Geometric and function analytic methods for flows of time- and parameter-dependent vector fields*

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Abstract

A framework is presented for ordinary differential equations with measurable time-dependence and with parameter-dependence in a general topological space. This framework has both geometric and function analytic aspects. The geometric aspect is reflected by the framework being that of vector fields and flows on manifolds. The function analytic aspect is reflected by the classes of vector fields and flows being characterised by function space topologies. These classes of vector fields and mappings are presented across a variety of regularity classes which includes Lipschitz, finitely differentiable, smooth, real analytic, and holomorphic. Special emphasis is placed on the rôle of composition operators, particularly in time- and parameter-dependent settings.

Keywords. Time-dependent vector fields, parameter-dependent vector fields, function spaces, composition operators

AMS Subject Classifications (2020). 34A12, 34A26, 34A34, 46E10, 46T20, 47H30, 53B05, 53C23

Contents

*This paper is dedicated to the memory of Professor Roger Brockett who sadly passed away in March 2023.

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1. Introduction

The essential subject of this paper, flows of time- and parameter-dependent vector fields, is a well-developed one, and one is justified in wondering, “What could possibly be new here?” One might like an answer to take a form like, “We give new results,” or, “We give new methods.” Both answers apply to this paper, although possibly the second one is the more applicable. In this introduction we give an overview of the main ideas of the paper, pointing out the new results and the new methods.

1.1. Review of what is known. While general results of the broad type we consider in this paper can be found in the standard reference texts dealing with ordinary differential equations, it is not easy, and sometimes not possible, to find the most general results. In this section we will summarise some of what is known, to the best of our understanding, and point out where current results find their limitations. To allow for a discussion with some context, for the moment we consider an initial value problem

$$\boldsymbol{\xi}'(t) = \boldsymbol{f}(t, \boldsymbol{\xi}(t), p), \quad \boldsymbol{\xi}(t_0) = \boldsymbol{x}_0$$

for a solution $t \mapsto \boldsymbol{\xi}(t) \in \mathbb{R}^n$ and p a parameter in a topological space $\mathcal{P}$. We write the solution of the initial value problem as

$$t \mapsto \Phi^{\boldsymbol{f}}(t, t_0, \boldsymbol{x}_0, p)$$

to include all of its dependencies. In the control theory literature, measurable time-dependence is required, e.g., for the consideration of certain optimal control problems. Thus, some of the most general results regarding existence, uniqueness, and continuous dependence can be found in that literature. For example, Sontag [1998, Theorem 55] proves a quite general theorem for continuous dependence of the final state on initial state $\boldsymbol{x}_0$ and (in a certain sense) on the right-hand side in the presence of measurable time-dependence. Hestenes [1966, Theorem 3.1 in Appendix] similarly proves a general theorem for continuity of the final state as it depends on initial and final times t_0 and t, initial state $\boldsymbol{x}_0$, and parameter p, again for measurable time-dependence. Parameters in Hestenes's results are assumed to lie in normed vector spaces, but similar ideas apply for metric spaces. Hartman [1964, Theorem II.3.2] proves the continuous dependence of the final state on the right-hand side if the space of right-hand sides is given the compact-open topology; in these results, time-dependence is continuous. Most slightly serious books on ordinary differential equations contain results of this nature, and we have only tried to point out the most general of these.

In the situation where $\boldsymbol{x} \mapsto \boldsymbol{f}(t, \boldsymbol{x}, p)$ has some regularity, the matter of when this regularity is shared by $\boldsymbol{x} \mapsto \Phi^{\boldsymbol{f}}(t, t_0, \boldsymbol{x}, p)$ is dealt with more sparsely in the ordinary differential equation literature. For continuous time-dependence and no parameter-dependence, these questions are considered for finite-order differentiability by Hartman [1964, §V.3, V.4]. A complicated statement for differentiability of order one in a control theoretic setting (where one also wants to consider the derivative with respect to control) is given by [Sontag 1998, Theorem 1]. Here measurable time-dependence is allowed, but parameter-dependence is not considered. A simpler statement (simpler because control is not considered) in a similar vein is given by [Hestenes 1966, Theorem 6.1 in Appendix]. The problem of differentiable dependence on parameter, when the parameter is assumed to be in an L^p-space, is considered in [Klose and Schuricht 2011]. We do not consider here the problem of differentiable dependence on parameters, as our immediate concerns are for making the parameter spaces as general as possible, e.g., not necessarily being able to support a theory of differentiation. That being said, we do envision the tools we give here as being useful for a theory where differentiability with respect to parameter can be handled naturally. Schuricht and von der Mosel [2000] carefully consider differentiable dependence on state with continuous dependence on parameters. These authors also point out the paucity of results in these directions concerning differential equations with measurable time dependence.

Smooth dependence of flows on initial conditions for smooth right-hand sides is often used, including with measurable time-dependence. For example, this arises in the chronological calculus approach of Agrachev and Gamkrelidze [1978], an approach which is given a nice outline in [Agrachev and Sachkov 2004]. A consequence of this chronological calculus approach is the smooth dependence of solutions on initial condition. For real analytic regularity, the desired result was obtained by Jafarpour and Lewis [2014], using a framework very closely aligned with those that we use here. Agrachev and Gamkrelidze [1978] also give results in the real analytic case, although their approach is different than ours; see Section 1.3. An approach where results for real analytic regularity are framed using the infinite-dimensional Lie group of real analytic diffeomorphisms is presented by Glöckner [2023]; see the discussion at the end of Section 5.3.

1.2. An outline of the approach and the results. The reader will have noticed that we have refrained from clearly stating any of the results above. Indeed statements can be very detailed, as can be see in the list of hypotheses from [Hestenes 1966] shown in Figure 1. One of the contributions of our work is to provide simple—well, compact, at

To each point (α,β) in $\mathscr{F}$ there is a constant $\delta > 0$ and two integrable functions $M(t)$, $K(t)$ such that

1. The δ-neighborhood of (α,β) is in $\mathscr{F}$;

2. For each x in the δ-neighborhood β_δ of β and for each λ in Λ, the functions $f^i(t,x,\lambda)$ are measurable in t on the δ-neighborhood α_δ of α and satisfy the inequality

$$|f(t,x,\lambda)| \leq M(t) \tag{2.1}$$

on α_δ. Hence $f(t,x,\lambda)$ is integrable on $\alpha - \delta < t < \alpha + \delta$ for each x in β_δ and λ in Λ;

3. For each x and y in the δ-neighborhood β_δ of β and each λ in Λ, the inequality

$$|f(t,x,\lambda) - f(t,y,\lambda)| \leq K(t)\,|x - y| \tag{2.2}$$

holds on the δ-neighborhood α_δ of α;

4. For each x in β_δ and λ_0 in Λ we have

$$\lim_{\lambda=\lambda_0} \int_{\alpha-\delta}^{\alpha+\delta} |f(t,x,\lambda) - f(t,x,\lambda_0)|\, dt = 0. \tag{2.3}$$

Figure 1. Hypotheses for existence, uniqueness, and continuous dependence from [Hestenes 1966]

least—descriptions of the right-hand side that give very general results, e.g., that subsume all results mentioned above (with the single exception of differentiability with respect to parameter), as well as give new results. We mention an alternative approach to these questions taken by Heunis [1984], where topologies for the space of right-hand sides are described which ensure continuous dependence on initial condition. Another piece of related work is that of Filippov [1996], who considers topological properties of right-hand sides that allow one to deal with singularities. The emphasis here is to arrive at those properties that ensure that the usual sorts of convergence arguments for existence of solutions apply.

At the heart of our approach is a unified theory across many regularity classes. To establish our notation for this, let $m \in \mathbb{Z}_{\geq 0}$, let $m' \in \{0, \mathrm{lip}\}$, and let $\nu \in \{m+m', \infty, \omega, \mathrm{hol}\}$. For us, this means the following:

1. $\nu = 0$: continuous;
2. $\nu = m$: m-times continuously differentiable;

3. $\nu = m + \text{lip}$: m-times continuously differentiable with locally Lipschitz top derivative;
4. $\nu = \infty$: infinitely differentiable;
5. $\nu = \omega$: real analytic;
6. $\nu = \text{hol}$: holomorphic.

We shall consider vector fields of class C^ν. The manifolds we consider will be of suitable regularity, i.e., of class C^κ with

1. $\kappa = \infty$ if $\nu \in \{m + m', \infty\}$,
2. $\kappa = \omega$ if $\nu = \omega$, and
3. $\kappa = \text{hol}$ if $\nu = \text{hol}$.

As we allow the holomorphic case in our analysis, we also must allow for consideration of both real and complex scalar fields. To this end, we consider scalar fields $\mathbb{F}$ with

1. $\mathbb{F} = \mathbb{R}$ if $\kappa \in \{\infty, \omega\}$ and
2. $\mathbb{F} = \mathbb{C}$ if $\kappa = \text{hol}$.

Suppose that $\pi\colon \mathsf{E} \to \mathsf{M}$ is a C^κ-vector bundle, $\kappa \in \{\infty, \omega, \text{hol}\}$. We denote by $\Gamma^\nu(\mathsf{E})$ the $\mathbb{F}$-vector space of C^ν-sections of E. In Section 2 we quickly summarise descriptions of locally convex topologies for these spaces of sections, following Jafarpour and Lewis [2014]. All topologies, except the cases $\nu = m + \text{lip}$ and $\nu = \omega$, are more or less classical and are some variation of a compact-open type topology. Topologies such as this, at least in the cases $\nu \in \{m, \infty\}$, are called "weak topologies" by Hirsch [1976].[1] The topology in the locally Lipschitz case is fairly easily described, but we shall see that it has features different than the other cases. The topology in the real analytic case is not classical, and indeed is not fairly easy to describe.

Once we have these topologies at hand, we can easily consider time-dependent vector fields or, more generally, sections of a vector bundle as in [Jafarpour and Lewis 2014]. We denote by $\mathbb{T} \subseteq \mathbb{R}$ a nonempty interval. Denote by

$$\mathrm{L}^1_{\mathrm{loc}}(\mathbb{T}; \Gamma^\nu(\mathsf{E}))$$

the set of locally integrable functions with values in the locally convex space $\Gamma^\nu(\mathsf{E})$. Here, "locally integrable" means that the application of any continuous seminorm to the restriction to any compact subinterval of $\mathbb{T}$ of the vector-valued functions yields a function in the usual scalar L^1-space; we shall be more careful about this in Section 3.1. We note that $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{T}; \Gamma^\nu(\mathsf{E}))$ is itself a locally convex space with seminorms defined by

$$p_{\mathbb{K}}(\xi) = \int_{\mathbb{K}} p \circ \xi(t)\,\mathrm{d}t, \qquad p \text{ a continuous seminorm, } \mathbb{K} \subseteq \mathbb{T} \text{ compact.}$$

Note that, given $\xi \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{T}; \Gamma^\nu(\mathsf{E}))$, we have a map

$$\begin{aligned} \widehat{\xi}\colon \mathbb{T} \times \mathsf{M} &\to \mathsf{E} \\ (t, x) &\mapsto \xi(t)(x); \end{aligned}$$

[1] While these are not "weak" topologies in the usual sense of the word—i.e., locally convex initial topologies for some collection of linear maps—the use of the word "weak" in [Hirsch 1976] is to contrast with "strong," by which Hirsch means topologies of Whitney type.

thus we see that this is a class of time-dependent sections, with the time-dependence characterised in a particular, and particularly succinct, way. The main contribution of [Jafarpour and Lewis 2014] is then to show that, if $X \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{T};\Gamma^\nu(\mathsf{TM}))$ is a time-dependent vector field with flow $t \mapsto \Phi^X(t,t_0,x_0)$, then the map

$$x \mapsto \Phi^X(t_1,t_0,x)$$

is a class C^ν-local diffeomorphism for fixed t_1 and t_0, defined on some suitable open subset of M. Moreover, these classes of vector fields contain natural extensions of the usual hypotheses for existence and uniqueness (e.g., that are used in the classical Carathéodory existence and uniqueness theory) in the sense that these usual hypotheses are precisely those for membership in $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{T};\Gamma^{\mathrm{lip}}(\mathsf{TM}))$. The theory we develop here is more unified, more elegant, and more general than that of [Jafarpour and Lewis 2014], to which generalisation we now turn.

Let us indicate how we prescribe parameter-dependence; to our knowledge, this is a new description. We let $\mathcal{P}$ be a topological space. If $\xi \in \mathrm{C}^0(\mathcal{P};\mathrm{L}^1_{\mathrm{loc}}(\mathbb{T};\Gamma^\nu(\mathsf{E})))$,[2] then we have a mapping

$$\begin{aligned}\widehat{\xi}\colon \mathbb{T}\times\mathsf{M}\times\mathcal{P} &\to \mathsf{E}\\ (t,x,p) &\mapsto \xi(p)(t)(x);\end{aligned}$$

thus the continuous mappings from the parameter space into our class of time-dependent sections yield time-dependent, parameter-dependent sections, with the precise nature of the time- and parameter-dependence, again, succinctly characterised. The purpose of the paper is to consider continuity (more generally, regularity) properties of

$$(t,t_0,x_0,p) \mapsto \Phi^X(t,t_0,x_0,p)$$

for $X \in \mathrm{C}^0(\mathcal{P};\mathrm{L}^1_{\mathrm{loc}}(\mathbb{T};\Gamma^\nu(\mathsf{TM})))$. We do this by considering a general class of time- and parameter-dependent mappings between manifolds M and N. The class of mappings is, as with our description of time- and parameter-dependent vector fields just preceding, characterised by continuous mappings whose domain is $\mathcal{P}$. The codomain for these mappings is the set of locally absolutely continuous mappings from $\mathbb{T}$ to $\mathrm{C}^\nu(\mathsf{M};\mathsf{N})$; the precise definition for absolute continuity is given in Section 3.3. We devote some effort to a careful study of this class of time- and parameter-dependent mappings, as the class exhibits many of the deep properties of flows that one may like to prove. In this way, the proofs of the main results concerning flows follow from showing that flows belong to this class of mappings, in an appropriate sense.

The restriction to "an appropriate sense" in the preceding sentence is required simply because flows are not defined on all of $\mathbb{T}\times\mathbb{T}\times\mathsf{M}\times\mathcal{P}$, i.e., not for all sets of final times, initial times, initial conditions, and parameter values. What *is* true is that flows are *locally* defined, i.e., in a neighbourhood of $(t_0,t_0,x_0,p_0) \in \mathbb{T}\times\mathbb{T}\times\mathsf{M}\times\mathcal{P}$. In subsequent work, we carefully carry out the extension of these local results to their "as global as possible" counterparts; see the discussion in Section 5.3.

[2]One can also consider parameter-dependence with no time-dependence, and this gives a useful setting for control systems [Jafarpour and Lewis 2016].

1.3. A comparison with exponential series representations for flows. Let us compare our framework for time-dependent vector fields to that of Agrachev and Gamkrelidze [1978], as there are important similarities and differences. Our summary will be a little heuristic here for the sake of exposition, and the reader can refer to the development in the text for the precise statements. The work of Agrachev and Gamkrelidzedoes not include parameter-dependence. Therefore, the comparison we give here is made with our approach in its parameter-independent formulation, e.g., without its full level of generality.

In both our approach and that of Agrachev and Gamkrelidze [1978], a flow is represented by the use of C^κ-functions by the formula

$$f \circ \Phi^X(t, t_0, x_0) = f(x_0) + \int_{t_0}^{t} Xf(s, \Phi^X(s, t_0, x_0))\,\mathrm{d}s, \qquad f \in \mathrm{C}^\kappa(\mathsf{M}).$$

In [Agrachev and Gamkrelidze 1978], the smooth and real analytic cases are considered in Euclidean space, and the framework is presented on manifolds in the smooth case in [Agrachev and Sachkov 2004]. The idea in this work is to think of both vector fields and mappings as being continuous linear mappings on $\mathrm{C}^\kappa(\mathsf{M})$ by the formulae

$$f \mapsto Xf, \qquad f \mapsto f \circ \Phi,$$

respectively. A time-dependent vector field, in this framework, is then a locally integrable function from $\mathbb{T}$ to the space $\mathrm{L}(\mathrm{C}^\kappa(\mathsf{M}); \mathrm{C}^\kappa(\mathsf{M}))$ of continuous linear mappings. Then Agrachev and Gamkrelidze [1978] set up a scheme of Picard-like iterates in this space of integrable functions that resembles what one does in the classical theory. The result of the iterative scheme is a Volterra series representative for the flow composed with a function:

$$f \circ \Phi^X_t(x) = \left(\sum_{k=0}^{\infty} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{k-1}} X(t_1)X(t_2)\cdots X(t_k)f\,\mathrm{d}t_k \cdots \mathrm{d}t_2 \mathrm{d}t_1 \right)(x).$$

We note that the integrand for the iterated integral is not just a continuous linear mapping from $\mathrm{L}(\mathrm{C}^\kappa(\mathsf{M}); \mathrm{C}^\kappa(\mathsf{M}))$, it is a particular sort of such continuous linear mapping: namely, it is a linear partial differential operator. The iterative procedure has one peculiar property: it starts with the identity mapping on $\mathrm{C}^\kappa(\mathsf{M})$ and converges (if it converges at all) to the flow

$$t \mapsto \Phi^X(t, t_0, x_0) \in \mathrm{C}^\kappa(\mathsf{M}; \mathsf{M}) \subseteq \mathrm{L}(\mathrm{C}^\kappa(\mathsf{M}); \mathrm{C}^\kappa(\mathsf{M}))$$

(ignoring technicalities of the flow not being globally defined, for the moment). Thus both the initial data for the iterates and the limit of the iterates are mappings. However, while the iterates remain in $\mathrm{L}(\mathrm{C}^\kappa(\mathsf{M}); \mathrm{C}^\kappa(\mathsf{M}))$ (a little more precisely, they remain in the set of linear partial differential operators), they are generally *not* linear mappings that are themselves obtained from mappings of M, i.e., they are not of the form $f \mapsto f \circ \Phi$. Indeed, the iterates will themselves have no geometric meaning, apart from being linear partial differential operators. Additionally, one of the features (and limitations) of the iterative process is that one iterates Lie derivatives of X, infinitely many times in the limit, and so the method relies on a setting that is at least infinitely differentiable. The final limitation of this iterative scheme is that it does not converge in the smooth case.[3] Thus we see that the

[3] To use their method to arrive at the flow in the smooth case, Agrachev and Gamkrelidze [1978] use a completely different proof strategy from the Volterra series above in this case.

Picard-like iterative scheme of Agrachev and Gamkrelidze [1978] is really only applicable to the real analytic case.

Our approach is related, but different in important ways. First of all, as we indicate above, it includes parameter-dependence as well as time-dependence. Second, while we use an iterative scheme that closely resembles the Picard scheme from the standard theory and from its adaptation by Agrachev and Gamkrelidze [1978], the scheme starts in the class of mappings and remains in the class of mappings, also converging in the class of mappings. A cartoon depicting how this all fits together is given in Figure 2. Moreover, since the

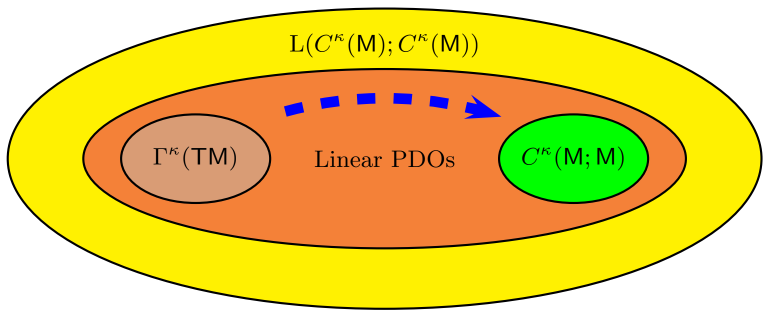


Figure 2. Vector fields are in the brown blob. The iterative scheme of Agrachev and Gamkrelidze [1978] starts in the brown blob, remains in the orange blob, and converges (when it converges) to something in the green blob. Our iterative scheme starts, stays in, and ends in the green blob.

iterations are not constructed using iterated Lie derivatives, the scheme makes sense for data that is not infinitely differentiable. The cost of an iterative scheme where the iterates remain in the space of mappings is that one must carefully analyse a nonlinear operator, related to the so-called superposition operator, that arises in defining the iterative procedure. We do this in Section 4, and the results of this section are where we make the most substantial use of the geometric function analytic tools developed in preceding sections. We note that the iterative scheme used by [Agrachev and Gamkrelidze 1978] proceeds making use only of linear analysis. Thus there is a tradeoff in the simplicity (if one wishes to call it this) of this iterative scheme versus the power of the results one gets.

1.4. Outline of paper. In Section 2, we develop the background for the paper. This background includes topological, differential geometric, and function analytic topics. In particular, we develop the topologies we use for spaces of vector fields of the various regularity classes described above.

In Section 3 we present the spaces of functions, sections, and mappings we use in the paper, in the presence of time- and parameter-dependence. We carefully develop the fundamental properties of these spaces, giving special attention to the locally Lipschitz case, since many of the classical properties of solutions of ordinary differential equations arise in a more general setting of locally absolutely continuous, parameter-dependent, locally Lipschitz mappings; see the results of Section 3.7.

A key aspect to our approach, and the one that lies behind the character of our theory as seen in Figure 2, is the use of composition operators, including the nonlinear superposition operator. This is developed in Section 4. The results we give concerning time- and parameter-dependent superposition operators constitute some of the main new results in the paper.

In Section 5, we give a few preliminary results concerning local flows for our class of time- and parameter-dependent vector fields in the classical case of Lipschitz regularity. First we define Picard operators for time- and parameter-dependent ordinary differential equations. This constitutes a careful development of the usual Picard iteration procedure in the classical existence and uniqueness theory, now adapted to the function analytic framework of the paper. After we do this, we prove our result concerning local flows in the Lipschitz case. This is meant to be an introduction to the methods outlined in the paper. We close the paper in Section 5.3 with a discussion of ongoing work along the lines of the paper that will appear in subsequent publications.

2. Geometric and function analytic tools

In this section we give a brief outline of the tools we use in the main body of the paper. We shall mainly give definitions and establish the facts that we require, referring the reader to the references for details.

2.1. Elementary notation. We shall use the slightly unconventional, but perfectly rational, notation of writing $A \subseteq B$ to denote set inclusion, and when we write $A \subset B$ we mean that $A \subseteq B$ and $A \neq B$. By id_A we denote the identity map on a set A. For a product $\prod_{i\in I} X_i$ of sets, $\mathrm{pr}_j \colon \prod_{i\in I} X_i \to X_j$ is the projection onto the jth component. By $\mathbb{Z}$ we denote the set of integers, with $\mathbb{Z}_{\geq 0}$ denoting the set of nonnegative integers and $\mathbb{Z}_{>0}$ denoting the set of positive integers. We denote by $\mathbb{R}$ the set of real numbers. By $\mathbb{R}_{\geq 0}$ we denote the set of nonnegative real numbers and by $\mathbb{R}_{>0}$ the set of positive real numbers. The set of complex numbers is denoted by $\mathbb{C}$.

For a topological space $\mathcal{X}$ and $A \subseteq \mathcal{X}$, $\mathrm{int}(A)$ denotes the interior of A and $\mathrm{cl}(A)$ denotes the closure of A. Neighbourhoods will always be open sets. By $\mathscr{K}(\mathcal{X})$ we denote the set of compact subsets of $\mathcal{X}$.

Elements of $\mathbb{R}^n$ are typically denoted with a bold font, e.g., "$\boldsymbol{x}$." Similarly, matrices are written using a bold font, e.g., "$\boldsymbol{A}$." By $\|\cdot\|$ we denote the Euclidean norm for $\mathbb{R}^n$ or $\mathbb{C}^n$. By $\mathsf{B}(r, \boldsymbol{x}) \subseteq \mathbb{R}^n$ we denote the open ball of radius r and centre $\boldsymbol{x}$. In like manner, $\overline{\mathsf{B}}(r, \boldsymbol{x})$ denotes the closed ball.

If $\mathcal{U} \subseteq \mathbb{R}^n$ is open and if $\Phi \colon \mathcal{U} \to \mathbb{R}^m$ is differentiable at $\boldsymbol{x} \in \mathcal{U}$, we denote its derivative by $\boldsymbol{D}\Phi(\boldsymbol{x})$. Higher-order derivatives, when they exist, are denoted by $\boldsymbol{D}^k\Phi(\boldsymbol{x})$, k being the order of differentiation. Note that $\boldsymbol{D}^k\Phi(\boldsymbol{x}) \in \mathrm{L}^k_{\mathrm{sym}}(\mathbb{R}^n; \mathbb{R}^m)$, this latter being the set of symmetric k-multilinear mappings from $(\mathbb{R}^n)^k$ to $\mathbb{R}^m$.

2.2. Manifolds and vector bundles. Our basic notation concerning differential geometry mostly follows [Abraham, Marsden, and Ratiu 1988]. We shall assume all manifolds to be of class C^κ, where $\kappa \in \{\infty, \omega, \mathrm{hol}\}$, and are Hausdorff, second countable, and connected; the choice of κ is adapted to the regularity of the other data as we explained in Section 1.2. When $\kappa = \mathrm{hol}$, we shall frequently ask that M be a Stein manifold; this means that there is a proper embedding of M in $\mathbb{C}^N$ for a suitable $N \in \mathbb{Z}_{>0}$ [Remmert 1954]. A typical C^κ-vector bundle we will denote by $\pi\colon \mathsf{E} \to \mathsf{M}$. The dual bundle we denote by E^*. The tangent bundle of a manifold M we denote by $\pi_{\mathsf{TM}}\colon \mathsf{TM} \to \mathsf{M}$ and the cotangent bundle by $\pi_{\mathsf{T}^*\mathsf{M}}\colon \mathsf{T}^*\mathsf{M} \to \mathsf{M}$.

As we have already indicated in Section 1.2, if $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, and if $\nu \in \{m + m', \infty, \omega, \mathrm{hol}\}$, then the C^ν-sections of E are denoted by $\Gamma^\nu(\mathsf{E})$. By $\mathrm{C}^\nu(\mathsf{M})$ we denote the set of C^ν-functions on M, noting that these are $\mathbb{F}$-valued, i.e., $\mathbb{C}$-valued when $\nu = \mathrm{hol}$ but $\mathbb{R}$-valued otherwise. If M and N are C^κ-manifolds, $\mathrm{C}^\nu(\mathsf{M};\mathsf{N})$ denotes the set of C^ν-mappings from M to N. If $f \in \mathrm{C}^{\nu+1}(\mathsf{M})$ and $X \in \Gamma^\nu(\mathsf{TM})$, we denote by $\mathscr{L}_X f \in \mathrm{C}^\nu(\mathsf{M})$ or $Xf \in \mathrm{C}^\nu(\mathsf{M})$ the Lie derivative of f with respect to X. If $\Phi \in \mathrm{C}^1(\mathsf{M};\mathsf{N})$, $T\Phi\colon \mathsf{TM} \to \mathsf{TN}$ denotes the derivative of Φ. We denote $T_x\Phi = T\Phi|\mathsf{T}_x\mathsf{M}$.

2.3. The rôle of jet bundles. Jet bundles arise in various ways in the paper.

For a C^κ-vector bundle $\pi\colon \mathsf{E} \to \mathsf{M}$, $\kappa \in \{\infty, \omega, \mathrm{hol}\}$, we denote by $\pi_m\colon \mathsf{J}^m\mathsf{E} \to \mathsf{M}$ the vector bundle of m-jets of sections of E; see [Kolář, Michor, and Slovák 1993, §12.17] and [Saunders 1989]. For C^κ-manifolds M and N, we denote by $\rho_0^m\colon \mathsf{J}^m(\mathsf{M};\mathsf{N}) \to \mathsf{M} \times \mathsf{N}$ the bundle of m-jets of mappings from M to N. We also have a fibre bundle

$$\rho_m \triangleq \mathrm{pr}_1 \circ \rho_0^m\colon \mathsf{J}^m(\mathsf{M};\mathsf{N}) \to \mathsf{M},$$

where pr_1 is the projection onto the first component. As a special case, $\mathsf{J}^m(\mathsf{M};\mathbb{R})$ denotes the bundle of m-jets of functions.

We signify the m-jet of a section, function, or mapping by use of the prefix j_m, i.e., $j_m\xi$, $j_m f$, or $j_m\Phi$. The set of jets of sections at x we denote by $\mathsf{J}^m_x\mathsf{E}$ and the set of jets of mappings at $(x,y) \in \mathsf{M} \times \mathsf{N}$ we denote by $\mathsf{J}^m(\mathsf{M};\mathsf{N})_{(x,y)}$. We denote by $\mathsf{T}^{*m}_x\mathsf{M} = \mathsf{J}^m(\mathsf{M};\mathbb{R})_{(x,0)}$ the jets of functions with value 0 at x, and $\mathsf{T}^{*m}\mathsf{M} = \cup_{x\in\mathsf{M}}\mathsf{T}^{*m}_x\mathsf{M}$. The space $\mathsf{T}^{*m}_x\mathsf{M}$ has the structure of a $\mathbb{R}$-algebra specified by requiring that

$$\mathfrak{m}_x^\kappa \ni f \mapsto j_m f(x) \in \mathsf{T}^{*m}_x\mathsf{M}$$

be a $\mathbb{R}$-algebra homomorphism, with $\mathfrak{m}_x^\kappa \subseteq \mathrm{C}^\kappa(\mathsf{M})$ being the ideal of functions vanishing at x. We then note [Kolář, Michor, and Slovák 1993, Proposition 12.9] that $\mathsf{J}^m(\mathsf{M};\mathsf{N})_{(x,y)}$ is identified with the set of $\mathbb{R}$-algebra homomorphisms from $\mathsf{T}^{*m}_y\mathsf{N}$ to $\mathsf{T}^{*m}_x\mathsf{M}$ according to

$$j_m\Phi(x)(j_m g(y)) = j_m(\Phi^* g)(x) \tag{2.1}$$

for Φ a smooth mapping defined in some neighbourhood of x and satisfying $\Phi(x) = y$.

