}(\Phi^X(t, t_0, x_1, p), \Phi^X(t, t_0, x_2, p)) \le C_x\, \mathrm{d}_{\mathbb{G}}(x_1, x_2), \quad t \in |t_0, t_1|,$$

taking

$$C_x = \max\{C_{(t_{x,1},x)}, \dots, C_{(t_{x,m_x},x)}\}^{m_x+1}$$

Choose $x_1, \dots, x_m \in K$ so that $K = \cup_{j=1}^m (K \cap \mathcal{U}_{x_j})$. Let $\mathcal{O} = \cap_{j=1}^m \mathcal{O}_{x_j}$. If necessary and by Corollary 3.11, shrink $\mathcal{O}$ so that

$$\{\Phi^X(t, t_0, x, p) \mid t \in |t_0, t_1|, x \in K, p \in \mathcal{O}\}$$

is precompact. Let $M \in \mathbb{R}_{>0}$ and $x_0 \in K$ be such that

$$\mathrm{d}_{\mathbb{G}}(x, x_0) \le M, \quad x \in K.$$

Note that

$$\{\Phi^X(t, t_0, x, p) \mid t \in |t_0, t_1|, x \in K, p \in \mathcal{O}\} \subseteq \bigcup_{j=1}^m \mathcal{V}_{x_j}.$$

By the Lebesgue Number Lemma (D. Burago, Y. Burago, and Ivanov, 2001, Theorem 1.6.11), let $r \in \mathbb{R}_{>0}$ be such that, if $x_1, x_2 \in K$ satisfy $\mathrm{d}_{\mathbb{G}}(x_1, x_2) < r$, then there exists $j \in \{1, \dots, m\}$ such that $x_1, x_2 \in \mathcal{U}_{x_j}$. Let

$$C = \max\left\{C_{x_1}, \dots, C_{x_m}, \frac{2M}{r}\right\}.$$

Finally, let $t \in |t_0, t_1|$, let $x_1, x_2 \in K$, and let $p \in \mathcal{O}$. If $\mathrm{d}_{\mathbb{G}}(x_1, x_2) < r$, let $j \in \{1, \dots, m\}$ be such that $x_1, x_2 \in \mathcal{U}_{x_j}$. Then we have

$$\mathrm{d}_{\mathbb{G}}(\Phi^X(t, t_0, x_1, p), \Phi^X(t, t_0, x_2, p)) \le C_{x_j}\, \mathrm{d}_{\mathbb{G}}(x_1, x_2) \le C\, \mathrm{d}_{\mathbb{G}}(x_1, x_2).$$

If $\mathrm{d}_{\mathbb{G}}(x_1, x_2) \ge r$, then

$$\begin{aligned}
\mathrm{d}_{\mathbb{G}}(\Phi^X(t, t_0, x_1, p), &\Phi^X(t, t_0, x_2, p)) \\
&\le \mathrm{d}_{\mathbb{G}}(\Phi^X(t, t_0, x_1, p), x_0) + \mathrm{d}_{\mathbb{G}}(\Phi^X(t, t_0, x_2, p), x_0) \\
&\le \frac{2M}{r} r \le C\, \mathrm{d}_{\mathbb{G}}(x_1, x_2),
\end{aligned}$$

as desired. □

3.3. Continuous dependence of fixed-time flow on parameter

Theorem 3.13. *Let $m \in \mathbb{Z}_{\geq 0}$, let $\nu \in \{m, \infty, \mathrm{hol}\}$ satisfy $\nu \geq \mathrm{lip}$, and let $r \in \{\infty, \mathrm{hol}\}$ as appropriate. Let M be a C^r-manifold, let $\mathbb{T} \subseteq \mathbb{R}$ be an interval, and let $\mathcal{P}$ be a topological space. For $X \in \Gamma^{\nu}_{\mathrm{PLI}}(\mathbb{T}; TM; \mathcal{P})$, let $t_0, t_1 \in \mathbb{T}$ and $p_0 \in \mathcal{P}$ be such that there exists a precompact open set $\mathcal{U} \subseteq M$ such that $\mathrm{cl}(\mathcal{U}) \subseteq \mathrm{D_X}(\mathrm{t_1}, \mathrm{t_0}, \mathrm{p_0})$. Then there exists a neighbourhood $\mathcal{O} \subseteq \mathcal{P}$ of p_0 such that mapping*

$$\mathcal{O} \ni p \mapsto \Phi^{X^p}_{t_1, t_0} \in C^{\nu}(\mathcal{U}; M)$$

is well-defined and continuous.

Proof. We break down the proof into the various classes of regularity. The proofs bear a strong resemblance to one another, so we go through the details carefully in the first case we prove, the locally Lipschitz case, and then merely outline where the arguments differ for the other regularity classes.

3.3.1. The C^0-case. Note that $\mathrm{cl}(\mathcal{U})$ is compact. Therefore, by Corollary 3.11, there exists a compact set $K' \subseteq M$ and a neighbourhood $\mathcal{O}$ of p_0 such that

$$\Phi^{X^p}_{t_1, t_0}(x) \in \mathrm{int}(K'), \quad (x, p) \in \mathrm{cl}(\mathcal{U}) \times \mathcal{O}.$$

This gives the well-definedness assertion of the theorem. Note, also, that it gives the well-definedness assertion for all $\nu \in \{m, \infty, \omega, \mathrm{hol}\}$, and so we need not revisit this for the remainder of the proof. The compact set $K' \subseteq M$ and the neighbourhood $\mathcal{O}$ of p_0 will be used in all parts of the proof without necessarily referring to our constructions here.

For continuity, first we show that the mapping

$$\mathcal{O} \ni p \mapsto \Phi^{X^p}_{t_1, t_0} \in C^0(\mathcal{U}; M)$$

is continuous. The topology for $C^0(\mathcal{U}; M)$ is the uniform topology defined by the semimetrics

$$\mathrm{d}^0_{K,f}(\Phi_1, \Phi_2) = \sup\{|f \circ \Phi_1(x) - f \circ \Phi_2(x)| \mid x \in K\}, \quad f \in C^{\infty}(M), \ K \subseteq \mathcal{U} \text{ compact}.$$

Thus, we must show that, for $f_1, ..., f_m \in C^{\infty}(M)$, for $K_1, ..., K_m \subseteq \mathcal{U}$ compact, and for $\epsilon_1, ..., \epsilon_m \in \mathbb{R}_{>0}$, there exists a neighborhood $\mathcal{O}$ of p_0 such that

$$|f_j \circ \Phi^{X^p}_{t_1, t_0}(x_j) - f_j \circ \Phi^{X^{p_0}}_{t_1, t_0}(x_j)| < \epsilon_j, \quad x_j \in K_j, \ p \in \mathcal{O}, \ j \in \{1, ..., m\}.$$

It will suffice to show that, for $f \in C^{\infty}(M)$, for $K \subseteq \mathcal{U}$ compact, and for $\epsilon \in \mathbb{R}_{>0}$, we have

$$|f \circ \Phi^{X^p}_{t_1, t_0}(x) - f \circ \Phi^{X^{p_0}}_{t_1, t_0}(x)| < \epsilon, \quad x \in K, \ p \in \mathcal{O}.$$

Indeed, if we show that then, taking $K = \cup_{j=1}^{k} K_j$ and $\epsilon = \min\{\epsilon_1, ..., \epsilon_m\}$, we have

$$|f_j \circ \Phi^{X^p}_{t_1, t_0}(x) - f_j \circ \Phi^{X^{p_0}}_{t_1, t_0}(x)| < \epsilon, \quad x \in K, \ p \in \mathcal{O}, \ j \in \{1, ..., m\}$$

for a suitable $\mathcal{O}$. This suffices to give the desired conclusion.

It is useful to consider the space $C^0(\mathbb{T}; M)$ with the topology (indeed, uniformity) defined by the family of semimetrics

$$\mathrm{d}_{\mathbb{S},M}(\gamma_1, \gamma_2) = \{d_{\mathbb{G}}(\gamma_1(t), \gamma_2(t)) \mid t \in \mathbb{S}\}, \quad \mathbb{S} \subseteq \mathbb{T} \text{ a compact interval.}$$

For $g \in C^0_{\mathrm{LI}}(|t_0, t_1|; M)$, we also have the mapping

$$\begin{aligned} \Psi_{|t_0,t_1|,M,g} : C^0(|t_0, t_1|; M) &\to L^1_{\mathrm{loc}}(|t_0, t_1|; \mathbb{R}) \\ \gamma &\mapsto (t \mapsto g_t(\gamma(t))). \end{aligned}$$

By Corollary 2.7 and Lemma 2.8, this map is well-defined and continuous. Now consider the following mapping

$$\begin{aligned} \Phi_{\mathbb{T},K,\mathcal{O}} : K \times \mathcal{O} &\to C^0(\mathbb{T}; M) \\ (x, p) &\mapsto (t \mapsto \Phi^X(t, t_0, x, p)). \end{aligned}$$

By Theorem 3.5(x), this mapping is continuous. We also have the continuous mapping

$$\begin{aligned} \iota_{|t_0,t_1|} : C^0(\mathbb{T}; M) &\to C^0(|t_0, t_1|; M) \\ \gamma &\mapsto \gamma||t_0, t_1|. \end{aligned}$$

Let $f \in C^\infty(M)$, let $\epsilon > 0$, and let $x \in K$. Combining the observations of two previous paragraphs, the mapping

$$\Psi_{|t_0,t_1|,M,X^{p_0}f} \circ \iota_{|t_0,t_1|} \circ \Phi_{\mathbb{T},K,\mathcal{O}} : K \times \mathcal{O} \to L^1(|t_0, t_1|; \mathbb{R})$$

is continuous. Thus there exists a relative neighbourhood $\mathcal{V}_x \subseteq K$ of x and a neighbourhood $\mathcal{O}_x \subseteq \mathcal{O}$ of p_0 such that

$$\int_{|t_0,t_1|} |X^{p_0} f(s, \Phi^{X^p}_{s,t_0}(x')) - X^{p_0} f(s, \Phi^{X^{p_0}}_{s,t_0}(x'))| \, ds < \frac{\epsilon}{2} \qquad x' \in \mathcal{V}_x, \ p \in \mathcal{O}_x.$$

Let $x_1, \dots, x_m \in K$ be such that $K = \cup_{j=1}^m \mathcal{V}_{x_j}$ and define a neighbourhood $\mathcal{O}_1 = \cap_{j=1}^k \mathcal{O}_{x_j}$ of p_0. Then we have

$$\int_{|t_0,t_1|} |X^{p_0} f(s, \Phi^{X^p}_{s,t_0}(x)) - X^{p_0} f(s, \Phi^{X^{p_0}}_{s,t_0}(x))| \, ds < \frac{\epsilon}{2}, \qquad x \in K, \ p \in \mathcal{O}_1. \tag{3.1}$$

By (2.4), we can further shrink $\mathcal{O}_1$ if necessary so that

$$\int_{|t_0,t_1|} |X^p f(s, x) - X^{p_0} f(s, x)| \, ds < \frac{\epsilon}{2} \qquad x' \in K, \ p \in \mathcal{O}_1.$$

Then we have

$$|f \circ \Phi^{X^p}_{t_1,t_0}(x) - f \circ \Phi^{X^{p_0}}_{t_1,t_0}(x)|$$

$$\begin{aligned}
&\leq \int_{|t_0,t_1|} |X^p f(s,\Phi^{X^p}_{s,t_0}(x)) - X^{p_0} f(s,\Phi^{X^{p_0}}_{s,t_0}(x))|\,ds \\
&\leq \int_{|t_0,t_1|} |X^p f(s,\Phi^{X^p}_{s,t_0}(x)) - X^{p_0} f(s,\Phi^{X^p}_{s,t_0}(x))|\,ds \\
&\quad + \int_{|t_0,t_1|} |X^{p_0} f(s,\Phi^{X^p}_{s,t_0}(x)) - X^{p_0} f(s,\Phi^{X^{p_0}}_{s,t_0}(x))|\,ds \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,
\end{aligned}$$

for $x \in K$ and $p \in \mathcal{O}_1$, as desired.

Therefore, for every compact $K \subseteq \mathcal{U}$, every $f \in C^\infty(M)$, and every $\epsilon \in \mathbb{R}_{>0}$, if $p \in \mathcal{O}_1 \cap \mathcal{O}_2$, then we ave

$$p^0_K(f \circ \Phi^{X^p}_{t_1,t_0} - f \circ \Phi^{X^{p_0}}_{t_1,t_0}) < \epsilon,$$

which gives the desired result.

3.3.2. The C^m-case. The topology for $C^m(\mathcal{U}; M)$ is the uniform topology defined by the semimetrics

$$\mathrm{d}^m_{K,f}(\Phi_1,\Phi_2) = \sup\{\|j_m(f\circ\Phi_1)(x) - j_m(f\circ\Phi_2)(x)\|_{\mathbb{G}_{M,m}} \mid x \in K\}, \\ f \in C^\infty(M),\ K \subseteq \mathcal{U} \text{ compact}.$$

As in the preceding section when we proved C^0 continuity, it suffices to show that, for $f \in C^\infty(M)$, $K \subseteq \mathcal{U}$ compact, and for $\epsilon \in \mathbb{R}_{>0}$, there exists a neighbourhood $\mathcal{O}'$ of p_0 such that

$$\|j_m(f\circ\Phi^{X^p}_{t_1,t_0})(x) - j_m(f\circ\Phi^{X^{p_0}}_{t_1,t_0})(x)\|_{\mathbb{G}_{M,m}} < \epsilon, \quad x \in K,\ p \in \mathcal{O}'.$$

Thus let $f \in C^\infty(M)$, $K \subseteq \mathcal{U}$ compact, and let $\epsilon \in \mathbb{R}_{>0}$. Consider the mapping

$$\begin{aligned}
\Phi_{|t_0,t_1|,K,\mathcal{O}} : K \times \mathcal{O} &\to C^0(|t_0,t_1|; J^m(\mathcal{U}; M)) \\
(x,p) &\mapsto (t \mapsto j_m\Phi^{X^p}_{t,t_0}(x)),
\end{aligned}$$

which is well-defined and continuous, c.f. Theorem 3.5(x). For $(x,p) \in K \times \mathcal{O}$ and for $t \in |t_0,t_1|$, we can think of $j_m\Phi^{X^p}_{t,t_0}(x))$ as a linear mapping

$$\begin{aligned}
j_m\Phi^{X^p}_{t,t_0}(x) : J^m(M;\mathbb{R})_{\Phi^{X^p}_{t,t_0}(x)} &\to J^m(M;\mathbb{R})_x \\
j_m g(\Phi^{X^p}_{t,t_0}(x)) &\mapsto j_m(g\circ\Phi^{X^p}_{t,t_0})(x).
\end{aligned}$$

Now, fixing $(x,p) \in \mathcal{U} \times \mathcal{O}$ for the moment, recall the constructions of Section 2.3.2, particularly those preceding the statement of Lemma 2.6. We consider the notation from those constructions with

1. $N = M$,

2. $E = F = J^m(M;\mathbb{R})$,
3. $\Gamma(t) = j_m\Phi^{X^p}_{t,t_0}(x) \in \mathrm{Hom}_{\mathbb{R}}(J^m(M,\mathbb{R})_{\Phi^{X^p}_{t,t_0}(x)}; J^m(M;\mathbb{R})_x)$, and
4. $\xi = j_m(X^{p_0}f)$.