Of particular interest is the bundle $\mathsf{J}^1(\mathsf{M};\mathsf{N})$ of 1-jets of mappings from M to N, which is a vector bundle over $\mathsf{M} \times \mathsf{N}$:

$$\mathsf{J}^1(\mathsf{M};\mathsf{N}) \simeq \mathrm{pr}_1^*\mathsf{T}^*\mathsf{M} \otimes \mathrm{pr}_2^*\mathsf{TN}, \tag{2.2}$$

where pr_a, $a \in \{1,2\}$, are the projections associated with the product $\mathsf{M}\times\mathsf{N}$ [Saunders 1989, Proposition 4.1.17]. If $\Phi \in \mathrm{C}^1(\mathsf{M};\mathsf{N})$, then we note that $T\Phi$ and $j_1\Phi$ are "the same," thought of in the right way. Precisely, $T_x\Phi$ and $j_1\Phi(x)$ agree as elements of $\mathrm{Hom}_{\mathbb{F}}(\mathsf{T}_x\mathsf{M};\mathsf{T}_{\Phi(x)}\mathsf{N})$. This can be thought of as $T\Phi$ and $j_1\Phi$ agreeing as vector bundle mappings over Φ:

$$\begin{array}{ccc} \mathsf{TM} & \xrightarrow{T\Phi = j_1\Phi} & \mathsf{TN} \\ \downarrow{\scriptstyle \pi_{\mathsf{TM}}} & & \downarrow{\scriptstyle \pi_{\mathsf{TN}}} \\ \mathsf{M} & \xrightarrow[\Phi]{} & \mathsf{N} \end{array} \tag{2.3}$$

We note that $\Gamma^\nu(\mathsf{J}^m\mathsf{E})$ can be thought of in the usual way since $\pi_m\colon \mathsf{J}^m\mathsf{E} \to \mathsf{M}$ is a C^κ-vector bundle. However, $\mathsf{J}^m(\mathsf{M};\mathsf{N})$ is not, generally, a vector bundle; nonetheless, we shall denote by $\Gamma^\nu(\mathsf{J}^m(\mathsf{M};\mathsf{N}))$ the set of C^ν-sections of the bundle $\rho_m\colon \mathsf{J}^m(\mathsf{M};\mathsf{N}) \to \mathsf{M}$.

A final piece of jet bundle structure of which we shall make use is the inclusion

$$\begin{aligned} \iota_{l,m}\colon \mathsf{J}^{l+m}\mathsf{E} &\to \mathsf{J}^l\mathsf{J}^m\mathsf{E} \\ j_{l+m}\xi(x) &\mapsto j_l j_m\xi(x), \end{aligned} \qquad l,m \in \mathbb{Z}_{\geq 0},$$

of jets associated with a vector bundle $\pi\colon \mathsf{E} \to \mathsf{M}$ and the inclusion

$$\begin{aligned} \iota_{l,m}\colon \mathsf{J}^{l+m}(\mathsf{M};\mathsf{N}) &\to \mathsf{J}^l(\mathsf{M};\mathsf{J}^m(\mathsf{M};\mathsf{N})) \\ j_{l+m}\Phi(x) &\mapsto j_l j_m\Phi(x), \end{aligned} \qquad l,m \in \mathbb{Z}_{\geq 0}, \tag{2.4}$$

of jets of mappings of manifolds M and N.

2.4. Linear and affine vector fields. Next we turn to vector fields on the total space of a vector bundle.

Let $m \in \mathbb{Z}_{\geq 0}$, let $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m+m', \infty, \omega, \mathrm{hol}\}$, and let $\kappa \in \{\infty, \omega, \mathrm{hol}\}$, as required. Let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a vector bundle of class C^κ.

A C^ν-vector field $X \in \Gamma^\nu(\mathsf{TE})$ is a ***linear vector field*** if

1. X is π-projectable, i.e., there exists a vector field $X_0 \in \Gamma^\nu(\mathsf{TM})$ such that $T_{e_x}\pi(X(e_x)) = X_0(x)$ for every $x \in \mathsf{M}$ and $e_x \in \mathsf{E}_x$, and
2. X is a vector bundle mapping for which the diagram

$$\begin{array}{ccc} \mathsf{E} & \xrightarrow{X} & \mathsf{TE} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle T\pi} \\ \mathsf{M} & \xrightarrow[X_0]{} & \mathsf{TM} \end{array}$$

commutes.

We say that X is a linear vector field ***over*** $\boldsymbol{X_0}$.

A C^ν-vector field $X \in \Gamma^\nu(\mathsf{TE})$ is an ***affine vector field*** if

1. X is β-projectable, i.e., there exists a vector field $X_0 \in \Gamma^\nu(\mathsf{TM})$ such that $T_{e_x}\beta(X(e_x)) = X_0(x)$ for every $x \in \mathsf{M}$ and $e_x \in \mathsf{E}_x$, and
2. X is an affine bundle mapping for which the diagram

$$\begin{array}{ccc} \mathsf{E} & \xrightarrow{X} & \mathsf{TE} \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle T\beta} \\ \mathsf{M} & \xrightarrow[X_0]{} & \mathsf{TM} \end{array}$$

commutes.

We say that X is an affine vector field ***over*** $\boldsymbol{X_0}$.

2.5. Prolongation of vector fields. The ***tangent lift*** of a vector field $X \in \Gamma^1(\mathsf{TM})$ is the vector field $X^T \in \Gamma^0(\mathsf{TTM})$ defined by the formula

$$X^T(v_x) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left(T_x\Phi_t^X(v_x)\right),$$

with $t \mapsto \Phi_t^X$ being the local flow of (time- and parameter-independent vector field) X.

The idea of the tangent lift is related to a lift to 1-jets by the identification depicted in (2.3). This can be generalised to a lift to m-jets. Let $X \in \Gamma^m(\mathsf{TM})$ be a C^m-vector field on a manifold M. Since Φ_t^X is a C^m-local diffeomorphism from standard results [e.g., Abraham, Marsden, and Ratiu 1988, Lemma 4.1.8], we have $j_m\Phi_t^X \in \mathsf{J}^m(\mathcal{U};\mathsf{M})$ for t fixed and small, and for a suitable open subset $\mathcal{U} \subseteq \mathsf{M}$. Thus we can define a vector field $\nu_m X$ on $\mathsf{J}^m(\mathsf{M};\mathsf{M})$ by

$$\nu_m X(j_m\Psi(x)) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} j_m(\Phi_t^X \circ \Psi)(x).$$

In particular,

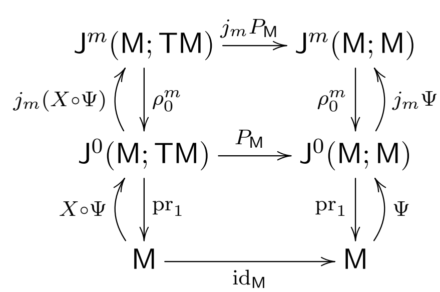
$$\nu_m X(j_m\Phi_t^X(x)) = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} j_m(\Phi_s^X \circ \Phi_t^X)(x) = \frac{\mathrm{d}}{\mathrm{d}t} j_m\Phi_t^X(x),$$

i.e., $t \mapsto j_m\Phi_t^X(x)$ is the integral curve of $\nu_m X$ with initial state $j_m \mathrm{id}_{\mathsf{M}}(x)$.

We wish to better understand this vector field $\nu_m X$. In order to do this, we note that $\mathsf{M} \times \mathsf{TM}$ has (at least) two bundle structures:

$$\mathrm{pr}_1 \colon \mathsf{M} \times \mathsf{TM} \to \mathsf{M}, \quad P_{\mathsf{M}} \triangleq \mathrm{id}_{\mathsf{M}} \times \pi_{\mathsf{TM}} \colon \mathsf{M} \times \mathsf{TM} \to \mathsf{M} \times \mathsf{M}.$$

The first gives the structure of a fibre bundle and the second the structure of a vector bundle. We shall denote the jets associated with these two different bundle structures by $\mathsf{J}^m(\mathsf{M};\mathsf{TM})$ and $\widehat{\mathsf{J}}^m(\mathsf{M};\mathsf{TM})$, respectively. We note that $\widehat{\mathsf{J}}^m(\mathsf{M};\mathsf{TM})$ is a vector bundle over $\mathsf{J}^m(\mathsf{M};\mathsf{M})$, by virtue the (easily verified to be commutative) diagram



which we can draw for every $\Psi \in \mathrm{C}^\infty(\mathsf{M};\mathsf{M})$. In the diagram, we think of P_{M} as being a morphism of fibred manifolds over M, with $j_m P_{\mathsf{M}}$ as its prolongation [Saunders 1989, §4.2]. If we recall that $\rho_m \colon \mathsf{J}^m(\mathsf{M};\mathsf{M}) \to \mathsf{M}$ is the projection onto the source space, then we denote by

$$\mathsf{VJ}^m(\mathsf{M};\mathsf{M}) = \ker(T\rho_m)$$

the vertical bundle, noting that this is a subbundle of $\mathsf{TJ}^m(\mathsf{M};\mathsf{M})$.

Note that, if $\overline{X} \in \mathrm{C}^m(\mathsf{M};\mathsf{TM})$, then there is an induced mapping $\Phi \in \mathrm{C}^m(\mathsf{M};\mathsf{M})$ for which the diagram

$$\begin{array}{ccc} & & \mathsf{TM} \\ & \overset{\overline{X}}{\nearrow} & \downarrow{\scriptstyle \pi_{\mathsf{TM}}} \\ \mathsf{M} & \xrightarrow[\Phi]{} & \mathsf{M} \end{array}$$

commutes. Thus we can think of mappings from M to TM as "vector fields over a mapping $\Phi\colon \mathsf{M} \to \mathsf{M}$." In particular, if $X \in \Gamma^m(\mathsf{TM})$ and if $\Phi\colon \mathsf{M} \to \mathsf{M}$, then $X \circ \Phi$ gives such a mapping from M to TM.

The following lemma relates the preceding constructions, and also characterises the vector field $\nu_m X$.

2.1 Lemma: (A jet bundle identification) *Let M be a C^∞-manifold and let $m \in \mathbb{Z}_{>0}$. Then the following statements hold:*

(i) there is a canonical isomorphism

$$\alpha_m \colon \mathsf{J}^m(\mathsf{M};\mathsf{TM}) \to \mathsf{VJ}^m(\mathsf{M};\mathsf{M})$$

of fibre bundles over M;

(ii) there is a canonical isomorphism

$$\widehat{\alpha}_m \colon \widehat{\mathsf{J}}^m(\mathsf{M};\mathsf{TM}) \to \mathsf{VJ}^m(\mathsf{M};\mathsf{M})$$

of vector bundles over $\mathsf{J}^m(\mathsf{M};\mathsf{M})$.

Now, if $X \in \Gamma^m(\mathsf{TM})$, then the following statements hold:

(iii) $\nu_m X(j_m\Psi(x)) = \alpha_m \circ j_m(X \circ \Psi)(x)$;

(iv) the diagram

$$\begin{array}{ccc} \mathsf{J}^m(\mathsf{M};\mathsf{M}) & \xrightarrow{\nu_m X} & \mathsf{TJ}^m(\mathsf{M};\mathsf{M}) \\ {\scriptstyle \rho^m_{m-1}}\downarrow & & \downarrow{\scriptstyle T\rho^m_{m-1}} \\ \mathsf{J}^{m-1}(\mathsf{M};\mathsf{M}) & \xrightarrow[\nu_{m-1}X]{} & \mathsf{TJ}^{m-1}(\mathsf{M};\mathsf{M}) \end{array}$$

commutes and gives $\nu_m X$ as a C^0-affine vector field over $\nu_{m-1}X$.

Proof: (i) We describe the diffeomorphism α_m, and then note that the verification that it is, in fact, a diffeomorphism is a fact easily checked in jet bundle coordinates.

Let $I \subseteq \mathbb{R}$ be an interval with $0 \in \mathrm{int}(I)$ and consider a smooth time-varying mapping $\psi\colon I \times \mathsf{M} \to \mathsf{M}$. We let $\psi_t(x) = \psi^x(t) = \psi(t,x)$. We then have maps

$$\begin{aligned} j^x_m\psi\colon I &\to \mathsf{J}^m(\mathsf{M};\mathsf{M}) \\ t &\mapsto j_m\psi_t(x) \end{aligned}$$

and

$$\begin{aligned} \psi'\colon \mathsf{M} &\to \mathsf{TM} \\ x &\mapsto \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \psi^x(t). \end{aligned}$$

Note that the curve $j^x_m\psi$ is a curve in the fibre of $\rho_m\colon \mathsf{J}^m(\mathsf{M};\mathsf{M})\to\mathsf{M}$. Thus we can sensibly define α_m by

$$\alpha_m(j_m\psi'(x)) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} j^x_m\psi(t).$$

We denote jet bundle coordinates for $\mathsf{J}^m(\mathsf{M};\mathsf{TM})$ by

$$((\boldsymbol{x}_1,(\boldsymbol{x}_2,\boldsymbol{A}_0)),(\boldsymbol{B}_1,\boldsymbol{A}_1,\dots,\boldsymbol{B}_m,\boldsymbol{A}_m)),$$

where $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ reside in some open subset of $\mathbb{R}^n$, and where

$$\begin{aligned}\boldsymbol{A}_j &\in \mathrm{L}^j_{\mathrm{sym}}(\mathbb{R}^n;\mathbb{R}^n), \qquad && j\in\{0,1,\dots,m\},\\ \boldsymbol{B}_j &\in \mathrm{L}^j_{\mathrm{sym}}(\mathbb{R}^n;\mathbb{R}^n), \qquad && j\in\{1,\dots,m\},\end{aligned}$$

reside in suitable spaces of symmetric multilinear mappings. In like manner, we denote jet bundle coordinates for $\mathsf{VJ}^m(\mathsf{M};\mathsf{M})$ by

$$((\boldsymbol{x}_1,(\boldsymbol{x}_2,\boldsymbol{B}_1,\dots,\boldsymbol{B}_m)),(\boldsymbol{A}_0,\boldsymbol{A}_1,\dots,\boldsymbol{A}_m)),$$

for

$$\begin{aligned}\boldsymbol{A}_j &\in \mathrm{L}^j_{\mathrm{sym}}(\mathbb{R}^n;\mathbb{R}^n), \qquad && j\in\{0,1,\dots,m\},\\ \boldsymbol{B}_j &\in \mathrm{L}^j_{\mathrm{sym}}(\mathbb{R}^n;\mathbb{R}^n), \qquad && j\in\{1,\dots,m\}.\end{aligned}$$

One can then check that α_m has the local representative

$$((\boldsymbol{x}_1,(\boldsymbol{x}_2,\boldsymbol{A}_0)),(\boldsymbol{B}_1,\boldsymbol{A}_1,\dots,\boldsymbol{B}_m,\boldsymbol{A}_m)) \mapsto ((\boldsymbol{x}_1,(\boldsymbol{x}_2,\boldsymbol{B}_1,\dots,\boldsymbol{B}_m)),(\boldsymbol{A}_0,\boldsymbol{A}_1,\dots,\boldsymbol{A}_m)),$$

showing that α_m is indeed a diffeomorphism.

(ii) We denote jet bundle coordinates for $\widehat{\mathsf{J}}^m(\mathsf{M};\mathsf{TM})$ by

$$((\boldsymbol{x}_1,\boldsymbol{x}_2),(\boldsymbol{B}_1,\dots,\boldsymbol{B}_m),(\boldsymbol{A}_0,\boldsymbol{A}_1,\dots,\boldsymbol{A}_m))$$

for

$$\begin{aligned}\boldsymbol{A}_j &\in \mathrm{L}^j_{\mathrm{sym}}(\mathbb{R}^n;\mathbb{R}^n), \qquad && j\in\{0,1,\dots,m\},\\ \boldsymbol{B}_j &\in \mathrm{L}^j_{\mathrm{sym}}(\mathbb{R}^n;\mathbb{R}^n), \qquad && j\in\{1,\dots,m\}.\end{aligned}$$

In this case, the mapping $\widehat{\alpha}_m$ has the local representative

$$((\boldsymbol{x}_1,(\boldsymbol{x}_2,\boldsymbol{B}_1,\dots,\boldsymbol{B}_m)),(\boldsymbol{A}_0,\boldsymbol{A}_1,\dots,\boldsymbol{A}_m)),\mapsto((\boldsymbol{x}_1,\boldsymbol{x}_2),(\boldsymbol{B}_1,\dots,\boldsymbol{B}_m),(\boldsymbol{A}_0,\widehat{\boldsymbol{A}}_1,\dots,\boldsymbol{A}_m))$$

which shows that $\widehat{\alpha}_m$ is a vector bundle isomorphism.

(iii) Define $\psi(t,x)=\Phi^X_t\circ\Psi(x)$. Then, in our notation above, $\psi'(x)=X\circ\Psi(x)$. Thus we have

$$\begin{aligned}\alpha_m\circ j_m(X\circ\Psi)(x) = \alpha_m\circ j_m\psi'(x) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} j^x_m\psi(t)\\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} j_m(\Phi^X_t\circ\Psi)(x) = \nu_m X(j_m\Psi(x)),\end{aligned}$$

as desired.

(iv) By Lemma 2.2 below, it suffices to consider the case of $m = 1$. In our coordinates above, the local representative of $\nu_0 X$ is

$$(\boldsymbol{x}_1, \boldsymbol{x}_2) \mapsto ((\boldsymbol{x}_1, \boldsymbol{x}_2), (\boldsymbol{0}, \boldsymbol{X}(\boldsymbol{x}_2)))$$

and the local representative of $\nu_1 X$ is

$$((\boldsymbol{x}_1, \boldsymbol{x}_2), \boldsymbol{A}) \mapsto (((\boldsymbol{x}_1, \boldsymbol{x}_2), \boldsymbol{A}), ((\boldsymbol{0}, \boldsymbol{X}(\boldsymbol{x}_2)), \boldsymbol{D}X(\boldsymbol{x}_2) \circ \boldsymbol{A}).$$

The result in the case $m = 1$ follows from these formulae. ■

Following the diagram (2.3) relating $T\Phi$ and $j_1\Phi$, we can provide an understanding of the relationship between the tangent lift X^T and the first prolongation $\nu_1 X$ of $X \in \Gamma^1(\mathsf{TM})$. Let $x \in \mathsf{M}$, let $v \in \mathsf{T}_x\mathsf{M}$, and let $\Psi \in \mathrm{C}^1(\mathsf{M}; \mathsf{M})$. Denote $y = \Psi(x)$. For t small, we have

$$\mathsf{T}_x\mathsf{M} \xrightarrow{j_1\Psi} \mathsf{T}_y\mathsf{M} \xrightarrow{j_1\Phi_t^X} \mathsf{T}_{\Phi_t^X(y)}\mathsf{M}$$

Thus

$$t \mapsto j_1\Phi_t^X \circ j_1\Psi(v) = j_1(\Phi_t^X \circ \Psi)(v)$$

is a curve in $\{x\} \times \mathsf{TM} \subseteq \mathsf{J}^0(\mathsf{M}; \mathsf{TM})$. Let us denote

$$j_1\Psi(v) \lrcorner\, \nu_1 X = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} j_1(\Phi_t^X \circ \Psi)(v).$$

Since

$$T(\Phi_t^X \circ \Psi)(v) = j_1(\Phi_t^X \circ \Psi)(v),$$

according to (2.3), we then have

$$X^T(T\Psi(v)) = j_1\Psi(v) \lrcorner\, \nu_1 X.$$

It is useful to see how the vector field $\nu_m X$ interacts with the inclusion (2.4) of jet bundles.

2.2 Lemma: (“$\boldsymbol{\nu_{l+m}X = \nu_l\nu_m X}$”) *Let $l, m \in \mathbb{Z}_{\geq 0}$, let M be a C^∞-manifold and let $X \in \Gamma^{l+m}(\mathsf{TM})$. Then*

$$T\iota_{l,m}(\nu_{l+m}X(j_{l+m}\Psi(x))) = \nu_l\nu_m X(j_m\Psi(x)).$$

Proof: This is a direct computation:

$$\begin{aligned}
T\iota_{l,m}(\nu_{l+m}X(j_{l+m}\Psi(x))) &= T\iota_{l,m}\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} j_{l+m}(\Phi_t^X \circ \Psi)(x)\right) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} j_l(j_m(\Phi_t^X \circ \Psi))(x) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} j_l\Phi_t^{\nu_m X}(j_m\Psi(x)) \\
&= \nu_l\nu_m X(j_m\Psi(x)),
\end{aligned}$$

as desired. ■

2.6. The rôle of Riemannian and fibre metrics. We shall make use of Riemannian and fibre metrics for the reason of convenience. Many definitions we give use a specific choice for such metrics, although none of the results depend on these choices. A similar state of affairs exists for choices of connections, as we will discuss in Section 2.7. This independence of the results on metrics and connections is more or less easy to see in all cases, the main exception being in the real analytic case. Here one must undertake some effort to show that, in fact, all constructions do not depend essentially on these choices. Thankfully, we can just take this for granted as the work here has been done by Lewis [2023, §4.3].

For $\kappa \in \{\infty, \omega\}$, we let $\pi \colon \mathsf{E} \to \mathsf{M}$ be a C^κ-vector bundle.

We denote by $\mathbb{G}_\mathsf{M}$ a C^κ-Riemannian metric on M and by $\mathbb{G}_\pi$ a C^κ-metric for the fibres of E. The existence of these in the real analytic case is verified by Jafarpour and Lewis [2014, Lemma 2.4]. The metrics $\mathbb{G}_\mathsf{M}$ and $\mathbb{G}_\pi$ then induce metrics in all tensor products of TM and E and their duals. For simplicity, we denote any such metric simply by $\mathbb{G}_{\mathsf{M},\pi}$.

We will frequently make use of the distance function on M associated with a Riemannian metric $\mathbb{G}$. In order to have constructions involving $\mathbb{G}$ make sense—in terms of not depending on the choice of Riemannian metric—we should verify that such constructions do not depend on the choice of this metric. Of course, this is not true for all manner of general assertions. However, the following lemma captures what we need.

2.3 Lemma: (Comparison of Riemannian distance for different Riemannian metrics) *If $\mathbb{G}_1$ and $\mathbb{G}_2$ are C^∞-Riemannian metrics on a C^∞-manifold M with metrics d_1 and d_2, respectively, and if $K \subseteq \mathsf{M}$ is compact, then there exists $C \in \mathbb{R}_{>0}$ such that*

$$C^{-1}\mathrm{d}_1(x_1, x_2) \le \mathrm{d}_2(x_1, x_2) \le C\mathrm{d}_1(x_1, x_2)$$

for every $x_1, x_2 \in K$.

Proof: See [Lewis 2023, Lemma 4.21]. ■

2.7. The rôle of affine and linear connections. For convenience we shall make use of connections in representing certain objects that do not actually require a connection for their description. While this is done as a convenience in some way, it can introduce its own set of complications. However, the complications do not bother us here as they have been dealt with elsewhere, especially in [Lewis 2023].

For $\kappa \in \{\infty, \omega\}$, we let $\pi \colon \mathsf{E} \to \mathsf{M}$ be a C^κ-vector bundle.

We let ∇^M be a C^κ-affine connection on M and we let ∇^π denote a C^κ-linear connection in the vector bundle. The existence of these in the real analytic case is proved by [Jafarpour and Lewis 2014, Lemma 2.4]. Almost always we will not require ∇^M to be the Levi-Civita connection for the Riemannian metric $\mathbb{G}_\mathsf{M}$, nor do we typically require there to be any metric relationship between ∇^π and $\mathbb{G}_\pi$. However, in our constructions for the Lipschitz topology, it is sometimes convenient to assume that ∇^M is the Levi-Civita connection for $\mathbb{G}_\mathsf{M}$ and that ∇^π is $\mathbb{G}_\pi$-orthogonal, i.e., parallel transport consists of inner product preserving mappings. Thus, a safety-minded reader may wish to make these assumptions in all cases.

Following Jafarpour and Lewis [2014, Lemma 2.1], we make a decomposition of the jet bundles $\pi_m \colon \mathsf{J}^m\mathsf{E} \to \mathsf{M}$, using the connections ∇^M and ∇^π. Indeed, these connections enable isomorphisms

$$\mathsf{J}^m\mathsf{E} \simeq \bigoplus_{j=0}^m \mathrm{S}^j(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{E}, \qquad m \in \mathbb{Z}_{\ge 0}. \tag{2.5}$$

Here $\mathrm{S}^j(\mathsf{V})$ is the set of symmetric tensors of degree j for a vector space V. We denote the projection of $j_m\xi(x)$ onto the jth component of the direct sum by $D^j_{\nabla^{\mathsf{M}},\nabla^\pi}(\xi)(x)$.

In the case of functions, which we can regard as sections of the trivial line bundle, by using the flat connection on this trivial bundle we reduce the data required for a decomposition of jet bundles to an affine connection ∇^{M}. Thus, if $\mathsf{J}^m(\mathsf{M};\mathbb{R})$ denotes the bundle of m-jets of functions, then we have

$$\mathsf{J}^m(\mathsf{M};\mathbb{R}) \simeq \bigoplus_{j=0}^m \mathrm{S}^j(\mathsf{T}^*\mathsf{M}).$$

In this case, we denote the projection of $j_mf(x)$ onto the jth component of the direct sum by $D^j_{\nabla^{\mathsf{M}}}(f)(x)$.

Our metrics $\mathbb{G}_{\mathsf{M}}$ and $\mathbb{G}_\pi$ then allow us to define fibre metrics for $\mathsf{J}^m\mathsf{E}$ using the above decomposition. That is, we define a fibre metric for $\mathsf{J}^m\mathsf{E}$ by

$$\mathbb{G}_{\mathsf{M},\pi,m}(j_m\xi(x), j_m\eta(x)) = \sum_{j=0}^m \mathbb{G}_{\mathsf{M},\pi}\left(\frac{1}{j!}D^j_{\nabla^{\mathsf{M}},\nabla^\pi}(\xi)(x), \frac{1}{j!}D^j_{\nabla^{\mathsf{M}},\nabla^\pi}(\eta)(x)\right). \tag{2.6}$$

for jet bundles, with the associated fibre norm $\|\cdot\|_{\mathbb{G}_{\mathsf{M},\pi,m}}$. In the special case of functions, as sections of the trivial line bundle, we denote

$$\mathbb{G}_{\mathsf{M},m}(j_mf(x), j_mg(x)) = \sum_{j=0}^m \mathbb{G}_{\mathsf{M}}\left(\frac{1}{j!}D^j_{\nabla^{\mathsf{M}}}(f)(x), \frac{1}{j!}D^j_{\nabla^{\mathsf{M}}}(g)(x)\right). \tag{2.7}$$

The reader may wonder whether the factorials are necessary in these formulae; they are, thanks to the complications of proving independence of topologies on metrics and connections in the real analytic case.

2.8. Locally Lipschitz sections of vector bundles. As we are interested in ordinary differential equations with well-defined flows, we must, according to the usual theory, consider locally Lipschitz sections of vector bundles. In particular, we will find it essential to topologise the space of locally Lipschitz sections of $\pi\colon \mathsf{E} \to \mathsf{M}$. To define the seminorms for this topology, we make use of a "local least Lipschitz constant."

We let $\xi\colon \mathsf{M} \to \mathsf{E}$ be such that $\xi(x) \in \mathsf{E}_x$ for every $x \in \mathsf{M}$. For a piecewise differentiable curve $\gamma\colon [0,T] \to \mathsf{M}$, we denote by $\tau_{\gamma,t}\colon \mathsf{E}_{\gamma(0)} \to \mathsf{E}_{\gamma(t)}$ the isomorphism of parallel translation along γ for each $t \in [0,T]$. We then define, for $K \subseteq \mathsf{M}$ compact,

$$l_K(\xi) = \sup\left\{\frac{\|\tau^{-1}_{\gamma,1}(\xi\circ\gamma(1)) - \xi\circ\gamma(0)\|_{\mathbb{G}_\pi}}{\ell_{\mathbb{G}_{\mathsf{M}}}(\gamma)} \;\middle|\; \gamma\colon [0,1]\to\mathsf{M},\ \gamma(0),\gamma(1)\in K,\ \gamma(0)\neq\gamma(1)\right\},$$

which is the ***K*-sectional dilatation** of ξ. Here $\ell_{\mathbb{G}_{\mathsf{M}}}$ is the length function on piecewise differentiable curves. We also define

$$\begin{aligned} &\operatorname{dil}\xi\colon \mathsf{M} \to \mathbb{R}_{\geq 0} \\ &\qquad x \mapsto \inf\{l_{\mathrm{cl}(\mathcal{U})}(\xi) \mid \mathcal{U} \text{ is a precompact neighbourhood of } x\}, \end{aligned}$$

which is the ***local sectional dilatation*** of ξ. Note that, while the values taken by $\operatorname{dil}\xi$ will depend on the choice of a Riemannian metric $\mathbb{G}$, the property $\operatorname{dil}\xi(x) < \infty$ for $x \in \mathsf{M}$ is independent of $\mathbb{G}$, thanks to Lemma 2.3. And, since $\xi \in \Gamma^{\mathrm{lip}}(\mathsf{E})$ if and only if $\operatorname{dil}\xi(x) < \infty$ for every $x \in \mathsf{M}$, [Jafarpour and Lewis 2014, Lemma 3.10], this is what is important.