Thus, again in the notation from Section 2.3.2, we have

$$\gamma_M(t) = \Phi^{X^p}_{t,t_0}(x), \quad \gamma_N(t) = x.$$

We then have the integrable section of $E = J^m(M;\mathbb{R})$ given by

$$\begin{aligned} \xi_\Gamma : |t_0,t_1| &\to E \\ t &\mapsto (t \mapsto j_m(X^p_t f \circ \Phi^{X^p}_{t,t_0})(x)) \end{aligned}$$

to obtain continuity of the mapping

$$\begin{aligned} \Psi_{|t_0,t_1|,J^m(M;\mathbb{R}),j_m(X^{p_0}f)} : C^0(|t_0,t_1|;J^m(\mathcal{U};M)) &\to L^1_{\mathrm{loc}}(|t_0,t_1|;J^m(M;\mathbb{R})) \\ \Gamma &\mapsto (t \mapsto \Gamma(t)(j_m(X^{p_0}_t f)(\gamma_M(t)))), \end{aligned}$$

and so of the composition

$$\Psi_{|t_0,t_1|,J^m(M;\mathbb{R}),j_m(X^{p_0}f)} \circ \Phi_{|t_0,t_1|,K,\mathcal{O}} : K \times \mathcal{O} \to L^1_{\mathrm{loc}}(|t_0,t_1|;J^m(M;\mathbb{R})).$$

Note that this is precisely the continuity of the mapping

$$K \times \mathcal{O} \ni (x,p) \mapsto (t \mapsto j_m(X^{p_0}f \circ \Phi^{X^p}_{t,t_0}(x))) \in L^1_{\mathrm{loc}}(|t_0,t_1|;J^m(M;\mathbb{R})).$$

In order to convert this continuity into a continuity statement involving the fibre norm for $J^m(M;\mathbb{R})$, we note that for $x \in K$, there exists a neighbourhood $\mathcal{V}_x$ and affine functions $F^1_x,\dots,F^{n+k}_x \in \mathrm{Aff}^\infty(J^m(M;\mathbb{R}))$ which are coordinates for $\rho_m^{-1}(\mathcal{V}_x)$. We can choose a Riemannian metric for $J^m(M;\mathbb{R})$, whose restriction to fibres agrees with the fibre metric (2.1) (Lewis, 2020, §4.1). It follows, therefore, from Lemma A.1 that there exists $C_x \in \mathbb{R}_{>0}$ such that

$$\|j_m g_1(x') - j_m g_2(x')\|_{\mathbb{G}_{M,m}} \le C_x |F^l_x \circ j_m g_1(x') - F^l_x \circ j_m g_2(x')|,$$

for $g_1, g_2 \in C^\infty(M)$, $x' \in \mathcal{V}_x$, $l \in \{1,\dots,n+k\}$. By the continuity proved in the preceding paragraph, we can take a relative neighbourhood $\mathcal{V}_x \subseteq K$ of x sufficiently small and a neighbourhood $\mathcal{O}_x \subseteq \mathcal{O}$ of p_0 such that

$$\int_{|t_0,t_1|} |F^l_x \circ j_m(X^{p_0}f \circ \Phi^{X^p}_{t,t_0}(x')) - F^l_x \circ j_m(X^{p_0}f \circ \Phi^{X^{p_0}}_{t,t_0}(x'))| dt < \frac{\epsilon}{2C_x},$$

for all $x' \in \mathcal{V}_x$, $p \in \mathcal{O}_x$, and $l \in \{1,\dots,n+k\}$, recalling from Section 2.3.2 the definition of the topology for $L^1(|t_0,t_1|;J^m(M;\mathbb{R})$. Therefore,

$$\int_{|t_0,t_1|} \|j_m(X^{p_0}f \circ \Phi^{X^p}_{t,t_0}(x')) - j_m(X^{p_0}f \circ \Phi^{X^{p_0}}_{t,t_0}(x'))\|_{\mathbb{G}_{M,m}} ds < \frac{\epsilon}{2}$$

for all $x' \in \mathcal{V}_x$, $p \in \mathcal{O}_x$. Now let $x_1, ..., x_s \in K$ be such that $K = \cup_{r=1}^s \mathcal{V}_{x_r}$ and define a neighbourhood $\mathcal{O}' = \cap_{r=1}^s \mathcal{O}_{x_r}$ of p_0. Then we have

$$\int_{|t_0,t_1|} \|j_m(X_s^{p_0} f \circ \Phi_{s,t_0}^{X^p})(x') - j_m(X_s^{p_0} f \circ \Phi_{s,t_0}^{X^{p_0}})(x')\|_{\mathbb{G}_{M,m}} ds < \frac{\epsilon}{2} \tag{3.2}$$

for all $x' \in K$, $p \in \mathcal{O}'$. By (2.4), we can further shrink $\mathcal{O}'$ if necessary so that

$$\int_{|t_0,t_1|} \|j_m(X^p f)(s,y) - j_m(X^{p_0} f)(s,y)\|_{\mathbb{G}_{M,m}} \mathrm{d}s < \frac{\epsilon}{2} \qquad y' \in K', \; p \in \mathcal{O}'.$$

Then we have

$$\begin{aligned}
&\|j_m(f \circ \Phi_{t_1,t_0}^{X^p})(x) - j_m(f \circ \Phi_{t_1,t_0}^{X^{p_0}})(x)\|_{\mathbb{G}_{M,m}} \\
&\qquad \le \int_{|t_0,t_1|} \|j_m(X^p f \circ \Phi_{t,t_0}^{X^p})(x) - j_m(X^{p_0} f \circ \Phi_{t,t_0}^{X^{p_0}})(x)\|_{\mathbb{G}_{M,m}} ds \\
&\qquad \le \int_{|t_0,t_1|} \|j_m(X^p f \circ \Phi_{t,t_0}^{X^p})(x) - j_m(X^{p_0} f \circ \Phi_{t,t_0}^{X^p})(x)\|_{\mathbb{G}_{M,m}} ds \\
&\qquad\quad + \int_{|t_0,t_1|} \|j_m(X^{p_0} f \circ \Phi_{t,t_0}^{X^p})(x) - j_m(X^{p_0} f \circ \Phi_{t,t_0}^{X^{p_0}})(x)\|_{\mathbb{G}_{M,m}} ds \\
&\qquad \le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,
\end{aligned}$$

for $x \in K$ and $p \in \mathcal{O}'$, as desired.

3.3.3. The C^∞-case. From the result in the C^m-case for $m \in \mathbb{Z}_{\ge 0}$, the mapping

$$\mathcal{O} \ni p \mapsto \Phi_{t_1,t_0}^{X^p} \in C^m(\mathcal{U}; M)$$

is continuous for each $m \in \mathbb{Z}_{\ge 0}$. From the diagram

$$\begin{tikzcd}
C^\infty(\mathcal{U}; M) \arrow[r, hookrightarrow] & C^m(\mathcal{U}; M) \\
\mathcal{O} \arrow[u, "p \mapsto \Phi_{t_1,t_0}^{X^p}"] \arrow[ur, "p \mapsto j_m \Phi_{t_1,t_0}^{X^p}"'] &
\end{tikzcd}$$

and noting that the diagonal mappings in the diagram are continuous, we obtain the continuity of the vertical mapping as a result of the fact that the C^∞-topology is the initial topology induced by the C^m-topologies, $m \in \mathbb{Z}_{\ge 0}$.

3.3.4. The C^{hol}-case. Since the C^{hol}-topology is the C^0-topology, with the scalars extended to be complex and the functions restricted to be holomorphic, the analysis in Section 3.3.1 can be carried out verbatim to give the theorem in the holomorphic case. ∇

Chapter 4

The exponential map

For the development of the exponential map, we shall use the language of category theory and sheaf theory. For both vector fields and flows, we will work with presheaves defined by prescribing local sections over a basis for the topology. We also wish to talk about classes of vector fields with various properties, e.g., certain regularity or geometric properties.

In the last section, we have shown that exponential maps are defined locally on open cubes $\mathbb{S}' \times \mathbb{S} \times \mathcal{U} \subseteq \mathbb{T} \times \mathbb{T} \times M$. More precisely, we established the well-definedness of the map

$$\overline{\exp} : L^1_{\mathrm{loc}}(\mathbb{T}; \Gamma^\nu(TM)) \supseteq \mathcal{V}^\nu(\mathbb{S}' \times \mathbb{S} \times \mathcal{U}) \to \mathrm{LocFlow}^\nu(\mathbb{S}'; \mathbb{S}; \mathcal{U})$$

from $\mathcal{V}^\nu(\mathbb{S}' \times \mathbb{S} \times \mathcal{U})$, the space of time-varying sections whose flows are defined on $\mathbb{S}' \times \mathbb{S} \times \mathcal{U}$, to the space of their local flows of regularity ν. This plays a key role in constructing the ultimate localisation of the map $\overline{\exp}$ between the presheaves of such spaces,

$$\exp : \{\text{presheaves of vector fields}\} \to \{\text{presheaves of local flows}\}.$$

4.1. Categories of time-varying sections

We will define what we shall call a "time-varying local section." This is nothing more than a time-varying section, defined locally.

Definition 4.1 (C^ν-time-varying local section)**.** Let $m \in \mathbb{Z}_{\geq 0}$, let $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega, \mathrm{hol}\}$, and let $r \in \{\infty, \omega, \mathrm{hol}\}$, as required. A ***(locally) integrally bounded local section of class*** $\mathbf{C}^\nu$ is a quadruple $(\mathbb{T}, E, \mathcal{U}, \xi)$ where

(i) $\mathbb{T} \subseteq \mathbb{R}$ is an interval,

(ii) $\pi : E \to M$ is a C^r-vector bundle,

(iii) $\mathcal{U} \subseteq M$ is open, and

(iv) $\xi \in \Gamma^{\nu}_{\mathrm{I}}(\mathbb{T}; E|\mathcal{U})$ ($\in \Gamma^{\nu}_{\mathrm{LI}}(\mathbb{T}; E|\mathcal{U})$).

We denote $L^1_{\mathrm{loc}}(\mathbb{T}; \Gamma^{\nu}(E|\mathcal{U}))$ the set of integrally bounded local sections of class C^{ν}. The reader may notice that we have duplicated constructions we have already made in Section 2.3, the only difference being that we have introduced an open subset $\mathcal{U} \subseteq M$. The reason for this is simply for the purpose of pedagogy so that we have symmetry with our categories of flows in 4.3.

Equipped with the seminorms defined by (2.1), the category we are building in this section is a subcategory of the category $\mathscr{LCTVS}$ of locally convex topological vector spaces.

Now let us consider morphisms in the category we are building. We give the definition in the locally integrable case, but the obvious definition can also be made in the integrable case.

Definition 4.2 (Morphisms of sets of local time-varying sections)**.** Let $m \in \mathbb{Z}_{\geq 0}$, let $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega, \mathrm{hol}\}$, and let $r \in \{\infty, \omega, \mathrm{hol}\}$, as required. Let $\mathbb{T}, \mathbb{S} \subseteq \mathbb{R}$ be intervals. A mapping

$$\alpha : L^1_{\mathrm{loc}}(\mathbb{T}; \Gamma^{\nu}(E|\mathcal{U})) \to L^1_{\mathrm{loc}}(\mathbb{S}; \Gamma^{\nu}(F|\mathcal{V}))$$

is a morphism of sets of C^{ν}-local time-varying sections if

(i) α is continuous, and

(ii) there exists a mapping $\tau_\alpha : \mathbb{T} \to \mathbb{S}$ and a C^r-vector bundle mapping $\Phi_\alpha \in \mathrm{VB}^r(E; F)$ over $\Phi_{\alpha,0} \in C^r(M; N)$ such that:

 (a) τ_α is the restriction to $\mathbb{T}$ of a nonconstant affine mapping,

 (b) $\Phi_{\alpha,0}(\mathcal{U}) \subseteq \mathcal{V}$, and

 (c) $\Phi_\alpha \circ \xi(t, x) = \alpha(\xi)(t, \Phi_{\alpha,0}(x))$ for $t \in \mathbb{T}$ and $x \in \mathcal{U}$.

Let us give some examples of morphisms of sets of time-varying local sections.

Example 4.3 (Morphisms of sets of local time-varying sections)**.** If $E = F$, and if $\mathbb{S} \subseteq \mathbb{T}$ and $\mathcal{V} \subseteq \mathcal{U}$, then the "restriction morphism"

$$\rho_{\mathbb{T}\times\mathbb{U},\mathbb{S}\times\mathcal{V}} : L^1_{\mathrm{loc}}(\mathbb{T}; \Gamma^{\nu}(E|\mathcal{U})) \to L^1_{\mathrm{loc}}(\mathbb{S}; \Gamma^{\nu}(E|\mathcal{V}))$$

is given by

$$\rho_{\mathbb{T}\times\mathbb{U},\mathbb{S}\times\mathcal{V}}(\xi)(t, x) = \xi(t, x), \quad (t, x) \in \mathbb{S} \times \mathcal{V}.$$

In this case we have "$\tau_\alpha = \iota_{\mathbb{T},\mathbb{S}}$" and "$\Phi_\alpha = \iota_{E|\mathcal{U},E|\mathcal{V}}$" where $\iota_{\mathbb{T},\mathbb{S}}$ and $\iota_{E|\mathcal{U},E|\mathcal{V}}$ are the inclusions. Of course, an entirely similar construction holds in the integrable case.

One can directly verify that morphisms can be composed and that composition is associative. One also has an identity morphism

$$\mathrm{id} : L^1_{\mathrm{loc}}(\mathbb{T}; \Gamma^{\nu}(E|\mathcal{U})) \to L^1_{\mathrm{loc}}(\mathbb{T}; \Gamma^{\nu}(E|\mathcal{U}))$$

given by "$\tau_{\mathrm{id}} = \mathrm{id}_{\mathbb{T}}$" and "$\Phi_{\mathrm{id}} = \mathrm{id}_{E|\mathcal{U}}$." This, then, gives the category $\mathcal{G}^{\nu}_{\mathrm{LI}}$ of locally integrally bounded local sections of class C^{ν} whose objects are the sets $L^1_{\mathrm{loc}}(\mathbb{T};\Gamma^{\nu}(E|\mathcal{U}))$. of locally integrable local sections of class C^{ν} and whose morphisms are as defined above. This category is a subcategory of $\mathcal{LCTVS}$. As we shall see, we like the category $\mathcal{LCTVS}$ because direct limits exist in this category. One similarly denotes by $\mathcal{G}^{\nu}_{\mathrm{I}}$ the category of integrable local sections of class C^{ν}.

4.2. Presheaves of time-varying vector fields

Throughout the following studies, we will investigate the open and connected subsets (thus a domain) $\mathcal{W} \subseteq \mathbb{T}\times\mathbb{T}\times M$ such that all the points $p_1 = (t_1, t_0, x_0) \in \mathcal{W}$ satisfy the following properties:

(a) if $t_1 = t_0$, then there exists $p_2 = (t_2, t_0, x_0) \in \mathcal{W}$ where $t_2 \neq t_0$, and the segment from p_1 to p_2 must also be contained in $\mathcal{W}$;

(b) if $t_1 \neq t_0$, then there exists $p_0 = (t_0, t_0, x_0) \in \mathcal{W}$, and the segment from p_1 to p_0 must also be contained in $\mathcal{W}$.

For convenience, we will call the subsets of $\mathbb{T}\times\mathbb{T}\times M$ with this property **flow admissible**. Note that for cubes $\mathbb{S}'\times\mathbb{S}\times\mathcal{U} \subseteq \mathbb{T}\times\mathbb{T}\times M$, flow admissible simply means $\mathbb{S}\subseteq\mathbb{S}'$.

We have the following lemma.

Lemma 4.4. *Let $\mathcal{W} \subseteq \mathbb{T}\times\mathbb{T}\times M$ be open and flow admissible. Then there exists a countable cover $\{\mathbb{S}'_i\times\mathbb{S}_i\times\mathcal{U}_i\}_{i\in\mathbb{Z}_{>0}}$ that is flow admissible for each $\mathbb{S}'_i\times\mathbb{S}_i\times\mathcal{U}_i$, $i\in\mathbb{Z}_{>0}$.*

Proof. Let $p_1 = (t_1, t_0, x_0) \in \mathcal{W}$ be an arbitrary point, then p_1 is an interior point as $\mathcal{W}$ is open. Hence there exists an open ball $B_r(p_1)$ of radius r cantered at p_1 that lies inside $\mathcal{W}$.

(i) When $t_1 = t_0$, then p_1 can be covered by an open cube $T\times T\times U \subset \mathbb{T}\times\mathbb{T}\times\mathcal{U}$ that lies inside $B_r(p_1)$, and this cube is flow admissible.

(ii) When $t_1 \neq t_0$, since $\mathcal{W}$ is flow admissible, there exists a $p_0 = (t_0, t_0, x_0) \in \mathcal{W}$ and the line segment from p_1 to p_0 lies inside $\mathcal{W}$. Since this line segment is compact, there is a finite open cover $\{T'_k\times T_k\times U_k\}_{k=1}^n$ such that $\bigcup_{k=1}^n T'_k\times T_k\times U_k \subset \mathcal{W}$.

Now denote
$$T' = \bigcup_{k=1}^{n} T'_k, \quad T = \bigcap_{k=1}^{n} T_k, \quad U = \bigcap_{k=1}^{n} U_k.$$
Hence $T'\times T\times U$ is flow admissible, and covers the line segment from p_0 to p_1.

□

Now consider the product space $\prod_{i\in\mathbb{Z}_{>0}} \mathcal{V}^{\nu}_{\mathbb{S}'_i\times\mathbb{S}_i\times\mathcal{U}_i}$ with the initial topology, i.e., the coarsest topology such that the following map

$$\pi_k : \prod_{i\in\mathbb{Z}_{>0}} \mathcal{V}^{\nu}_{\mathbb{S}'_i\times\mathbb{S}_i\times\mathcal{U}_i} \to \mathcal{V}^{\nu}_{\mathbb{S}'_k\times\mathbb{S}_k\times\mathcal{U}_k}$$

is continuous for each $k \in \mathbb{Z}_{>0}$.

Properties of this topology:

(i) It is Hausdorff (as the product of Hausdorff spaces is Hausdorff).

(ii) It is complete (as the product of complete spaces is complete).

(iii) It is separable (as the countable product of separable spaces is separable).

(iv) It is Suslin (as the countable product of Suslin spaces is Suslin).

For any open subset $\mathcal{W} \subseteq \mathbb{T}\times\mathbb{T}\times M$, we will define the presheaf of sets over $\mathcal{W}$, denoted by $\mathscr{G}^{\nu}_{\mathrm{LI}}(\mathbb{T};TM)(\mathcal{W})$. Let $\mathcal{W}$ be covered by a family $\{\mathbb{S}'_i\times\mathbb{S}_i\times\mathcal{U}_i\}_{i\in\mathbb{Z}_{>0}}$ that is flow admissible for each $i \in \mathbb{Z}_{>0}$. We assign

$$\mathscr{G}^{\nu}_{\mathrm{LI}}(\mathbb{T};TM)(\mathcal{W}) \subseteq \prod_{i\in\mathbb{Z}_{>0}} \mathcal{V}^{\nu}_{\mathbb{S}'_i\times\mathbb{S}_i\times\mathcal{U}_i}$$

consisting of all sequences $(X_i)_{i\in\mathbb{Z}_{>0}}$, $X_i \in \mathcal{V}^{\nu}_{\mathbb{S}'_i\times\mathbb{S}_i\times\mathcal{U}_i}$ for which $\Phi^{X_j}|_R = \Phi^{X_k}|_R$ whenever $R = (\mathbb{S}'_j\times\mathbb{S}_j\times\mathcal{U}_j)\cap(\mathbb{S}'_k\times\mathbb{S}_k\times\mathcal{U}_k)$ for some $j,k\in\mathbb{Z}_{>0}$. For convenience, we call this the **"overlap condition."**

Lemma 4.5. *Let $\mathbb{S}'\times\mathbb{S}\times\mathcal{U}$, $\tilde{\mathbb{S}}'\times\tilde{\mathbb{S}}\times\tilde{\mathcal{U}} \subseteq \mathbb{T}\times\mathbb{T}\times M$ be flow admissible with nonempty intersection. Let $X \in \mathcal{V}^{\nu}_{\mathbb{S}'\times\mathbb{S}\times\mathcal{U}}$ and $Y \in \mathcal{V}^{\nu}_{\tilde{\mathbb{S}}'\times\tilde{\mathbb{S}}\times\tilde{\mathcal{U}}}$ satisfy the overlap condition. Then there exists a $Z \in \mathcal{V}^{\nu}_{(\mathbb{S}'\times\mathbb{S}\times\mathcal{U})\cup(\tilde{\mathbb{S}}'\times\tilde{\mathbb{S}}\times\tilde{\mathcal{U}})}$ such that $\Phi^Z|_{\mathbb{S}'\times\mathbb{S}\times\mathcal{U}} = \Phi^X$ and $\Phi^Z|_{\tilde{\mathbb{S}}'\times\tilde{\mathbb{S}}\times\tilde{\mathcal{U}}} = \Phi^Y$.*

Proof. Denote $(\mathbb{S}'\times\mathbb{S}\times\mathcal{U})\cap(\tilde{\mathbb{S}}'\times\tilde{\mathbb{S}}\times\tilde{\mathcal{U}}) = R$. Since $X \in \mathcal{V}^{\nu}_{\mathbb{S}'\times\mathbb{S}\times\mathcal{U}}$ and $Y \in \mathcal{V}^{\nu}_{\tilde{\mathbb{S}}'\times\tilde{\mathbb{S}}\times\tilde{\mathcal{U}}}$, the flow for X is the mapping

$$\begin{aligned} \Phi^X : \mathbb{S}'\times\mathbb{S}\times\mathcal{U} &\to M \\ (t,t_0,x_0) &\mapsto \xi(t), \end{aligned}$$

where ξ is a locally absolutely continuous curve $\xi : \mathbb{I}\subseteq\mathbb{T}\to M$ satisfying $\xi(t_0) = x_0$ for every $(t_0,x_0)\in\mathbb{S}\times\mathcal{U}$ and $\xi'(t) = X(t,\xi(t))$ for almost every $t\in\mathbb{S}'$; the flow for Y is the mapping

$$\begin{aligned} \Phi^Y : \tilde{\mathbb{S}}'\times\tilde{\mathbb{S}}\times\tilde{\mathcal{U}} &\to M \\ (t,t_0,x_0) &\mapsto \eta(t), \end{aligned}$$

where η is a locally absolutely continuous curve $\eta : \mathbb{I}'\subseteq\mathbb{T}\to M$ satisfying $\eta(t_0) = x_0$ for every $(t_0,x_0)\in\tilde{\mathbb{S}}\times\tilde{\mathcal{U}}$ and $\eta'(t) = Y(t,\eta(t))$ for almost every $t\in\tilde{\mathbb{S}}'$.

Since X and Y satisfy the overlap condition, $\Phi^X|_R = \Phi^Y|_R$, i.e., $\xi(t) = \eta(t)$ for each $(t, t_0, x_0) \in R$. Now let Ψ be a flow such that

$$\begin{aligned} \Psi : (\mathbb{S}' \times \mathbb{S} \times \mathcal{U}) \cup (\tilde{\mathbb{S}}' \times \tilde{\mathbb{S}} \times \tilde{\mathcal{U}}) &\to M \\ (t, t_0, x_0) &\mapsto \zeta(t), \end{aligned}$$

and $\Psi|_{\mathbb{S}' \times \mathbb{S} \times \mathcal{U}} = \Phi^X$ and $\Psi|_{\tilde{\mathbb{S}}' \times \tilde{\mathbb{S}} \times \tilde{\mathcal{U}}} = \Phi^Y$, i.e., ζ is a locally absolutely continuous curve $\zeta : \mathbb{T}' \subseteq \mathbb{T} \to M$ satisfying

(a) if $(t_0, x_0) \in \mathbb{S} \times \mathcal{U}$, then $\zeta(t_0) = x_0$ and $\zeta(t) = \xi(t)$ for almost every $t \in \mathbb{S}'$;

(b) if $(t_0, x_0) \in \tilde{\mathbb{S}} \times \tilde{\mathcal{U}}$, then $\zeta(t_0) = x_0$ and $\zeta(t) = \eta(t)$ for almost every $t \in \tilde{\mathbb{S}}'$.

Then let Z be a vector field such that $\zeta'(t) = Z(t, \zeta(t))$ for almost every $t \in \tilde{\mathbb{T}}'$. Then it is obvious that Ψ is the flow for Z such that $\Psi|_{\mathbb{S}' \times \mathbb{S} \times \mathcal{U}} = \Phi^X$ and $\Psi|_{\tilde{\mathbb{S}}' \times \tilde{\mathbb{S}} \times \tilde{\mathcal{U}}} = \Phi^Y$. □

Lemma 4.6. *Let $\mathbb{S}' \times \mathbb{S} \times \mathcal{U}$, $\tilde{\mathbb{S}}' \times \tilde{\mathbb{S}} \times \tilde{\mathcal{U}} \subseteq \mathbb{T} \times \mathbb{T} \times M$ be flow admissible and such that $\tilde{\mathbb{S}}' \times \tilde{\mathbb{S}} \times \tilde{\mathcal{U}} \subseteq \mathbb{S}' \times \mathbb{S} \times \mathcal{U}$. Then the map*

$$\rho_{\mathbb{S}' \times \mathbb{S} \times \mathcal{U}, \tilde{\mathbb{S}}' \times \tilde{\mathbb{S}} \times \tilde{\mathcal{U}}} : \mathcal{V}^\nu_{\mathbb{S}' \times \mathbb{S} \times \mathcal{U}} \to \mathcal{V}^\nu_{\tilde{\mathbb{S}}' \times \tilde{\mathbb{S}} \times \tilde{\mathcal{U}}}$$

given by

$$\rho_{\mathbb{S}' \times \mathbb{S} \times \mathcal{U}, \tilde{\mathbb{S}}' \times \tilde{\mathbb{S}} \times \tilde{\mathcal{U}}}(X) = X$$

is an homeomorphism onto its image.

Proof. Let $X \in \mathcal{V}^\nu_{\mathbb{S}' \times \mathbb{S} \times \mathcal{U}}$, i.e., $\mathbb{S}' \times \mathbb{S} \times \mathcal{U} \subseteq D_X$. Then $\tilde{\mathbb{S}}' \times \tilde{\mathbb{S}} \times \tilde{\mathcal{U}} \subseteq D_X$, whence $\mathcal{V}^\nu_{\mathbb{S}' \times \mathbb{S} \times \mathcal{U}} \subseteq \mathcal{V}^\nu_{\tilde{\mathbb{S}}' \times \tilde{\mathbb{S}} \times \tilde{\mathcal{U}}}$. Therefore, $\rho_{\mathbb{S}' \times \mathbb{S} \times \mathcal{U}, \tilde{\mathbb{S}}' \times \tilde{\mathbb{S}} \times \tilde{\mathcal{U}}}$ is an inclusion map, hence a homeomorphism onto its image. □

Theorem 4.7. *$\mathscr{G}^\nu_{LI}(\mathbb{T}; TM)(\mathcal{W})$ is unique up to homeomorphisms, i.e., it is independent of the choices of the covers for $\mathcal{W}$.*

Proof. Let $\{\mathbb{S}'_i \times \mathbb{S}_i \times \mathcal{U}_i\}_{i \in \mathbb{Z}_{>0}}$ and $\{\tilde{\mathbb{S}}'_j \times \tilde{\mathbb{S}}_j \times \tilde{\mathcal{U}}_j\}_{j \in \mathbb{Z}_{>0}}$ be two flow admissible open covers of $\mathcal{W}$, and let $\mathcal{P} \subseteq \prod_{i \in \mathbb{Z}_{>0}} \mathcal{V}^\nu_{\mathbb{S}'_i \times \mathbb{S}_i \times \mathcal{U}_i}$ and $\mathcal{Q} \subseteq \prod_{j \in \mathbb{Z}_{>0}} \mathcal{V}^\nu_{\tilde{\mathbb{S}}'_j \times \tilde{\mathbb{S}}_j \times \tilde{\mathcal{U}}_j}$ be the subsets that satisfy the overlap condition. Consider the map

$$\begin{aligned} T : \mathcal{P} &\to \mathcal{Q} \\ (X_1, X_2, \ldots) &\mapsto (Y_1, Y_2, \ldots), \end{aligned}$$

where $X_i \in \mathcal{V}^\nu_{\mathbb{S}'_i \times \mathbb{S}_i \times \mathcal{U}_i}$, $i \in \mathbb{Z}_{>0}$, and $Y_j \in \mathcal{V}^\nu_{\tilde{\mathbb{S}}'_j \times \tilde{\mathbb{S}}_j \times \tilde{\mathcal{U}}_j}$, $j \in \mathbb{Z}_{>0}$, satisfy the overlap condition, and if $R = (\mathbb{S}'_k \times \mathbb{S}_k \times \mathcal{U}_k) \cap (\tilde{\mathbb{S}}'_l \times \tilde{\mathbb{S}}_l \times \tilde{\mathcal{U}}_l) \neq \varnothing$, then $\Phi^{X_k}|_R = \Phi^{Y_l}|_R$. Consider the map

$$\begin{aligned} T' : \mathcal{Q} &\to \mathcal{P} \\ (Y_1, Y_2, \ldots) &\mapsto (X_1, X_2, \ldots), \end{aligned}$$

where $X_i \in \mathcal{V}^{\nu}_{\mathbb{S'}_i \times \mathbb{S}_i \times \mathcal{U}_i}$, $i \in \mathbb{Z}_{>0}$ and $Y_j \in \mathcal{V}^{\nu}_{\tilde{\mathbb{S}'}_j \times \tilde{\mathbb{S}}_j \times \tilde{\mathcal{U}}_j}$, $j \in \mathbb{Z}_{>0}$ satisty the overlap condition, and if $R = (\mathbb{S'}_k \times \mathbb{S}_k \times \mathcal{U}_k) \cap (\tilde{\mathbb{S}'}_l \times \tilde{\mathbb{S}}_l \times \tilde{\mathcal{U}}_l) \neq \varnothing$, then $\Phi^{X_k}|_R = \Phi^{Y_l}|_R$.

By the overlap condition, $T \circ T' = T' \circ T = \mathrm{Id}$, thus T is an isomorphism with $T^{-1} = T'$. Now we need to show that T is a homeomorphism.

Let $\{\tilde{\mathbb{S}'}_{n_l} \times \tilde{\mathbb{S}}_{n_l} \times \tilde{\mathcal{U}}_{n_l}\}_{l \in \mathbb{Z}_{>0}}$ be an open cover for $\mathbb{S'}_k \times \mathbb{S}_k \times \mathcal{U}_k$ for some $n_l \in \{1, 2, ...\}$, such that $(\mathbb{S'}_k \times \mathbb{S}_k \times \mathcal{U}_k) \cap (\tilde{\mathbb{S}'}_{n_l} \times \tilde{\mathbb{S}}_{n_l} \times \tilde{\mathcal{U}}_{n_l}) \neq \varnothing$ for each $l \in \mathbb{Z}_{>0}$. Denote $R_{kl} = (\mathbb{S'}_k \times \mathbb{S}_k \times \mathcal{U}_k) \cap (\tilde{\mathbb{S}'}_{n_l} \times \tilde{\mathbb{S}}_{n_l} \times \tilde{\mathcal{U}}_{n_l})$. Consider the following diagram

$$\begin{tikzcd}
\prod_{i \in \mathbb{Z}_{>0}} \mathcal{V}^{\nu}_{\mathbb{S'}_i \times \mathbb{S}_i \times \mathcal{U}_i} \supseteq \mathcal{P} \arrow[r, "\pi_k"] \arrow[d, dashed, shift left, "\phi"] & \pi_k(\mathcal{P}) \subseteq \mathcal{V}^{\nu}_{\mathbb{S'}_k \times \mathbb{S}_k \times \mathcal{U}_k} \\
\prod_{j \in \mathbb{Z}_{>0}} \mathcal{V}^{\nu}_{\tilde{\mathbb{S}'}_j \times \tilde{\mathbb{S}}_j \times \tilde{\mathcal{U}}_j} \supseteq \mathcal{Q} \arrow[u, dashed, shift left, "\phi'"] \arrow[ur, "\psi"] &
\end{tikzcd}$$

where ψ is the map

$$\begin{array}{rcl}
\psi : \quad \mathcal{Q} & \to & \pi_k(\mathcal{P}) \subseteq \mathcal{V}^{\nu}_{\mathbb{S'}_k \times \mathbb{S}_k \times \mathcal{U}_k} \\
(X_1, X_2, ...) & \mapsto & Y
\end{array}$$

such that $Y \in \mathcal{V}^{\nu}_{\mathbb{S'}_k \times \mathbb{S}_k \times \mathcal{U}_k}$ satisfies $\Phi^Y|_{R_{kl}} = \Phi^{X_{n_l}}|_{R_{kl}}$ for all $l \in \mathbb{Z}_{>0}$. Lemma 4.5 shows that ψ is well-defined.