The following characterisations of the local sectional dilatation are useful.

2.4 Lemma: (Local sectional dilatation using derivatives) *For a C^∞-vector bundle $\pi\colon \mathsf{E} \to \mathsf{M}$ and for $\xi \in \Gamma^{\mathrm{lip}}(\mathsf{E})$, we have*

$$\operatorname{dil}\xi(x) = \inf\{\sup\{\|\nabla^{\pi}_{v_y}\xi\|_{\mathbb{G}_{\mathsf{M},\pi}} \mid y \in \mathrm{cl}(\mathcal{U}),\ \|v_y\|_{\mathbb{G}_{\mathsf{M}}} = 1,\ \xi \text{ differentiable at } y\} \mid \mathcal{U} \text{ is a precompact neighbourhood of } x\}.$$

Proof: [Jafarpour and Lewis 2014, Lemma 3.12]. ■

2.5 Lemma: (Local sectional dilatation and sectional dilatation) *Let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a C^∞-vector bundle. Then, for each $x_0 \in \mathsf{M}$, there exists a precompact neighbourhood $\mathcal{U}$ of x_0 such that*

$$l_{\mathrm{cl}(\mathcal{U})}(\xi) = \sup\{\operatorname{dil}\xi(x) \mid x \in \mathrm{cl}(\mathcal{U})\}, \qquad \xi \in \Gamma^{\mathrm{lip}}(\mathsf{E}).$$

Proof: Making reference to the proof of Lemma 2.4 given in [Jafarpour and Lewis 2014], we let $\mathcal{U}$ be a geodesically convex neighbourhood of x_0 so that

$$l_{\mathrm{cl}(\mathcal{U})}(\xi) = \sup\{\|\nabla^{\pi}_{v_y}\xi\|_{\mathbb{G}_{\mathsf{M},\pi}} \mid y \in \mathrm{cl}(\mathcal{U}),\ \|v_y\|_{\mathbb{G}_{\mathsf{M}}} = 1,\ \xi \text{ differentiable at } y\}.$$

Thus $l_{\mathrm{cl}(\mathcal{U})}(\xi)$ is an upper bound for

$$\{\operatorname{dil}\xi(x) \mid x \in \mathrm{cl}(\mathcal{U})\}.$$

Next, let $\epsilon \in \mathbb{R}_{>0}$. Let $x \in \mathcal{U}$ and $v_x \in \mathsf{T}_x\mathsf{M}$ be such that (1) ξ is differentiable at x, (2) $\|v_x\|_{\mathbb{G}_{\mathsf{M}}} = 1$, and (3) $l_{\mathrm{cl}(\mathcal{U})}(\xi) - \|\nabla^{\pi}_{v_x}\xi\|_{\mathbb{G}_{\mathsf{M},\pi}} < \frac{\epsilon}{2}$. Then let $\mathcal{V}$ be a geodesically convex neighbourhood of x such that $\mathrm{cl}(\mathcal{V}) \subseteq \mathcal{U}$ and such that

$$\sup\{\|\nabla^{\pi}_{v_y}\xi\|_{\mathbb{G}_{\mathsf{M},\pi}} \mid y \in \mathrm{cl}(\mathcal{V}),\ \|v_y\|_{\mathbb{G}_{\mathsf{M}}} = 1,\ \xi \text{ differentiable at } y\} - \operatorname{dil}\xi(x) < \frac{\epsilon}{2}.$$

We have

$$\begin{aligned} l_{\mathrm{cl}(\mathcal{U})}(\xi) - \frac{\epsilon}{2} &< \|\nabla^{\pi}_{v_x}\xi\|_{\mathbb{G}_{\mathsf{M},\pi}} \\ &\leq \sup\{\|\nabla^{\pi}_{v_y}\xi\|_{\mathbb{G}_{\mathsf{M},\pi}} \mid y \in \mathrm{cl}(\mathcal{V}),\ \|v_y\|_{\mathbb{G}_{\mathsf{M}}} = 1,\ \xi \text{ differentiable at } y\} \leq l_{\mathrm{cl}(\mathcal{U})}(\xi). \end{aligned}$$

Therefore,

$$\begin{aligned} l_{\mathrm{cl}(\mathcal{U})}(\xi) - \epsilon &= l_{\mathrm{cl}(\mathcal{U})}(\xi) - \frac{\epsilon}{2} - \frac{\epsilon}{2} \\ &\leq \sup\{\|\nabla^{\pi}_{v_y}\xi\|_{\mathbb{G}_{\mathsf{M},\pi}} \mid y \in \mathrm{cl}(\mathcal{V}),\ \|v_y\|_{\mathbb{G}_{\mathsf{M}}} = 1,\ \xi \text{ differentiable at } y\} \\ &\quad + \operatorname{dil}\xi(x) - \sup\{\|\nabla^{\pi}_{v_y}\xi\|_{\mathbb{G}_{\mathsf{M},\pi}} \mid y \in \mathrm{cl}(\mathcal{V}),\ \|v_y\|_{\mathbb{G}_{\mathsf{M}}} = 1,\ \xi \text{ differentiable at } y\} \\ &= \operatorname{dil}\xi(x). \end{aligned}$$

This shows that $l_{\mathrm{cl}(\mathcal{U})}(\xi)$ is the least upper bound for

$$\{\operatorname{dil}\xi(x) \mid x \in \mathrm{cl}(\mathcal{U})\},$$

as required. ■

2.9. Topological and function analytical background. Throughout the paper, we make use of ideas and results from topology and functional analysis that perhaps are not standard for some researchers in differential equations. As a reference for topics in topology, we recommend [Willard 1970]. As a reference for topics in functional analysis, we recommend [Jarchow 1981]. A slightly more fulsome, but definitely not comprehensive, summary of what we need in the paper can be found in Sections 1.7 and 1.8 of [Lewis 2023].

In topology, we frequently will make reference to "initial" topologies. These are defined as follows. We let $((\mathcal{X}_i, \mathscr{O}_i))_{i \in I}$ be a family of topological spaces, let $\mathcal{Y}$ be a set, and let $\Phi_i \colon \mathcal{Y} \to \mathcal{X}_i$, $i \in I$, be a familt of mappings. The ***initial topology*** for $\mathcal{Y}$ defined by the mappings Φ_i, $i \in I$, is the coarsest topology for $\mathcal{Y}$ such that each of the mappings Φ_i, $i \in I$, is continuous. The subsets $\Phi_i^{-1}(\mathcal{O}_i)$, $\mathcal{O}_i \in \mathscr{O}_i$, $i \in I$, are a base for the initial topology. The initial topology is characterised by the following fact. If $(\mathcal{Z}, \mathscr{O})$ is a topological space, a mapping $\Psi \colon \mathcal{Z} \to \mathcal{Y}$ is continuous if and only if the diagram

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{\Phi_i} & \mathcal{X}_i \\ {\scriptstyle\Psi}\uparrow & \nearrow_{\Phi_i \circ \Psi} & \\ \mathcal{Z} & & \end{array}$$

is a commutative diagram of topological spaces for each $i \in I$.

The fact that our important spaces of sections are "Suslin" spaces is important. The meaning of "Suslin" is as follows. A ***Polish space*** is a topological space $(\mathcal{X}, \mathscr{O})$ for which the topology is separable and is the complete metric topology for some metric d on $\mathcal{X}$. A ***Suslin space*** is a topological space $(\mathcal{X}, \mathscr{O})$ for which there exists a Polish space $(\mathcal{X}', \mathscr{O}')$ and a continuous surjective mapping $\Phi \in \mathrm{C}^0(\mathcal{X}'; \mathcal{X})$, with $\mathrm{C}^0(\mathcal{X}'; \mathcal{X})$ the space of continuous mappings from $(\mathcal{X}', \mathscr{O}')$ to $(\mathcal{X}, \mathscr{O})$.

We shall make reference to the notion of a uniform space. There are at least three ways to define what is meant by a uniform space. For our purposes, we choose the most concrete but least popular definition, where a ***uniform space*** is a topological space $(\mathcal{X}, \mathscr{O})$ whose topology is defined by a family $(\mathrm{d}_i)_{i \in I}$ of semimetrics, a semimetric being a metric, absent the property that it is zero only when its two arguments agree. When we say that the topology is "defined by" the semimetrics, we mean that the balls

$$\mathsf{B}_i(r, x_0) = \{x \in \mathcal{X} \mid \mathrm{d}_i(x, x_0) < r\}, \qquad x_0 \in \mathcal{X},\ r \in \mathbb{R}_{>0},$$

are a subbase for the topology $\mathscr{O}$, i.e., open sets are unions of finite intersections of these subbasic sets.

In terms of functional analysis, we make reference to and use of properties of locally convex topological vector spaces, or just "locally convex spaces." Normed vector spaces are examples of locally convex spaces, but almost all of our examples of locally convex spaces will not be normed. A locally convex space has a topology defined by a family $(p_i)_{i \in I}$ of seminorms, a seminorm being a norm, absent the definiteness property. Some of our locally convex spaces will be ***Fréchet spaces***, meaning that their locally convex topology is that defined by a translation-invariant metric. However, in the real analytic setting, one must contend with locally convex spaces that are not metrisable.

We shall make use of the notion of an inverse limit of a directed family of locally convex spaces, this being an important special case where initial topologies arise. We refer to [Jarchow 1981, §2.6] for details.

If $(\mathsf{U},\mathscr{O}_{\mathsf{U}})$ and $(\mathsf{V},\mathscr{O}_{\mathsf{V}})$ are locally convex spaces, we denote by $\mathrm{L}(\mathsf{U};\mathsf{V})$ the set of continuous linear mappings. By V' we denote the topological dual of a locally convex space $(\mathsf{V},\mathscr{O})$.

2.10. Topologies for spaces of sections of vector bundles. We can now quickly describe the topologies for $\Gamma^{\nu}(\mathsf{E})$ that we use in the paper. We refer to [Jafarpour and Lewis 2014] for details.

Let $\kappa \in \{\infty,\omega,\mathrm{hol}\}$ and let $\pi\colon \mathsf{E}\to\mathsf{M}$ be a C^{κ}-vector bundle, let $m\in\mathbb{Z}_{\geq 0}$ and let $m'\in\{0,\mathrm{lip}\}$, let $\nu\in\{m+m',\infty,\omega,\mathrm{hol}\}$. We denote by $\mathrm{c}_0(\mathbb{Z}_{\geq 0};\mathbb{R}_{>0})$ the positive sequences in $\mathbb{R}$ converging to 0. We define seminorms for $\Gamma^{\nu}(\mathsf{E})$ as follows:

1. $\nu = m$: For compact $K\subseteq\mathsf{M}$, denote

$$p^m_K(\xi) = \sup\{\|j_m\xi(x)\|_{\mathsf{M},\pi,m} \mid x\in K\};$$

2. $\nu = m+\mathrm{lip}$: For compact $K\subseteq\mathsf{M}$, denote

$$p^m_K(\xi) = \sup\{\|j_m\xi(x)\|_{\mathsf{M},\pi,m} \mid x\in K\}, \qquad \lambda^m_K(\xi) = \sup\{\mathrm{dil}\,\xi(x) \mid x\in K\},$$

and assimilate these into a single seminorm by

$$p^{m+\mathrm{lip}}_K(\xi) = \max\{p^m_K(\xi),\lambda^m_K(\xi)\};$$

3. $\nu=\infty$: For compact $K\subseteq\mathsf{M}$ and $m\in\mathbb{Z}_{\geq 0}$, denote

$$p^{\infty}_{K,m}(\xi) = \sup\{\|j_m\xi(x)\|_{\mathsf{M},\pi,m} \mid x\in K\};$$

4. $\nu=\omega$: For compact $K\subseteq\mathsf{M}$ and $\boldsymbol{a}\in\mathrm{c}_0(\mathbb{Z}_{\geq 0};\mathbb{R}_{>0})$, denote

$$p^{\omega}_{K,\boldsymbol{a}}(\xi) = \sup\left\{a_0a_1\cdots a_m\|j_m\xi(x)\|_{\mathsf{M},\pi,m} \mid x\in K,\ m\in\mathbb{Z}_{\geq 0}\right\};$$

5. $\nu=\mathrm{hol}$: For compact $K\subseteq\mathsf{M}$, denote

$$p^{\mathrm{hol}}_K(\xi) = \sup\{\|\xi(x)\|_{\pi} \mid x\in K\}.$$

Because these seminorms have a similar character, we shall often denote, for a compact $K\subseteq\mathsf{M}$, a seminorm for $\Gamma^{\nu}(\mathsf{E})$ by $p^{\nu}_{K,*}$, with the ornamentation (when required) associated with a specific ν bundled into the $*$. This will allow us to treat all regularity classes simultaneously when it is convenient to do so. There are times, however, when the precise nature of the seminorm for a specific regularity class becomes important, such as for real analyticity in [Lewis 2023].

The following property of the seminorms $p^{\nu}_{K,*}$ will be frequently used without mention.

2.6 Lemma: (0-base for $\Gamma^{\nu}(\mathsf{E})$ from seminorms) *Let $m\in\mathbb{Z}_{\geq 0}$, let $m'\in\{0,\mathrm{lip}\}$, let $\nu\in\{m+m',\infty,\omega,\mathrm{hol}\}$, and let $\kappa\in\{\infty,\omega,\mathrm{hol}\}$ as required. If $\pi\colon\mathsf{E}\to\mathsf{M}$ be a C^{κ}-vector bundle, then the sets of the following form are a 0-base for the C^{ν}-topology for $\Gamma^{\nu}(\mathsf{E})$:*

(i) $\nu=m\in\mathbb{Z}_{\geq 0}$: $\{\xi\in\Gamma^m(\mathsf{E}) \mid p^m_K(\xi)<r\}$, *$K\subseteq\mathsf{M}$ compact, $r\in\mathbb{R}_{>0}$;*

(ii) $\nu=m+\mathrm{lip}$, $m\in\mathbb{Z}_{\geq 0}$: $\{\xi\in\Gamma^{m+\mathrm{lip}}(\mathsf{E}) \mid p^{m+\mathrm{lip}}_K(\xi)<r\}$, *$K\subseteq\mathsf{M}$ compact, $r\in\mathbb{R}_{>0}$;*

(iii) $\nu=\infty$: $\{\xi\in\Gamma^{\infty}(\mathsf{E}) \mid p^{\infty}_{K,m}(\xi)<r\}$, *$K\subseteq\mathsf{M}$ compact, $m\in\mathbb{Z}_{\geq 0}$, $r\in\mathbb{R}_{>0}$;*

(iv) $\nu = \omega$*:* $\{\xi \in \Gamma^\omega(\mathsf{E}) \mid p^\omega_{K,\boldsymbol{a}}(\xi) < r\}$*,* $K \subseteq \mathsf{M}$ *compact,* $\boldsymbol{a} \in \mathrm{c}_0(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0})$*,* $r \in \mathbb{R}_{>0}$*;*

(v) $\nu = \mathrm{hol}$*:* $\{\xi \in \Gamma^{\mathrm{hol}}(\mathsf{E}) \mid p^{\mathrm{hol}}_K(\xi) < r\}$*,* $K \subseteq \mathsf{M}$ *compact.*

Proof: Since the locally convex topology is defined by the given seminorms, by definition this means that the collection of finite intersections of the given sets is a 0-base for the topology of $\Gamma^\nu(\mathsf{E})$. Thus the result is that, for every such finite intersection of these sets, there is a set of this form contained in this finite intersection.[4] Said otherwise, we know that the sets in the statement of the lemma form a 0-subbase, and we will show that they form a 0-base. Thus our proof will consist of an explanation for why this is so.

(i) In this case, a finite collection of seminorms is defined by a finite collection $K_1, \dots, K_l \subseteq \mathsf{M}$ of compact sets and a finite collection of $r_1, \dots, r_l \in \mathbb{R}_{>0}$. If we take $K = \cup_{j=1}^l K_j$ and $r = \min\{r_1, \dots, r_l\}$, then we have

$$\{\xi \in \Gamma^m(\mathsf{E}) \mid \ p^m_K(\xi) < r\} \subseteq \bigcap_{j=1}^l \{\xi \in \Gamma^m(\mathsf{E}) \mid p^m_{K_j}(\xi) < r_j\},$$

which gives the result in this case.

(ii) The idea here is the same as that in part (i).

(iii) In this case, a finite collection of seminorms is defined by a finite collection $K_1, \dots, K_l \subseteq \mathsf{M}$ of compact sets, a finite collection of $m_1, \dots, m_l \in \mathbb{Z}_{\geq 0}$, and a finite collection $r_1, \dots, r_l \in \mathbb{R}_{>0}$. If we take $K = \cup_{j=1}^l K_j$, $m = \max\{m_1, \dots, m_l\}$, and $r = \min\{r_1, \dots, r_l\}$, then we have

$$\{\xi \in \Gamma^\infty(\mathsf{E}) \mid p^\infty_{K,m}(\xi) < r\} \subseteq \bigcap_{j=1}^l \{\xi \in \Gamma^\infty(\mathsf{E}) \mid p^\infty_{K_j,m_j}(\xi) < r_j\},$$

which gives the result in this case.

(iv) Here, if we have a finite collection $K_1, \dots, K_l \subseteq \mathsf{M}$ of compact sets, a finite collection of $\boldsymbol{a}_1, \dots, \boldsymbol{a}_l \in \mathrm{c}_0(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0})$, and a finite collection $r_1, \dots, r_l \in \mathbb{R}_{>0}$, we take $K = \cup_{j=1}^l K_j$, $a_n = \max\{a_{1,n}, \dots, a_{l,n}\}$, $n \in \mathbb{Z}_{\geq 0}$, and $r = \min\{r_1, \dots, r_l\}$. We then have $\boldsymbol{a} \in \mathrm{c}_0(\mathbb{Z}_{\geq 0}, \mathbb{R}_{>0})$ and

$$\{\xi \in \Gamma^\omega(\mathsf{E}) \mid p^\omega_{K,\boldsymbol{a}}(\xi) < r\} \subseteq \bigcap_{j=1}^l \{\xi \in \Gamma^\omega(\mathsf{E}) \mid p^\omega_{K_j,\boldsymbol{a}_j}(\xi) < r_j\},$$

which gives the result in this case.

(v) The idea here is the same as that in part (i). ■

These seminorms define a locally convex topology for $\Gamma^\nu(\mathsf{E})$ which we simply call the ***C***$^{\boldsymbol{\nu}}$***-topology***. Let us make some observations about these topologies, for whose verifications we refer to [Jafarpour and Lewis 2014] and the references cited therein.

1. The topology for $\Gamma^{\mathrm{hol}}(\mathsf{E})$ is the compact-open topology, i.e., the topology of uniform convergence on compact sets, in this case.

[4] This property of a collection of seminorms defining a locally convex topology is that of being ***saturated***.

2. The topology for $\Gamma^m(\mathsf{E})$ is the topology of uniform convergence of the sections and their first m derivatives on compact sets.
3. The topology for $\Gamma^\infty(\mathsf{E})$ is the initial topology induced by the inclusions $\Gamma^\infty(\mathsf{E}) \to \Gamma^m(\mathsf{E})$, $m \in \mathbb{Z}_{\geq 0}$.
4. The C^ν-topologies are all Hausdorff, complete, Suslin topologies.
5. All C^ν-topologies are metrisable, except when $\nu = \omega$.
6. All C^ν-topologies are nuclear, except when $\nu \in \{m, m + \mathrm{lip}\}$.
7. All C^ν-topologies are webbed, barrelled, and ultrabornological.
8. All C^ν-topologies are sequential.

These properties make these friendly topologies to work with, for the most part.

We comment that the seminorms we have defined make it clear that we have an ordering of the regularity classes as

$$m_1 < m_1 + \mathrm{lip} < \cdots < m_2 < m_2 + \mathrm{lip} < \cdots < \infty < \omega < \mathrm{hol}$$

from least regular (coarser topology) to more regular (finer topology), and where $m_1 < m_2$. There is also an obvious "arithmetic" of degrees of regularity that we will use without feeling the need to explain it.

2.11. Topologies for spaces of mappings. Let $\kappa \in \{\infty, \omega, \mathrm{hol}\}$ and let M and N be C^κ-manifolds. For $m \in \mathbb{Z}_{\geq 0}$, $m' \in \{0, \mathrm{lip}\}$, and $\nu \in \{m + m', \infty, \omega, \mathrm{hol}\}$, we have the set $\mathrm{C}^\nu(\mathsf{M}; \mathsf{N})$ of C^ν-mappings. One might topologise this space in a direct way, e.g., by using the topologies of uniform convergence of derivatives on compact subsets, sometimes called the weak topology [Hirsch 1976, §2.1]. However, we choose to do this in an indirect manner that is equivalent where both descriptions exist.

To this end, the ***weak-PB C^ν-topology*** for $\mathrm{C}^\nu(\mathsf{M}; \mathsf{N})$ is the initial topology defined by the mappings

$$\begin{aligned} \Psi_f \colon \mathrm{C}^\nu(\mathsf{M}; \mathsf{N}) &\to \mathrm{C}^\nu(\mathsf{M}) \\ \Phi &\mapsto \Phi^* f, \end{aligned} \qquad f \in \mathrm{C}^\kappa(\mathsf{N}).$$

(We can see that "PB" is, of course, for "pull-back.) That this topology agrees with the usual "weak" topology is a consequence of the fact that we are considering cases where there exist coordinate functions around any point that are globally defined (making the assumption that M and N are Stein in the case $\nu = \mathrm{hol}$).

One way to view the weak-PB topology is as the uniform topology defined by the family of semimetrics

$$\mathrm{d}^\nu_{K,*,f}(\Phi_1, \Phi_2) = p^\nu_{K,*}(f \circ \Phi_1 - f \circ \Phi_2), \qquad f \in \mathrm{C}^\kappa(\mathsf{N}),\ K \subseteq \mathsf{M} \text{ compact},$$

and where $p^\nu_{K,*}$ is one of seminorms defined above for $\mathrm{C}^\nu(\mathsf{M})$.

The following result gives a useful alternative characterisation of the above topology for the spaces of mappings. The result relies on embedding theorems in the C^∞-case [Whitney 1936], in the C^ω-case [Grauert 1958], and in the case of Stein manifolds [Remmert 1954].

2.7 Lemma: (Characterisation of topology for spaces of mappings) *Let* $m \in \mathbb{Z}_{\geq 0}$*, let* $m' \in \{0, \mathrm{lip}\}$*, let* $\nu \in \{m+m', \infty, \omega, \mathrm{hol}\}$*, and let* $\kappa \in \{\infty, \omega, \mathrm{hol}\}$*, as appropriate. Let* M *and* N *be* C^κ*-manifolds, Stein if* $\nu = \mathrm{hol}$*. Let*

$$\begin{aligned} \boldsymbol{\chi}\colon \mathsf{N} &\to \mathbb{F}^N \\ y &\mapsto (\chi^1(y), \dots, \chi^N(y)) \end{aligned}$$

be a proper C^κ*-embedding. Then the following topologies for* $\mathrm{C}^\nu(\mathsf{M};\mathsf{N})$ *agree:*

(i) *the initial topology associated with the family of mappings*

$$\begin{aligned} \Psi_f\colon \mathrm{C}^\nu(\mathsf{M};\mathsf{N}) &\to \mathrm{C}^\nu(\mathsf{M}) \\ \Phi &\mapsto \Phi^* f, \end{aligned} \qquad f \in \mathrm{C}^\kappa(\mathsf{M});$$

(ii) *the initial topology associated with the family of mappings*

$$\begin{aligned} \Psi_{\chi^j}\colon \mathrm{C}^\nu(\mathsf{M};\mathsf{N}) &\to \mathrm{C}^\nu(\mathsf{M}) \\ \Phi &\mapsto \Phi^* \chi^j, \end{aligned} \qquad j \in \{1, \dots, N\};$$

(iii) *the topology induced on* $\mathrm{C}^\nu(\mathsf{M};\mathsf{N}) \subseteq \mathrm{C}^\nu(\mathsf{M};\mathbb{F}^N)$ *by the* C^ν*-topology for*

$$\mathrm{C}^\nu(\mathsf{M};\mathbb{F}^N) \simeq \oplus_{j=1}^N \mathrm{C}^\nu(\mathsf{M}).$$

Proof: The result is proved in the real analytic case as Theorem 2.25 in [Lewis 2023]. The constructions in that proof remain valid after noting the following facts:

1. Lemma 2.1 in [Lewis 2023] is valid for $\kappa = \mathrm{hol}$ when $\pi\colon \mathsf{E} \to \mathsf{M}$ is an holomorphic vector bundle over a Stein manifold. This is a consequence of the vanishing of the cohomology of the sheaf of holomorphic sections in this case, this itself being a result of Cartan [1951-52].
2. Lemma 1 from the proof of Theorem 2.25 in [Lewis 2023] is valid for smooth regularity and holomorphic regularity for Stein manifolds, since it relies on Lemma 2.1 from the same book, and since the spaces $\mathrm{C}^\kappa(\mathsf{M})$ are ultrabornological and webbed. ■

2.12. Continuity of geometric operations. An essential element of our global geometric functional analytic treatment is that our topologies are such that most of the standard geometric operations one encounters are continuous. This will allow us to easily give integrability and parameter-continuity attributes to objects constructed from other objects with these attributes. A few such results are proved by Jafarpour and Lewis [2014] in an ad hoc fashion. The tools for doing this systematically in the real analytic case are developed by Lewis [2023], and these tools are easily applied to give the results for the other regularity classes. The exception to this blanket statement is the Lipschitz topology, which must be handled separately. Here we simply summarise the operations of whose continuity we shall make use, and sketch proofs that do not immediately follow from the methods of [Lewis 2023].

2.8 Lemma: (Continuous operators) *Let $m \in \mathbb{Z}_{\geq 0}$, let $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega, \mathrm{hol}\}$, and let $\kappa \in \{\infty, \omega, \mathrm{hol}\}$, as appropriate. Let M, N, and P be C^κ-manifolds, and let $\pi_{\mathsf{E}} \colon \mathsf{E} \to \mathsf{M}$ and $\pi_{\mathsf{F}} \colon \mathsf{F} \to \mathsf{M}$ be C^κ-vector bundles. When $\nu = \mathrm{hol}$, assume that M, N, and P are Stein manifolds. Let $\Phi \in \mathrm{C}^\nu(\mathsf{M}; \mathsf{N})$. When $j \neq \mathrm{hol}$, let $\nabla^{\pi_{\mathsf{E}}}$ be a C^κ-linear connection in E. Then the following mappings are continuous:*

(i) $\Gamma^\nu(\mathsf{E}) \oplus \Gamma^\nu(\mathsf{E})(\xi, \eta) \mapsto \xi + \eta \in \Gamma^\nu(\mathsf{E})$;

(ii) $\Gamma^\nu(\mathsf{E}) \times \Gamma^\nu((\mathsf{F} \otimes \mathsf{E}^*)) \ni (\xi, L) \mapsto L(\xi) \in \Gamma^\nu(\mathsf{F})$;

(iii) $\mathrm{C}^\nu(\mathsf{N}) \ni g \mapsto \Phi^* g \in \mathrm{C}^\nu(\mathsf{M})$;

(iv) $\Gamma^{m+\nu}(\mathsf{E}) \ni \xi \mapsto j_m \xi \in \Gamma^\nu(\mathsf{J}^m \mathsf{E})$, $m \in \mathbb{Z}_{>0}$;

(v) $\Gamma^\nu(\mathsf{TM}) \times \mathrm{C}^{\nu+1}(\mathsf{M}) \ni (X, f) \mapsto \mathscr{L}_X f \in \mathrm{C}^\nu(\mathsf{M})$;

(vi) $\Gamma^\nu(\mathsf{TM}) \times \Gamma^{\nu+1}(\mathsf{E}) \ni (X, \xi) \mapsto \nabla^{\pi_{\mathsf{E}}}_X \xi \in \Gamma^\nu(\mathsf{E})$.

Proof: We only prove those parts of the lemma that do not follow immediately or easily from the results of [Lewis 2023].

Parts (i) and (ii) hold in the Lipschitz case since addition and multiplication are continuous in this case, essentially by [Weaver 1999, Proposition 1.5.2(b)] and [Weaver 1999, Proposition 1.5.3(a)], respectively.