We then claim that ψ is continuous. To prove this claim, consider the following diagram:

$$\begin{tikzcd}
\prod_{j \in \mathbb{Z}_{>0}} \mathcal{V}^{\nu}_{\tilde{\mathbb{S}'}_j \times \tilde{\mathbb{S}}_j \times \tilde{\mathcal{U}}_j} \supseteq \mathcal{Q} \arrow[r, "\pi'_k"] \arrow[rr, bend left, "T"] \arrow[drr, "\psi"'] & \mathcal{V}^{\nu}_{\tilde{\mathbb{S}'}_{n_l} \times \tilde{\mathbb{S}}_{n_l} \times \tilde{\mathcal{U}}_{n_l}} \arrow[r, "\rho_{\tilde{\mathbb{S}'}_{n_l} \times \tilde{\mathbb{S}}_{n_l} \times \tilde{\mathcal{U}}_{n_l}, R_{kl}}"] & \mathcal{V}^{\nu}_{R_{kl}} \\
& & \mathcal{V}^{\nu}_{\mathbb{S'}_k \times \mathbb{S}_k \times \mathcal{U}_k} \arrow[u, "\rho_{\mathbb{S'}_k \times \mathbb{S}_k \times \mathcal{U}_k, R_{kl}}"']
\end{tikzcd}$$

Let us describe the map $\rho_{\mathbb{S'}_k \times \mathbb{S}_k \times \mathcal{U}_k, R_{kl}}$,

$$\begin{array}{rcl}
\rho_{\mathbb{S'}_k \times \mathbb{S}_k \times \mathcal{U}_k, R_{kl}} : \mathcal{V}^{\nu}_{\mathbb{S'}_k \times \mathbb{S}_k \times \mathcal{U}_k} & \to & \mathcal{V}^{\nu}_{R_{kl}} \\
X & \mapsto & Y,
\end{array}$$

where $\Phi^X|_{R_{kl}} = \Phi^Y$. It is clear that this diagram commutes. Let $\mathcal{O} \subseteq \mathrm{Im}(\psi) \subseteq \mathcal{V}^{\nu}_{\mathbb{S'}_k \times \mathbb{S}_k \times \mathcal{U}_k}$ be open. Since $\rho_{\mathbb{S'}_k \times \mathbb{S}_k \times \mathcal{U}_k, R_{kl}}$ is an open map, $\rho_{\mathbb{S'}_k \times \mathbb{S}_k \times \mathcal{U}_k, R_{kl}}(\mathcal{O})$ is open in $\mathcal{V}^{\nu}_{R_{kl}}$. Denote $T := \rho_{\tilde{\mathbb{S}'}_{n_l} \times \tilde{\mathbb{S}}_{n_l} \times \tilde{\mathcal{U}}_{n_l}, R_{kl}} \circ \pi'_k$. Since $\rho_{\tilde{\mathbb{S}'}_{n_l} \times \tilde{\mathbb{S}}_{n_l} \times \tilde{\mathcal{U}}_{n_l}, R_{kl}}$ and π'_k are both continuous, so is T. Then there exists an open set $U \subseteq \mathcal{Q}$ such that $T(U) \subseteq \rho_{\mathbb{S'}_k \times \mathbb{S}_k \times \mathcal{U}_k, R_{kl}}(\mathcal{O})$. Since for any map $f : A \to B$, $D \subseteq f^{-1}(f(D))$ for all subsets $D \subseteq A$, we have

$$U \subseteq T^{-1} \circ T(U) \subseteq T^{-1} \circ \rho_{\mathbb{S'}_k \times \mathbb{S}_k \times \mathcal{U}_k, R_{kl}}(\mathcal{O}).$$

Since $T = \rho_{\mathbb{S}'_k\times\mathbb{S}_k\times\mathcal{U}_k,R_{kl}} \circ \psi$, $T^{-1} = \psi^{-1} \circ (\rho_{\mathbb{S}'_k\times\mathbb{S}_k\times\mathcal{U}_k,R_{kl}})^{-1}$. Hence

$$\begin{aligned}
\psi(U) &\subseteq \psi \circ T^{-1} \circ \rho_{\mathbb{S}'_k\times\mathbb{S}_k\times\mathcal{U}_k,R_{kl}}(\mathcal{O}) \\
&= \psi \circ (\psi^{-1} \circ (\rho_{\mathbb{S}'_k\times\mathbb{S}_k\times\mathcal{U}_k,R_{kl}})^{-1}) \circ \rho_{\mathbb{S}'_k\times\mathbb{S}_k\times\mathcal{U}_k,R_{kl}}(\mathcal{O}) \\
&= (\psi \circ \psi^{-1}) \circ [(\rho_{\mathbb{S}'_k\times\mathbb{S}_k\times\mathcal{U}_k,R_{kl}})^{-1} \circ \rho_{\mathbb{S}'_k\times\mathbb{S}_k\times\mathcal{U}_k,R_{kl}}](\mathcal{O}) \\
&= (\rho_{\mathbb{S}'_k\times\mathbb{S}_k\times\mathcal{U}_k,R_{kl}})^{-1} \circ \rho_{\mathbb{S}'_k\times\mathbb{S}_k\times\mathcal{U}_k,R_{kl}}(\mathcal{O}) = \mathcal{O},
\end{aligned}$$

provided that $(\rho_{\mathbb{S}'_k\times\mathbb{S}_k\times\mathcal{U}_k,R_{kl}})^{-1} \circ \rho_{\mathbb{S}'_k\times\mathbb{S}_k\times\mathcal{U}_k,R_{kl}}(\mathcal{O}) = \mathrm{Id}(\mathcal{O})$. Indeed, this is true under the condition that $\rho_{\mathbb{S}'_k\times\mathbb{S}_k\times\mathcal{U}_k,R_{kl}}(\mathcal{O})$ is injective. Hence ψ is continuous.

Since ι, ι', π_k, ψ are continuous, by the universal property of products and subspaces, there exist ϕ and ϕ' such that

$$\psi \circ \iota' \circ \phi = \pi_k \circ \iota \text{ and} \tag{4.1}$$

$$\pi_k \circ \iota \circ \phi' = \psi \circ \iota'. \tag{4.2}$$

Now we need to show that $T = \phi$ and $T' = \phi'$. Indeed, T has the property that satisfies (4.1), hence $T = \phi$ by the uniqueness of the map ϕ. Similarly, T' has the property that satisfies (4.2), hence $T = \phi'$ by the uniqueness of the map ϕ'. □

4.3. Category of time-varying local flows

We have thoroughly developed in Section 4.1 and 4.2 our presheaf theoretic notion of what a time-dependent vector field is. In this section we begin to develop the presheaf point of view for flows. What we do in this section is develop the space that will be the codomain of the exponential map. We do this by developing a vector field independent theory of flows that is analogous to our theory of vector fields in Section 4.1, we develop a notion of "category of flows" that we will use as the basis for defining the "flow presheaf."

Definition 4.8 (C^ν-local flow)**.** Let $m \in \mathbb{Z}_{\geq 0}$, let $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m+m', \infty, \omega, \mathrm{hol}\}$, and let $r \in \{\infty, \omega, \mathrm{hol}\}$, as required. A "$C^\nu$-local flow" is a quintuple $(\mathbb{T}', \mathbb{T}, M, \mathcal{U}, \Phi)$ where

(i) $\mathbb{T} \subseteq \mathbb{T}' \subseteq \mathbb{R}$ are intervals,

(ii) M is a C^r-manifold,

(iii) $\mathcal{U} \subseteq M$ is open, and

(iv) $\Phi : \mathbb{T}' \times \mathbb{T} \times \mathcal{U} \to M$ is such that:

  (a) $\Phi(t_0, t_0, x) = x$, $(t_0, t_0, x) \in \mathbb{T}' \times \mathbb{T} \times \mathcal{U}$;

  (b) $\Phi(t_2, t_1, \Phi(t_1, t_0, x)) = \Phi(t_2, t_0, x)$, $t_0, t_1, \in \mathbb{T}$, $t_2 \in \mathbb{T}'$ $x \in \mathcal{U}$;

(c) the map $x \mapsto \Phi(t_1, t_0, x)$ is a C^ν-diffeomorphism onto its image for every $t_0 \in \mathbb{T}$ and $t_1 \in \mathbb{T}'$;

(d) the map $t_0 \mapsto \Phi_{t_1,t_0} \in C^\nu(\mathcal{U}; M)$ is continuous for a fixed $t_1 \in \mathbb{T}'$, and the map $t_1 \mapsto \Phi_{t_1,t_0} \in C^\nu(\mathcal{U}; M)$ is absolutely continuous for a fixed $t_0 \in \mathbb{T}$, where $\Phi_{t_1,t_0}(x) = \Phi(t_1, t_0, x)$.

We denote

$$\mathrm{LocFlow}^\nu(\mathbb{S}'; \mathbb{S}; \mathcal{U}) = \{\Phi : \mathbb{S}' \times \mathbb{S} \times \mathcal{U} \to M \mid (\mathbb{S}', \mathbb{S}, M, \mathcal{U}, \Phi) \text{ is a } C^\nu\text{-local flow}\},$$

We will work with a category whose objects are the sets of local flows $\mathrm{LocFlow}^\nu(\mathbb{S}'; \mathbb{S}; \mathcal{U})$. We shall think of this category as a subcategory of the category $\mathscr{Top}$ of topological spaces and continuous maps. In particular, we shall place topologies on the spaces $\mathrm{LocFlow}^\nu(\mathbb{S}'; \mathbb{S}; \mathcal{U})$. This we do as follows.

Let M be a C^r-manifold and let $\mathbb{T} \subseteq \mathbb{R}$ be an interval. Let $\mathbb{S}, \mathbb{S}' \subseteq \mathbb{T}$ be subintervals, $\mathbb{S} \subseteq \mathbb{S}'$, and let $\mathcal{U} \subseteq M$ be open. Note that a local flow $\Phi \in \mathrm{LocFlow}^\nu(\mathbb{S}'; \mathbb{S}; \mathcal{U})$ defines absolutely continuous curves with respect to its final time,

$$\hat{\Phi} \in \mathrm{AC}(\mathbb{S}'; \mathrm{C}^0(\mathbb{S}; \mathrm{C}^\nu(\mathcal{U}; \mathrm{M})))$$

by $\hat{\Phi}(t)(t_0)(x) = \Phi(t, t_0, x)$. Then we topologise $\mathrm{LocFlow}^\nu(\mathbb{S}'; \mathbb{S}; \mathcal{U})$ as follows.

First, we give $C^0(\mathbb{S}; C^\nu(\mathcal{U}))$ the topology defined by the seminorms

$$p^\nu_{K,\mathbb{I}}(g) = \sup\{p^\nu_K \circ g(t_0) \mid t_0 \in \mathbb{I})\},$$

$\mathbb{I} \subseteq \mathbb{S}$ a compact interval, p^ν_K is the appropriate seminorm defined by (2.3) for $C^\nu(\mathcal{U})$.

Second, we give $\mathrm{AC}(\mathbb{S}'; \mathrm{C}^0(\mathbb{S}; \mathrm{C}^\nu(\mathcal{U})))$ the topology defined by the seminorms

$$q^\nu_{K,\mathbb{I},\mathbb{I}'}(g) = \max\{p^\nu_{K,\mathbb{I},\mathbb{I}',\infty}(g),\ \hat{p}^\nu_{K,\mathbb{I},\mathbb{I}',1}(g)\}$$

where

$$p^\nu_{K,\mathbb{I},\mathbb{I}',\infty}(g) = \sup\{p^\nu_{K,\mathbb{I}} \circ g(t) \mid t \in \mathbb{I}'\} \qquad \text{and} \qquad \hat{p}^\nu_{K,\mathbb{I},\mathbb{I}',1}(g) = \int_{\mathbb{I}'} p^\nu_{K,\mathbb{I}} \circ \frac{\mathrm{d}g}{\mathrm{d}t}(t)\,\mathrm{d}t,$$

$\mathbb{I} \subseteq \mathbb{S}$, $\mathbb{I}' \subseteq \mathbb{S}'$ compact intervals.

Finally, we give $\mathrm{AC}(\mathbb{S}'; \mathrm{C}^0(\mathbb{S}; \mathrm{C}^\nu(\mathcal{U}; \mathrm{M})))$ the initial topology associated with the mappings

$$\begin{aligned} \Psi_f : \mathrm{AC}(\mathbb{S}'; \mathrm{C}^0(\mathbb{S}; \mathrm{C}^\nu(\mathcal{U}; \mathrm{M}))) &\to \mathrm{AC}(\mathbb{S}'; \mathrm{C}^0(\mathbb{S}; \mathrm{C}^\nu(\mathcal{U}))) \\ \Phi &\mapsto f \circ \Phi, \end{aligned}$$

for $f \in C^\nu(M)$.

More explicitly for the topology of the space $\mathrm{AC}(\mathbb{S}'; \mathrm{C}^0(\mathbb{S}; \mathrm{C}^\nu(\mathcal{U})))$, given $f \in C^\nu(M)$ and $\Phi \in \mathrm{LocFlow}^\nu(\mathbb{S}'; \mathbb{S}; \mathcal{U})$,

$$p^\nu_{K,\mathbb{I},\mathbb{I}',\infty}(f \circ \Phi) = \sup\{p^\nu_K(f \circ \Phi_{t_1,t_0}) \mid (t_1, t_0) \in \mathbb{I}' \times \mathbb{I}\}$$

and

$$\begin{aligned}\hat{p}^{\nu}_{K,\mathbb{I},\mathbb{I}',1}(f\circ\Phi) &= \int_{\mathbb{I}'} p^{\nu}_{K,\mathbb{I}}\left(\frac{\mathrm{d}}{\mathrm{d}t}(f\circ\Phi_{t,t_0})\right)\,\mathrm{d}t\\ &= \int_{\mathbb{I}'} p^{\nu}_{K,\mathbb{I}}\left(\langle \mathrm{d}f(\Phi_{t,t_0}),\frac{\mathrm{d}}{\mathrm{d}t}\Phi_{t,t_0}\rangle\right)\,\mathrm{d}t\\ &= \int_{\mathbb{I}'} \sup\left\{p^{\nu}_{K}\left(\langle \mathrm{d}f(\Phi_{t,t_0}),\frac{\mathrm{d}}{\mathrm{d}t}\Phi_{t,t_0}\rangle\right)\;\middle|\; t_0\in\mathbb{I}\right\}\,\mathrm{d}t,\end{aligned}$$

where $K\subseteq M$ and $\mathbb{I}\subseteq\mathbb{I}'\subseteq\mathbb{T}$ are compact. The topology defined above for flows is important as it allows us to consider such spaces in the category of topological spaces, which allows us to construct the presheaf of local flows.

Definition 4.9 (Morphisms of sets of local flows)**.** Let $m\in\mathbb{Z}_{\geq 0}$, let $m'\in\{0,\mathrm{lip}\}$, let $\nu\in\{m+m',\infty,\omega,\mathrm{hol}\}$, and let $r\in\{\infty,\omega,\mathrm{hol}\}$, as required. Let M and N be C^r-manifolds, and let $\mathbb{T}\subseteq\mathbb{R}$ be a time interval. Let $\mathbb{S}'\times\mathbb{S}\times\mathcal{U}\subseteq\mathbb{T}\times\mathbb{T}\times M$ and $\tilde{\mathbb{S}}'\times\tilde{\mathbb{S}}\times\tilde{\mathcal{U}}\subseteq\mathbb{T}\times\mathbb{T}\times N$ be flow admissible. A mapping

$$\alpha:\mathrm{LocFlow}^{\nu}(\mathbb{S}';\mathbb{S};\mathcal{U})\to\mathrm{LocFlow}^{\nu}(\tilde{\mathbb{S}}';\tilde{\mathbb{S}};\tilde{\mathcal{U}})$$

is a morphism of sets of C^{ν}-local flows if

(i) α is continuous, and

(ii) there exist mappings $\tau_\alpha:\mathbb{S}\to\tilde{\mathbb{S}}$, $\tau'_\alpha:\mathbb{S}'\to\tilde{\mathbb{S}}'$, and $\phi_\alpha\in C^{\nu}(M;N)$ such that:

  (a) τ_α and τ'_α are the restrictions to $\tilde{\mathbb{S}}$ and $\tilde{\mathbb{S}}'$ of nonconstant affine mappings, respectively;

  (b) $\phi_\alpha(\mathcal{U})\subseteq\mathcal{V}$;

  (c) $\phi_\alpha\circ\Phi(t_1,t_0,x)=\alpha(\Phi)(\tau'_\alpha(t_1),\tau_\alpha(t_0),\phi_\alpha(x))$ for every $\Phi\in\mathrm{LocFlow}^{\nu}(\mathbb{T};\mathcal{U})$ for $t_0,t_1\in\mathbb{T}$ and $x\in\mathcal{U}$.

Let us give some examples of morphisms of sets of flows.

Example 4.10. (Morphisms of sets of local time-varying sections) If $N=M$, and if $\tilde{\mathbb{S}}'\times\tilde{\mathbb{S}}\times\tilde{\mathcal{U}}\subseteq\mathbb{S}'\times\mathbb{S}\times\mathcal{U}$, then the "restriction morphism"

$$\rho_{\mathbb{S}'\times\mathbb{S}\times\mathcal{U},\tilde{\mathbb{S}}'\times\tilde{\mathbb{S}}\times\tilde{\mathcal{U}}}:\mathrm{LocFlow}^{\nu}(\mathbb{S}';\mathbb{S};\mathcal{U})\to\mathrm{LocFlow}^{\nu}(\tilde{\mathbb{S}}';\tilde{\mathbb{S}};\tilde{\mathcal{U}})$$

given by

$$\rho_{\mathbb{S}'\times\mathbb{S}\times\mathcal{U},\tilde{\mathbb{S}}'\times\tilde{\mathbb{S}}\times\tilde{\mathcal{U}}}(\Phi)(t_1,t_0,x)=\Phi(t_1,t_0,x),\quad (t_1,t_0,x)\in\tilde{\mathbb{S}}'\times\tilde{\mathbb{S}}\times\tilde{\mathcal{U}}.$$

In this case we have "$\tau_\alpha=\iota_{\mathbb{S}',\tilde{\mathbb{S}}'}$" and "$\phi_\alpha=\iota_{E|\mathcal{U},E|\mathcal{V}}$" where $\iota_{\mathbb{T},\mathbb{S}}$ and $\iota_{E|\mathcal{U},E|\tilde{\mathcal{U}}}$ are the inclusions.

4.4. Presheaves of time-varying flows

The manner in which we construct the presheaves of time-varying flows is similar to the manner in which we constructed the presheaves of vector fields in Section 4.2. For any open flow admissible subset $\mathcal{W} \subseteq \mathbb{T} \times \mathbb{T} \times M$, let $\{\mathbb{S}'_i \times \mathbb{S}_i \times \mathcal{U}_i\}_{i \in \mathbb{Z}_{>0}}$ be an open cover of $\mathcal{W}$ and is flow admissible for each $i \in \mathbb{Z}_{>0}$. We will define the presheaf of sets over $\mathcal{W}$, denoted by $\mathscr{LF}^\nu(\mathbb{T};\mathbb{T};M)(\mathcal{W})$, the subset of $\prod_{i \in \mathbb{Z}_{>0}} \mathrm{LocFlow}^\nu(\mathbb{S}'_i;\mathbb{S}_i;\mathcal{U}_i)$ consisting of all sequences $(\Phi_i)_{i \in \mathbb{Z}_{>0}}$, $\Phi_i \in \mathrm{LocFlow}^\nu(\mathbb{S}'_i;\mathbb{S}_i;\mathcal{U}_i)$ for which $\Phi_j|_R = \Phi_k|_R$ whenever $R = (\mathbb{S}'_j \times \mathbb{S}_j \times \mathcal{U}_j) \cap (\mathbb{S}'_k \times \mathbb{S}_k \times \mathcal{U}_k)$ for some $j, k \in \mathbb{Z}_{>0}$.

Theorem 4.11. *$\mathscr{LF}^\nu(\mathbb{T};\mathbb{T};M)(\mathcal{W})$ is unique up to homeomorphisms, i.e., it is independent of the choices of the covers for $\mathcal{W}$.*

Proof. Same argument as Theorem 4.7. □

4.5. The exponential map

To establish the exponential map properly, we will use the language of category theory. Consider the following commutative diagram in the category of topological spaces:

$$\begin{tikzcd}
\mathscr{G}^\nu_{\mathrm{LI}}(\mathbb{T};TM)(\mathcal{W}) \arrow[r, "\exp"] \arrow[d, "\pi_j"'] \arrow[dr, "\overline{\exp}_{\mathbb{S}'_j \times \mathbb{S}_j \times \mathcal{U}_j} \circ \pi_j"] & \mathscr{LF}^\nu(\mathbb{T};\mathbb{T};M)(\mathcal{W}) \arrow[d, "\mathrm{pr}_j"] \\
\mathcal{V}^\nu_{\mathbb{S}'_j \times \mathbb{S}_j \times \mathcal{U}_j} \supseteq Q_j \arrow[r, "\overline{\exp}_{\mathbb{S}'_j \times \mathbb{S}_j \times \mathcal{U}_j}"] & P_j \subseteq \mathrm{LocFlow}^\nu(\mathbb{S}'_j;\mathbb{S}_j;\mathcal{U}_j)
\end{tikzcd}$$

where π_j and pr_j are the canonical projections hence continuous. Suppose that $\overline{\exp}_{\mathbb{S}'_j \times \mathbb{S}_j \times \mathcal{U}_j}$ are continuous, then by the universal property of the subspace of product spaces, there exists a well-defined continuous mapping

$$\exp : \prod_{\mathrm{i} \in \mathbb{Z}_{>0}} \mathcal{V}^\nu_{\mathbb{S}'_{\mathrm{i}} \times \mathbb{S}_{\mathrm{i}} \times \mathcal{U}_{\mathrm{i}}} \supseteq \mathscr{G}^\nu_{\mathrm{LI}}(\mathbb{T};\mathrm{M})(\mathcal{W}) \longrightarrow \mathscr{LF}^\nu(\mathbb{T};\mathbb{T};M)(\mathcal{W}).$$

Moreover, we have an explicit description of this map by

$$\exp((\mathrm{X}_1, \ldots, \mathrm{X}_{\mathrm{k}}, \ldots)) = (\overline{\exp}_{\mathbb{S}'_1 \times \mathbb{S}_1 \times \mathcal{U}_1}(\mathrm{X}_1), \ldots, \overline{\exp}_{\mathbb{S}'_{\mathrm{k}} \times \mathbb{S}_{\mathrm{k}} \times \mathcal{U}_{\mathrm{k}}}(\mathrm{X}_{\mathrm{k}}), \ldots)$$

We have shown in Section 3.2 that, for each $(t_0, x_0) \in \mathbb{T} \times M$, there exist $\mathbb{S}', \mathbb{S} \subseteq \mathbb{T}$, $\mathcal{U} \subseteq M$ such that $(t_0, x_0) \in \mathbb{S} \times \mathcal{U}$, and an open subset $\mathcal{V}^\nu_{\mathbb{S}' \times \mathbb{S} \times \mathcal{U}} \subseteq \Gamma^\nu_{\mathrm{LI}}(\mathbb{T};TM)$ such that the map

$$\overline{\exp}_{\mathbb{S}' \times \mathbb{S} \times \mathcal{U}} : \mathcal{V}^\nu_{\mathbb{S}' \times \mathbb{S} \times \mathcal{U}} \to \mathrm{LocFlow}^\nu(\mathbb{S}';\mathbb{S};\mathcal{U})$$

$$X \mapsto \Phi^X \tag{4.1}$$

is well-defined. We will show that this map is a homeomorphism onto its image by showing it is a continuous map with an continuous inverse. Then the continuity of exp will follow by the following elementary lemma.

Lemma 4.12. *Let $\{A_i\}_{i\in\mathbb{Z}_{>0}}$ and $\{B_i\}_{i\in\mathbb{Z}_{>0}}$ be collections of topological spaces, let $f_i : A_i \to B_i$ be continuous maps. Then the map*

$$\prod_{i\in\mathbb{Z}_{>0}} f_i : \prod_{i\in\mathbb{Z}_{>0}} A_i \to \prod_{i\in\mathbb{Z}_{>0}} B_i$$

given by

$$\Big(\prod_{i\in\mathbb{Z}_{>0}} f_i\Big)(x_1, x_2, ...) = (f_1(x_1), f_2(x_2), ...)$$

is continuous.

We, then, consider the continuity on cubes $\mathbb{S}' \times \mathbb{S} \times \mathcal{U} \subseteq \mathbb{T} \times \mathbb{T} \times M$.

4.5.1. Continuity.

Proposition 4.13. *Let $m \in \mathbb{Z}_{\geq 0}$, let $\nu \in \{m, \infty, hol\}$, and let $r \in \{\infty, hol\}$, as required. Let M be a C^r-manifold and let $\mathbb{T} \subseteq \mathbb{R}$ be an interval. Let $\mathbb{S}$, $\mathbb{S}' \subseteq \mathbb{T}$ and $\mathcal{U} \subseteq M$ be open. The map*

$$\begin{aligned} \exp_{\mathbb{S}'\times\mathbb{S}\times\mathcal{U}} : \mathcal{N} \subseteq \mathcal{V}^{\nu}_{\mathbb{S}'\times\mathbb{S}\times\mathcal{U}} &\to LocFlow^{\nu}(\mathbb{S}'; \mathbb{S}; \mathcal{U}) \\ X &\mapsto \Phi^X \end{aligned}$$

is continuous.

Proof. Consider the following mappings

$$\mathcal{N} \subseteq \mathcal{V}^{\nu}_{\mathbb{S}'\times\mathbb{S}\times\mathcal{U}} \xrightarrow{\exp_{\mathbb{S}'\times\mathbb{S}\times\mathcal{U}}} \mathrm{LocFlow}^{\nu}(\mathbb{S}'; \mathbb{S}; \mathcal{U}) \xrightarrow{\Psi_f} \mathrm{AC}(\mathbb{S}'; \mathrm{C}^0(\mathbb{S}; \mathrm{C}^{\nu}(\mathcal{U}))).$$

Since $\mathrm{LocFlow}^{\nu}(\mathbb{S}'; \mathbb{S}; \mathcal{U})$ is equipped with the initial topology with Ψ_f, it is enough to show the continuity of $\Psi_f \circ \exp$ for a fixed $f \in C^{\nu}(M)$.

Let $\{K_i\}_{i\in\mathbb{Z}_{>0}} \subset M$ be compact neighborhoods of x_0 and such that $K_j \subset \mathrm{int}(K_{j+1})$ and $x_0 \in \mathrm{int}(K_1)$, and $M = \bigcup_{i\in\mathbb{Z}_{>0}} K_i$. Denote $K = \bigcap_{i\in\mathbb{Z}_{>0}} K_i$. Similarly, let $\{\mathbb{I}_i\}_{i\in\mathbb{Z}_{>0}}$ be compact neighborhoods of t_0 and such that $\mathbb{I}_j \subset \mathrm{int}(\mathbb{I}_{j+1})$ and $t_0 \in \mathrm{int}(\mathbb{I}_1)$, and $\mathbb{T} = \bigcup_{i\in\mathbb{Z}_{>0}} \mathbb{I}_i$.

For each $f \in C^{\nu}(M)$ and $X \in \mathcal{V}^{\nu}_{\mathbb{S}'\times\mathbb{S}\times\mathcal{U}}$, let $\mathcal{R}$ be a neighborhood of $\exp(X)$. Then there exist increasing sequences $\{i_1, i_2, ..., i_m\} \subset \mathbb{Z}_{>0}$, $\{j'_1, j'_2, ..., j'_n\} \subset \mathbb{Z}_{>0}$ and $\{j_1, j_2, ..., j_n\} \subset \mathbb{Z}_{>0}$ such that $\mathbb{I}_{j_k} \subseteq \mathbb{I}_{j'_k}$ for all $k \in \{1, 2, ..., n\}$ and such that

$$\bigcap_{k=1}^{m}\bigcap_{l=1}^{n}\Big\{\Phi \in \mathrm{LocFlow}^{\nu}(\mathbb{S}'; \mathbb{S}; \mathcal{U}) \;\big|\; q^{\nu}_{K_{i_k}, \mathbb{I}_{j_l}, \mathbb{I}_{j'_l}, f}(\Phi - \Phi^X) < r\Big\} \subseteq \mathcal{R}$$

forms a neighborhood of Φ^X.

Observe, $i_a < i_b$ implies $K_{i_a} \subset K_{i_b}$ which gives

$$(q^\nu_{K_{i_b},\mathbb{I}_{j_l},\mathbb{I}_{j'_l},f})^{-1}([0,r)) \subseteq (q^\nu_{K_{i_a},\mathbb{I}_{j_l},\mathbb{I}_{j'_l},f})^{-1}([0,r)),$$

and $j_a < j_b$ (thus $j'_a < j'_b$) implies $\mathbb{I}_{j_a} \subset \mathbb{I}_{j_b}$ and $\mathbb{I}_{j'_a} \subset \mathbb{I}_{j'_b}$, which gives

$$(q^\nu_{K_{i_k},\mathbb{I}_{j_b},\mathbb{I}_{j'_b},f})^{-1}([0,r)) \subseteq (q^\nu_{K_{i_k},\mathbb{I}_{j_a},\mathbb{I}_{j'_a},f})^{-1}([0,r)).$$

Observe that

$$\begin{array}{ccccccc} \mathbb{I}_{j_1} & \subseteq & \mathbb{I}_{j_2} & \subseteq & \dots & \subseteq & \mathbb{I}_{j_n} \\ \cap\!\!\!| & & \cap\!\!\!| & & \cap\!\!\!| & & \cap\!\!\!| \\ \mathbb{I}_{j'_1} & \subseteq & \mathbb{I}_{j'_2} & \subseteq & \dots & \subseteq & \mathbb{I}_{j'_n} \end{array}$$

and

$$K_{i_1} \subseteq K_{i_2} \subseteq \dots \subseteq K_{i_m}.$$

Let $K := K_{i_m}$, $\mathbb{I} := \mathbb{I}_{j_n}$ and $\mathbb{I}' := \mathbb{I}_{j'_n}$, then $\mathbb{I} \subseteq \mathbb{I}'$. Now, let

$$Q := \left\{\Phi \in \mathrm{LocFlow}^\nu(\mathbb{S}';\mathbb{S};\mathcal{U}) \mid q^\nu_{K,\mathbb{I},\mathbb{I}',f}(\Phi - \Phi^X) < r\right\}.$$

By (3.1-3.2), there exist a neighborhood $\mathcal{N}$ of X and a compact $K' \subseteq M$ such that $\Phi^Y(t_1,t_0,x) \in K'$ for all $(t_1,t_0,x) \in \mathbb{I}' \times \mathbb{I} \times K$ and $Y \in \mathcal{N}$, and that

$$\sup\left\{\int_{\mathbb{I}'} p^\nu_K\left(Xf(s,\Phi^X_{s,t_0}) - Xf(s,\Phi^Y_{s,t_0})\right)\mathrm{d}s \;\middle|\; t_0 \in \mathbb{I}\right\} \leq \frac{r}{2}$$

for all $Y \in \mathcal{N}$. Let

$$\mathcal{N}' := \left\{Y \in \mathcal{V}^\nu_{\mathbb{S}'\times\mathbb{S}\times\mathcal{U}} \mid p^\nu_{K',\mathbb{I}',f}(Y - X) < \frac{r}{2}\right\},$$

and denote $\mathcal{O} := \mathcal{N} \cap \mathcal{N}'$. We claim that $\exp_{\mathbb{S}'\times\mathbb{S}\times\mathcal{U}}(\mathcal{O}) \subseteq Q$. Indeed, for any $Y \in \mathcal{O}$,

$$\begin{aligned} & p^\nu_{K,\mathbb{I},\mathbb{I}',\infty}(f\circ\Phi^X - f\circ\Phi^Y) \\ &\quad = \sup\{p^\nu_K(f\circ(\Phi^X - \Phi^Y)(t_1,t_0)) \mid (t_1,t_0) \in \mathbb{I}'\times\mathbb{I}\} \\ &\quad = \sup\left\{p^\nu_K\left(\int_{|t_0,t_1|} Xf(s,\Phi^X_{s,t_0}) - Yf(s,\Phi^Y_{s,t_0})\,\mathrm{d}s\right) \;\middle|\; (t_1,t_0) \in \mathbb{I}'\times\mathbb{I}\right\} \\ &\quad \leq \sup\left\{p^\nu_K\left(\int_{|t_0,t_1|} Xf(s,\Phi^X_{s,t_0}) - Xf(s,\Phi^Y_{s,t_0})\,\mathrm{d}s\right) \;\middle|\; (t_1,t_0) \in \mathbb{I}'\times\mathbb{I}\right\} \\ &\qquad + \sup\left\{p^\nu_K\left(\int_{|t_0,t_1|} Xf(s,\Phi^Y_{s,t_0}) - Yf(s,\Phi^Y_{s,t_0})\,\mathrm{d}s\right) \;\middle|\; (t_1,t_0) \in \mathbb{I}'\times\mathbb{I}\right\} \\ &\quad \leq \sup\left\{\int_{\mathbb{I}'} p^\nu_K\left(Xf(s,\Phi^X_{s,t_0}) - Xf(s,\Phi^Y_{s,t_0})\right)\mathrm{d}s \;\middle|\; t_0 \in \mathbb{I}\right\} \end{aligned}$$