We shall prove part (iii) in the case that $\nu = m + \mathrm{lip}$ as this does not fall under the umbrella of the techniques of [Lewis 2023]. It will suffice to prove the result for $\nu = \mathrm{lip}$ since the result for $m > 0$ follows by the addition of more notation. Let $K \subseteq \mathsf{M}$ be compact and, for $x \in K$, let $\mathcal{U}_x \subseteq \mathsf{M}$ and $\mathcal{V}_x \subseteq \mathsf{N}$ be precompact neighbourhoods of x and $\Phi(x)$, respectively, such that $\Phi(\mathcal{U}_x) \subseteq \mathcal{V}_x$ (simply by continuity of Φ) and such that

$$\lambda^0_{\mathrm{cl}(\mathcal{U}_x)}(f) = l_{\mathrm{cl}(\mathcal{U}_x)}(f), \qquad f \in \mathrm{C}^{\mathrm{lip}}(\mathsf{M}),$$

and

$$\lambda^0_{\mathrm{cl}(\mathcal{V}_x)}(g) = l_{\mathrm{cl}(\mathcal{U}_x)}(g), \qquad g \in \mathrm{C}^{\mathrm{lip}}(\mathsf{N}),$$

these by Lemma 2.5. By compactness of K, let $x_1, \dots, x_k \in K$ be such that $K \subseteq \cup_{j=1}^k \mathcal{U}_{x_j}$. Then $\Phi(K) \subseteq \cup_{j=1}^k \mathcal{V}_{x_j}$. Now, if $x \in K$, then $x \in \mathcal{U}_{x_j}$ for some $j \in \{1, \dots, k\}$. Note also that, if $x'_1 \neq x'_2$ but $\Phi(x'_1) = \Phi(x'_2)$, then obviously

$$\frac{|g \circ \Phi(x'_1) - g \circ \Phi(x'_2)|}{\mathrm{d}_{\mathsf{M}}(x'_1, x'_2)} = 0.$$

Thus, for $g \in \mathrm{C}^{\mathrm{lip}}(\mathsf{N})$ and $x \in K$,

$$\begin{aligned} \operatorname{dil} \Phi^* g(x) &\leq \lambda^0_{\mathrm{cl}(\mathcal{U}_{x_j})}(\Phi^* g) = l_{\mathrm{cl}(\mathcal{U}_{x_j})}(\Phi^* g) \\ &= \sup \left\{ \frac{|g \circ \Phi(x'_1) - g \circ \Phi(x'_2)|}{\mathrm{d}_{\mathsf{M}}(x'_1, x'_2)} \;\middle|\; x'_1, x'_2 \in \mathrm{cl}(\mathcal{U}_{x_j}),\ x'_1 \neq x'_2 \right\} \\ &\leq \sup \left\{ \frac{|g \circ \Phi(x'_1) - g \circ \Phi(x'_2)|}{\mathrm{d}_{\mathsf{N}}(\Phi(x'_1), \Phi(x'_2))} \frac{\mathrm{d}_{\mathsf{N}}(\Phi(x'_1), \Phi(x'_2))}{\mathrm{d}_{\mathsf{M}}(x'_1, x'_2)} \;\middle|\right. \\ &\qquad\qquad \left. x'_1, x'_2 \in \mathrm{cl}(\mathcal{U}_{x_j}),\ x'_1 \neq x'_2,\ \Phi(x'_1) \neq \Phi(x'_2) \right\} \\ &\leq l_{\mathrm{cl}(\mathcal{V}_{x_j})}(g) \sup \left\{ \frac{\mathrm{d}_{\mathsf{N}}(\Phi(x'_1), \Phi(x'_2))}{\mathrm{d}_{\mathsf{M}}(x'_1, x'_2)} \;\middle|\; x'_1, x'_2 \in \mathrm{cl}(\mathcal{U}_{x_j}),\ x'_1 \neq x'_2 \right\}. \end{aligned}$$

Since $\Phi \in \mathrm{C}^{\mathrm{lip}}(\mathsf{M};\mathsf{N})$, this last supremum is finite. Then we compute

$$\begin{aligned}\lambda^0_K(\Phi^*g) &= \sup\{\operatorname{dil}\Phi^*g(x) \mid x \in K\}\\ &\le \sup\left\{\frac{\mathrm{d}_{\mathsf{N}}(\Phi(x'_1),\Phi(x'_2))}{\mathrm{d}_{\mathsf{M}}(x'_1,x'_2)}\;\middle|\; x'_1,x'_2 \in \mathrm{cl}(\mathcal{U}_{x_j}),\ x'_1 \neq x'_2,\ j \in \{1,\dots,k\}\right\}\\ &\quad\times \max\{\lambda^0_{\mathrm{cl}(\mathcal{V}_1)}(g),\dots,\lambda^0_{\mathrm{cl}(\mathcal{V}_k)}(g)\},\end{aligned}$$

which gives the desired continuity.

Part (iv) can be proved in the Lipschitz case by combining the proofs of [Lewis 2023, Theorem 5.13] and Lemma 2.4.

Parts (v) and (vi) follow from part (iv), cf. Corollaries 5.13 and 5.14 of [Lewis 2023]. ■

We shall have a great deal more to say about the continuity of composition in Section 4.

A consequence of part (v) of the preceding lemma is the following.

2.9 Lemma: (Characterising vector field topologies in terms of functions) *Let $m \in \mathbb{Z}_{\ge 0}$, let $m' \in \{0,\mathrm{lip}\}$, let $\nu \in \{m+m',\infty,\omega,\mathrm{hol}\}$ satisfy $\nu \ge 1$, and let $\kappa \in \{\infty,\omega,\mathrm{hol}\}$, as required. Let M be a C^κ-manifold and, when $\nu = \mathrm{hol}$, assume that M is Stein. Then the C^ν-topology for $\Gamma^\nu(\mathsf{TM})$ is the initial topology associated with the mappings*

$$\begin{aligned}\Psi_f\colon \Gamma^\nu(\mathsf{TM}) &\to \mathrm{C}^\nu(\mathsf{M})\\ X &\mapsto Xf,\end{aligned}$$

$f \in \mathrm{C}^\kappa(\mathsf{M})$.

Proof: From part (v) of the preceding lemma, it follows that the C^ν-topology is finer than the initial topology from the statement of the lemma. For the converse, we show that the identity mapping on $\Gamma^\nu(\mathsf{TM})$ is continuous if the domain has the initial topology and the codomain has the C^ν-topology.

Let $K \subseteq \mathsf{M}$ be compact and denote by $p^\nu_{K,*}$ a seminorm for the C^ν-topology, appropriate for ν. Let $x \in K$ and let $\mathcal{U}_x$ be a precompact neighbourhood of x and let $\chi^1_x,\dots,\chi^n_x \in \mathrm{C}^\kappa(\mathsf{M})$ form a local coordinate system on $\mathrm{cl}(\mathcal{U}_x)$. Then we have

$$X(x) = \sum_{j=1}^n \Psi_{\chi^j_x}(X)(x), \qquad x \in \mathrm{cl}(\mathcal{U}_x),\ X \in \Gamma^\nu(\mathsf{TM}).$$

By compactness of K, let $x_1,\dots,x_s \in K$ be such that $K \subseteq \cup_{r=1}^s \mathcal{U}_{x_r}$. Then we have (allowing the symbol $p^\nu_{K,*}$ to be overused)

$$p^\nu_{K,*}(X) \le \sum_{j=1}^n \sum_{r=1}^s p^\nu_{K,*}(\Psi_{\chi^j_{x_s}}(X)),$$

which suffices to show that the identity map is continuous is the asserted topologies. ■

We note that the lemma is generally false in the case that $r = \mathrm{hol}$ and M is not a Stein manifold.

3. Time- and parameter-dependent functions, vector fields, and mappings

In this section we introduce the classes of vector fields, depending on both time and parameter, that we work with. Since we will typically convert statements about vector fields into statements about functions using Lemma 2.9, we give a formulation for sections of vector bundles rather than just for vector fields. This setup will also allow us to introduce classes of time- and parameter-dependent mappings, using the weak-PB topologies of Section 2.11. The classes of sections we introduce include both locally integrable (as discussed in Section 1.2) and locally absolutely continuous time-dependence. A great deal of the technical development of the paper takes place in this section since, especially in the parameter-dependent locally absolutely continuous case, (a) our constructions are new and (b) we make substantial use of the detailed properties arising from these constructions.

3.1. Locally integrable time-dependent sections. We carefully introduce in this section the class of time-dependent sections we consider, and which were quickly introduced in Section 1.2. In our presentation, we shall make use of measurable and locally integrable functions with values in a locally convex topological vector space. This is classical in the case of Banach spaces, but is not as fleshed out in the general case. We refer to [Lewis 2022] for details and further references. Here we quickly give the outlines of the development.

We begin with definitions.

3.1 Definition: (Measurable, integrable, locally integrable sections) Let $m \in \mathbb{Z}_{\geq 0}$, let $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega, \mathrm{hol}\}$, and let $\kappa \in \{\infty, \omega, \mathrm{hol}\}$, as required. Let $\pi \colon \mathsf{E} \to \mathsf{M}$ be a C^κ-vector bundle and let $\mathbb{T} \subseteq \mathbb{R}$ be an interval. A mapping $\xi \colon \mathbb{T} \to \Gamma^\nu(\mathsf{E})$ is:

(i) ***measurable*** if $\lambda \circ \xi$ is measurable for every $\lambda \in \Gamma^\nu(\mathsf{E})'$;

(ii) ***integrable*** if $p^\nu_{K,*} \circ \xi \in \mathrm{L}^1(\mathbb{T}; \mathbb{R}_{\geq 0})$ for any of the seminorms $p^\nu_{K,*}$ from Section 2.10 for the C^ν-topology;

(iii) ***locally integrable*** if $\xi|\mathbb{K}$ is integrable for every compact interval $\mathbb{K} \subseteq \mathbb{T}$.

3.2 Remarks and notation: 1. The notion of measurability we give is typically called "weak measurability." Because all locally convex spaces we consider are Suslin spaces, all standard notions of measurability coincide [Thomas 1975, Theorem 1]. Thus, for example, one can take as one's notion of measurability the naïve one that preimages of Borel sets are measurable. Even more explicitly, [Jafarpour and Lewis 2014] show that measurability of ξ as we have defined it is equivalent to measurability of the mappings

$$\mathbb{T} \ni t \mapsto \xi(t, x) \in \mathsf{E}_x, \qquad x \in \mathsf{M}.$$

As Jafarpour and Lewis show, this is a consequence of the fact that the family of continuous functions $\xi \mapsto \xi(x)$, $x \in \mathsf{M}$ is point-separating.

2. The notion of integrability we use is "integrability by seminorm," and is a generalisation to the locally convex case of the quite classical notion of Bochner integrability [Diestel and Uhl, Jr. 1977]. This locally convex extension seems to originate in [Garnir, De Wilde, and Schmets 1972].

3. If $\pi_{\mathsf{E}}\colon \mathsf{E} \to \mathsf{M}$ and $\pi_{\mathsf{F}}\colon \mathsf{F} \to \mathsf{N}$ are C^κ-vector bundles, $\nu_1, \nu_2 \in \{m+m', \infty, \omega, \mathrm{hol}\}$ are two regularity classes, and if $\phi\colon \Gamma^{\nu_1}(\mathsf{E}) \to \Gamma^{\nu_2}(\mathsf{F})$ is a continuous linear map, then
$$\phi \circ \xi \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{T}; \Gamma^{\nu_1}(\mathsf{F})), \qquad \xi \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{T}; \Gamma^{\nu_2}(\mathsf{E})).$$
This is directly seen since continuous linear maps preserve continuity of seminorms.

We denote by $\mathrm{L}^1(\mathbb{T}; \Gamma^\nu(\mathsf{E}))$ the set of integrable sections and by $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{T}; \Gamma^\nu(\mathsf{E}))$ the set of locally integrable sections. We are primarily interested in the locally integrable case, and so use the abbreviation
$$\Gamma^\nu_{\mathrm{LI}}(\mathbb{T}; \mathsf{E}) = \mathrm{L}^1_{\mathrm{loc}}(\mathbb{T}; \Gamma^\nu(\mathsf{E})). \qquad \bullet$$

We give a locally convex topology for the set of integrable sections by the seminorms
$$p^\nu_{K,*,\mathbb{T}}(\xi) = \int_{\mathbb{T}} p^\nu_{K,*}(\xi(t))\,\mathrm{d}t, \qquad K \subseteq \mathsf{M} \text{ compact},$$
where $p^\nu_{K,*}$ is one of the seminorms for the C^ν-topology as defined in Section 2.10. As shown by [Lewis 2022, Theorem 3.2], we have a topological isomorphism
$$\mathrm{L}^1(\mathbb{T}; \Gamma^\nu(\mathsf{E})) \simeq \mathrm{L}^1(\mathbb{T}; \mathbb{F}) \overline{\otimes}_\pi \Gamma^\nu(\mathsf{E}),$$
where $\overline{\otimes}_\pi$ is the completion of the projective tensor product [Jarchow 1981, Chapter 15].

There is then an associated locally convex topology for the set of locally integrable sections using the seminorms
$$p^\nu_{K,*,\mathbb{K}}(\xi) = \int_{\mathbb{K}} p^\nu_{K,*}(\xi(t))\,\mathrm{d}t, \qquad K \subseteq \mathsf{M},\ \mathbb{K} \subseteq \mathbb{T} \text{ compact}, \tag{3.1}$$
where, again, $p^\nu_{K,*}$ is one of the seminorms for the C^ν-topology as defined in Section 2.10. If we denote by $\mathscr{K}(\mathbb{T})$ the set of compact intervals in $\mathbb{T}$, then we have a topological isomorphism
$$\mathrm{L}^1_{\mathrm{loc}}(\mathbb{T}; \Gamma^\nu(\mathsf{E})) \simeq \varprojlim_{\mathbb{K} \in \mathscr{K}(\mathbb{T})} \mathrm{L}^1(\mathbb{K}; \Gamma^\nu(\mathsf{E})),$$
where the inverse limit is defined by the continuous inclusions
$$\mathrm{L}^1(\mathbb{K}_1; \Gamma^\nu(\mathsf{E})) \hookrightarrow \mathrm{L}^1(\mathbb{K}_2; \Gamma^\nu(\mathsf{E}))$$
for $\mathbb{K}_1, \mathbb{K}_2 \in \mathscr{K}(\mathbb{T})$ satisfying $\mathbb{K}_2 \subseteq \mathbb{K}_1$. Therefore, we have topological isomorphisms
$$\Gamma^\nu_{\mathrm{LI}}(\mathbb{T}; \mathsf{E}) \simeq \varprojlim_{\mathbb{K} \in \mathscr{K}(\mathbb{T})} \mathrm{L}^1(\mathbb{K}; \Gamma^\nu(\mathsf{E})) \simeq \varprojlim_{\mathbb{K} \in \mathscr{K}(\mathbb{T})} \mathrm{L}^1(\mathbb{K}; \mathbb{F}) \overline{\otimes}_\pi \Gamma^\nu(\mathsf{E}) \simeq \mathrm{L}^1_{\mathrm{loc}}(\mathbb{T}; \mathbb{F}) \overline{\otimes}_\pi \Gamma^\nu(\mathsf{E}), \tag{3.2}$$
with the final "$\simeq$" being a consequence of [Jarchow 1981, Theorem 15.4.2].

In the introductory discussion of Section 1.2, we distinguished between $\xi \in \Gamma^\nu(\mathsf{E})$ and $\hat{\xi}\colon \mathbb{T} \times \mathsf{M} \to \mathsf{E}$. We will not generally do this, and so we will conflate $\xi(t)(x)$ and $\xi(t,x)$ when convenient. We shall also use the notation $\xi_t(x) = \xi(t,x)$.

Our method of working with vector fields and their flows is to use general globally defined functions to replace local coordinates. As such, functions assume an important rôle in our presentation. Bearing in mind that functions are sections of the trivial line bundle, the above general definitions for sections of vector bundles apply specifically to functions, and yield the spaces $\mathrm{C}^\nu_{\mathrm{LI}}(\mathbb{T}; \mathsf{M})$ of locally integrally bounded time-dependent functions $f\colon \mathbb{T} \to \mathrm{C}^\nu(\mathsf{M})$.

The following lemma indicates how we will convert vector fields into functions.

3.3 Lemma: (Time-dependent functions from time-dependent vector fields) *Let $m \in \mathbb{Z}_{\geq 0}$, let $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m+m', \infty, \omega, \mathrm{hol}\}$, and let $\kappa \in \{\infty, \omega, \mathrm{hol}\}$, as required. Let M be a C^κ-manifold, assumed Stein if $\nu = \mathrm{hol}$. Then the topology for $\Gamma^\nu_{\mathrm{LI}}(\mathbb{T}; \mathsf{TM})$ is the initial topology associated with the mappings*

$$\begin{aligned} \Psi_f \colon \Gamma^\nu_{\mathrm{LI}}(\mathsf{TM}) &\to \mathrm{C}^\nu_{\mathrm{LI}}(\mathsf{M}) \\ X &\mapsto Xf, \end{aligned}$$

$f \in \mathrm{C}^\kappa(\mathsf{M})$.

Proof: This follows from Lemma 2.9, the definition of local integrability, and Remark 3.2–3. ■

3.2. Locally absolutely continuous time-dependent sections. The notion of absolute continuity we use for time-dependent sections of a vector bundle echoes the classical theorem where the ϵ-δ definition of absolute continuity is shown to be equivalent to the function being the indefinite integral of a locally integrable function [Cohn 2013, Proposition 4.4.6].

3.4 Definition: (Locally absolutely continuous section) Let $m \in \mathbb{Z}_{\geq 0}$, let $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega, \mathrm{hol}\}$, and let $\kappa \in \{\infty, \omega, \mathrm{hol}\}$, as required. Let $\pi \colon \mathsf{E} \to \mathsf{M}$ be a C^κ-vector bundle and let $\mathbb{T} \subseteq \mathbb{R}$ be an interval. We say that $\Xi \colon \mathbb{T} \to \Gamma^\nu(\mathsf{E})$ is ***locally absolutely continuous*** if there exists $\xi \in \Gamma^\nu_{\mathrm{LI}}(\mathsf{E})$ and $t_0 \in \mathbb{T}$ such that

$$\Xi(t) = \Xi(t_0) + \int_{t_0}^t \xi(s)\,\mathrm{d}s, \qquad t \in \mathbb{T}. \tag{3.3}$$

We denote by $\Gamma^\nu_{\mathrm{LAC}}(\mathbb{T}; \mathsf{E})$ the space of locally absolutely continuous sections of class C^ν. •

The usual properties of the integral ensure that $\Xi(t)$ is independent of t_0. Lewis [2022, Theorem 4.2] shows that, if Ξ is locally absolutely continuous, then, for almost every $t \in \mathbb{T}$, we have

$$\Xi'(t) \triangleq \lim_{h \to 0} \frac{\Xi(t+h) - \Xi(t)}{h} = \xi(t),$$

just as in the scalar case [Cohn 2013, Theorem 6.3.6]. One readily verifies that this space of time-varying sections is a subspace of the set $\Gamma^\nu(\mathsf{E})^{\mathbb{T}}$ of functions on $\mathbb{T}$ with values in $\Gamma^\nu(\mathsf{E})$.

We topologise $\Gamma^\nu_{\mathrm{LAC}}(\mathbb{T}; \mathsf{E})$ by seminorms

$$\overline{p}^\nu_{K,*,t_0,t}(\Xi) = p^\nu_{K,*}(\Xi(t_0)) + p^\nu_{K,*,|t_0,t|}(\xi), \qquad K \subseteq \mathsf{M},\ t \in \mathbb{T}, \tag{3.4}$$

for some (it matter not which) $t_0 \in \mathbb{T}$, where $p^\nu_{K,*}$ is one of the seminorms for the C^ν-topology from Section 2.10, and where $p^\nu_{K,*,|t_0,t|}(\xi)$ is as in (3.1). We make use of the notation

$$|a, b| = \begin{cases} [a, b], & a \leq b, \\ [b, a], & a > b, \end{cases} \tag{3.5}$$

for $a, b \in \mathbb{R}$. One sees readily that this gives $\Gamma^\nu_{\mathrm{LAC}}(\mathbb{T}; \mathsf{E})$ an Hausdorff locally convex topology.

There are many equivalent ways to topologise $\Gamma^\nu_{\mathrm{LAC}}(\mathbb{T}; \mathsf{E})$, the following results giving one such equivalence that is especially useful.

3.5 Lemma: (Topology of locally absolutely continuous sections) *Let $m \in \mathbb{Z}_{\geq 0}$, let $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega, \mathrm{hol}\}$, and let $\kappa \in \{\infty, \omega, \mathrm{hol}\}$, as required. Let $\pi \colon \mathsf{E} \to \mathsf{M}$ be a C^κ-vector bundle and let $\mathbb{T} \subseteq \mathbb{R}$ be an interval. Let $t_0 \in \mathbb{T}$. The topology of $\Gamma^\nu_{\mathrm{LAC}}(\mathbb{T}; \mathsf{E})$ is defined by the seminorms*

$$p^\nu_{K,*,\mathbb{K},0}(\Xi) = \sup\{p^\nu_{K,*}(\Xi(t)) \mid t \in \mathbb{K}\},$$
$$p^\nu_{K,*,\mathbb{K},t_0,1}(\Xi) = p^\nu_{K,*,\mathbb{K}}(\xi),$$

where ξ satisfies (3.3), where $K \subseteq \mathsf{M}$ and $\mathbb{K} \subseteq \mathbb{T}$ are compact, and where $p^\nu_{K,*}$ is one of the seminorms from Section 2.10 and $p^\nu_{K,*,\mathbb{K}}$ is the seminorm (3.1).

Proof: Let $\Xi \in \Gamma^\nu_{\mathrm{LAC}}(\mathbb{T}; \mathsf{E})$.

Let $K \subseteq \mathsf{M}$ and let $\mathbb{K} \subseteq \mathbb{T}$ be compact, and let $t \in \mathbb{T}$. Let $\mathbb{K}' \subseteq \mathbb{T}$ be the smallest compact interval containing $\{t_0, t\} \cup \mathbb{K}$. We then have

$$\overline{p}^\nu_{K,*,t_0,t}(\Xi) \leq p^\nu_{K,*,\mathbb{K}',0}(\Xi) + p^\nu_{K,*,\mathbb{K}',1}(\Xi).$$

Let $K \subseteq \mathsf{M}$ and $\mathbb{K} \subseteq \mathbb{T}$ be compact. Let

$$a = \inf\{t_0\} \cup \mathbb{K}, \quad b = \sup\{t_0\} \cup \mathbb{K}.$$

Immediately from (3.3), we have

$$p^\nu_{K,*}(\Xi(t)) \leq p^\nu_{K,*}(\Xi(t_0)) + \int_{|t_0,t|} p^\nu_{K,*}(\xi(s))\,\mathrm{d}s, \qquad t \in \mathbb{K},$$

and so

$$p^\nu_{K,*,\mathbb{K},0}(\Xi) \leq \overline{p}^\nu_{K,*,t_0,a}(\Xi) + \overline{p}^\nu_{K,*,t_0,b}(\Xi).$$

Also,

$$p^\nu_{K,*,\mathbb{K},t_0,1}(\Xi) \leq \overline{p}^\nu_{K,*,t_0,a}(\Xi) + \overline{p}^\nu_{K,*,t_0,b}(\Xi).$$

Combining the preceding paragraphs gives the lemma. ■

The following simple property of locally absolutely continuous sections will be useful.

3.6 Lemma: (Locally absolutely continuous sections are continuous) *Let $m \in \mathbb{Z}_{\geq 0}$, let $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega, \mathrm{hol}\}$, and let $\kappa \in \{\infty, \omega, \mathrm{hol}\}$, as required. Let $\pi \colon \mathsf{E} \to \mathsf{M}$ be a C^κ-vector bundle and let $\mathbb{T} \subseteq \mathbb{R}$ be an interval. If $\nu = \mathrm{hol}$, we assume that M is Stein. Then*

$$\Gamma^\nu_{\mathrm{LAC}}(\mathbb{T}; \mathsf{E}) \subseteq \mathrm{C}^0(\mathbb{T} \times \mathsf{M}; \mathsf{E}).$$

Proof: We first claim that

$$\Gamma^\nu_{\mathrm{LAC}}(\mathbb{T}; \mathsf{E}) \subseteq \mathrm{C}^0(\mathbb{T}; \Gamma^\nu(\mathsf{E})).$$

Let $\Xi \in \Gamma^\nu_{\mathrm{LAC}}(\mathbb{T}; \mathsf{E})$, let $t \in \mathbb{T}$, and let $\epsilon \in \mathbb{R}_{>0}$. Let $\xi \in \Gamma^\nu_{\mathrm{LI}}(\mathbb{T}; \mathsf{E})$ satisfy (3.3). We consider a seminorm $p^\nu_{K,*}$ for $\Gamma^\nu(\mathsf{E})$. Since $p^\nu_{K,*} \circ \xi \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{T}; \mathbb{R})$, there exists $\delta \in \mathbb{R}_{>0}$ such that

$$\int_t^{t+h} p^\nu_{K,*} \circ \xi(s)\,\mathrm{d}s < \epsilon, \qquad h \in [t - \delta, t + \delta] \cap \mathbb{T}.$$

Then, for $h \in [t-\delta, t+\delta] \cap \mathbb{T}$, we have

$$p^{\nu}_{K,*}(\Xi(t+h) - \Xi(t)) = p^{\nu}_{K,*}\left(\int_t^{t+h} \xi(s)\,\mathrm{d}s\right) \leq \int_t^{t+h} p^{\nu}_{K,*} \circ \xi(s)\,\mathrm{d}s < \epsilon.$$

Thus $\lim_{h\to 0} \Xi(t+h) = \Xi(t)$, which suffices to prove continuity of $t \mapsto \Xi(t)$. Moreover, we note that the seminorms $p^{\nu}_{K,*,\mathbb{K},0}$ of Lemma 3.5 are precisely the seminorms defining the compact-open topology for $\mathrm{C}^0(\mathbb{T};\Gamma^{\nu}(\mathsf{E}))$.

Let $\Xi \in \Gamma^{\nu}_{\mathrm{LAC}}(\mathbb{T};\mathsf{E})$ and let $(t_0, x_0) \in \mathbb{T} \times \mathsf{E}$. Let $\mathcal{W} \subseteq \mathsf{E}$ be a neighbourhood of $\Xi(t_0, x_0)$. Let $\mathcal{U}$ be a precompact neighbourhood of x_0 and let $\epsilon \in \mathbb{R}_{>0}$ be such that

$$\pi^{-1}(\mathcal{U}) \cap \{e \in \mathsf{E} \mid \ \|e - \Xi(t_0, \pi(e))\|_{\pi} < \epsilon\} \subseteq \mathcal{W}.$$

Let $\mathbb{K} \subseteq \mathbb{T}$ be compact and such that $t_0 \in \mathrm{int}(\mathbb{K})$ and

$$p^{\nu}_{\mathrm{cl}(\mathcal{U}),*,\mathbb{K},0}(\Xi_t - \Xi_{t_0}) < \epsilon, \qquad t \in \mathbb{K},$$

this by the first paragraph of the proof. Then we have

$$\|\Xi(t,x) - \Xi(t_0,x)\|_{\pi} < \epsilon, \qquad (t,x) \in \mathbb{K} \times \mathrm{cl}(\mathcal{U}).$$

Then, if $(t,x) \in \mathrm{int}(\mathbb{K}) \times \mathcal{U}$, we have $\pi(\Xi(t,x)) = x \in \mathcal{U}$ and so $\Xi(t,x) \in \mathcal{W}$, giving the desired conclusion. ■

3.3. Locally absolutely continuous time-dependent mappings. Our main application of the notion of local absolute continuity will be as it pertains to mappings between manifolds.

3.7 Definition: (Locally absolution continuous mapping) Let $m \in \mathbb{Z}_{\geq 0}$, let $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m+m', \infty, \omega, \mathrm{hol}\}$, and let $\kappa \in \{\infty, \omega, \mathrm{hol}\}$, as required. Let M and N be C^{κ}-manifolds, Stein if $\nu = \mathrm{hol}$, and let $\mathbb{T} \subseteq \mathbb{R}$ be an interval. A mapping $\Phi\colon \mathbb{T} \to \mathrm{C}^{\nu}(\mathsf{M};\mathsf{N})$ is ***locally absolutely continuous*** if $(t \mapsto \Phi(t)^* g) \in \mathrm{C}^{\nu}(\mathsf{M})$ is locally absolutely continuous for every $g \in \mathrm{C}^{\kappa}(\mathsf{N})$. We denote by $\mathrm{C}^{\nu}_{\mathrm{LAC}}(\mathbb{T};(\mathsf{M};\mathsf{N}))$ the set of locally absolutely continuous mappings from M to N of class C^{ν}. •

We topologise the space of locally absolutely continuous mappings by the family of semimetrics

$$\mathrm{d}^{\nu}_{K,*,\mathbb{K},0,g}(\Phi_1,\Phi_2) = p^{\nu}_{K,*,\mathbb{K},0}(g\circ\Phi_1 - g\circ\Phi_2), \tag{3.6}$$

$$\mathrm{d}^{\nu}_{K,*,\mathbb{K},t_0,1,g}(\Phi_1,\Phi_2) = p^{\nu}_{K,*,\mathbb{K},t_0,1}(g\circ\Phi_1 - g\circ\Phi_2), \tag{3.7}$$

for $K \subseteq \mathsf{M}$ and $\mathbb{K} \subseteq \mathbb{T}$ compact, and for $g \in \mathrm{C}^{\kappa}(\mathsf{N})$, and where $p^{\nu}_{K,*,\mathbb{K},0}$ and $p^{\nu}_{K,*,\mathbb{K},1}$ are the seminorms from Lemma 3.5. This makes $\mathrm{C}^{\nu}_{\mathrm{LAC}}(\mathbb{T};(\mathsf{M};\mathsf{N}))$ a uniform space.

Locally absolutely continuous mappings have useful regularity and compactness properties from which will follow analogous properties for flows. We initiate this here with a basic description of locally absolutely continuous mappings in the parameter-independent case. For $\Phi \in \mathrm{C}^{\nu}_{\mathrm{LAC}}(\mathbb{T};(\mathsf{M};\mathsf{N}))$ and for fixed $x \in \mathsf{M}$, we define a curve

$$\begin{aligned}\Phi_x\colon \mathbb{T} &\to \mathsf{N}\\ t &\mapsto \Phi(t,x).\end{aligned}$$

We shall prove absolute continuity of this curve, for which we use the following definition.

3.8 Definition: (Locally absolutely continuous curve) Let $\mathbb{T} \subseteq \mathbb{R}$ be an interval and let M be a C^∞-manifold. A curve $\xi\colon \mathbb{T} \to \mathsf{M}$ is ***locally absolutely continuous*** if $g \circ \xi\colon \mathbb{T} \to \mathbb{R}$ is locally absolutely continuous for every $g \in \mathrm{C}^\infty(\mathsf{M})$. •

3.9 Remarks: (On local absolute continuity) There are a few easily proved consequences of this definition that we list.

1. A curve is locally absolutely continuous if and only if its local representative in any coordinate chart is locally absolutely continuous in the usual sense. This is a consequence of the fact that, for a C^∞-manifold, one can always find around any point coordinate functions that are the restrictions of globally defined functions.
2. The previous statement also has repercussions for the characterisation of local absolute continuity for curves with values in a real analytic or Stein manifold. Indeed, the existence of real analytic or holomorphic local coordinate functions in these cases means that ξ is locally absolutely continuous if and only if $f \circ \xi$ is locally absolutely continuous for every $f \in \mathrm{C}^\kappa(\mathsf{M})$, $\kappa \in \{\infty, \omega, \mathrm{hol}\}$, as appropriate. •

We then have the following assertions.

3.10 Lemma: (Regularity of locally absolutely continuous mappings) *Let $m \in \mathbb{Z}_{\geq 0}$, let $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega, \mathrm{hol}\}$, and let $\kappa \in \{\infty, \omega, \mathrm{hol}\}$, as required. Let M and N be C^κ-manifolds, they being Stein if $\nu = \mathrm{hol}$. Let $\mathbb{T} \subseteq \mathbb{R}$ be an interval, and let $\Phi \in \mathrm{C}^0_{\mathrm{LAC}}(\mathbb{T}; (\mathsf{M}; \mathsf{N}))$. Then*

(i) $\Phi \in \mathrm{C}^0(\mathbb{T} \times \mathsf{M}; \mathsf{N})$,

(ii) Φ_x *is locally absolutely continuous for* $x \in \mathsf{M}$.

Proof: (i) By Lemma 3.6, $g \circ \Phi \in \mathrm{C}^0(\mathbb{T} \times \mathsf{M}; \mathbb{F})$ for every $g \in \mathrm{C}^\kappa(\mathsf{N})$. We claim that this implies that $\Phi \in \mathrm{C}^0(\mathbb{T} \times \mathsf{M}; \mathsf{N})$. Let $(t_0, x_0) \in \mathbb{T} \times \mathsf{M}$ and denote $y_0 = \Phi(t_0, x_0)$. Let $(\mathcal{U}, \boldsymbol{\chi})$ be a coordinate chart for M about x_0 whose coordinate functions are restrictions to $\mathcal{U}$ of globally defined functions $\chi^1, \dots, \chi^n$ of class C^κ. Let $(\mathcal{V}, \boldsymbol{\eta})$ be a coordinate chart for N about y_0 whose coordinate functions $\eta^1, \dots, \eta^k$ are, again, are restrictions of globally defined functions of class C^κ. The mapping

$$\begin{aligned} \boldsymbol{\eta}\colon \mathsf{N} &\to \mathbb{F}^k \\ y &\mapsto (\eta^1(y), \dots, \eta^k(y)) \end{aligned}$$

is a C^κ-diffeomorphism from a neighbourhood $\mathcal{V}' \subseteq \mathcal{V}$ of y_0 to a neighbourhood $\mathcal{W}$ of $\boldsymbol{\eta}(y_0) \in \mathbb{F}^k$. Since $\Phi^*\boldsymbol{\eta}$ is continuous by hypothesis, there is a neighbourhood $\mathcal{U}'$ of x_0 and an interval $\mathbb{S} \subseteq \mathbb{T}$ about t_0 such that $\Phi^*\boldsymbol{\eta}(\mathbb{S} \times \mathcal{U}') \subseteq \mathcal{W}$. Thus $\Phi(\mathbb{S} \times \mathcal{U}') \subseteq \mathcal{V}'$. Therefore, we can assume, without loss of generality, that $\Phi(\mathbb{S} \times \mathcal{U}) \subseteq \mathcal{V}$. We denote

$$\begin{aligned} \boldsymbol{\chi}\colon \mathsf{M} &\to \mathbb{F}^n \\ x &\mapsto (\chi^1(x), \dots, \chi^n(x)), \end{aligned}$$

making an abuse of notation by using $\boldsymbol{\chi}$ to represent a mapping and its restriction to $\mathcal{U}$. Note that the local representative of Φ in the charts $(\mathcal{U}, \boldsymbol{\chi})$ and $(\mathcal{V}, \boldsymbol{\eta})$ is

$$\begin{aligned} \boldsymbol{\Phi}\colon \mathbb{T} \times \boldsymbol{\chi}(\mathcal{U}) &\to \boldsymbol{\eta}(\mathcal{V}) \\ (t, \boldsymbol{x}) &\mapsto \boldsymbol{\eta} \circ \Phi \circ (\boldsymbol{\chi}|\mathcal{U})^{-1}(t, \boldsymbol{x}, p). \end{aligned}$$

Since $\boldsymbol{\eta}\circ\Phi$ is continuous (by hypothesis) and $(\boldsymbol{\chi}|\mathcal{U})^{-1}$ is of class C^κ, the local representative of Φ is continuous, and this shows that Φ is continuous.

(ii) Since Φ is locally absolutely continuous, $g\circ\Phi \in \mathrm{C}^\nu_{\mathrm{LAC}}(\mathbb{T};\mathsf{M})$ for every $g\in\mathrm{C}^\kappa(\mathsf{N})$. Thus there exists $F\in\mathrm{C}^\nu_{\mathrm{LI}}(\mathbb{T};\mathsf{M})$ and $t_0\in\mathbb{T}$ such that

$$g\circ\Phi(t) = g\circ\Phi(t_0) + \int_{t_0}^t F(s)\,\mathrm{d}s \implies g\circ\Phi_x(t) = g\circ\Phi(t_0,x) + \int_{t_0}^t F(s,x)\,\mathrm{d}s$$

since the map

$$\begin{aligned}\mathrm{ev}_x\colon \mathrm{C}^\nu(\mathsf{M}) &\to \mathsf{M}\\ g &\mapsto g(x)\end{aligned}$$

is continuous. Since

$$|F(s,x)| \le p^\nu_{\{x\},*}(F_s), \qquad s\in\mathbb{T},$$

we can conclude that $(t\mapsto F(t,x))\in\mathrm{L}^1_{\mathrm{loc}}(\mathbb{T};\mathbb{R})$, and so $g\circ\Phi_x$ is the indefinite integral of a locally integrable function, and so is locally absolutely continuous. Local absolute continuity of Φ_x follows from Remark 3.9–1. ■

3.4. Locally integrable time- and parameter-dependent sections. Now we turn our attention to sections depending on both parameter and locally integrably time, as we outlined in Section 1.2. Thus we let $m\in\mathbb{Z}_{\ge0}$, let $m'\in\{0,\mathrm{lip}\}$, let $\nu\in\{m+m',\infty,\omega,\mathrm{hol}\}$, and let $\kappa\in\{\infty,\omega,\mathrm{hol}\}$, as required. As in the preceding section, we let $\pi\colon\mathsf{E}\to\mathsf{M}$ be a C^κ-vector bundle and let $\mathbb{T}\subseteq\mathbb{R}$ be an interval, and now we add to the mix a topological space $\mathcal{P}$. We require no assumptions of $\mathcal{P}$. We consider time- and parameter-dependent sections as being prescribed by continuous functions from $\mathcal{P}$ to $\Gamma^\nu_{\mathrm{LI}}(\mathbb{T};\mathsf{E})$, and we abbreviate

$$\Gamma^\nu_{\mathrm{PLI}}(\mathbb{T};\mathsf{E};\mathcal{P}) = \mathrm{C}^0(\mathcal{P};\Gamma^\nu_{\mathrm{LI}}(\mathbb{T};\mathsf{E})).$$

As we indicated with time-dependent sections, we shall not distinguish between $\xi\in\Gamma^\nu_{\mathrm{PLI}}(\mathbb{T};\mathsf{E};\mathcal{P})$ and the corresponding mapping $\xi\colon\mathbb{T}\times\mathsf{M}\times\mathcal{P}\to\mathsf{E}$. We shall also make use of the notation

$$\xi^p(t,x) = \xi_t(p,x) = \xi^p_t(x) = \xi(p)(t,x) = \xi(t,x,p),$$

when we feel as if it is in our interests to do so. We will still be especially interested in functions, and denote the corresponding spaces of time- and parameter-dependent functions by $\mathrm{C}^\nu_{\mathrm{PLI}}(\mathbb{T};\mathsf{M};\mathcal{P})$.

To give a slightly explicit characterisation of membership in $\Gamma^\nu_{\mathrm{PLI}}(\mathbb{T};\mathsf{E};\mathcal{P})$, we note that the conditions for such membership on ξ are, just by definition: for each $p_0\in\mathcal{P}$, for each compact $K\subseteq\mathsf{M}$ and $\mathbb{K}\subseteq\mathbb{T}$, and for each $\epsilon\in\mathbb{R}_{>0}$, there exists a neighbourhood $\mathcal{O}\subseteq\mathcal{P}$ of p_0 such that

$$\int_{\mathbb{K}} p^\nu_{K,*}(\xi^p_t - \xi^{p_0}_t)\,\mathrm{d}t < \epsilon, \qquad p\in\mathcal{O}, \tag{3.8}$$

where $p^\nu_{K,*}$ is the seminorm for $\Gamma^\nu(\mathsf{E})$, chosen appropriately for ν. The matter of topologising $\Gamma^\nu_{\mathrm{PLI}}(\mathbb{T};\mathsf{E};\mathcal{P})$ is problematic. One might, for instance, use the compact-open topology; however, this topology is not complete without some hypotheses on $\mathcal{P}$ [cf. Jarchow 1981, Proposition 16.6.2].

The following result characterises time- and parameter-dependent vector fields using functions.

3.11 Lemma: (Time- and parameter-dependent functions from time- and parameter-dependent vector fields) *Let $m \in \mathbb{Z}_{\geq 0}$, let $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega, \mathrm{hol}\}$, and let $\kappa \in \{\infty, \omega, \mathrm{hol}\}$, as required. Let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a C^κ-vector bundle, let $\mathbb{T} \subseteq \mathbb{R}$ be an interval, and let $\mathcal{P}$ be a topological space. When $\nu = \mathrm{hol}$, assume that M is a Stein manifold. Then, for a mapping $X\colon \mathcal{P} \to \Gamma^\nu_{\mathrm{LI}}(\mathbb{T}; \mathsf{TM})$, the following are equivalent:*

(i) $X \in \Gamma^\nu_{\mathrm{PLI}}(\mathbb{T}; \mathsf{TM}; \mathcal{P})$*;*

(ii) for any $f \in \mathrm{C}^\kappa(\mathsf{M})$, $(p \mapsto (t \mapsto X^p_t f)) \in \mathrm{C}^\nu_{\mathrm{PLI}}(\mathbb{T}; \mathsf{E}; \mathcal{P})$.

Proof: This follows from Lemma 3.3, making use of the universal property of the initial topology. Indeed, for $f \in \mathrm{C}^\kappa(\mathsf{M})$, we have the diagram

$$\begin{array}{ccc} \Gamma^\nu_{\mathrm{LI}}(\mathbb{T}; \mathsf{TM}) & \xrightarrow{Y \mapsto Yf} & \mathrm{C}^\nu_{\mathrm{LI}}(\mathbb{T}; \mathsf{M}) \\ {\scriptstyle X}\uparrow & \nearrow_{p \mapsto (t \mapsto X^p_t f)} & \\ \mathcal{P} & & \end{array}$$

and the aforementioned universal property means that the vertical arrow is continuous if and only if the diagonal arrow is continuous for every $f \in \mathrm{C}^\kappa(\mathsf{M})$. ■

3.5. Locally absolutely continuous time- and parameter-dependent sections. Now we consider sections that depend locally absolutely continuously on time and which have a suitable dependence on parameters in a topological space. The manner in which this is characterised follows in the obvious way from the manner in which we defined parameter-dependence in the locally integrable case. Thus we let $m \in \mathbb{Z}_{\geq 0}$, let $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega, \mathrm{hol}\}$, and let $\kappa \in \{\infty, \omega, \mathrm{hol}\}$, as required. We let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a C^κ-vector bundle, let $\mathbb{T} \subseteq \mathbb{R}$ be an interval, and let $\mathcal{P}$ be a topological space. We then denote

$$\Gamma^\nu_{\mathrm{PLAC}}(\mathbb{T}; \mathsf{E}; \mathcal{P}) = \mathrm{C}^0(\mathcal{P}; \Gamma^\nu_{\mathrm{LAC}}(\mathbb{T}; \mathsf{E})),$$

and this determines the character of the dependence on parameters.

From the definitions and from Lemma 3.5, we see that $\Xi \in \Gamma^\nu_{\mathrm{PLAC}}(\mathbb{T}; \mathsf{E}; \mathcal{P})$ if and only if, for every $p_0 \in \mathcal{P}$, for every compact $K \subseteq \mathsf{M}$, every compact interval $\mathbb{K} \subseteq \mathbb{T}$, and every $\epsilon \in \mathbb{R}_{>0}$, there exists a neighbourhood $\mathcal{O} \subseteq \mathcal{P}$ such that

$$\sup\{p^\nu_{K,*}(\Xi^p(t) - \Xi^{p_0}(t)) \mid t \in \mathbb{K}\} < \epsilon, \tag{3.9}$$

$$\int_{\mathbb{K}} p^\nu_{K,*}(\xi^p(t) - \xi^{p_0}(t))\, \mathrm{d}t < \epsilon, \tag{3.10}$$

for $p \in \mathcal{O}$, and with ξ being as defined by (3.3). The following quite simple continuity result will be frequently useful for us.

3.12 Lemma: (Locally absolutely continuous parameter-dependent sections are continuous) *Let $m \in \mathbb{Z}_{\geq 0}$, let $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega, \mathrm{hol}\}$, and let $\kappa \in \{\infty, \omega, \mathrm{hol}\}$, as required. We let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a C^κ-vector bundle, let $\mathbb{T} \subseteq \mathbb{R}$ be an interval, and let $\mathcal{P}$ be a topological space. Then*

$$\Gamma^\nu_{\mathrm{PLAC}}(\mathbb{T}; \mathsf{E}; \mathcal{P}) \subseteq \mathrm{C}^0(\mathbb{T} \times \mathsf{M} \times \mathcal{P}; \mathsf{E}).$$

Proof: Let $\mathsf{E}_{\mathbb{T}} = \mathbb{T} \times \mathsf{E}$, which we consider as a C^0-vector bundle over $\mathbb{T} \times \mathsf{M}$ with projection

$$\begin{aligned} \pi_{\mathbb{T}} \colon \mathsf{E}_{\mathbb{T}} &\to \mathbb{T} \times \mathsf{M} \\ (t, e) &\mapsto (t, \pi(e)). \end{aligned}$$

By Lemma 3.6, we have

$$\Gamma^{\nu}_{\mathrm{LAC}}(\mathbb{T}; \mathsf{E}) \subseteq \Gamma^0(\mathsf{E}_{\mathbb{T}}).$$

We claim that the preceding inclusion is continuous if $\Gamma^0(\mathsf{E}_{\mathbb{T}})$ is given the C^0-topology. This, however, follows since, for $\Xi \in \Gamma^{\nu}_{\mathrm{LAC}}(\mathbb{T}; \mathsf{E})$,

$$p^0_{\mathbb{K} \times K}(\Xi) \le p^{\nu}_{K,*,\mathbb{K},0}(\Xi),$$

where $p^{\nu}_{K,*,\mathbb{K},0}$ is as in the proof of Lemma 3.5. Thus we have

$$\mathrm{C}^0(\mathcal{P}; \Gamma^{\nu}_{\mathrm{LAC}}(\mathbb{T}; \mathsf{E})) \subseteq \mathrm{C}^0(\mathcal{P}; \Gamma^0(\mathsf{E}_{\mathbb{T}}))$$

Let $\Xi \in \Gamma^{\nu}_{\mathrm{PLAC}}(\mathbb{T}; \mathsf{E}; \mathcal{P})$ and let $(t_0, x_0, p_0) \in \mathbb{T} \times \mathsf{E} \times \mathcal{P}$. Let $\mathcal{W} \subseteq \mathsf{E}$ be a neighbourhood of $\Xi(t_0, x_0, p_0)$. Let $\mathcal{U}$ be a precompact neighbourhood of x_0, let $\mathbb{S} \subseteq \mathbb{T}$ be a precompact interval with $t_0 \in \mathbb{S}$, and let $\epsilon \in \mathbb{R}_{>0}$ be such that

$$\pi_{\mathbb{T}}^{-1}(\mathbb{S} \times \mathcal{U}) \cap \{(t, e) \in \mathbb{S} \times \mathsf{E} \mid \ \|(t, e) - \Xi(t, \pi(e), p_0)\|_{\pi} < \epsilon\} \subseteq \mathcal{W}.$$

Let $\mathcal{O} \subseteq \mathcal{P}$ be a neighbourhood of p_0 such that

$$p^0_{\mathrm{cl}(\mathcal{U}) \times \mathrm{cl}(\mathbb{S})}(\Xi^p - \Xi^{p_0}) < \epsilon, \qquad p \in \mathcal{O},$$

this by the first paragraph of the proof. Then we have

$$\|\Xi(t, x, p) - \Xi(t, x, p_0)\|_{\pi} < \epsilon, \qquad (t, x, p) \in \mathrm{cl}(\mathbb{S}) \times \mathrm{cl}(\mathcal{U}) \times \mathcal{O}.$$

Then, if $(t, x, p) \in \mathbb{S} \times \mathcal{U} \times \mathcal{O}$, we have $\pi(\Xi(t, x, p)) = (t, x) \in \mathbb{S} \times \mathcal{U}$ and so $\Xi(t, x, p) \in \mathcal{W}$, giving the desired conclusion. ■

3.6. Locally absolutely continuous time- and parameter-dependent mappings. As is the situation with time-dependent local absolute continuity, in the time- and parameter-dependent case we are principally interested in time- and parameter-dependent locally absolutely continuous mappings. Thus let $m \in \mathbb{Z}_{\ge 0}$, let $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m+m', \infty, \omega, \mathrm{hol}\}$, and let $\kappa \in \{\infty, \omega, \mathrm{hol}\}$, as required. Let M and N be C^{κ}-manifolds, let $\mathbb{T} \subseteq \mathbb{R}$ be an interval, and let $\mathcal{P}$ be a topological space. We denote

$$\mathrm{C}^{\nu}_{\mathrm{PLAC}}(\mathbb{T}; (\mathsf{M}; \mathsf{N}); \mathcal{P}) = \mathrm{C}^0(\mathcal{P}; \mathrm{C}^{\nu}_{\mathrm{LAC}}(\mathbb{T}; (\mathsf{M}; \mathsf{N}))),$$

this serving as the space of locally absolutely continuous families of C^{ν}-mappings depending on parameter. This space of mappings has many of the properties of flows, and we will enumerate some of these as they will, when applied to flows, give many of the useful regularity, uniformity, and compactness properties of flows. We give a few such properties here; further such properties are given in Section 3.7.

We consider an extension of Lemma 3.10 to the parameter-dependent setting. We fix $\Phi \in \mathrm{C}^{\nu}_{\mathrm{PLAC}}(\mathbb{T}; (\mathsf{M}; \mathsf{N}); \mathcal{P})$ and, for $(x, p) \in \mathsf{M} \times \mathcal{P}$, denote

$$\begin{aligned} \Phi^p_x \colon \mathbb{T} &\to \mathsf{N} \\ t &\mapsto \Phi(t, x, p). \end{aligned}$$

With this notation, we have the following result.

3.13 Lemma: (Regularity of locally absolutely continuous parameter-dependent mappings) *Let* $m \in \mathbb{Z}_{\geq 0}$*, let* $m' \in \{0, \mathrm{lip}\}$*, let* $\nu \in \{m + m', \infty, \omega, \mathrm{hol}\}$*, and let* $\kappa \in \{\infty, \omega, \mathrm{hol}\}$*, as required. Let* M *and* N *be* C^κ*-manifolds, they being Stein if* $\nu = \mathrm{hol}$*. Let* $\mathbb{T} \subseteq \mathbb{R}$ *be an interval, let* $\mathcal{P}$ *be a topological space, and let* $\Phi \in \mathrm{C}^0_{\mathrm{PLAC}}(\mathbb{T}; (\mathsf{M}; \mathsf{N}); \mathcal{P})$*. Then*

(i) $\Phi \in \mathrm{C}^0(\mathbb{T} \times \mathsf{M} \times \mathcal{P}; \mathsf{N})$ *and*

(ii) Φ^p_x *is locally absolutely continuous for* $(x, p) \in \mathsf{M} \times \mathcal{P}$*.*

Proof: (i) By Lemma 3.12, $f \circ \Phi \in \mathrm{C}^0(\mathbb{T} \times \mathsf{M} \times \mathcal{P}; \mathbb{F})$ for every $f \in \mathrm{C}^\kappa(\mathsf{N})$. We claim that this implies that $\Phi \in \mathrm{C}^0(\mathbb{T} \times \mathsf{M} \times \mathcal{P}; \mathsf{N})$. Let $(t_0, x_0, p_0) \in \mathbb{T} \times \mathsf{M} \times \mathcal{P}$ and denote $y_0 = \Phi(t_0, x_0, p_0)$. Let $(\mathcal{U}, \boldsymbol{\chi})$ be a coordinate chart for M about x_0 whose coordinate functions are restrictions to $\mathcal{U}$ of globally defined functions $\chi^1, \dots, \chi^n$ of class C^κ. Let $(\mathcal{V}, \boldsymbol{\eta})$ be a coordinate chart for N about y_0 whose coordinate functions $\eta^1, \dots, \eta^k$ are, again, restrictions of globally defined functions of class C^κ. The mapping

$$\begin{aligned} \boldsymbol{\eta} \colon \mathsf{N} &\to \mathbb{F}^k \\ y &\mapsto (\eta^1(y), \dots, \eta^k(y)) \end{aligned}$$

(abusing notation) is a C^κ-diffeomorphism from a neighbourhood $\mathcal{V}' \subseteq \mathcal{V}$ of y_0 to a neighbourhood $\mathcal{W}$ of $\boldsymbol{\eta}(y_0) \in \mathbb{F}^k$. Since $\Phi^* \boldsymbol{\eta}$ is continuous by hypothesis, there is a neighbourhood $\mathcal{U}'$ of x_0, an interval $\mathbb{S} \subseteq \mathbb{T}$ about t_0, and a neighbourhood $\mathcal{O}$ of p_0 such that $\Phi^* \boldsymbol{\eta}(\mathbb{S} \times \mathcal{U}' \times \mathcal{O}) \subseteq \mathcal{W}$. Thus $\Phi(\mathbb{S} \times \mathcal{U}' \times \mathcal{O}) \subseteq \mathcal{V}'$. Therefore, we can assume, without loss of generality, that $\Phi(\mathbb{S} \times \mathcal{U} \times \mathcal{O}) \subseteq \mathcal{V}$. We denote

$$\begin{aligned} \boldsymbol{\chi} \colon \mathsf{M} &\to \mathbb{F}^n \\ x &\mapsto (\chi^1(x), \dots, \chi^n(x)) \end{aligned}$$

(abusing notation again). Note that the local representative of Φ in the charts $(\mathcal{U}, \boldsymbol{\chi})$ and $(\mathcal{V}, \boldsymbol{\eta})$ is

$$\begin{aligned} \boldsymbol{\Phi} \colon \mathbb{T} \times \boldsymbol{\chi}(\mathcal{U}) \times \mathcal{P} &\to \boldsymbol{\eta}(\mathcal{V}) \\ (t, \boldsymbol{x}, p) &\mapsto \boldsymbol{\eta} \circ \Phi \circ (\boldsymbol{\chi}|\mathcal{U})^{-1}(t, \boldsymbol{x}, p). \end{aligned}$$

Since $\boldsymbol{\eta} \circ \Phi$ is continuous (by hypothesis) and $(\boldsymbol{\chi}|\mathcal{U})^{-1}$ is of class C^κ, the local representative of Φ is continuous, and this shows that Φ is continuous.

(ii) This follows from Lemma 3.10(ii). ■

3.7. Particular properties of locally Lipschitz time- and parameter-dependent sections and mappings. It turns out that time- and parameter-dependent sections and mappings of local Lipschitz regularity have many rich properties, many of which show up as properties of flows of locally Lipschitz vector fields. We have given a few such properties for general regularity in Lemmata 3.10 and 3.13. In this section, we concentrate particularly on those properties that follow in the Lipschitz (and greater) regularity class.

First we give characterisations of time- and parameter-dependent functions in the case of Lipschitz regularity. These characterisations have an appearance that more closely resembles the hypotheses one sees in standard statements of existence and uniqueness theorems for ordinary differential equations.

We first consider the time-dependent case. In the statement of the results, bear in mind our policy of introducing a Riemannian metric whenever it is convenient; in the statement, the Riemannian metric is denoted by $\mathbb{G}$. First we have the locally integrable case.

3.14 Lemma: (Property of locally integrable locally Lipschitz functions) *Let* M *be a* C^∞*-manifold, let* $\mathbb{T}$ *be an interval, and let* $f \in \mathrm{C}^{\mathrm{lip}}_{\mathrm{LI}}(\mathbb{T};\mathsf{M})$*. If* $K \subseteq \mathsf{M}$ *is compact, then there exists* $\ell \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{T};\mathbb{R}_{\geq 0})$ *such that*

$$|f(t,x_1) - f(t,x_2)| \leq \ell(t)\mathrm{d}_{\mathbb{G}}(x_1,x_2), \qquad t \in \mathbb{T},\ x_1,x_2 \in K.$$

Proof: Since functions are to be thought of as sections of the trivial line bundle $\mathbb{R}_{\mathsf{M}} = \mathsf{M}\times\mathbb{R}$ and since we use the flat connection on this bundle, we have, for any compact set $K \subseteq \mathsf{M}$ and for $g \in \mathrm{C}^{\mathrm{lip}}(\mathsf{M})$,

$$\begin{aligned} l_K(g) &= \sup\left\{ \frac{|g\circ\gamma(1) - g\circ\gamma(0)|}{\ell_{\mathbb{G}}(\gamma)} \;\middle|\; \gamma\colon [0,1] \to \mathsf{M},\ \gamma(0),\gamma(1) \in K,\ \gamma(0) \neq \gamma(1)\right\} \\ &= \sup\left\{ \frac{|g(x_1) - g(x_2)|}{\mathrm{d}_{\mathbb{G}}(x_1,x_2)} \;\middle|\; x_1,x_2 \in K,\ x_1 \neq x_2\right\}. \end{aligned}$$

Let $K \subseteq \mathsf{M}$ be compact. Let $x \in K$ and let $\mathcal{U}_x$ be a neighbourhood of x such that, by Lemma 2.5, for $g \in \mathrm{C}^{\mathrm{lip}}(\mathsf{M})$, we have $\lambda^0_{\mathrm{cl}(\mathcal{U}_x)}(g) = l_{\mathrm{cl}(\mathcal{U}_x)}(g)$. Since $f \in \mathrm{C}^{\mathrm{lip}}_{\mathrm{LI}}(\mathbb{T};\mathsf{M})$, there exists $\ell_x \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{T};\mathbb{R}_{\geq 0})$ such that

$$\operatorname{dil} f(t,y) \leq \ell_x(t), \qquad (t,y) \in \mathbb{T}\times\mathrm{cl}(\mathcal{U}_x).$$

Thus

$$l_{\mathrm{cl}(\mathcal{U}_x)}(f_t) = \lambda^0_{\mathrm{cl}(\mathcal{U}_x)}(f_t) \leq \ell_x(t), \qquad t \in \mathbb{T}.$$

Thus, for $x_1,x_2 \in \mathcal{U}_x$, we have

$$|f(t,x_1) - f(t,x_2)| \leq \ell_x(t)\mathrm{d}_{\mathbb{G}}(x_1,x_2), \qquad t \in \mathbb{T}. \tag{3.11}$$

By compactness of K, there exist $x_1,\dots,x_m \in K$ such that $K \subseteq \cup_{j=1}^m \mathcal{U}_{x_j}$. By the Lebesgue Number Lemma [Burago, Burago, and Ivanov 2001, Theorem 1.6.11], there exists $r \in \mathbb{R}_{>0}$ with the property that, if $x_1,x_2 \in K$ satisfy $\mathrm{d}_{\mathbb{G}}(x_1,x_2) < r$, then there exists $j \in \{1,\dots,m\}$ such that $x_1,x_2 \in \mathcal{U}_{x_j}$. Since $\mathrm{C}^{\mathrm{lip}}_{\mathrm{LI}}(\mathbb{T};\mathsf{M}) \subseteq \mathrm{C}^0_{\mathrm{LI}}(\mathbb{T};\mathsf{M})$, there exists $\beta \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{T};\mathbb{R}_{\geq 0})$ such that $|f(t,x)| \leq \beta(t)$ for $(t,x) \in \mathbb{T}\times K$. Let

$$\ell(t) = \max\left\{\ell_{x_1}(t),\dots,\ell_{x_m}(t), \frac{2\beta(t)}{r}\right\}, \qquad t \in \mathbb{T},$$

noting that $\ell \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{T};\mathbb{R}_{\geq 0})$. Let $x_1,x_2 \in K$. If $\mathrm{d}_{\mathbb{G}}(x_1,x_2) < r$, then let $j \in \{1,\dots,m\}$ be such that $x_1,x_2 \in \mathcal{U}_{x_j}$, and then we have

$$|f(t,x_1) - f(t,x_2)| \leq \ell_{x_j}(t)\mathrm{d}_{\mathbb{G}}(x_1,x_2) \leq \ell(t)\mathrm{d}_{\mathbb{G}}(x_1,x_2), \qquad t \in \mathbb{T}.$$

If $\mathrm{d}_{\mathbb{G}}(x_1,x_2) \geq r$, then

$$|f(t,x_1) - f(t,x_2)| \leq |f(t,x_1)| + |f(t,x_2)| \leq 2\beta(t) \leq \frac{2\beta(t)}{r}\mathrm{d}_{\mathbb{G}}(x_1,x_2) \leq \ell(t)\mathrm{d}_{\mathbb{G}}(x_1,x_2),$$

which gives the result. ■

We also have the following characterisation in the locally absolutely continuous case.