$$
\begin{aligned}
&\quad + \sup\left\{ p^{\nu}_{K'}\left(\int_{|t_0,t_1|} Xf(s,y) - Yf(s,y)\,\mathrm{d}s \right) \;\middle|\; (t_1,t_0) \in \mathbb{I}' \times \mathbb{I} \right\} \\
&\le \frac{r}{2} + \int_{\mathbb{I}'} p^{\nu}_{K'}(Xf(s,y) - Yf(s,y))\,\mathrm{d}s \qquad (\mathbb{I} \subseteq \mathbb{I}') \\
&\le \frac{r}{2} + p^{\nu}_{K',\mathbb{I}',f}(X - Y). \\
&< \frac{r}{2} + \frac{r}{2} = r
\end{aligned}
$$

and

$$
\begin{aligned}
&\hat{p}^{\nu}_{K,\mathbb{I},\mathbb{I}',1}(f \circ \Phi^X - f \circ \Phi^Y) \\
&= \int_{\mathbb{I}'} p^{\nu}_{K,\mathbb{I}}\left(\frac{d}{dt}(f \circ \Phi^X(t,t_0,x_0) - f \circ \Phi^Y(t,t_0,x_0)) \right)\mathrm{d}t \\
&= \int_{\mathbb{I}'} \sup\left\{ p^{\nu}_{K}\left(\langle df(\Phi^X_{t,t_0}), \frac{d}{dt}\Phi^X_{t,t_0}\rangle - \langle df(\Phi^Y_{t,t_0}), \frac{d}{dt}\Phi^Y_{t,t_0}\rangle \right) \;\middle|\; t_0 \in \mathbb{I} \right\}\mathrm{d}t \\
&= \int_{\mathbb{I}'} \sup\left\{ p^{\nu}_{K}\left(Xf(t,\Phi^X_{t,t_0}) - Yf(t,\Phi^Y_{t,t_0}) \right) \;\middle|\; t_0 \in \mathbb{I} \right\}\mathrm{d}t \\
&\le \int_{\mathbb{I}'} \sup\left\{ p^{\nu}_{K}\left(Xf(t,\Phi^X_{t,t_0}) - Xf(t,\Phi^Y_{t,t_0}) \right) \;\middle|\; t_0 \in \mathbb{I} \right\}\mathrm{d}t \\
&\quad + \int_{\mathbb{I}'} \sup\left\{ p^{\nu}_{K}\left(Xf(t,\Phi^Y_{t,t_0}) - Yf(t,\Phi^Y_{t,t_0}) \right) \;\middle|\; t_0 \in \mathbb{I} \right\}\mathrm{d}t \\
&\le \frac{r}{2} + \int_{\mathbb{I}'} p^{\nu}_{K'}(Xf(s,y) - Yf(s,y))\,\mathrm{d}s \\
&\le \frac{r}{2} + p^{\nu}_{K',\mathbb{I}',f}(X - Y). \\
&< \frac{r}{2} + \frac{r}{2} = r.
\end{aligned}
$$

Hence

$$
q^{\nu}_{K,\mathbb{I},\mathbb{I}',f}(f \circ \Phi^X - f \circ \Phi^Y) = \max\{ p^{\nu}_{K,\mathbb{I},\mathbb{I}',\infty}(f \circ \Phi^X - f \circ \Phi^Y),\ \hat{p}^{\nu}_{K,\mathbb{I},\mathbb{I}',1}(f \circ \Phi^X - f \circ \Phi^Y) \} < r
$$

for all $Y \in \mathcal{O}$. Therefore $\exp_{\mathbb{S}'\times\mathbb{S}\times\mathcal{U}}(Y) \in Q$, whence the continuity of $\Psi_f \circ \exp_{\mathbb{S}'\times\mathbb{S}\times\mathcal{U}}$ holds true. □

4.5.2. Openness. To show the openness, it is useful to consider the parameter-dependent local flows, i.e., the continuous maps

$$
\mathcal{P} \ni p \mapsto \Phi^p \in \mathrm{LocFlow}^{\nu}(\mathbb{S}';\mathbb{S};\mathcal{U}),
$$

where $\mathcal{P}$ is a topological space. We denote this set by $\mathrm{LocFlow}^{\nu}(\mathbb{S}';\mathbb{S};\mathcal{U};\mathcal{P})$.

We start by showing the following lemmata which helps us to establish the continuity of the inverse of (4.1).

Lemma 4.14. *Let $f \in C^\infty(M)$ and $p_0 \in \mathcal{P}$ be fixed. Define a map*

$$\begin{aligned} g : \mathbb{T} \times M &\to \mathbb{R} \\ (t, x) &\mapsto \left.\frac{d}{d\tau}\right|_{\tau = t} f \circ \Phi^{p_0}_{\tau, t_0}(x). \end{aligned}$$

Then, for $\gamma \in C^0(\mathbb{T}; M)$, the mapping

$$\begin{aligned} \Psi_{\mathbb{T},M,g} : C^0(\mathbb{T}; M) &\to L^1_{\mathrm{loc}}(\mathbb{T}; \mathbb{R}) \\ \gamma &\mapsto (s \mapsto g(s, \gamma(s))) \end{aligned}$$

is well-defined and continuous.

Proof. Denote $g_x : t \mapsto g(t, x)$ and $g_t : x \mapsto g(t, x)$. We first note that $g_x \in L^1_{\mathrm{loc}}(\mathbb{T}; \mathbb{R})$ since $f \circ \Phi^{p_0}_{\tau, t_0}$ is locally absolutely continuous. We first show that $t \mapsto g(t, \gamma(t))$ is measurable on $\mathbb{T}$. Note that

$$t \mapsto g(t, \gamma(s))$$

is measurable for each $s \in \mathbb{T}$ and that

$$s \mapsto g(t, \gamma(s)) \tag{4.2}$$

is continuous for each $t \in \mathbb{T}$ (this since both $x \mapsto g_t(x)$ and γ are continuous). Let $[a, b] \subseteq \mathbb{T}$ be compact, let $k \in \mathbb{Z}_{>0}$, and denote

$$t_{k,j} = a + \frac{j-1}{k}(b - a), \quad j \in \{1, \ldots, k+1\}.$$

Also denote

$$\mathbb{T}_{k,j} = [t_{k,j}, t_{k,j+1}), \quad j \in \{1, \ldots, k-1\},$$

and $\mathbb{T}_{k,k} = [t_{k,k}, t_{k,k+1}]$. Then define $g_k : \mathbb{T} \to \mathbb{R}$ by

$$g_k(t) = \sum_{j=1}^{k} g(t, \gamma(t_{k,j}))\chi_{t_{k,j}}.$$

Note that g_k is measurable, being a sum of products of measurable functions (Cohn, 2013, Proposition 2.1.7). By continuity of (4.2) for each $t \in \mathbb{T}$, we have

$$\lim_{k \to \infty} g_k(t) = g(t, \gamma(t)), \quad t \in [a, b],$$

showing that $t \mapsto g(t, \gamma(t))$ is measurable on $[a, b]$, as pointwise limits of measurable functions are measurable (Cohn, 2013, Proposition 2.1.5). Since the compact interval $[a, b] \subseteq \mathbb{T}$ is arbitrary, we conclude that $t \mapsto g(t, \gamma(t))$ is measurable on $\mathbb{T}$.

Let $\mathbb{S} \subseteq \mathbb{T}$ be compact and let $K \subseteq M$ be a compact set for which $\gamma(\mathbb{S}) \subseteq K$. Since $g_x \in L^1_{\mathrm{loc}}(\mathbb{T}; \mathbb{R})$, there exists $C \in \mathbb{R}_{>0}$ be such that

$$\int_{\mathbb{S}} |g(t, x)| \, \mathrm{d}t \le C \qquad x \in K.$$

In particular, this shows that $t \mapsto g(t, \gamma(t))$ is integrable on $\mathbb{S}$ and so locally integrable on $\mathbb{T}$. This gives the well-definedness of $\Psi_{\mathbb{T},M,g}$.

For continuity, let $\gamma_j \in C^0(\mathbb{S}; M)$, $j \in \mathbb{Z}_{>0}$, be a sequence of curves converging uniformly to $\gamma \in C^0(\mathbb{S}; M)$. Let $\mathbb{S} \subseteq \mathbb{T}$ be a compact interval and let $K \subseteq M$ be compact. Since $\operatorname{image}(\gamma) \cup K$ is compact and M is locally compact, we can find a precompact neighbourhood $\mathcal{U}$ of $\operatorname{image}(\gamma) \cup K$. Then for $N \in \mathbb{Z}_{>0}$ 0 sufficiently large, we have $\operatorname{image}(\gamma_j) \subseteq \mathcal{U}$ for all $j \geq N$ by uniform convergence. Therefore, we can find a compact set $K' \subseteq M$ such that $\operatorname{image}(\gamma_j) \subseteq K'$ for all $j \geq N$ and $\operatorname{image}(\gamma) \subseteq K'$. Then for fixed $t \in \mathbb{S}$, continuity of $x \mapsto g(t,x)$ ensures that $\lim_{j\to\infty} g(t, \gamma_j(t)) = g(t, \gamma(t))$. We also have

$$\int_{\mathbb{S}} |g(t, \gamma_j(t))| \, \mathrm{d}t \leq C \qquad t \in \mathbb{S}$$

for some $C \in \mathbb{R}_{>0}$. Therefore, by the Dominated Convergence Theorem

$$\lim_{j\to\infty} \int_{\mathbb{S}} g(t, \gamma_j(t)) \, dt = \int_{\mathbb{S}} g(t, \gamma(t)) \, dt,$$

which gives the desired continuity. □

Lemma 4.15. *Let $m \in \mathbb{Z}_{\geq 0}$, let $\nu \in \{m, \infty, \mathrm{hol}\}$ satisfy $\nu \geq \mathrm{lip}$, and let $r \in \{\infty, \mathrm{hol}\}$ as appropriate. Let M be a C^r-manifold, let $\mathbb{T} \subseteq \mathbb{R}$ be an interval, and let $\mathcal{P}$ be a topological space. Let $\mathbb{S} \subseteq \mathbb{S}' \subseteq \mathbb{T}$ and $\mathcal{U} \subseteq M$ be open. Let $\Phi \in \mathrm{LocFlow}^\nu(\mathbb{S}'; \mathbb{S}; \mathcal{U}; \mathcal{P})$, let $f \in C^r(M)$, and let $(t_1, t_0, p_0) \in \mathbb{S}' \times \mathbb{S} \times \mathcal{P}$ be fixed and $t_0 < t_1$. Then, for any $\epsilon \in \mathbb{R}_{>0}$, there exists a neighborhood $\mathcal{O} \subseteq \mathcal{P}$ of p_0 and a compact $K \subseteq M$ such that $\Phi^p_{t,t_0}(x) \in \mathrm{int}(\mathrm{K})$ for all $(x,p) \in \mathcal{U} \times \mathcal{O}$ and that*

$$\int_{|t_0,t_1|} p^\nu_K \left(\frac{d}{d\tau}\bigg|_{\tau=s} f \circ \Phi^{p_0}_{\tau,t_0} \circ (\Phi^p_{s,t_0})^{-1}(x) - \frac{d}{d\tau}\bigg|_{\tau=s} f \circ \Phi^{p_0}_{\tau,t_0} \circ (\Phi^{p_0}_{s,t_0})^{-1}(x) \right) \mathrm{d}s < \epsilon$$
$$x \in K, \ p \in \mathcal{O}.$$

Or, equivalently, the map

$$\mathcal{O} \ni p \mapsto \frac{d}{d\tau}\bigg|_{\tau=s} f \circ \Phi^{p_0}_{\tau,t_0} \circ (\Phi^p_{s,t_0})^{-1} \in C^\nu(\mathcal{U}; \mathbb{R})$$

is continuous.

Proof. **(The C^0-case)**. Now consider the following mapping

$$\begin{aligned} \Phi_{\mathbb{T},M,\mathcal{P}} : M \times \mathcal{P} &\to C^0(\mathbb{T}; M) \\ (x,p) &\mapsto (t \mapsto \Phi^p(t, t_0, x)). \end{aligned}$$

By Theorem 3.5(x), this mapping is continuous. It is obvious that

$$\begin{aligned} \Phi^*_{\mathbb{T},M,\mathcal{P}} : M \times \mathcal{P} &\to C^0(\mathbb{T}; M) \\ (x,p) &\mapsto (t \mapsto (\Phi^p_{t,t_0})^{-1}(x)) \end{aligned}$$

is also continuous.

Denote the following continuous mapping

$$\begin{aligned} \iota_{|t_0,t_1|} : C^0(\mathbb{T}; M) &\to C^0(|t_0,t_1|; M) \\ \gamma &\mapsto \gamma||t_0,t_1|. \end{aligned}$$

Then the mapping

$$\Psi_{|t_0,t_1|,M,g} \circ \iota_{|t_0,t_1|} \circ \Phi_{\mathbb{T},M,\mathcal{P}} : M \times \mathcal{P} \to L^1_{\mathrm{loc}}(|t_0,t_1|; \mathbb{R})$$

is continuous, being a composition of continuous maps.

Let $K \subseteq M$ be compact. Then, for $f \in C^\infty(M)$, $\epsilon > 0$, and $x \in K$, there exist a neighbourhood $\mathcal{V}_x \subseteq K$ of x and a neighbourhood $\mathcal{O}_x \subseteq \mathcal{O}$ of p_0 such that

$$\int_{|t_0,t_1|} \left| \frac{d}{d\tau}\bigg|_{\tau=s} f \circ \Phi^{p_0}_{\tau,t_0} \circ (\Phi^{p}_{s,t_0})^{-1}(x') - \frac{d}{d\tau}\bigg|_{\tau=s} f \circ \Phi^{p_0}_{\tau,t_0} \circ (\Phi^{p_0}_{s,t_0})^{-1}(x') \right| ds < \epsilon,$$
$$x' \in \mathcal{V}_x,\ p \in \mathcal{O}_x.$$

Let $x_1, ..., x_m \in K$ be such that $K = \cup_{j=1}^m \mathcal{V}_{x_j}$ and define a neighbourhood $\mathcal{O} = \cap_{j=1}^k \mathcal{O}_{x_j}$ of p_0. Then we have

$$\int_{|t_0,t_1|} \left| \frac{d}{d\tau}\bigg|_{\tau=s} f \circ \Phi^{p_0}_{\tau,t_0} \circ (\Phi^{p}_{s,t_0})^{-1}(x) - \frac{d}{d\tau}\bigg|_{\tau=s} f \circ \Phi^{p_0}_{\tau,t_0} \circ (\Phi^{p_0}_{s,t_0})^{-1}(x) \right| ds < \epsilon \tag{4.3}$$

for $x \in K,\ p \in \mathcal{O}$. Hence, from (4.3) we ascertain that, for every compact $K \subseteq \mathcal{U}$, every $f \in C^\infty(M)$, and every $\epsilon \in \mathbb{R}_{>0}$, if $p \in \mathcal{O}$, then we ave

$$\int_{|t_0,t_1|} p^0_K \left(\frac{d}{d\tau}\bigg|_{\tau=s} f \circ \Phi^{p_0}_{\tau,t_0} \circ (\Phi^{p}_{s,t_0})^{-1} - \frac{d}{d\tau}\bigg|_{\tau=s} f \circ \Phi^{p_0}_{\tau,t_0} \circ (\Phi^{p_0}_{s,t_0})^{-1} \right) \mathrm{d}s < \epsilon$$

which gives the desired result.