3.15 Lemma: (Property of locally absolutely continuous locally Lipschitz functions) *Let* M *be a* C^∞*-manifold, let* $\mathbb{T}$ *be an interval, and let* $f \in \mathrm{C}^{\mathrm{lip}}_{\mathrm{LAC}}(\mathbb{T};\mathsf{M})$*. If* $K \subseteq \mathsf{M}$ *is compact, then there exists* $\ell \in \mathrm{C}^0(\mathbb{T};\mathbb{R}_{\geq 0})$ *such that*

$$|f(t,x_1) - f(t,x_2)| \leq \ell(t)\mathrm{d}_{\mathbb{G}}(x_1,x_2), \qquad t \in \mathbb{T},\ x_1,x_2 \in K.$$

Proof: The proof follows exactly like that of Lemma 3.14, except that the functions ℓ_x, $x \in K$, that appear in the proof can be taken to be continuous and the function β can also be replaced by a continuous function. ■

Now we give similar statements in the time- and parameter-dependent case. The reader may wish to compare the form of the characterisation here with the usual hypotheses one sees for existence, uniqueness, and continuous-dependence on parameters for ordinary differential equations. One way to understand how our hypotheses are different from the usual ones is that we consider continuity "outside the integral."

3.16 Lemma: (Property of locally integrable and parameter-dependent locally Lipschitz functions) *Let* M *be a* C^∞*-manifold, let* $\mathbb{T}$ *be an interval, let* $\mathcal{P}$ *be a topological space, and let* $f \in \mathrm{C}^{\mathrm{lip}}_{\mathrm{PLI}}(\mathbb{T};\mathsf{M};\mathcal{P})$*. If* $K \subseteq \mathsf{M}$ *is compact, if* $\mathbb{K} \subseteq \mathbb{T}$ *is a compact interval, and if* $p_0 \in \mathcal{P}$*, then there exists* $C \in \mathbb{R}_{>0}$ *and a neighbourhood* $\mathcal{O}$ *of* p_0 *such that*

$$\int_{\mathbb{K}} |f(t,x_1,p) - f(t,x_2,p)|\,\mathrm{d}t \leq C\mathrm{d}_{\mathbb{G}}(x_1,x_2), \qquad x_1,x_2 \in K,\ p \in \mathcal{O}.$$

Proof: Let $K \subseteq \mathsf{M}$ and $\mathbb{K} \subseteq \mathbb{T}$ be compact, and let $p_0 \in \mathcal{P}$. Let $x \in K$ and, as in the proof of Lemma 3.14, let $\mathcal{U}_x$ be a neighbourhood of x and let $\ell_x \in \mathrm{L}^1(\mathbb{K};\mathbb{R}_{\geq 0})$ be such that

$$\operatorname{dil} f(t,y,p_0) \leq \ell_x(t)\mathrm{d}_{\mathbb{G}}(x_1,x_2), \qquad (t,y) \in \mathbb{K} \times \operatorname{cl}(\mathcal{U}_x).$$

According to (3.8), there exists a neighbourhood $\mathcal{O}_x$ of p_0 such that

$$\int_{\mathbb{K}} \operatorname{dil}(f^p - f^{p_0})(t,y)\,\mathrm{d}t < 1, \qquad (t,y,p) \in \mathbb{K} \times \mathcal{U}_x \times \mathcal{O}_x.$$

Therefore, by the triangle inequality,

$$\int_{\mathbb{K}} \operatorname{dil} f^p(t,y)\,\mathrm{d}t \leq \int_{\mathbb{K}} \operatorname{dil}(f^p - f^{p_0})(t,y)\,\mathrm{d}t + \int_{\mathbb{K}} \operatorname{dil} f^{p_0}(t,y)\,\mathrm{d}t < \underbrace{1 + \int_{\mathbb{K}} \ell_x(t)\,\mathrm{d}t}_{C_x}$$

for all $(t,y,p) \in \mathbb{K} \times \mathcal{U}_x \times \mathcal{O}_x$. Thus, by Lemma 2.5 and with C_x as indicated on the right in the preceding equation, we have

$$\int_{\mathbb{K}} \frac{|f(t,x_1,p) - f(t,x_2,p)|}{\mathrm{d}_{\mathbb{G}}(x_1,x_2)}\,\mathrm{d}t \leq \int_{\mathbb{K}} \lambda^0_{\operatorname{cl}(\mathcal{U}_x)}(f^p_t)\,\mathrm{d}t \leq C_x \tag{3.12}$$

for $x_1,x_2 \in \mathcal{U}_x$ distinct and for $p \in \mathcal{O}_x$.

By compactness of K, there exists $x_1,\dots,x_m \in K$ such that $K \subseteq \cup_{j=1}^m \mathcal{U}_{x_j}$. By the Lebesgue Number Lemma [Burago, Burago, and Ivanov 2001, Theorem 1.6.11], there exists $r \in \mathbb{R}_{>0}$ with the property that, if $x_1,x_2 \in K$ satisfy $\mathrm{d}_{\mathbb{G}}(x_1,x_2) < r$, then there exists

$j \in \{1, \dots, m\}$ such that $x_1, x_2 \in \mathcal{U}_{x_j}$. Since $\mathrm{C}^0_{\mathrm{LI}}(\mathbb{T}; \mathsf{M}; \mathcal{P}) \subseteq \mathrm{C}^{\mathrm{lip}}_{\mathrm{LI}}(\mathbb{T}; \mathsf{M}; \mathcal{P})$, by (3.8) there exists a neighbourhood $\mathcal{O}'$ of p_0 such that

$$\int_{\mathbb{K}} |f(t, x, p)| \, \mathrm{d}t < 1, \qquad (t, x, p) \in \mathbb{K} \times K \times \mathcal{O}'.$$

Let

$$C = \max\left\{C_{x_1}, \dots, C_{x_m}, \frac{2}{r}\right\}$$

and let $\mathcal{O} = \mathcal{O}' \cap (\cap_{j=1}^m \mathcal{O}_{x_j})$. Let $x_1, x_2 \in K$ and $p \in \mathcal{O}$. If $\mathrm{d}_{\mathbb{G}}(x_1, x_2) < r$, then let $j \in \{1, \dots, m\}$ be such that $x_1, x_2 \in \mathcal{U}_{x_j}$, and then we have

$$\int_{\mathbb{K}} |f(t, x_1, p) - f(t, x_2, p)| \, \mathrm{d}t \le C_{x_j} \mathrm{d}_{\mathbb{G}}(x_1, x_2) \le C \mathrm{d}_{\mathbb{G}}(x_1, x_2).$$

If $\mathrm{d}_{\mathbb{G}}(x_1, x_2) \ge r$, then

$$\begin{aligned} \int_{\mathbb{K}} |f(t, x_1, p) - f(t, x_2, p)| \, \mathrm{d}t \le \int_{\mathbb{K}} |f(t, x_2, p)| \, \mathrm{d}t + \int_{\mathbb{K}} f(t, x_2, p) \, \mathrm{d}t \\ < 2 \le \frac{2}{r} \mathrm{d}_{\mathbb{G}}(x_1, x_2) \le C \mathrm{d}_{\mathbb{G}}(x_1, x_2), \end{aligned}$$

which gives the result. ■

For locally absolutely continuous parameter-dependent locally Lipschitz functions, we have the following analogue of Lemma 3.16 in the locally integrable case.

3.17 Lemma: (Property of locally absolutely continuous and parameter-dependent locally Lipschitz functions) *Let M be a C^∞-manifold, let $\mathbb{T}$ be an interval, let $\mathcal{P}$ be a topological space, and let $f \in \mathrm{C}^{\mathrm{lip}}_{\mathrm{PLAC}}(\mathbb{T}; \mathsf{M}; \mathcal{P})$. If $K \subseteq \mathsf{M}$ is compact, if $\mathbb{K} \subseteq \mathbb{T}$ is a compact interval, and if $p_0 \in \mathcal{P}$, then there exists $C \in \mathbb{R}_{>0}$ and a neighbourhood $\mathcal{O}$ of p_0 such that*

$$|f(t, x_1, p) - f(t, x_2, p)| \le C \mathrm{d}_{\mathbb{G}}(x_1, x_2), \qquad t \in \mathbb{K}, \ x_1, x_2 \in K, \ p \in \mathcal{O}.$$

Proof: Just as Lemma 3.15 follows in the same manner as Lemma 3.14, replacing locally integrable functions by continuous functions, we can execute the proof of Lemma 3.16 with continuous functions in place of locally integrable functions and sup-norms in place of L^1-norms. Since continuous functions are bounded on compact integrals, the result follows. ■

Now we turn to particular properties of locally absolutely continuous mappings in the Lipschitz regularity class. First we consider a uniformity in time property of the family locally absolutely continuous curves Φ^p_x, $(x, p) \in \mathsf{M} \times \mathcal{P}$. This property, when translated to flows, gives a stronger than usual continuity property of flows as a joint function of state and parameter. To state the result, we consider the space $\mathrm{C}^0(\mathbb{T}; \mathsf{M})$ with the topology (indeed, uniformity) defined by the family of semimetrics

$$\mathrm{d}_{\mathbb{K}, \mathsf{M}}(\gamma_1, \gamma_2) = \sup\{\mathrm{d}_{\mathbb{G}}(\gamma_1(t), \gamma_2(t)) \mid t \in \mathbb{K}\}, \qquad \mathbb{K} \subseteq \mathbb{T} \text{ a compact interval}, \tag{3.13}$$

making use of a Riemannian metric $\mathbb{G}$ for M.

Our result is then the following, recalling the notion of uniform convergence in uniform spaces as in [Willard 1970, §42].

3.18 Lemma: (Uniform convergence property of locally absolutely continuous parameter-dependent mappings) *Let $m \in \mathbb{Z}_{\geq 0}$, let $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega, \mathrm{hol}\}$, and let $\kappa \in \{\infty, \omega, \mathrm{hol}\}$, as required. Let M and N be C^κ-manifolds, they being Stein if $\nu = \mathrm{hol}$. Let $\mathbb{T} \subseteq \mathbb{R}$ be an interval, let $\mathcal{P}$ be a topological space, and let $\Phi \in \mathrm{C}^\nu_{\mathrm{PLAC}}(\mathbb{T}; (\mathsf{M}; \mathsf{N}); \mathcal{P})$. If $\nu \geq \mathrm{lip}$, then the family of curves Φ^p_x, $(x, p) \in \mathsf{M} \times \mathcal{P}$, converges uniformly to $\Phi^{p_0}_{x_0}$ as $(x, p) \to (x_0, p_0)$.*

Proof: Let $\mathbb{G}_{\mathsf{M}}$ be a Riemannian or Hermitian metric for M and let $\mathbb{G}_{\mathsf{N}}$ be a Riemannian or Hermitian for N. Let us denote by d_{M} and d_{N} the distance functions associated with these metrics.

We begin by noting that

$$\mathrm{d}_{\mathsf{N}}(\Phi(t, x, p), \Phi(t, x_0, p_0)) \leq \mathrm{d}_{\mathsf{N}}(\Phi(t, x, p_0), \Phi(t, x_0, p_0)) + \mathrm{d}_{\mathsf{N}}(\Phi(t, x, p), \Phi(t, x, p_0)). \quad (3.14)$$

We shall, therefore, break the proof into two parts, the first being concerned with estimating the first term on the right and the second the second.

Before we embark upon these estimates, let us introduce some preliminary notation. Let $f \in \mathrm{C}^\kappa(\mathsf{M})$. Since $f \circ \Phi \in \mathrm{C}^\nu_{\mathrm{PLAC}}(\mathbb{T}; \mathsf{M}; \mathcal{P})$, for $t_0 \in \mathbb{T}$ there exists $F_{f,t_0} \in \mathrm{C}^\nu_{\mathrm{PLI}}(\mathbb{T}; \mathsf{M}; \mathcal{P})$ such that

$$f \circ \Phi^p(t) = f \circ \Phi^p(t_0) + \int_{t_0}^t F^p_{f,t_0}(s)\, \mathrm{d}s \implies f \circ \Phi^p_x(t) = f \circ \Phi^p_x(t_0) + \int_{t_0}^t F^p_{f,t_0}(s, x)\, \mathrm{d}s.$$

First we show that, for $t_0 \in \mathbb{T}$, there exists an interval $\mathbb{T}_{t_0,1}$ about t_0 with the following property: for any $\epsilon \in \mathbb{R}_{>0}$, there exists a neighbourhood $\mathcal{U}'_{t_0,1}$ of x_0 such that

$$\mathrm{d}_{\mathsf{N}}(\Phi(t, x, p_0), \Phi(t, x_0, p_0)) < \epsilon, \qquad (t, x) \in \mathbb{T}_{t_0,1} \times \mathcal{U}'_{t_0,1}. \quad (3.15)$$

Thus we fix, for the moment, $t_0 \in \mathbb{T}$. Let $\eta^1, \dots, \eta^k \in \mathrm{C}^\kappa(\mathsf{N})$ be such that they form a coordinate system defined on a precompact neighbourhood $\mathcal{V}_{t_0}$ about $\Phi(t_0, x_0, p_0)$. By Lemma 2.3, let $C \in \mathbb{R}_{>0}$ be such that

$$\begin{aligned} &C^{-1} \max\{|\eta^j(y_1) - \eta^j(y_2)| \mid j \in \{1, \dots, k\}\} \\ &\qquad \leq \mathrm{d}_{\mathsf{N}}(y_1, y_2) \leq C \max\{|\eta^j(y_1) - \eta^j(y_2)| \mid j \in \{1, \dots, k\}\}, \qquad y_1, y_2 \in \mathcal{V}_{t_0}, \end{aligned}$$

making use of the fact that

$$(y_1, y_2) \mapsto \left(\sum_{j=1}^k |\eta^j(y_1) - \eta^j(y_2)|^2 \right)^{1/2}$$

is a Riemannian or Hermitian metric on $\mathcal{V}_{t_0}$, along with the standard relationship between the 2- and ∞-norms for $\mathbb{F}^k$. By Lemma 3.13(i), let $\mathbb{T}_{t_0,1} \subseteq \mathbb{T}$ be an interval with $t_0 \in \mathrm{int}(\mathbb{T}_{t_0,1})$ and let $\mathcal{U}_{t_0,1} \subseteq \mathsf{M}$ be a geodesically convex neighbourhood of x_0 such that

$$\Phi(\mathbb{T}_{t_0,1} \times \mathcal{U}_{t_0,1} \times \{p_0\}) \subseteq \mathcal{V}_{t_0}.$$

Since $F^{p_0}_{\eta^j,t_0} \in \mathrm{C}^{\mathrm{lip}}_{\mathrm{LI}}(\mathbb{T}; \mathsf{M})$, there exists $\alpha \in \mathbb{R}_{>0}$ such that

$$\int_{\mathbb{T}_{t_0,1}} \mathrm{dil}\,(F^{p_0}_{\eta^j,t_0})(s, x)\, \mathrm{d}s \leq \alpha, \qquad j \in \{1, \dots, k\},\ x \in \mathcal{U}_{t_0,1}.$$

Since $\mathcal{U}_{t_0,1}$ is assumed to be geodesically convex, we see from the proof of Lemma 2.5 that

$$\sup\left\{\int_{\mathbb{T}_{t_0,1}} \frac{|F^{p_0}_{\eta^j,t_0}(s,x_1)-F^{p_0}_{\eta^j,t_0}(s,x_2)|}{\mathrm{d}_{\mathsf{M}}(x_1,x_2)}\,\mathrm{d}s \;\middle|\; x_1,x_2\in\mathrm{cl}(\mathcal{U}_{t_0,1}),\ x_1\neq x_2\right\}\\ = \sup\left\{\int_{\mathbb{T}_{t_0,1}} \mathrm{dil}\,F^{p_0}_{\eta^j,t_0}(s,x)\,\mathrm{d}s \;\middle|\; x\in\mathrm{cl}(\mathcal{U}_{t_0,1})\right\}\leq\alpha, \qquad j\in\{1,\dots,k\}.$$

Therefore,

$$\int_{\mathbb{T}_{t_0,1}} |F^{p_0}_{\eta^j,t_0}(s,x_1)-F^{p_0}_{\eta^j,t_0}(s,x_2)|\,\mathrm{d}s\leq\alpha\mathrm{d}_{\mathsf{M}}(x_1,x_2), \qquad j\in\{1,\dots,k\},\ x_1,x_2\in\mathcal{U}_{t_0,1}.$$

Since

$$\eta^j\circ\Phi^{p_0}_x(t)-\eta^j\circ\Phi^{p_0}_{x_0}(t)=\eta^j\circ\Phi^{p_0}_x(t_0)-\eta^j\circ\Phi^{p_0}_{x_0}(t_0)+\int_{t_0}^t(F^{p_0}_{\eta^j,t_0}(s,x)-F^{p_0}_{\eta^j,t_0}(s,x_0))\,\mathrm{d}s,\\ j\in\{1,\dots,k\},\ (t,x)\in\mathbb{T}_{t_0,1}\times\mathcal{U}_{t_0,1},$$

we have

$$\begin{aligned}\mathrm{d}_{\mathsf{N}}(\Phi^{p_0}_x(t),\Phi^{p_0}_{x_0}(t)) &\leq C\max\{|\eta^j\circ\Phi^{p_0}_x(t)-\eta^j\circ\Phi^{p_0}_{x_0}(t)| \mid j\in\{1,\dots,k\}\}\\ &\leq C\max\{|\eta^j\circ\Phi^{p_0}_x(t_0)-\eta^j\circ\Phi^{p_0}_{x_0}(t_0)| \mid j\in\{1,\dots,k\}\}\\ &\quad + C\alpha\mathrm{d}_{\mathsf{M}}(x,x_0),\end{aligned} \tag{3.16}$$

for $t\in\mathbb{T}_{t_0,1}$ and $x\in\mathcal{U}_{t_0,1}$. Now note that

$$\lim_{x\to x_0}\Phi(t_0,x,p_0)=\Phi(t_0,x_0,p_0),$$

simply by continuity of Φ which is proved in Lemma 3.13(i). By this fact, along with continuity of d_{N}, for $\epsilon\in\mathbb{R}_{>0}$, there exists a neighbourhood $\mathcal{U}'_{t_0,1}\subseteq\mathcal{U}_{t_0,1}$ of x_0 such that

$$\mathrm{d}_{\mathsf{N}}(\Phi^{p_0}_x(t),\Phi^{p_0}_{x_0}(t))<\epsilon, \qquad (t,x)\in\mathbb{T}_{t_0,1}\times\mathcal{U}'_{t_0,1}.$$

This gives (3.15).

Next we show that, for $t_0\in\mathbb{T}$, there exist an interval $\mathbb{T}_{t_0,2}$ about t_0 and a precompact neighbourhood $\mathcal{U}_{t_0,2}$ about x_0 with the following property: for any $\epsilon\in\mathbb{R}_{>0}$, there exists a neighbourhood $\mathcal{O}_{t_0,2}$ of p_0 such that

$$\mathrm{d}_{\mathsf{N}}(\Phi(t,x,p),\Phi(t,x,p_0))<\epsilon, \qquad (t,x,p)\in\mathbb{T}_{t_0,2}\times\mathcal{U}_{t_0,2}\times\mathcal{O}_{t_0,2}. \tag{3.17}$$

Thus we fix, for the moment, $t_0\in\mathbb{T}$. We take the data

1. $\mathcal{V}_{t_0}\subseteq\mathsf{N}$,
2. $\eta^1,\dots,\eta^k\in\mathrm{C}^\kappa(\mathsf{N})$,
3. $C\in\mathbb{R}_{>0}$,

as in the preceding part of the proof. By Lemma 3.13(i), we take an interval $\mathbb{T}_{t_0,2} \subseteq \mathbb{T}$ with $t_0 \in \operatorname{int}(t_{\mathbb{T}_{t_0,2}})$, a precompact neighbourhood $\mathcal{U}_{t_0,2} \subseteq \mathsf{M}$ of x_0, and a neighbourhood $\mathcal{O}_{t_0,2} \subseteq \mathcal{P}$ of p_0 such that

$$\Phi(\mathbb{T}_{t_0,2} \times \mathcal{U}_{t_0,2} \times \mathcal{O}_{t_0,2}) \subseteq \mathcal{V}_{t_0}.$$

We then immediately have

$$\begin{aligned} \mathrm{d}_{\mathsf{N}}(\Phi^p_x(t), \Phi^{p_0}_x(t)) &\leq C \max\{\eta^i \circ \Phi^p_x(t) - \eta^j \circ \Phi^{p_0}_x(t) \mid j \in \{1, \dots, m\}\} \\ &\leq C \max\{|\eta^j \circ \Phi^p_x(t_0) - \eta^j \circ \Phi^{p_0}_x(t_0)| \mid j \in \{1, \dots, k\}\} \\ &\leq C \max\left\{ \int_{\mathbb{T}_{t_0,2}} |F^p_{\eta^j,t_0}(s,x) - F^{p_0}_{\eta^j,t_0}(s,x)| \,\mathrm{d}s \;\middle|\; j \in \{1, \dots, k\} \right\}, \end{aligned}$$

for $(t,x,p) \in \mathbb{T}_{t_0,2} \times \mathcal{U}_{t_0,2} \times \mathcal{O}_{t_0,2}$. By (3.9) and (3.10), for $\epsilon \in \mathbb{R}_{>0}$, there is a neighbourhood $\mathcal{O}'_{t_0,2} \subseteq \mathcal{O}_{t_0,2}$ of p_0 such that

$$\mathrm{d}_{\mathsf{N}}(\Phi(t,x,p), \Phi(t,x,p_0)) < \epsilon, \qquad (t,x,p) \in \mathbb{T}_{t_0,2} \times \mathcal{U}_{t_0,2} \times \mathcal{O}'_{t_0,2}.$$

This gives (3.17).

Combining the previous two parts of the proof and (3.14), we see that, for each $t_0 \in \mathbb{T}$, there exists an interval $\mathbb{T}_{t_0} \subseteq \mathbb{T}$ with $t_0 \in \operatorname{int}(\mathbb{T}_{t_0})$, and with the following property: for each $\epsilon \in \mathbb{R}_{>0}$, there is a neighbourhood $\mathcal{U}_{t_0}$ of x_0 and a neighbourhood $\mathcal{O}_{t_0}$ of p_0 such that

$$\mathrm{d}_{\mathsf{N}}(\Phi^x_p(t), \Phi^{p_0}_{x_0}(t)) < \epsilon, \qquad (t,x,p) \in \mathbb{T}_{t_0} \times \mathcal{U}_{t_0} \times \mathcal{O}_{t_0}.$$

Now, if $\mathbb{K} \subseteq \mathbb{T}$ is a compact subinterval and if $\epsilon \in \mathbb{R}_{>0}$, let $t_1, \dots, t_l \in \mathbb{K}$ be such that $\mathbb{K} \subseteq \cup_{j=1}^l \mathbb{T}_{t_j}$, and define

$$\mathcal{U} = \bigcap_{j=1}^{l} \mathcal{U}_{t_j}, \quad \mathcal{O} = \bigcap_{j=1}^{l} \mathcal{O}_{t_j}.$$

Then

$$\mathrm{d}_{\mathsf{N}}(\Phi^p_x(t), \Phi^{p_0}_{x_0}(t)) < \epsilon, \qquad (t,x,p) \in \mathbb{K} \times \mathcal{U} \times \mathcal{O},$$

giving the lemma. ■

Next we give a useful result which has, as a consequence, that compact images under parameter-dependent locally absolutely continuous maps are precompact, even under a variation of parameters.

3.19 Lemma: (Robustness of compactness by variations of parameters) *Let $m \in \mathbb{Z}_{\geq 0}$, let $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m+m', \infty, \omega, \mathrm{hol}\}$, and let $\kappa \in \{\infty, \omega, \mathrm{hol}\}$, as required. Let M and N be C^κ-manifolds, they being Stein if $\nu = \mathrm{hol}$. Let $\mathbb{T} \subseteq \mathbb{R}$ be an interval, let $\mathcal{P}$ be a topological space, and let $\Phi \in \mathrm{C}^\nu_{\mathrm{PLAC}}(\mathbb{T}; (\mathsf{M}; \mathsf{N}); \mathcal{P})$. Let $K \subseteq \mathsf{M}$ be compact, let $\mathbb{K} \subseteq \mathbb{T}$ be a compact interval, and let $p_0 \in \mathcal{P}$. Denote*

$$K_0 = \bigcup_{(t,x) \in \mathbb{K} \times K} \Phi(t,x,p_0).$$

Then, if $\nu \geq \mathrm{lip}$, for any neighbourhood $\mathcal{V}$ of K_0, there exists a neighbourhood $\mathcal{O} \subseteq \mathcal{P}$ of p_0 such that

$$\bigcup_{(t,x,p) \in \mathbb{K} \times K \times \mathcal{O}} \Phi(t,x,p) \subseteq \mathcal{V}.$$

Proof: By Lemma 3.18, for $x \in K$, let $\mathcal{U}_x$ be a neighbourhood of x and let $\mathcal{O}_x$ be a neighbourhood of p_0 such that

$$\bigcup_{(t,x,p)\in\mathbb{K}\times(K\cap\mathcal{U}_x)\times\mathcal{O}_x} \Phi(t,x,p) \subseteq \mathcal{V}.$$

By compactness of K, let $x_1,\dots,x_k \in K$ be such that $K = \cup_{j=1}^k K \cap \mathcal{U}_{x_j}$ and let $\mathcal{O} = \cap_{j=1}^k \mathcal{O}_{x_j}$. Then

$$\Phi(t,x,p) \subseteq \mathcal{V}, \qquad (t,x,p) \in \mathbb{K}\times K\times\mathcal{O},$$

as desired. ■

The final result we record establishes a global (on compact sets) Lipschitz constant for locally absolutely continuous mappings.

3.20 Lemma: (Uniform Lipschitz character of locally absolutely continuous mappings) *Let $m \in \mathbb{Z}_{\geq 0}$, let $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m+m', \infty, \omega, \mathrm{hol}\}$, and let $\kappa \in \{\infty, \omega, \mathrm{hol}\}$, as required. Let M and N be C^κ-manifolds, they being Stein if $\nu = \mathrm{hol}$. Let $\mathbb{T} \subseteq \mathbb{R}$ be an interval, let $\mathcal{P}$ be a topological space, and let $\Phi \in \mathrm{C}^\nu_{\mathrm{PLAC}}(\mathbb{T};(\mathsf{M};\mathsf{N});\mathcal{P})$. Let $K \subseteq \mathsf{M}$ be compact, let $\mathbb{K} \subseteq \mathbb{T}$ be a compact interval, and let $p_0 \in \mathcal{P}$. Let $\mathbb{G}_{\mathsf{M}}$ and $\mathbb{G}_{\mathsf{N}}$ be Riemannian or Hermitian metrics for M and N, respectively. Then, if $\nu \geq \mathrm{lip}$, there exists a neighbourhood $\mathcal{O} \subseteq \mathcal{P}$ of p_0 and $C \in \mathbb{R}_{>0}$ such that*

$$\mathrm{d}_{\mathsf{N}}(\Phi(t,x_1,p),\Phi(t,x_2,p)) \leq C\mathrm{d}_{\mathsf{M}}(x_1,x_2), \qquad t \in \mathbb{K},\ x_1,x_2 \in K,\ p \in \mathcal{O}.$$

Proof: For $(t,x) \in \mathbb{K}\times K$, let $\mathcal{V}_{(t,x)}$ be a neighbourhood of $\Phi(t,x,p_0)$. By Lemma 3.19, there exist an open interval $\mathbb{T}_{(t,x)} \subseteq \mathbb{K}$ containing t, a precompact geodesically convex neighbourhood $\mathcal{U}_{(t,x)} \subseteq \mathsf{M}$ of x, and a neighbourhood $\mathcal{O}_{(t,x)}$ of p_0 such that

$$\{\Phi(t',x',p) \mid (t',x',p) \in \mathbb{T}_{(t,x)}\times\mathcal{U}_{(t,x)}\times\mathcal{O}_{(t,x)}\} \subseteq \mathcal{V}_{(t,x)}.$$

Moreover, as we saw in the proof of Lemma 3.18, we can choose $\mathcal{V}_{(t,x)}$ to be a coordinate chart with globally defined coordinate functions $\eta^1,\dots,\eta^k \in \mathrm{C}^\kappa(\mathsf{M})$ and we can take $C'_{(t,x)} \in \mathbb{R}_{>0}$ so that

$$\begin{aligned} &C'^{-1}_{(t,x)} \max\{|\eta^j(y_1)-\eta^j(y_2)| \mid j \in \{1,\dots,k\}\} \\ &\qquad \leq \mathrm{d}_{\mathsf{N}}(y_1,y_2) \leq C'_{(t,x)} \max\{|\eta^j(y_1)-\eta^j(y_2)| \mid j\in\{1,\dots,k\}\}, \qquad y_1,y_2 \in \mathcal{V}_{(t,x)}. \end{aligned}$$

In this case, by following the steps leading to the formula (3.16), we arrive at

$$\mathrm{d}_{\mathsf{N}}(\Phi(t,x_1,p),\Phi(t,x_2,p)) \leq C_{(t,x)}\mathrm{d}_{\mathsf{M}}(x_1,x_2), \qquad t' \in \mathbb{T}_{(t,x)},\ x_1,x_2 \in \mathcal{U}_{(t,x)},\ p \in \mathcal{O}_{(t,x)}.$$

where $C_{(t,x)} = C'^2_{(t,x)}$.

By compactness of

$$L_x \triangleq \{\Phi(t,x,p_0) \mid t \in \mathbb{K}\},$$

there exist $t_{x,1},\dots,t_{x,k_x} \in \mathbb{K}$ such that

$$L_x \subseteq \bigcup_{j=1}^{k_x} \mathrm{int}(\mathcal{V}_{(t_{x,j},x)}).$$

Let $\mathcal{O}_x = \cap_{j=1}^{k_x} \mathcal{O}_{t_{x,j},x}$ and $\mathcal{U}_x = \cap_{j=1}^{k_x} \mathcal{U}_{(t_{x,j},x)}$, and define

$$C_x = \max\{C_{(t_{x,1},x)}, \dots, C_{(t_{x,k_x},x)}\}.$$

Choose $x_1, \dots, x_k \in K$ so that $K \subseteq \cup_{j=1}^{k} \mathcal{U}_{x_j}$. Let $\mathcal{O} = \cap_{j=1}^{k} \mathcal{O}_{x_j}$ and define

$$C = \max\{C_{x_1}, \dots, C_{x_k}\}.$$

If necessary and by Lemma 3.19, shrink $\mathcal{O}$ so that

$$L = \mathrm{cl}(\{\Phi(t, x, p) \mid t \in \mathbb{K},\ x \in K,\ p \in \mathcal{O}\})$$

is compact. Let $M \in \mathbb{R}_{>0}$ and $y_0 \in K$ be such that

$$\mathrm{d}_{\mathsf{M}}(y, y_0) \le M, \qquad y \in L.$$

By the Lebesgue Number Lemma [Burago, Burago, and Ivanov 2001, Theorem 1.6.11], let $r \in \mathbb{R}_{>0}$ be such that, if $x_1, x_2 \in K$ satisfy $\mathrm{d}_{\mathsf{M}}(x_1, x_2) < r$, then there exists $j \in \{1, \dots, k\}$ such that $x_1, x_2 \in \mathcal{U}_{x_j}$. Let

$$C = \max\left\{C_{x_1}, \dots, C_{x_k}, \frac{2M}{r}\right\}.$$

Now let $t \in \mathbb{K}$, let $x_1, x_2 \in K$, and let $p \in \mathcal{O}$. If $\mathrm{d}_{\mathsf{M}}(x_1, x_2) < r$, let $j \in \{1, \dots, k\}$ be such that $x_1, x_2 \in \mathcal{U}_{x_j}$. Let $l \in \{1, \dots, k_{x_j}\}$ be such that $t \in \mathbb{T}_{(t_{x_j,l},x_j)}$. Then we have

$$\mathrm{d}_{\mathsf{N}}(\Phi(t, x_1, p), \Phi(t, x_2, p)) \le C_{(t_l,x_j)} \mathrm{d}_{\mathsf{M}}(x_1, x_2) \le C \mathrm{d}_{\mathsf{M}}(x_1, x_2).$$

If $\mathrm{d}_{\mathsf{M}}(x_1, x_2) \ge r$, then

$$\begin{aligned} &\mathrm{d}_{\mathsf{N}}(\Phi(t, x_1, p), \Phi(t, x_2, p)) \\ &\qquad \le \mathrm{d}_{\mathsf{N}}(\Phi(t, x_1, p), y_0) + \mathrm{d}_{\mathsf{N}}(\Phi(t, x_2, p), y_0) \le \frac{2M}{r} r \le C \mathrm{d}_{\mathsf{M}}(x_1, x_2), \end{aligned}$$

as desired. ■

4. Time- and parameter-dependent composition operators

The Picard operator we introduce in Section 5.1 will be a mapping involving a certain sort of composition operator. As we discussed in Section 1.3, the operator we use differs from that of Agrachev and Gamkrelidze [1978], where the operator features a differentiation and has the benefit that differentiation is linear and continuous. However, the operator we define involves a *nonlinear* operator, and so proving continuity becomes possibly difficult.

We, therefore, discuss first the time- and parameter-*in*dependent setting where we are able to make use of the recent result of Lewis [2023, Theorem 5.29] in the real analytic case. After this, we consider the complications added by time and parameter dependence.

4.1. Time- and parameter-independent composition operators. Let $m \in \mathbb{Z}_{\geq 0}$, let $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega, \mathrm{hol}\}$, and let $\kappa \in \{\infty, \omega, \mathrm{hol}\}$, as required. Let M and N be C^κ-manifolds, let $\Phi_0 \in \mathrm{C}^\nu(\mathsf{M}; \mathsf{N})$ and $g_0 \in \mathrm{C}^\nu(\mathsf{N})$, and consider the mappings

$$\begin{aligned} \mathscr{C}^\nu_{\Phi_0} \colon \mathrm{C}^\nu(\mathsf{N}) &\to \mathrm{C}^\nu(\mathsf{M}) \\ g &\mapsto g \circ \Phi_0, \\ \mathscr{S}^\nu_{g_0} \colon \mathrm{C}^\nu(\mathsf{M}; \mathsf{N}) &\to \mathrm{C}^\nu(\mathsf{M}) \\ \Phi &\mapsto g_0 \circ \Phi, \\ \mathscr{C}^\nu_{\mathsf{M},\mathsf{N}} \colon \mathrm{C}^\nu(\mathsf{N}) \times \mathrm{C}^\nu(\mathsf{M}; \mathsf{N}) &\to \mathrm{C}^\nu(\mathsf{M}) \\ (g, \Phi) &\mapsto g \circ \Phi. \end{aligned}$$

We shall call the mapping $\mathscr{C}^\nu_{\Phi_0}$ the ***$\mathbf{C}^\nu$-composition operator*** associated with Φ_0, we shall call $\mathscr{S}^\nu_{g_0}$ the ***$\mathbf{C}^\nu$-superposition operator*** associated with g_0, and we shall call $\mathscr{C}^\nu_{\mathsf{M},\mathsf{N}}$ the ***$\mathbf{C}^\nu$-joint composition operator***. For the joint composition operator, the product topology for the domain is used. In some cases, one considers "nonautonomous" versions of these, for example given by

$$\mathrm{C}^\nu(\mathsf{M}; \mathsf{N}) \ni \Phi \mapsto (x \mapsto g_0(x, \Phi(x))) \in \mathrm{C}^\nu(\mathsf{M}),$$

for a mapping $g_0 \colon \mathsf{M} \times \mathsf{N} \to \mathbb{F}$ (with some regularity we do not specify here). In this setting, one frequently considers $\mathsf{M} = \mathsf{N} = \mathbb{F}$. We shall only consider the "autonomous" case as prescribed by the operators $\mathscr{C}^\nu_{\Phi_0}$, $\mathscr{S}^\nu_{g_0}$, and $\mathscr{C}^\nu_{\mathsf{M},\mathsf{N}}$ above. While we have made our definitions for manifolds and our particular regularity classes, other sorts of function and mapping spaces are studies, e.g., Lebesgue class, Hardy class, the class of functions of bounded variation, etc. In all cases, questions of concern with these problems of composition include the following.

1. *Well-definedness:* Are the domains and codomains of the operators such that the definitions of the operators make sense?
2. *Continuity:* Are the operators continuous for some topologies for the domain and codomain?
3. *Boundedness:* Are the operators bounded for some bornologies for the domain and codomain?

We refer an interested reader to [Appell and Zabrejko 1990] for a detailed treatment of such questions in some function spaces.

Let us address the questions of well-definedness and continuity in the situations of interest to us. First we deal with the cases of well-definedness, where the answers are elementary.

4.1 Proposition: (Well-definedness of composition and superposition operators) *Let $m \in \mathbb{Z}_{\geq 0}$, let $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega, \mathrm{hol}\}$, and let $\kappa \in \{\infty, \omega, \mathrm{hol}\}$, as required. Let M and N be C^κ-manifolds. Then $\mathscr{C}^\nu_{\mathsf{M},\mathsf{N}}$ is well-defined.*

Proof: For completeness, we give references.

For $\nu \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, the result is that composition of C^ν-mappings is of class C^ν. For $\nu = 0$ this is proved as Theorem 7.3 by Willard [1970]. The case of $\nu \in \mathbb{Z}_{>0}$ is given as Proposition 3.2.7 of [Abraham, Marsden, and Ratiu 1988]. Of course, the latter result implies the desired conclusion in the case $\nu = \infty$.

In the real analytic case, the fact that the composition of real analytic mappings is real analytic is proved by [Krantz and Parks 2002, Proposition 2.2.8]. For the holomorphic result, we refer to [Gunning and Rossi 1965, Theorem I.A.5].

Weaver [1999, Proposition 1.2.2] proves that the composition of Lipschitz mappings is Lipschitz. We can similarly prove that, if $g \in \mathrm{C}^{\mathrm{lip}}(\mathsf{N})$ and $\Phi \in \mathrm{C}^{\mathrm{lip}}(\mathsf{M};\mathsf{N})$, then $g \circ \Phi \in \mathrm{C}^{\mathrm{lip}}(\mathsf{M})$ as follows. We assume Riemannian metrics $\mathbb{G}_{\mathsf{M}}$ and $\mathbb{G}_{\mathsf{N}}$ with distance functions d_{M} and d_{N}, respectively. Let $K \subseteq \mathsf{M}$ be compact, so that $L = \Phi(K)$ is also compact. Then there exists $\lambda, \mu \in \mathbb{R}_{>0}$ such that

$$|g(y_1) - g(y_2)| \le \lambda \mathrm{d}_{\mathsf{N}}(y_1, y_2), \qquad y_1, y_2 \in L,$$

and

$$\mathrm{d}_{\mathsf{N}}(\Phi(x_1), \Phi(x_2)) \le \mu \mathrm{d}_{\mathsf{M}}(x_1, x_2), \qquad x_1, x_2 \in K.$$

Therefore,

$$|g \circ \Phi(x_1) - g \circ \Phi(x_2)| \le \ell \mathrm{d}_{\mathsf{N}}(\Phi(x_1), \Phi(x_2)) \le \ell\mu \mathrm{d}_{\mathsf{M}}(x_1, x_2), \qquad x_1, x_2 \in K,$$

and so $g \circ \Phi$ is locally Lipschitz. ■

Now let us consider continuity of composition.

4.2 Proposition: (Continuity of composition and superposition operators) *Let $m \in \mathbb{Z}_{\ge 0}$, let $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega, \mathrm{hol}\}$, and let $\kappa \in \{\infty, \omega, \mathrm{hol}\}$, as required. Let M and N be C^κ-manifolds. Let $\Phi_0 \in \mathrm{C}^\nu(\mathsf{M};\mathsf{N})$. Then*

(i) $\mathscr{C}^\nu_{\Phi_0}$ is continuous and

(ii) $\mathscr{C}^\nu_{\mathsf{M},\mathsf{N}}$ is continuous for $\nu \neq m + \mathrm{lip}$.

Proof: (ii) We give references for completeness.

For $\nu = 0$, the result relies on the fact that manifolds are locally compact topological spaces. In this case, the result is classical [e.g., Engelking 1989, Theorem 3.4.2].

For $\nu \in \mathbb{Z}_{>0}$, one can show that $\mathrm{C}^\nu(\mathsf{M};\mathsf{N})$ is topologically embedded in $\mathrm{C}^0(\mathsf{M}; \mathsf{J}^\nu\mathsf{MN})$ by the mapping $\Phi \mapsto j_\nu\Phi$ [Michor 1980, Lemma 4.2]. Thus the result follows from the continuous case referred to in the previous paragraph. The case $\nu = \infty$ follows from this, essentially since the C^∞-topology is the inverse limit of the C^m-topologies as $m \to \infty$.

The holomorphic case, essentially, follows from the continuous case since $\mathrm{C}^{\mathrm{hol}}(\mathsf{M};\mathsf{N})$ is topologically embedded as a subspace of $\mathrm{C}^0(\mathsf{M};\mathsf{N})$. A proof in the holomorphic case can be found in [Lewis 2023, Theorem 5.28].

The real analytic case is proved by Lewis [2023, Theorem 5.29].

The continuity of $\mathscr{C}^\nu_{\Phi_0}$ in the case $m + \mathrm{lip}$ is given in the corresponding part of the proof for Lemma 2.8(iii). ■

We have omitted a statement that the $\mathrm{C}^{\mathrm{lip}}$-superposition operator is continuous. This is because it is not continuous. This is shown in the Lipschitz case (as opposed to the locally Lipschitz case) by Drábek [1975, Theorem 2] who shows that S_{f_0} is continuous from the space of Lipschitz mappings to the space of Lipschitz functions if and only if f_0 is continuously differentiable. The following example shows that the superposition operator is generally not continuous in the locally Lipschitz case. It is probable that the sharper conclusion of Drábek extends to the locally Lipschitz case, but we have not proved this.[5]

4.3 Example: (Discontinuity of the $\mathrm{C}^{\mathrm{lip}}$-superposition operator) The example we give is not exotic. We take $\mathsf{M} = \mathsf{N} = \mathbb{R}$ and $g_0(y) = \min\{y, 1\}$. Consider the sequence $(\Phi_j)_{j\in\mathbb{Z}_{>0}}$ in $\mathrm{C}^{\mathrm{lip}}(\mathbb{R};\mathbb{R})$ defined by

$$\Phi_j(x) = x + j^{-1}, \qquad j \in \mathbb{Z}_{>0},\ x \in \mathbb{R}.$$

This sequence converges to $\mathrm{id}_{\mathbb{R}}$ in $\mathrm{C}^{\mathrm{lip}}(\mathbb{R};\mathbb{R})$. However, we claim that the sequence $(g_0 \circ \Phi_j)_{j\in\mathbb{Z}_{>0}}$ does not converge in $\mathrm{C}^{\mathrm{lip}}(\mathbb{R})$. To see this, we note that, if $(g_0\circ\Phi_j)_{j\in\mathbb{Z}_{>0}}$ converges in $\mathrm{C}^{\mathrm{lip}}(\mathbb{R})$, then it converges in $\mathrm{C}^0(\mathbb{R})$. Therefore, if the sequence $(g_0\circ\Phi_j)_{j\in\mathbb{Z}_{>0}}$ converges in $\mathrm{C}^{\mathrm{lip}}(\mathbb{R})$, it must converge to g_0. We easily see that $\operatorname{dil} g_0(0) = 1$ and $\operatorname{dil} g_0\circ\Phi_j(0) = 0$. Therefore, if $K \subseteq \mathbb{R}$ is any compact set containing 0, then $\lambda^0_K(g_0 - g_0\circ\Phi_j) \geq 1$ for every $j \in \mathbb{Z}_{>0}$. This ensures that $(g_0\circ\Phi_j)_{j\in\mathbb{Z}_{>0}}$ does not converge in $\mathrm{C}^{\mathrm{lip}}(\mathbb{R})$. •

We note that the joint composition operator is linear in its first argument. We use this to obtain the following sharper statement concerning the nature of continuity of this operator.

4.4 Proposition: (Seminorm bounds for the joint composition operator) *Let $m \in \mathbb{Z}_{\geq 0}$, let $\nu \in \{m, \infty, \omega, \mathrm{hol}\}$, and let $\kappa \in \{\infty, \omega, \mathrm{hol}\}$, as required. Let M and N be C^κ-manifolds. Then, for*

(i) $\Phi_0 \in \mathrm{C}^\nu(\mathsf{M};\mathsf{N})$,

(ii) compact $K \subseteq \mathsf{M}$, and

(iii) data $*$ required to define a seminorm $p^\nu_{K,*}$ for $\mathrm{C}^\nu(\mathsf{M})$,

there exist

(iv) a neighbourhood $\mathcal{O}$ of Φ_0 in the weak-PB topology,

(v) a constant $C \in \mathbb{R}_{>0}$,

(vi) and a compact set $L \subseteq \mathsf{N}$, and

(vii) data $\star$ required to define a seminorm $q^\nu_{L,\star}$ for $\mathrm{C}^\nu(\mathsf{N})$

such that

$$p^\nu_{K,*} \circ \mathscr{C}^\nu_{\mathsf{M},\mathsf{N}}(g, \Phi) \leq C q^\nu_{L,\star}(g), \qquad g \in \mathrm{C}^\nu(\mathsf{N}),\ \Phi \in \mathcal{O}.$$

Proof: Let $p^\nu_{K,*}$ be a seminorm for $\mathrm{C}^\nu(\mathsf{M})$ and, by continuity of $\mathscr{C}^\nu_{\mathsf{M},\mathsf{N}}$ at $(0, \Phi_0)$, let $\mathcal{N}$ be an absolutely convex 0-neighbourhood for $\mathrm{C}^\nu(\mathsf{N})$ and let $\mathcal{O}$ be a neighbourhood of Φ_0 such that $\mathscr{C}^\nu_{\mathsf{M},\mathsf{N}}(\mathcal{N}\times\mathcal{O}) \subseteq (p^\nu_{L,*})^{-1}([0, 1])$. Let $L \subseteq \mathsf{N}$ be compact and let $\star$ be data to define a seminorm $q^\nu_{L,\star}$ such that $(q^\nu_{L,\star})^{-1}([0, \delta]) \subseteq \mathcal{N}$ for some $\delta \in \mathbb{R}_{>0}$, this by Lemma 2.6.

We claim that

$$\mathscr{C}^\nu_{\mathsf{M},\mathsf{N}}((q^\nu_{L,\star})^{-1}(0) \times \mathcal{O}) \subseteq (p^\nu_{K,*})^{-1}(0).$$

Evidently, $\mathscr{C}^\nu_{\mathsf{M}}((q^\nu_{L,\star})^{-1}(0) \times \mathcal{O}) \subseteq (p^\nu_{K,*})^{-1}([0, 1])$. Suppose that $(g, \Phi) \in (q^\nu_{L,\star})^{-1}(0) \times \mathcal{O}$ satisfies $p^\nu_{K,*}\circ\mathscr{C}^\nu_{\mathsf{M}}(g, \Phi) \neq 0$. Then, for some $r \in \mathbb{R}_{>0}$ sufficiently large, by linearity we have $p^\nu_{K,*}\circ\mathscr{C}^\nu_{\mathsf{M}}(rg, \Phi) > 1$, giving a contradiction.

[5] Let us categorise this more as "an exercise for the reader" than "an important open problem."

Now let $(g,\Phi)\in \mathrm{C}^\nu(\mathsf{N})\times\mathcal{O}$. First suppose that $q^\nu_{L,\star}(g)=0$. Then, as we saw in the preceding paragraph,

$$p^\nu_{K,*}\circ\mathscr{C}^\nu_{\mathsf{M},\mathsf{N}}(g,\Phi)=0 \implies p^\nu_{K,*}\circ\mathscr{C}^\nu_{\mathsf{M},\mathsf{N}}(g,\Phi)\le q^\nu_{L,\star}(g).$$

If $q^\nu_{L,\star}(g)\neq 0$, then $\frac{g}{q^\nu_{L,\star}(g)}\in\mathcal{N}$ and so

$$p^\nu_{K,*}\circ\mathscr{C}^\nu_{\mathsf{M},\mathsf{N}}(\tfrac{g}{q(g)},\Phi)\le 1 \implies p^\nu_{K,*}\circ\mathscr{C}^\nu_{\mathsf{M},\mathsf{N}}(g,\Phi)\le q(g).$$

The proposition now follows from Lemma 2.6. ■

Now we extend composition and superposition operators to allow time and parameter dependence. First we consider just time-dependence.

4.2. The time-dependent joint composition operator.

The next class of joint composition operators we consider allows for the functions g and the mappings Φ in the operator $\Phi\mapsto g\circ\Phi$ to be time-dependent. The set up is as follows. Let $m\in\mathbb{Z}_{\ge 0}$, let $m'\in\{0,\mathrm{lip}\}$, let $\nu\in\{m+m',\infty,\omega,\mathrm{hol}\}$, and let $\kappa\in\{\infty,\omega,\mathrm{hol}\}$, as required. Let M and N be C^κ-manifolds and let $\mathbb{T}\subseteq\mathbb{R}$ be an interval. We consider the space $\mathrm{L}(\mathrm{C}^\nu(\mathsf{N});\mathrm{C}^\nu(\mathsf{M}))$ of continuous linear mappings, which we equip with the topology of simple convergence, i.e., the pointwise convergence topology. By virtue of Lemma 2.8(iii), we have a continuous mapping

$$\mathrm{C}^\nu(\mathsf{M};\mathsf{N})\subseteq \mathrm{L}(\mathrm{C}^\nu(\mathsf{N});\mathrm{C}^\nu(\mathsf{M})),$$

if $\mathrm{C}^\nu(\mathsf{M};\mathsf{N})$ has the weak-PB topology. This mapping is, in fact, a topological embedding since the topology inherited from the pointwise convergence topology is exactly the weak-PB topology. We consider the mapping

$$\mathrm{C}^0(\mathbb{T};\mathrm{L}(\mathrm{C}^\nu(\mathsf{N});\mathrm{C}^\nu(\mathsf{N})))\times\mathrm{C}^\nu_{\mathrm{LI}}(\mathbb{T};\mathsf{N})\ni(\Phi,g)\mapsto(t\mapsto\Phi_t(g_t))\in\mathrm{C}^\nu(\mathsf{M})^{\mathbb{T}}. \tag{4.1}$$

Here $\mathrm{C}^\nu(\mathsf{M})^{\mathbb{T}}$ is the set of mappings from $\mathbb{T}$ to $\mathrm{C}^\nu(\mathsf{M})$.

The following lemmata give the properties of the time-dependence of the mapping (4.1).

4.5 Lemma: (The time-dependent joint composition operator: measurability) *Let $m\in\mathbb{Z}_{\ge 0}$, let $m'\in\{0,\mathrm{lip}\}$, let $\nu\in\{m+m',\infty,\omega,\mathrm{hol}\}$, and let $\kappa\in\{\infty,\omega,\mathrm{hol}\}$, as required. Let M and N be C^κ-manifolds, Stein if $\nu=\mathrm{hol}$, let $\mathbb{T}\subseteq\mathbb{R}$ be an interval. For $\Phi\in\mathrm{C}^0(\mathbb{T};\mathrm{L}(\mathrm{C}^\nu(\mathsf{N});\mathrm{C}^\nu(\mathsf{M})))$ and $g\in\mathrm{C}^\nu_{\mathrm{LI}}(\mathsf{N})$, the mapping $t\mapsto\Phi_t(g_t)$ is measurable.*

Proof: Let $h\in\mathrm{C}^\kappa(\mathsf{N})$ so that

$$(s\mapsto\Phi_s(h))\in\mathrm{C}^0(\mathbb{T};\mathrm{C}^\nu(\mathsf{M})).$$

The definition of the pointwise convergence topology allows us to conclude that

$$(s\mapsto\Phi_s)\in\mathrm{C}^0(\mathbb{T};\mathrm{L}(\mathrm{C}^\nu(\mathsf{N});\mathrm{C}^\nu(\mathsf{M}))).$$

Thus, for fixed $t\in\mathbb{T}$,

$$(s\mapsto\Phi_s(g_t))\in\mathrm{C}^0(\mathbb{T};\mathrm{C}^\nu(\mathsf{M}))\subseteq\mathrm{L}^1_{\mathrm{loc}}(\mathbb{T};\mathrm{C}^\nu(\mathsf{M})),$$

cf. [Thomas 1975, Corollary 3.1]. For fixed $s \in \mathbb{T}$,

$$(t \mapsto \Phi_s(g_t)) \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{T}; \mathrm{C}^\nu(\mathsf{M}))$$

by virtue of Remark 3.2–3 and the continuity and linearity of the composition operator. Let $\lambda \in \mathrm{C}^\nu(\mathsf{M})'$ so that

$$s \mapsto \langle \lambda; \Phi_s(g_t) \rangle \tag{4.2}$$

is continuous for each $t \in \mathbb{T}$ and

$$t \mapsto \langle \lambda; \Phi_s(g_t) \rangle$$

is measurable for each $s \in \mathbb{T}$, the latter by measurability of $t \mapsto g_t$ and continuity of Φ_s and λ. Let $[a, b] \subseteq \mathbb{T}$ be compact, let $k \in \mathbb{Z}_{>0}$, and denote

$$s_{k,l} = a + \tfrac{l-1}{k}(b-a), \qquad l \in \{1, \dots, k+1\}.$$

Also denote

$$\mathbb{S}_{k,l} = [s_{k,l}, s_{k,l+1}), \qquad l \in \{1, \dots, k-1\},$$

and $\mathbb{S}_{k,k} = [s_{k,k}, s_{k,k+1}]$. Then define $h_{j,k} \colon [a, b] \to \mathbb{F}$ by

$$h_{j,k}(t) = \sum_{l=1}^{k} \langle \lambda; \Phi_{s_{k,l}}(f_t) \rangle \chi_{\mathbb{S}_{k,l}}(t).$$

Note that $h_{j,k}$ is measurable, being a sum of products of measurable functions [Cohn 2013, Proposition 2.1.7]. By continuity of (4.2), for each $t \in [a, b]$ we have

$$\lim_{k \to \infty} h_{j,k}(t) = \langle \lambda; \Phi_t(g_t) \rangle.$$

Thus $t \mapsto \langle \lambda; \Phi_t(g_t) \rangle$ is measurable on $[a, b]$, as pointwise limits of measurable functions are measurable [Cohn 2013, Proposition 2.1.5]. Since $[a, b]$ is arbitrary, we conclude that $t \mapsto \langle \lambda; \Phi_t(g_t) \rangle$ is measurable on $\mathbb{T}$, and so $t \mapsto \Phi_t(g_t)$ is weakly measurable. By [Thomas 1975, Theorem 1] and recalling that $\mathrm{C}^\nu(\mathsf{M})$ is Suslin, we conclude that $t \mapsto \Phi_t(g_t)$ is measurable. ■

4.6 Lemma: (The time-dependent joint composition operator: well-definedness) *Let $m \in \mathbb{Z}_{\geq 0}$, let $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega, \mathrm{hol}\}$, and let $\kappa \in \{\infty, \omega, \mathrm{hol}\}$, as required. Let M and N be C^κ-manifolds, Stein if $\nu = \mathrm{hol}$, let $\mathbb{T} \subseteq \mathbb{R}$ be an interval. For $\Phi \in \mathrm{C}^0(\mathbb{T}; \mathrm{C}^\nu(\mathsf{M}; \mathsf{N}))$ and $g \in \mathrm{C}^\nu_{\mathrm{LI}}(\mathsf{N})$, the mapping $t \mapsto \Phi_t(g_t)$ takes values in $\mathrm{C}^\nu_{\mathrm{LI}}(\mathbb{T}; \mathsf{M})$.*