(The C^m-case). The topology for $C^m(\mathcal{U}; M)$ is the uniform topology defined by the semimetrics

$$d^m_{K,f}(\Phi_1, \Phi_2) = \sup\{\|j_m(f \circ \Phi_1)(x) - j_m(f \circ \Phi_2)(x)\|_{\mathbb{G}_{M,m}} \mid x \in K\},$$
$$f \in C^\infty(M),\ K \subseteq \mathcal{U} \text{ compact}.$$

Consider the mapping

$$\begin{aligned} \Phi_{|t_0,t_1|,M,\mathcal{P}} : M \times \mathcal{P} &\to C^0(|t_0,t_1|; J^m(\mathcal{U}; M)) \\ (x,p) &\mapsto (t \mapsto j_m(\Phi^p_{t,t_0})^{-1}(x)), \end{aligned}$$

which is continuous, c.f. Theorem 3.5(x). For $(x,p) \in M \times \mathcal{P}$ and for $t \in |t_0,t_1|$, we can think of $j_m(\Phi^p_{t,t_0})^{-1}(x)$ as a linear mapping

$$j_m(\Phi^p_{t,t_0})^{-1}(x) : J^m(M;\mathbb{R})_{(\Phi^p_{t,t_0})^{-1}(x)} \to J^m(M;\mathbb{R})_x$$

$$j_m g((\Phi^p_{t,t_0})^{-1}(x)) \quad \mapsto \quad j_m(g \circ (\Phi^p_{t,t_0})^{-1})(x).$$

For a fixed $f \in C^\infty(M)$ and some $\gamma \in C^0(\mathbb{T}; M)$, define a map

$$\begin{aligned} g : \mathbb{T} \times M &\to \mathbb{R} \\ (s, x) &\mapsto j_m \left(\frac{d}{d\tau}\bigg|_{\tau=s} f \circ \Phi^{p_0}_{\tau,t_0} \right)(x). \end{aligned}$$

Denote $g_x : s \mapsto g(s,x)$ and $g_s : x \mapsto g(s,x)$. Then $g_x \in L^1_{\mathrm{loc}}(\mathbb{T}; \mathbb{R})$ since $f \circ \Phi^{p_0}_{\tau,t_0}$ is locally absolutely continuous, and $g_s \in C^0(M, \mathbb{R})$ since for a fixed $s \in \mathbb{T}$,

$$\frac{d}{d\tau}\bigg|_{\tau=s} f \circ \Phi^{p_0}_{\tau,t_0} \in C^m(M; \mathbb{R}).$$

Now, fixing $(x,p) \in M \times \mathcal{P}$ for the moment, recall the constructions of Section 2.3.2, particularly those preceding the statement of Lemma 2.6. We consider the notation from those constructions with

1. $N = M$,
2. $E = F = J^m(M; \mathbb{R})$,
3. $\Gamma(s) = j_m(\Phi^p_{s,t_0})^{-1}(x) \in \mathrm{Hom}_{\mathbb{R}}(J^m(M,\mathbb{R})_{(\Phi^p_{s,t_0})^{-1}(x)}; J^m(M;\mathbb{R})_x)$, and
4. $\xi = j_m(\frac{d}{d\tau}\big|_{\tau=s} f \circ \Phi^{p_0}_{\tau,t_0})$.

Thus, again in the notation from Section 2.3.2, we have

$$\gamma_M(s) = (\Phi^p_{s,t_0})^{-1}(x), \quad \gamma_N(s) = x.$$

We then have the integrable section of $E = J^m(M;\mathbb{R})$ given by

$$\begin{aligned} \xi_\Gamma : |t_0, t_1| &\to E \\ s &\mapsto \left(s \mapsto j_m \left(\frac{d}{d\tau}\bigg|_{\tau=s} f \circ \Phi^{p_0}_{\tau,t_0} \circ (\Phi^p_{s,t_0})^{-1} \right)(x) \right) \end{aligned}$$

to obtain continuity of the mapping

$$\begin{aligned} \Psi_{|t_0,t_1|, J^m(M;\mathbb{R}), j_m(\frac{d}{d\tau}|_{\tau=s} f \circ \Phi^{p_0}_{\tau,t_0})} &: C^0(|t_0,t_1|; J^m(\mathcal{U}; M)) \to L^1_{\mathrm{loc}}(|t_0,t_1|; J^m(M;\mathbb{R})) \\ \Gamma &\mapsto \left(s \mapsto \Gamma(s) \left(j_m \left(\frac{d}{d\tau}\Big|_{\tau=s} f \circ \Phi^{p_0}_{\tau,t_0} \right)(\gamma_M(s)) \right) \right), \end{aligned}$$

and so of the composition

$$\Psi_{|t_0,t_1|, J^m(M;\mathbb{R}), j_m(\frac{d}{d\tau}|_{\tau=s} f \circ \Phi^{p_0}_{\tau,t_0})} \circ \Phi_{|t_0,t_1|, M, \mathcal{P}} : M \times \mathcal{P} \to L^1_{\mathrm{loc}}(|t_0,t_1|; J^m(M;\mathbb{R})).$$

Note that this is precisely the continuity of the mapping

$$M \times \mathcal{P} \ni (x,p) \mapsto \left(t \mapsto j_m\left(\frac{d}{d\tau}\bigg|_{\tau=s} f \circ \Phi^{p_0}_{\tau,t_0} \circ (\Phi^{p}_{s,t_0})^{-1}\right)(x)\right) \in L^1_{\mathrm{loc}}(|t_0,t_1|; J^m(M;\mathbb{R})).$$

In order to convert this continuity into a continuity statement involving the fibre norm for $J^m(M;\mathbb{R})$, we note that, for $x \in K$, there exists a neighbourhood $\mathcal{V}_x$ and affine functions $F^1_x, \dots, F^{n+k}_x \in \mathrm{Aff}^\infty(J^m(M;\mathbb{R}))$ which are coordinates for $\rho_m^{-1}(\mathcal{V}_x)$. We can choose a Riemannian metric for $J^m(M;\mathbb{R})$, whose restriction to fibres agrees with the fibre metric (2.1) (Lewis, 2020, §4.1). It follows, therefore, from Lemma A.1 that there exists $C_x \in \mathbb{R}_{>0}$ such that

$$\|j_m g_1(x') - j_m g_2(x')\|_{\mathbb{G}_{M,m}} \le C_x |F^l_x \circ j_m g_1(x') - F^l_x \circ j_m g_2(x')|,$$

for $g_1, g_2 \in C^\infty(M)$, $x' \in \mathcal{V}_x$, $l \in \{1, \dots, n+k\}$. By the continuity proved in the preceding paragraph, we can take a relative neighbourhood $\mathcal{V}_x \subseteq K$ of x sufficiently small and a neighbourhood $\mathcal{O}_x \subseteq \mathcal{O}$ of p_0 such that

$$\int_{|t_0,t_1|} \left| F^l_x \circ j_m\left(\frac{d}{d\tau}\bigg|_{\tau=s} f \circ \Phi^{p_0}_{\tau,t_0} \circ (\Phi^{p}_{s,t_0})^{-1}\right)(x') \right.$$
$$\left. - F^l_x \circ j_m\left(\frac{d}{d\tau}\bigg|_{\tau=s} f \circ \Phi^{p_0}_{\tau,t_0} \circ (\Phi^{p_0}_{s,t_0})^{-1}\right)(x') \right| \mathrm{d}s < \frac{\epsilon}{2C_x},$$

for all $x' \in \mathcal{V}_x$, $p \in \mathcal{O}_x$, and $l \in \{1, \dots, n+k\}$. Therefore,

$$\int_{|t_0,t_1|} \left\| j_m\left(\frac{d}{d\tau}\bigg|_{\tau=s} f \circ \Phi^{p_0}_{\tau,t_0} \circ (\Phi^{p}_{s,t_0})^{-1}\right)(x') \right.$$
$$\left. - j_m\left(\frac{d}{d\tau}\bigg|_{\tau=s} f \circ \Phi^{p_0}_{\tau,t_0} \circ (\Phi^{p_0}_{s,t_0})^{-1}\right)(x') \right\|_{\mathbb{G}_{M,m}} \mathrm{d}s < \frac{\epsilon}{2}$$

for all $x' \in \mathcal{V}_x$, $p \in \mathcal{O}_x$. Now let $x_1, \dots, x_s \in K$ be such that $K = \cup_{r=1}^s \mathcal{V}_{x_r}$ and define a neighbourhood $\mathcal{O}' = \cap_{r=1}^s \mathcal{O}_{x_r}$ of p_0. Then we have

$$\int_{|t_0,t_1|} \left\| j_m\left(\frac{d}{d\tau}\bigg|_{\tau=s} f \circ \Phi^{p_0}_{\tau,t_0} \circ (\Phi^{p}_{s,t_0})^{-1}\right)(x') \right.$$
$$\left. - j_m\left(\frac{d}{d\tau}\bigg|_{\tau=s} f \circ \Phi^{p_0}_{\tau,t_0} \circ (\Phi^{p_0}_{s,t_0})^{-1}\right)(x') \right\|_{\mathbb{G}_{M,m}} ds < \frac{\epsilon}{2}$$

for all $x' \in K$, $p \in \mathcal{O}'$, as desired.

(The C^∞-case). From the result in the C^m-case for $m \in \mathbb{Z}_{\ge 0}$, the mapping

$$\mathcal{O} \ni p \mapsto \frac{d}{d\tau}\bigg|_{\tau=s} f \circ \Phi^{p_0}_{\tau,t_0} \circ (\Phi^{p}_{s,t_0})^{-1}(x) \in C^\nu(\mathcal{U};\mathbb{R})$$

is continuous for each $m \in \mathbb{Z}_{\ge 0}$. From the diagram

$$\begin{tikzcd}
C^\infty(\mathcal{U};\mathbb{R}) \arrow[rr, hook] & & C^m(\mathcal{U};\mathbb{R}) \\
\mathcal{O} \arrow[u, "p\mapsto \frac{d}{d\tau}\big|_{\tau=s} f\circ\Phi^{p_0}_{\tau,t_0}\circ(\Phi^{p}_{s,t_0})^{-1}"] \arrow[urr, swap, "p\mapsto j_m\left(\frac{d}{d\tau}\big|_{\tau=s} f\circ\Phi^{p_0}_{\tau,t_0}\circ(\Phi^{p}_{s,t_0})^{-1}\right)"] & &
\end{tikzcd}$$

and noting that the diagonal mappings in the diagram are continuous, we obtain the continuity of the vertical mapping as a result of the fact that the C^∞-topology is the initial topology induced by the C^m-topologies, $m \in \mathbb{Z}_{\geq 0}$.

(The $C^{\mathbf{hol}}$-case). Since the C^{hol}-topology is the C^0-topology, with the scalars extended to be complex and the functions restricted to be holomorphic, the analysis in Section 3.3.1 can be carried out verbatim to give the theorem in the holomorphic case. $\square$

Proposition 4.16. *Let $m \in \mathbb{Z}_{\geq 0}$, let $\nu \in \{m, \infty, hol\}$, and let $r \in \{\infty, hol\}$, as required. Let M be a C^r-manifold and let $\mathbb{T} \subseteq \mathbb{R}$ be an interval. Let $\mathbb{S} \subseteq \mathbb{S}' \subseteq \mathbb{T}$ and $\mathcal{U} \subseteq M$ be open. The map*

$$\begin{array}{rcl} exp_{\mathbb{S}'\times\mathbb{S}\times\mathcal{U}} : \mathcal{N} \subseteq \mathcal{V}^{\nu}_{\mathbb{S}'\times\mathbb{S}\times\mathcal{U}} & \to & LocFlow^{\nu}(\mathbb{S}';\mathbb{S};\mathcal{U}) \\ X & \mapsto & \Phi^X \end{array}$$

is open.

Proof. Since $\exp_{\mathbb{S}'\times\mathbb{S}\times\mathcal{U}}$ is one-to-one and onto its image, it is enough to show the continuity of the inverse map, denoted by

$$\begin{array}{rcl} \exp^{-1}_{\mathbb{S}'\times\mathbb{S}\times\mathcal{U}} : \mathrm{LocFlow}^{\nu}(\mathbb{S}';\mathbb{S};\mathcal{U}) & \to & \mathcal{V}^{\nu}_{\mathbb{S}'\times\mathbb{S}\times\mathcal{U}} \\ \Phi & \mapsto & X_\Phi \end{array}$$

where

$$\begin{aligned} X_\Phi(t,x) = X_\Phi(t,\Phi(t,t_0,x_0)) = \frac{d}{d\tau}\bigg|_{\tau=t} \Phi(\tau,t_0,x_0) &= \frac{d}{d\tau}\bigg|_{\tau=t} \Phi_{\tau,t_0}(x_0) \\ &= \frac{d}{d\tau}\bigg|_{\tau=t} \Phi_{\tau,t_0}\circ\Phi^{-1}_{t,t_0}(x) \end{aligned}$$

and $(t,t_0,x_0) \in \mathbb{S}'\times\mathbb{S}\times\mathcal{U}$. Hence $\exp^{-1}_{\mathbb{S}'\times\mathbb{S}\times\mathcal{U}} \circ \exp_{\mathbb{S}'\times\mathbb{S}\times\mathcal{U}} = \mathrm{Id}$.

The topology of $\mathcal{V}^{\nu}_{\mathbb{S}'\times\mathbb{S}\times\mathcal{U}}$ is generated by the family of seminorms

$$p^{\nu}_{K',\mathbb{I}',f}(X) = \int_{\mathbb{I}'} p^{\nu}_{K'}(X_t f)\ dt,$$

where $p^{\nu}_{K'}$ is the appropriate seminorm from (2.3). For a fixed $\Phi \in \mathrm{LocFlow}^{\nu}(\mathbb{S}';\mathbb{S};\mathcal{U})$, $X_\Phi \in \mathcal{V}^{\nu}_{\mathbb{S}'\times\mathbb{S}\times\mathcal{U}}$.

Let $\{K'_i\}_{i\in\mathbb{Z}_{>0}} \subset M$ be compact neighborhoods of x_0 and such that $K'_j \subset \operatorname{int}(K'_{j+1})$ and $x_0 \in \operatorname{int}(K'_1)$, and $M = \bigcup_{i\in\mathbb{Z}_{>0}} K'_i$. Denote $K' = \bigcap_{i\in\mathbb{Z}_{>0}} K'_i$. Similarly, let $\{\mathbb{I}'_i\}_{i\in\mathbb{Z}_{>0}}$ be compact neighborhoods of t_0 and such that $\mathbb{I}'_j \subset \operatorname{int}(\mathbb{I}'_{j+1})$ and $t_0 \in \operatorname{int}(\mathbb{I}'_1)$, and $\mathbb{T} = \bigcup_{i\in\mathbb{Z}_{>0}} \mathbb{I}'_i$.

For a fixed $\Phi \in \mathrm{LocFlow}^\nu(\mathbb{S}';\mathbb{S};\mathcal{U})$, let $\mathcal{R}$ be a neighborhood of $X_\Phi \in \mathcal{V}^\nu_{\mathbb{S}'\times\mathbb{S}\times\mathcal{U}}$. Then there exist increasing sequences $\{i_1, i_2, ..., i_m\} \subset \mathbb{Z}_{>0}$, $\{j_1, j_2, ..., j_n\} \subset \mathbb{Z}_{>0}$ and a finite collection of functions $f_1, f_2, ... f_p \in C^\nu(M)$ such that

$$\bigcap_{s=1}^{p}\bigcap_{k=1}^{m}\bigcap_{l=1}^{n}\left\{Y \in \mathcal{V}^\nu_{\mathbb{S}'\times\mathbb{S}\times\mathcal{U}} \mid p^\nu_{K'_{i_k},\mathbb{I}'_{j_l},f_s}(Y - X_\Phi) < r\right\} \subseteq \mathcal{R}$$

forms a neighborhood of X_Φ.

Observe, $i_a < i_b$ implies $K'_{i_a} \subset K'_{i_b}$ which gives

$$p^{\nu^{-1}}_{K'_{i_b},\mathbb{I}'_{j_l},f_s}([0,r)) \subseteq p^{\nu^{-1}}_{K'_{i_a},\mathbb{I}'_{j_l},f_s}([0,r)),$$

and $j_a < j_b$ implies $\mathbb{I}'_{j_a} \subset \mathbb{I}'_{j_b}$, which gives

$$p^{\nu^{-1}}_{K'_{i_k},\mathbb{I}'_{j_b},f_s}([0,r)) \subseteq p^{\nu^{-1}}_{K'_{i_k},\mathbb{I}'_{j_a},f_s}([0,r)).$$

Observe that $\mathbb{I}'_{j_1} \subseteq \mathbb{I}'_{j_2} \subseteq ... \subseteq \mathbb{I}'_{j_n}$ and $K'_{i_1} \subseteq K'_{i_2} \subseteq ... \subseteq K'_{i_m}$. Denote $K' := K'_{i_m}$ and $\mathbb{I}' := \mathbb{I}'_{j_n}$, and let

$$Q := \bigcap_{s=1}^{p}\left\{Y \in \mathcal{V}^\nu_{\mathbb{S}'\times\mathbb{S}\times\mathcal{U}} \mid p^\nu_{K',\mathbb{I}',f_s}(Y - X_\Phi) < r\right\}.$$

The topology on $\mathrm{LocFlow}^\nu(\mathbb{S}';\mathbb{S};\mathcal{U})$ is defined by the semi-metrics

$$q^\nu_{K,\mathbb{I},\mathbb{I}',f}(\Phi_1 - \Phi_2) = \max\left\{p^\nu_{K,\mathbb{I},\mathbb{I}',\infty}(f\circ(\Phi_1 - \Phi_2)),\ \hat{p}^\nu_{K,\mathbb{I},\mathbb{I}',1}(f\circ(\Phi_1 - \Phi_2))\right\}.$$

By Proposition (4.15), for a fixed $f_j \in \{f_1, ..., f_p\}$, there exist $K_j \subseteq M$ and $\mathbb{I}_j \subseteq \mathbb{I}'$ compact with t_0 as interior, and a neighborhood $\mathcal{N}_j$ of Φ, such that $\Psi^{-1}_{t,t_0}(x) \in \operatorname{int}(K_j)$ for all $\Psi \in \mathcal{N}_j$ and $(t, t_0, x) \in \mathbb{I}' \times \mathbb{I}_j \times K'$ and that

$$\int_{\mathbb{I}'} p^\nu_{K'}\left(\left.\frac{d}{d\tau}\right|_{\tau=s} f_j\circ\Psi_{\tau,t_0}\circ\Psi^{-1}_{s,t_0}(x) - \left.\frac{d}{d\tau}\right|_{\tau=s} f_j\circ\Psi_{\tau,t_0}\circ\Phi^{-1}_{s,t_0}(x)\right)\mathrm{d}s < \frac{r}{2}. \tag{4.4}$$

Now denote $K := \bigcap_{j=1}^{p} K_j$, $\mathbb{I} := \bigcap_{j=1}^{p}\mathbb{I}_j$. Let

$$\mathcal{N}' := \bigcap_{j=1}^{p}\left\{\Psi \in \mathrm{LocFlow}^\nu(\mathbb{S}';\mathbb{S};\mathcal{U}) \mid q^\nu_{K,\mathbb{I},\mathbb{I}',f_j}(\Psi - \Phi) < \frac{r}{2}\right\},$$

and denote $\mathcal{O} := (\bigcap_{j=1}^{p}\mathcal{N}_j) \cap \mathcal{N}'$. We claim that $\exp^{-1}_{\mathbb{S}'\times\mathbb{S}\times\mathcal{U}}(\mathcal{O}) \subseteq Q$. Indeed, for any $\Psi \in \mathcal{O}$ and for fixed $(t, t_0) \in \mathbb{I}' \times \mathbb{I}$, we have that for each $s \in \{1, ..., p\}$,

$$p^\nu_{K'}\left(X_\Psi f_s(t,x) - X_\Phi f_s(t,x)\right)$$

$$
\begin{aligned}
&= p^{\nu}_{K'}\left(\left(\frac{d}{d\tau}\bigg|_{\tau=t}\Psi_{\tau,t_0}\circ\Psi^{-1}_{t,t_0}-\frac{d}{d\tau}\bigg|_{\tau=t}\Phi_{\tau,t_0}\circ\Phi^{-1}_{t,t_0}\right)f_s\right)\\
&\le p^{\nu}_{K'}\left(\left(\frac{d}{d\tau}\bigg|_{\tau=t}\Psi_{\tau,t_0}(\Psi^{-1}_{t,t_0}(x))-\frac{d}{d\tau}\bigg|_{\tau=t}\Psi_{\tau,t_0}(\Phi^{-1}_{t,t_0}(x))\right)f_s\right)\\
&\quad+p^{\nu}_{K'}\left(\left(\frac{d}{d\tau}\bigg|_{\tau=t}\Psi_{\tau,t_0}(\Phi^{-1}_{t,t_0}(x))-\frac{d}{d\tau}\bigg|_{\tau=t}\Phi_{\tau,t_0}(\Phi^{-1}_{t,t_0}(x))\right)f_s\right)\\
&\le p^{\nu}_{K'}\left(\frac{d}{d\tau}\bigg|_{\tau=t}f_s\circ\Psi_{\tau,t_0}(\Psi^{-1}_{t,t_0}(x))-\frac{d}{d\tau}\bigg|_{\tau=t}f_s\circ\Psi_{\tau,t_0}(\Phi^{-1}_{t,t_0}(x))\right)\\
&\quad+p^{\nu}_{K}\left(\frac{d}{d\tau}\bigg|_{\tau=t}f_s\circ\Psi_{\tau,t_0}(y)-\frac{d}{d\tau}\bigg|_{\tau=t}f_s\circ\Phi_{\tau,t_0}(y)\right),
\end{aligned}
$$

which implies

$$
\begin{aligned}
&p^{\nu}_{K',\mathbb{I}',f_s}(X_\Psi-X_\Phi)\\
&=\int_{\mathbb{I}'}p^{\nu}_{K'}\left(X_\Psi f_s(t,x)-X_\Phi f_s(t,x)\right)\mathrm{d}t\\
&\le\int_{\mathbb{I}'}p^{\nu}_{K'}\left(\frac{d}{d\tau}\bigg|_{\tau=t}f_s\circ\Psi_{\tau,t_0}(\Psi^{-1}_{t,t_0}(x))-\frac{d}{d\tau}\bigg|_{\tau=t}f_s\circ\Psi_{\tau,t_0}(\Phi^{-1}_{t,t_0}(x))\right)\mathrm{d}t\\
&\quad+\int_{\mathbb{I}'}p^{\nu}_{K}\left(\frac{d}{d\tau}\bigg|_{\tau=t}f_s\circ\Psi_{\tau,t_0}(y)-\frac{d}{d\tau}\bigg|_{\tau=t}f_s\circ\Phi_{\tau,t_0}(y)\right)\mathrm{d}t\\
&\le\frac{r}{2}+\int_{\mathbb{I}'}\sup\left\{p^{\nu}_{K}\left(\frac{d}{d\tau}\bigg|_{\tau=t}f_s\circ\Psi_{\tau,t_0}(y)-\frac{d}{d\tau}\bigg|_{\tau=t}f_s\circ\Phi_{\tau,t_0}(y)\right)\,\bigg|\,t_0\in\mathbb{I}\right\}dt\\
&=\frac{r}{2}+\hat{p}^{\nu}_{K,\mathbb{I},\mathbb{I}',1}(f_s\circ\Psi-f_s\circ\Phi)\\
&\le\frac{r}{2}+q^{\nu}_{K,\mathbb{I},\mathbb{I}',f_s}(\Psi-\Phi)\\
&\le\frac{r}{2}+\frac{r}{2}=r,
\end{aligned}
$$

as desired. □

Chapter 5

Summary

The general question about the existence of the exponential map is addressed in this thesis by considering, not vector fields and diffeomorphisms, but presheaves of vector fields and presheaves of local diffeomorphisms of various of regularities, i.e., Lipschitz, finitely differentiable, smooth, and holomorphic. This allows for systematic localisation of the components of what becomes the exponential map, i.e.,

$$\exp : \{\text{presheaf of vector fields}\} \to \{\text{presheaf of local diffeomorphisms}\}$$

Moreover, the homeomorphism of this map is established upon the suitable topologies for sets of vector fields and flows using geometric decompositions of various jet bundles by various connections. This framework is interesting in that it allows an elegant and uniform treatment of vector fields across various regularity classes, understands vector fields and local flows from a topological perspective, and studies control systems in a more general and categorical point of view.

These results give us many applications in geometric control theory. The most important application is that one can define a control system from these presheaves, i.e., a control system, denoted by $\mathcal{C}$, is a sub-presheaf

$$\mathcal{C} \subseteq \mathscr{G}^{\nu}_{\mathrm{LI}}(\mathbb{T}; TM) \qquad \text{or} \qquad \mathcal{C} \subseteq \mathscr{L}\mathscr{F}^{\nu}(\mathbb{T}; \mathbb{T}; M)$$

with the integral curves for local sections over an open set as its controlled trajectory. This definition of control systems generalises the classical definition of a control system. It is convincing that it is easier to do all the control theoretic things one is used to using flows other than using vector fields, e.g., talk about controllability, optimality, and stabilisability. Now one can formulates theorems using properties of flows and diffeomorphisms, rather than vector fields. In the case of flows defined by vector fields, one translates conditions on flows to conditions on vector fields by the homeomorphism of the exponential map. Moreover, by studying the stuctures and topologies in the space of presheaves, one can obtain controllability under the certain assumptions of the structure of the presheaf given by the following conjecture.

Conjecture 5.1. *A control system $\mathcal{C}$ is controllable if and only if the control system generates a open neighborhood of the identity.*

This definition of control systems is more general in the sense that it captures both locally and globally defined control systems, whereas classical definition only captures the globally defined control systems.

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Appendix A

Riemannian metrics

The following results are used for the proof of different theorems in this thesis.

Lemma A.1 (Comparison of Riemannian distance for different Riemannian metrics))**.** *If $\mathbb{G}_1$ and $\mathbb{G}_2$ are smooth Riemannian metrics on M with metrics d_1 and d_2, respectively, and if $K \subseteq M$ is compact, then there exists $c \in \mathbb{R}_{>0}$ such that*

$$c^{-1} d_1(x_1, x_2) \leq d_2(x_1, x_2) \leq c d_1(x_1, x_2)$$

for every $x_1, x_2 \in K$.

Proof. We shall prove the result in increments. The first step is simple linear algebra.

Sublemma 1. *If $\mathbb{G}_1$ and $\mathbb{G}_2$ are inner products on a finite-dimensional $\mathbb{R}$-vector space V, then there exists $c \in \mathbb{R}_{>0}$ such that*

$$c^{-1}\mathbb{G}_1(v, v) \leq \mathbb{G}_2(v, v) \leq c\mathbb{G}_1(v, v)$$

for all $v \in V$.

Proof. Let $\mathbb{G}_j^\flat \in \mathrm{Hom}_{\mathbb{R}}(V; V^*)$ and $\mathbb{G}_j^\sharp \in \mathrm{Hom}_{\mathbb{R}}(V^*; V)$, $j \in \{1, 2\}$, be the induced linear maps. Note that

$$\mathbb{G}_1(\mathbb{G}_1^\sharp \circ \mathbb{G}_2^\flat(v_1), v_2) = \mathbb{G}_2(v_1, v_2) = \mathbb{G}_2(v_2, v_1) = \mathbb{G}_1(\mathbb{G}_1^\sharp \circ \mathbb{G}_2^\flat(v_2), v_1),$$

showing that $\mathbb{G}_1^\sharp \circ \mathbb{G}_2^\flat$ is $\mathbb{G}_1$-symmetric. Let $(e_1, \dots, e_n)$ be a $\mathbb{G}_1$-orthonormal basis for V that is also a basis of eigenvectors for $G_1^\sharp \circ \mathbb{G}_2^\flat$. The matrix representatives of $\mathbb{G}_1$ and $\mathbb{G}_2$ are then

$$[\mathbb{G}_1] = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}, \qquad [\mathbb{G}_2] = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix},$$

where $\lambda_1, \dots, \lambda_n \in \mathbb{R}_{>0}$. Let us assume without loss of generality that $\lambda_1 < \cdots < \lambda_n$. Then taking $c = \max\{\lambda_n, \lambda_1^{-1}\}$ gives the result, as one can verify directly. ∇

Next let us give the local version of the result.

Sublemma 2. *Let $\mathbb{G}_1$ and $\mathbb{G}_2$ be smooth Riemannian metrics on a manifold M with metrics d_1 and d_2, respectively. For each $x \in M$, there exists a neighbourhood $\mathcal{U}_x$ of x and $c_x \in \mathbb{R}_{>0}$ such that*

$$c_x^{-1} d_1(x_1, x_2) \leq d_2(x_1, x_2) \leq c_x d_1(x_1, x_2)$$

for every $x_1, x_2 \in \mathcal{U}_x$.

Proof. Let $x \in M$. Let $\mathcal{N}_1$ and $\mathcal{N}_2$ be geodesically convex neighbourhoods of x with respect to the Riemannian metrics $\mathbb{G}_1$ and $\mathbb{G}_2$, respectively [Kobayashi and Nomizu 1963, Proposition IV.3.4]. Thus every pair of points in $\mathcal{N}_1$ can be connected by a unique distanceminimising geodesic for $\mathbb{G}_1$ that remains in $\mathcal{N}_1$, and similarly with $\mathcal{N}_2$ and $\mathbb{G}_2$. By Sublemma 1, let $c_x \in \mathbb{R}_{>0}$ be such that

$$c_x^{-2} \mathbb{G}_1(v_x, v_x) \leq \mathbb{G}_2(v_x, v_x) \leq c_x^2 \mathbb{G}_1(v_x, v_x), \quad v_x \in T_x M.$$

By continuity of $\mathbb{G}_1$ and $\mathbb{G}_2$, we can choose $\mathcal{N}_1$ and $\mathcal{N}_2$ sufficiently small that

$$c_x^{-2} \mathbb{G}_1(v_y, v_y) \leq \mathbb{G}_2(v_y, v_y) \leq c_x^2 \mathbb{G}_1(v_y, v_y), \quad v_y \in \mathcal{N}_1 \cup \mathcal{N}_1.$$

Now define $\mathcal{U}_x = \mathcal{N}_1 \cap \mathcal{N}_1$. Then every pair of points in $\mathcal{U}_x$ can be connected with a unique distance-minimising geodesic of both $\mathbb{G}_1$ and $\mathbb{G}_2$ that remains in $\mathcal{N}_1 \cup \mathcal{N}_1$. Now let $x_1, x_2 \in \mathcal{U}_x$. Let $\gamma : [0, 1] \to M$ be the unique distance-minimising $\mathbb{G}_1$-geodesic connecting x_1 and x_2. Then

$$\begin{aligned} d_2(x_1, x_2) &\leq \ell_{\mathbb{G}_2}(\gamma) = \int_0^1 \sqrt{\mathbb{G}_2(\gamma'(t), \gamma'(t))}\, \mathrm{d}t \\ &\leq c_x \int_0^1 \sqrt{\mathbb{G}_1(\gamma'(t), \gamma'(t))}\, \mathrm{d}t \\ &\leq c_x \ell_{\mathbb{G}_1}(\gamma) = c_x d_1(x_1, x_2). \end{aligned}$$

One similarly shows that $d_1(x_1, x_2) \leq c_x d_2(x_1, x_2)$. ∇

Now let $K \subseteq M$ be compact and, for each $x \in K$, let $\mathcal{U}_x$ be a neighbourhood of x and let $c_x \in \mathbb{R}_{>0}$ be as in the preceding sublemma. Then $(\mathcal{U}_x)_{x \in K}$ is an open cover of K. Let $x_1, ..., x_k \in K$ be such that

$$K \subseteq \cup_{j=1}^k \mathcal{U}_{x_j}.$$

Let

$$D_a = \sup\{d_a(x, y) \mid x, y \in K\}, \quad a \in \{1, 2\}.$$

By the Lebesgue Number Lemma (D. Burago, Y. Burago, and Ivanov, 2001, Theorem 1.6.11), let $r_a \in \mathbb{R}_{>0}$ be such that, if $x \in K$, then there exists $j \in \{1, ..., k\}$ for which $B_a(r, x) \in \mathcal{U}_{x_j}$ ($B_a(r, x)$ is the ball with respect to the metric d_a). Let us denote

$$c = \max\left\{c_{x_1}, ..., c_{x_k}, \frac{D_1}{r_2}, \frac{D_2}{r_1}\right\}.$$

Now let $x_1, x_2 \in K$. If $d_1(x_1, x_2) < r_1$, then let $j \in \{1, ..., k\}$ be such that $x_1, x_2 \in \mathcal{U}_j$. Then

$$d_2(x_1, x_2) \le c d_1(x_1, x_2).$$

If $d_1(x_1, x_2) \ge r_1$, then

$$\frac{d_2(x_1, x_2) r_1}{D_2} \le \frac{d_2(x_1, x_2) r_1}{d_2(x_1, x_2)} \le d_1(x_1, x_2).$$

This gives $d_2(x_1, x_2) \le c d_1(x_1, x_2)$. Swapping the roles of $\mathbb{G}_1$ and $\mathbb{G}_2$ gives $d_1(x_1, x_2) \le c d_2(x_1, x_2)$, giving the lemma. ■