Proof: Let $\mathbb{K} \subseteq \mathbb{T}$ be a compact interval and let $K \subseteq \mathsf{M}$ be compact. Let $L \subseteq \mathsf{N}$ be such that

$$\Phi_t(x) \subseteq L, \qquad (t, x) \subseteq \mathbb{K} \times K,$$

this being possible since $(t, x) \mapsto \Phi_t(x)$ is continuous, cf. Lemma 3.10(i). Since $g \in \mathrm{C}^\nu_{\mathrm{LI}}(\mathbb{T}; \mathsf{N})$, there exists $h \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{T}; \mathbb{R}_{\geq 0})$ such that

$$p^\nu_{L,*}(g_t) \leq h(t), \qquad t \in \mathbb{T}.$$

Then we have

$$p^\nu_{K,*}(\Phi_t(g_t)) = p^\nu_{K,*}(g_t \circ \Phi_t) \leq h(t), \qquad t \in \mathbb{K},$$

whence

$$(t \mapsto \Phi_t(g_t)) \in \mathrm{L}^1(\mathbb{K}; \mathrm{C}^\nu(\mathsf{M})),$$

as desired. ■

4.7 Lemma: (The time-dependent joint composition operator: continuity) *Let $m \in \mathbb{Z}_{\geq 0}$, let $\nu \in \{m, \infty, \omega, \mathrm{hol}\}$, and let $\kappa \in \{\infty, \omega, \mathrm{hol}\}$, as required. Let M and N be C^κ-manifolds, Stein if $\nu = \mathrm{hol}$, let $\mathbb{T} \subseteq \mathbb{R}$ be an interval. Then the mapping*

$$\begin{aligned} \mathscr{C}^\nu_{\mathsf{M},\mathsf{N},\mathbb{T}} \colon \mathrm{C}^\nu_{\mathrm{LI}}(\mathbb{T}; \mathsf{N}) \times \mathrm{C}^0(\mathbb{T}; \mathrm{C}^\nu(\mathsf{M}; \mathsf{N})) &\to \mathrm{C}^\nu_{\mathrm{LI}}(\mathbb{T}; \mathsf{M}) \\ (g, \Phi) &\mapsto (t \mapsto \Phi_t(g_t)) \end{aligned}$$

is continuous. More precisely, for

(i) a compact set $K \subseteq \mathsf{M}$,

(ii) a compact interval $\mathbb{K} \subseteq \mathbb{T}$,

(iii) data $*$ required to define the seminorm $p^\nu_{K,*,\mathbb{K}}$ for $\mathrm{C}^\nu_{\mathrm{LI}}(\mathbb{T}; \mathsf{M})$, and

(iv) $\Phi_0 \in \mathrm{C}^0(\mathbb{T}; \mathrm{C}^\nu(\mathsf{M}; \mathsf{N}))$,

there exists

(v) a compact set $L \subseteq \mathsf{N}$,

(vi) data $\star$ required to define a seminorm $q^\nu_{L,\star,\mathbb{K}}$ for $\mathrm{C}^\nu_{\mathrm{LI}}(\mathbb{T}; \mathsf{N})$,

(vii) a neighbourhood $\widehat{\mathcal{O}}$ of Φ_0 in $\mathrm{C}^0(\mathbb{T}; \mathrm{C}^\nu(\mathsf{M}; \mathsf{N}))$, and

(viii) a constant $C \in \mathbb{R}_{>0}$

such that

$$p^\nu_{K,*,\mathbb{K}} \circ \mathscr{C}^\nu_{\mathsf{M},\mathsf{N},\mathbb{T}}(g, \Phi) \leq C q^\nu_{L,\star,\mathbb{K}}(g), \qquad g \in \mathrm{C}^\nu_{\mathrm{LI}}(\mathsf{N}),\ \Phi \in \widehat{\mathcal{O}}.$$

Proof: Let $\Phi_0 \in \mathrm{C}^0(\mathbb{T}; \mathrm{C}^\nu(\mathsf{M}; \mathsf{N}))$ and $g_0 \in \mathrm{C}^\nu_{\mathrm{LI}}(\mathbb{T}; \mathsf{N})$. Let $\mathcal{K} \subseteq \mathsf{M}$ and $\mathbb{K} \subseteq \mathbb{T}$ be compact. Choose suitable other data require to define a seminorm $p^\nu_{K,*}$ for $\mathrm{C}^\nu(\mathsf{M})$ and let $p_{\mathcal{K},*,\mathbb{K}}$ be the associated seminorm for $\mathrm{C}^\nu_{\mathrm{LI}}(\mathbb{T}; \mathsf{M})$. Let $t \in \mathbb{K}$, and let q_t be a continuous seminorm for $\mathrm{C}^\nu(\mathsf{N})$ and let $\mathcal{O}_t \subseteq \mathrm{C}^\nu(\mathsf{M}; \mathsf{N})$ be a neighbourhood of $\Phi_{0,t}$ such that

$$\mathscr{C}^\nu_{\mathsf{M},\mathsf{N}}(q_t^{-1}([0,1]) \times \mathcal{O}_t) \subseteq (p^\nu_{K,*})^{-1}([0,1)).$$

These exist by continuity of $C_{\mathsf{M},\mathsf{N}}$ at $(0, \Phi_{0,t})$. Let $\mathbb{K}_t \subseteq \mathbb{T}$ be a compact interval such that $t \in \mathbb{K}_t$ and

$$\Phi_0 \in \mathcal{B}(\mathbb{K}_t, \mathcal{O}_t) = \{\Psi \in \mathrm{C}^0(\mathbb{T}; \mathrm{C}^\nu(\mathsf{M}; \mathsf{N})) \mid \Psi(\mathbb{K}_t) \subseteq \mathcal{O}_t\}.$$

As in the proof of Proposition 4.4, we have

$$p^\nu_{K,*} \circ \mathscr{C}^\nu(g, \Phi_{0,t}) \leq q_t(g), \qquad t \in \mathbb{K}_t,\ g \in \mathrm{C}^\nu(\mathsf{N}).$$

Let $t_1, \dots, t_k \in \mathbb{K}$ be such that $\mathbb{K} \subseteq \cup_{j=1}^k \mathbb{K}_{t_j}$. Let

$$\widehat{\mathcal{O}} = \bigcap_{j=1}^k \mathcal{B}(\mathbb{K}_{t_j}; \mathcal{O}_{t_j})$$

and $q = \max\{q_{t_1}, \dots, q_{t_k}\}$. Then, if $t \in \mathbb{K}$, $g \in \mathrm{C}^\nu(\mathsf{N})$, and $\Phi \in \widehat{\mathcal{O}}$, we have $t \in \mathbb{K}_{t_j}$ for some $j \in \{1, \dots, k\}$, and so $\Phi_t \in \mathcal{O}_{t_j}$. Therefore,

$$p^\nu_{K,*} \circ \mathscr{C}^\nu_{\mathsf{M},\mathsf{N}}(g, \Phi_t) \leq q_{t_j}(g) \leq q(g).$$

By Lemma 2.6, there exists a constant $C \in \mathbb{R}_{>0}$, and a compact set $L \subseteq \mathsf{N}$ and data $\star$ defining a seminorm $q^{\nu}_{L,\star}$ for $\mathrm{C}^{\nu}(\mathsf{N})$ such that

$$p^{\nu}_{K,*} \circ \mathscr{C}^{\nu}_{\mathsf{M},\mathsf{N}}(g, \Phi_t) \leq C q^{\nu}_{L,\star}(g), \qquad t \in \mathbb{K},\ g \in \mathrm{C}^{\nu}(\mathsf{N}).$$

Therefore, if $g \in \mathrm{C}^{\nu}_{\mathrm{LI}}(\mathbb{T}; \mathsf{N})$, we have

$$\int_{\mathbb{K}} p^{\nu}_{K,*} \circ \mathscr{C}^{\nu}_{\mathsf{M},\mathsf{N}}(g_t, \Phi_t)\, \mathrm{d}t \leq C \int_{\mathbb{K}} q^{\nu}_{L,\star}(g_t)\, \mathrm{d}t,$$

which is the result. ■

4.3. The time- and parameter-dependent integral superposition operator. The next class of joint composition operator $(g, \Phi) \mapsto g \circ \Phi$ we consider allows both g and Φ to be time- and parameter-dependent. The precise setup is the following. Let $m \in \mathbb{Z}_{\geq 0}$, let $\nu \in \{m, \infty, \omega, \mathrm{hol}\}$, and let $\kappa \in \{\infty, \omega, \mathrm{hol}\}$, as required. Let M and N be C^{κ}-manifolds, Stein if $\nu = \mathrm{hol}$, let $\mathbb{T} \subseteq \mathbb{R}$ be an interval, and let $\mathcal{P}$ be a topological space. We let $g \in \mathrm{C}^{\nu}_{\mathrm{PLI}}(\mathbb{T}; \mathsf{N}; \mathcal{P})$ and $\Phi \in \mathrm{C}^{\nu}_{\mathrm{PLAC}}(\mathbb{T}; (\mathsf{M}; \mathsf{N}); \mathcal{P})$ and consider the time- and parameter-dependent function

$$\mathbb{T} \times \mathcal{P} \ni (t, p) \mapsto \Phi^p_t(g^p_t) \in \mathrm{C}^{\nu}(\mathsf{M}).$$

What shall be of interest to us here is the following integral variant of this superposition operator:

$$p \mapsto \left(t \mapsto \left(\int_{t_0}^{t} \Phi^p_s(g^p_s)\, \mathrm{d}s\right)\right).$$

We shall prove that this mapping is in $\mathrm{C}^{\nu}_{\mathrm{PLAC}}(\mathbb{T}; \mathsf{M}; \mathcal{P})$ for suitable ν; to do so requires us to evaluate continuity with respect to the conditions (3.9) and (3.10). However, because this map evaluates to 0 at $t = t_0$, there is a simplification that occurs, and we enunciate this in the following lemma.

4.8 Lemma: (Characterisation of continuous time- and parameter-dependent joint composition operators) *Let $m \in \mathbb{Z}_{\geq 0}$, let $\nu \in \{m, \infty, \omega, \mathrm{hol}\}$, and let $\kappa \in \{\infty, \omega, \mathrm{hol}\}$, as required. Let M and N be C^{κ}-manifolds, Stein if $\nu = \mathrm{hol}$, let $\mathbb{T} \subseteq \mathbb{R}$ be an interval, and let $\mathcal{P}$ be a topological space. Let $t_0 \in \mathbb{T}$. If $\Phi \in \mathrm{C}^{\nu}_{\mathrm{PLAC}}(\mathbb{T}; (\mathsf{M}; \mathsf{N}); \mathcal{P})$ and $g \in \mathrm{C}^{\nu}_{\mathrm{PLI}}(\mathbb{T}; \mathsf{N}; \mathcal{P})$, then the mapping*

$$\mathcal{P} \ni p \mapsto \left(t \mapsto \int_{t_0}^{t} \Phi^p_s(g^p_s)\, \mathrm{d}s\right) \in \mathrm{C}^{\nu}_{\mathrm{LAC}}(\mathbb{T}; \mathsf{M})$$

is continuous if and only if the mapping

$$\mathcal{P} \ni p \mapsto (t \mapsto \Phi^p_t(g^p_t)) \in \mathrm{C}^{\nu}_{\mathrm{LI}}(\mathbb{T}; \mathsf{M})$$

is continuous.

Proof: First note that, by Lemma 4.6, for fixed $p \in \mathcal{P}$,the mapping $t \mapsto \Phi^p_t(g^p_t)$ is indeed in $\mathrm{C}^{\nu}_{\mathrm{LI}}(\mathbb{T}; \mathsf{M})$.

First suppose that the mapping

$$\mathcal{P} \ni p \mapsto \left(t \mapsto \int_{t_0}^{t} \Phi^p_s(g^p_s)\, \mathrm{d}s\right) \in \mathrm{C}^{\nu}_{\mathrm{LAC}}(\mathbb{T}; \mathsf{M})$$

is continuous. Let $p_0 \in \mathcal{P}$, let $\epsilon \in \mathbb{R}_{>0}$, and let $K \subseteq \mathsf{M}$ and $\mathbb{K} \subseteq \mathbb{T}$ be compact. The condition (3.10) implies that there exists a neighbourhood $\mathcal{O}$ of p_0 such that

$$\int_{\mathbb{K}} p^{\nu}_{K,*}(\Phi^p_t(g^p_t) - \Phi^{p_0}_t(g^{p_0}_t))\,\mathrm{d}t < \epsilon, \qquad p \in \mathcal{O},$$

which implies that the mapping

$$\mathcal{P} \ni p \mapsto (t \mapsto \Phi^p_t(g^p_t)) \in \mathrm{C}^{\nu}_{\mathrm{LI}}(\mathbb{T}; \mathsf{M})$$

is continuous.

Next suppose that the mapping

$$\mathcal{P} \ni p \mapsto (t \mapsto \Phi^p_t(g^p_t)) \in \mathrm{C}^{\nu}_{\mathrm{LI}}(\mathbb{T}; \mathsf{M})$$

is continuous. Let $p_0 \in \mathcal{P}$, let $\epsilon \in \mathbb{R}_{>0}$, let $t \in \mathbb{T}$, and let $K \subseteq \mathsf{M}$ be compact. By hypothesis, there exists a neighbourhood $\mathcal{O}$ of p_0 such that

$$\int_{|t_0,t|} p^{\nu}_{K,*}(\Phi^p_t(g^p_t) - \Phi^{p_0}_t(g^{p_0}_t))\,\mathrm{d}t < \epsilon, \qquad p \in \mathcal{O}.$$

Since

$$\int_{t_0}^{t_0} \Phi^p_t(g^p_t)\,\mathrm{d}t = 0,$$

we can see that the mapping

$$\mathcal{P} \ni p \mapsto \left(t \mapsto \int_{t_0}^{t} \Phi^p_s(g^p_s)\,\mathrm{d}s\right) \in \mathrm{C}^{\nu}_{\mathrm{LAC}}(\mathbb{T}; \mathsf{M})$$

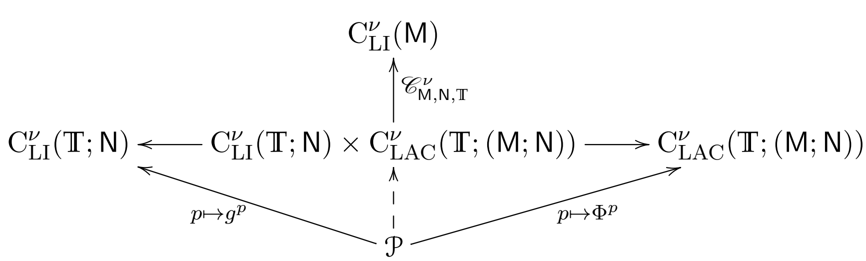
is continuous, using the seminorms for $\mathrm{C}^{\nu}_{\mathrm{LAC}}(\mathbb{T}; \mathsf{M})$ as in (3.4). ■

4.9 Lemma: (Continuity of an integral time- and parameter-dependent joint superposition operator) *Let $m \in \mathbb{Z}_{\geq 0}$, let $\nu \in \{m, \infty, \omega, \mathrm{hol}\}$, and let $\kappa \in \{\infty, \omega, \mathrm{hol}\}$, as required. Let M and N be C^{κ}-manifolds, Stein if $\nu = \mathrm{hol}$, let $\mathbb{T} \subseteq \mathbb{R}$ be an interval, and let $\mathcal{P}$ be a topological space. Let $t_0 \in \mathbb{T}$. If $\Phi \in \mathrm{C}^{\nu}_{\mathrm{PLAC}}(\mathbb{T}; (\mathsf{M}; \mathsf{N}); \mathcal{P})$ and $g \in \mathrm{C}^{\nu}_{\mathrm{PLI}}(\mathbb{T}; \mathsf{N}; \mathcal{P})$, then the mapping*

$$\mathcal{P} \ni p \mapsto \left(t \mapsto \int_{t_0}^{t} \Phi^p_s(g^p_s)\,\mathrm{d}s\right) \in \mathrm{C}^{\nu}_{\mathrm{LAC}}(\mathbb{T}; \mathsf{M})$$

is continuous.

Proof: Consider the diagram



where the horizontal arrows are the projections. By the definitions of $\mathrm{C}^{\nu}_{\mathrm{PLI}}(\mathbb{T};\mathsf{N};\mathsf{P})$ and $\mathrm{C}^{\nu}_{\mathrm{PLAC}}(\mathbb{T};(\mathsf{M};\mathsf{N});\mathcal{P})$, the two diagonal mappings are continuous. Therefore, by definition of the product topology, the lower vertical dashed arrow is continuous. By Lemma 4.7, the upper vertical arrow is continuous. The mapping

$$\mathcal{P} \ni p \mapsto (t \mapsto \Phi^p_t(g^p_t)) \in \mathrm{C}^{\nu}_{\mathrm{LI}}(\mathbb{T};\mathsf{M}),$$

is the composition of the two vertical mappings. Hence it is continuous. The continuity assertion of the lemma follows from Lemma 4.8. ■

5. Local flows for time- and parameter-dependent vector fields

In this section we give the most basic of results concerning the existence, uniqueness, and continuous dependence of local flows for our class of time- and parameter-dependent vector fields; the result in the classical locally Lipschitz case. In this case, even, there are a few important differences between what we do and what is classically done.

1. As we explained at some length in Section 1.1, our notion of parameter-dependence is not the classical one; more properly, it is not one of the many extant classical notions.
2. We see that there arises some consequences of our discussion in Section 4 of composition operators in the locally Lipschitz class. Namely, while we require the data to be locally Lipschitz to obtain the usual results, we are not able to prove that the flow shares the locally Lipschitz regularity for parameter-dependence, consistent with our observation in Example 4.3 that the locally Lipschitz superposition operator is not continuous. This is in contrast with other regularity classes where we anticipate that the local flow will share the regularity of the vector field.

5.1. Time- and parameter-dependent Picard operators. In this section we introduce standard constructions for the local existence and uniqueness theory for ordinary differential equations, but adapted to our framework for time- and parameter-dependent vector fields. Here we make essential use of our results from Section 4.

The following lemma characterises the basic data that we will use, and the properties we will require it to have. In the lemma, we use the following bit of notation. Let $\kappa \in \{\infty, \omega, \mathrm{hol}\}$, let M be a C^{κ}-manifold, let $x_0 \in \mathsf{M}$, and let $f^1, \dots, f^k \in \mathrm{C}^{\kappa}(\mathsf{M})$. For $a \in \mathbb{R}_{>0}$ and $x \in \mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, $\mathsf{B}(a,x)$ denotes the ball of radius a about x in $\mathbb{F}$, this is thus an interval when $\mathbb{F} = \mathbb{R}$ and a disk when $\mathbb{F} = \mathbb{C}$. We denote by $\mathcal{U}_{\boldsymbol{f}}(a, x_0)$ the connected component of the open set

$$\bigcap_{j=1}^{k} (f^j)^{-1}(\mathsf{B}(a, f^j(x_0)))$$

containing x_0. Also, if $\mathbb{T} \subseteq \mathbb{R}$ is an interval, if $t_0 \in \mathbb{T}$, and if $\alpha \in \mathbb{R}_{>0}$, denote

$$\mathbb{T}_{t_0,\alpha} = \mathbb{T} \cap [t_0 - \alpha, t_0 + \alpha].$$

Then we have the following result.

5.1 Lemma: (Properties of Picard data) *Let $m \in \mathbb{Z}_{\ge 0}$, let $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m+m', \infty, \omega, \mathrm{hol}\}$ satisfy $\nu \ge \mathrm{lip}$, and let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and $\kappa \in \{\infty, \omega, \mathrm{hol}\}$ be as required. Let M be a C^κ-manifold, assuming that it is a Stein manifold when $r = \mathrm{hol}$, let $\mathbb{T} \subseteq \mathbb{R}$ be an interval, and let $\mathcal{P}$ be a topological space. Let $\mathbb{G}$ be a Riemannian or Hermitian metric on M. Let $X \in \Gamma^\nu_{\mathrm{PLI}}(\mathbb{T}; \mathsf{TM}; \mathcal{P})$. For $(t_0, x_0, p_0) \in \mathbb{T} \times \mathsf{M} \times \mathcal{P}$, there exist*

(i) a neighbourhood $\mathcal{U}$ of x_0,

(ii) a neighbourhood $\mathcal{O}$ of p_0,

(iii) $\chi^1, \dots, \chi^n \in \mathrm{C}^\kappa(\mathsf{M})$,

(iv) $C, r, \alpha \in \mathbb{R}_{>0}$, and

(v) $\lambda \in (0,1)$

such that

(vi) $\mathcal{U} \ni x \mapsto \boldsymbol{\chi}(x) \triangleq (\chi^1(x), \dots, \chi^n(x)) \in \mathbb{F}^n$ is a coordinate chart for $\mathcal{U}$,

(vii) $\mathrm{cl}(\mathcal{U}_{\boldsymbol{\chi}}(r, x)) \subseteq \mathcal{U}_{\boldsymbol{\chi}}(2r, x_0) \subseteq \mathrm{cl}(\mathcal{U}_{\boldsymbol{\chi}}(2r, x_0)) \subseteq \mathcal{U}$ for $x \in \mathcal{U}_{\boldsymbol{\chi}}(r, x_0)$,

(viii) $\boldsymbol{\chi}|\mathcal{U}_{\boldsymbol{\chi}}(a, x)$ is a C^κ-diffeomorphism onto $\prod_{j=1}^n \mathsf{B}(a, \chi^j(x))$ for $x \in \mathcal{U}$ and $a \in (0, r]$ such that $\mathcal{U}_{\boldsymbol{\chi}}(a, x) \subseteq \mathcal{U}$,

(ix) we have

$$C^{-1} \sup\{|\chi^j(x_1) - \chi^j(x_2)| \mid j \in \{1, \dots, n\}\} \\ \le \mathrm{d}_{\mathbb{G}}(x_1, x_2) \le C \sup\{|\chi^j(x_1) - \chi^j(x_2)| \mid j \in \{1, \dots, n\}\}$$

for $x_1, x_2 \in \mathcal{U}$,

(x) $\displaystyle\int_{\mathbb{T}_{t_0,\alpha}} |X\chi^j(s, x, p)| \, \mathrm{d}s < r$ for $x \in \mathcal{U}$ and $p \in \mathcal{O}$, and

(xi) $\displaystyle\int_{\mathbb{T}_{t_0,\alpha}} |X\chi^j(s, x_1, p) - X\chi^j(s, x_2, p)| \, \mathrm{d}s < \frac{\lambda}{C} \mathrm{d}_{\mathbb{G}}(x_1, x_2)$ for $x_1, x_2 \in \mathcal{U}$ and $p \in \mathcal{O}$.

Proof: Let $\chi^1, \dots, \chi^n \in \mathrm{C}^\kappa(\mathsf{M})$ be such that $x \mapsto \boldsymbol{\chi}(x)$ is a coordinate chart in a neighbourhood of x_0 satisfying $\chi^j(x_0) = 0$, $j \in \{1, \dots, n\}$. The existence of such functions follows from the fact that M admits a proper C^κ-embedding into $\mathbb{F}^N$ for some $N \in \mathbb{Z}_{>0}$. We choose R sufficiently small that

$$\mathcal{U} = \{x \in \mathsf{M} \mid \; |\chi^j(x)| < R, \; j \in \{1, \dots, n\}\}$$

is such that $(\mathcal{U}, \boldsymbol{\chi})$ is a connected precompact chart for M. For $a \in \mathbb{R}_{>0}$ and $x \in \mathcal{U}$ for which $\mathcal{U}_{\boldsymbol{\chi}}(a, x) \subseteq \mathcal{U}$, note that $\mathcal{U}_{\boldsymbol{\chi}}(a, x)$ is a neighbourhood of x and that

$$\boldsymbol{\chi}(\mathcal{U}_{\boldsymbol{\chi}}(a, x)) = \prod_{j=1}^n \mathsf{B}(a, \chi^j(x)),$$

just by definition of $\mathcal{U}_{\boldsymbol{\chi}}(a, x)$. Since $\boldsymbol{\chi}$ is a C^κ-diffeomorphism, we can conclude that $\boldsymbol{\chi}|\mathcal{U}_{\boldsymbol{\chi}}(a, x)$ is a C^κ-diffeomorphism onto $\prod_{j=1}^n \mathsf{B}(a, \chi^j(x))$. Take $r = \frac{R}{3}$ so that $\mathcal{U}_{\boldsymbol{\chi}}(r, x_0)$ is also precompact. With $\mathcal{U}$ chosen thus, we have

$$\mathrm{cl}(\mathcal{U}_{\boldsymbol{\chi}}(r, x)) \subseteq \mathcal{U}_{\boldsymbol{\chi}}(2r, x_0) \subseteq \mathrm{cl}(\mathcal{U}_{\boldsymbol{\chi}}(2r, x_0)) \subseteq \mathcal{U}, \qquad x \in \mathcal{U}_{\boldsymbol{\chi}}(r, x_0).$$

By Lemma 2.3, there exists $C \in \mathbb{R}_{>0}$ such that

$$C^{-1} \sup\{|\chi^j(x_1) - \chi^j(x_2)| \mid j \in \{1, \dots, n\}\} \\ \le \mathrm{d}_{\mathbb{G}}(x_1, x_2) \le C \sup\{|\chi^j(x_1) - \chi^j(x_2)| \mid j \in \{1, \dots, n\}\}, \qquad x_1, x_2 \in \mathcal{U}.$$

Here we use the fact that

$$\mathrm{d}(x_1, x_2) = \left(\sum_{j=1}^{n} |\chi^j(x_1) - \chi^j(x_2)|^2\right)^{1/2}$$

defines a Riemannian or Hermitian metric on $\mathcal{U}$ and that the 2-norm and the ∞-norm on $\mathbb{F}^n$ are mutually bounded by constants depending on n; specifically

$$\|\boldsymbol{v}\|_\infty \le \|\boldsymbol{v}\| \le \sqrt{n}\|\boldsymbol{v}\|_\infty, \qquad \boldsymbol{v} \in \mathbb{F}^n.$$

Let $\lambda \in (0, 1)$.

By Lemma 3.11, $X\chi^j \in \mathrm{C}^{\mathrm{lip}}_{\mathrm{PLI}}(\mathbb{T}; \mathsf{M}, \mathcal{P})$, $j \in \{1, \dots, n\}$. Thus there exists $\alpha \in \mathbb{R}_{>0}$ such that

$$\int_{\mathbb{T}_{t_0,\alpha}} |X\chi^j(s, x, p_0)| \,\mathrm{d}s < \frac{r}{2}$$

and

$$\int_{\mathbb{T}_{t_0,\alpha}} \mathrm{dil}\,(X\chi^j)(s, x, p_0) \,\mathrm{d}s < \frac{\lambda}{2C},$$

for $x \in \mathcal{U}$. By (3.8), there exists a neighbourhood $\mathcal{O}$ of p_0 such that

$$\int_{\mathbb{T}_{t_0,\alpha}} |X\chi^j(s, x, p) - X\chi^j(s, x, p_0)| \,\mathrm{d}s < \frac{r}{2}$$

and

$$\int_{\mathbb{T}_{t_0,\alpha}} \mathrm{dil}\,((X\chi^j)^p - (X\chi^j)^{p_0})(s, x) \,\mathrm{d}s < \frac{\lambda}{2C}$$

for $x \in \mathcal{U}$, $p \in \mathcal{O}$, and $j \in \{1, \dots, n\}$. The triangle inequality then gives

$$\int_{\mathbb{T}_{t_0,\alpha}} |X\chi^j(s, x, p)| \,\mathrm{d}s < r$$

and

$$\int_{\mathbb{T}_{t_0,\alpha}} \mathrm{dil}\,(X\chi^j)(s, x, p) \,\mathrm{d}s < \frac{\lambda}{C}$$

for $x \in \mathcal{U}$, $p \in \mathcal{O}$, and $j \in \{1, \dots, n\}$. The last inequality, with the aid of Lemma 3.16 (see, especially, (3.12)), gives

$$\int_{\mathbb{T}_{t_0,\alpha}} |X\chi^j(s, x_1, p) - X\chi^j(s, x_2, p)| \,\mathrm{d}s < \frac{\lambda}{C} \mathrm{d}_{\mathbb{G}}(x_1, x_2)$$

for $x_1, x_2 \in \mathcal{U}$ and $p \in \mathcal{O}$.

One then readily verifies that $\mathcal{U}$, $\mathcal{O}$, $\boldsymbol{\chi}$, C, r, α, and λ satisfies any of the properties not explicitly proved in the preceding paragraphs. ■

As the constructions of the lemma will be drawn upon on several occasions, it is worth giving them a title.

5.2 Definition: (Picard data) Let $m \in \mathbb{Z}_{\geq 0}$, let $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m+m', \infty, \omega, \mathrm{hol}\}$ satisfy $\nu \geq \mathrm{lip}$, and let $\kappa \in \{\infty, \omega, \mathrm{hol}\}$ be as required. Let M be a C^κ-manifold, let $\mathbb{T} \subseteq \mathbb{R}$ be an interval, and let $\mathcal{P}$ be a topological space. Let $X \in \Gamma^\nu_{\mathrm{PLI}}(\mathbb{T}; \mathsf{TM}; \mathcal{P})$ and let $(t_0, x_0, p_0) \in \mathbb{T} \times \mathsf{M} \times \mathcal{P}$. A septuple $\mathscr{P} = (\mathcal{U}, \mathcal{O}, \boldsymbol{\chi}, C, r, \alpha, \lambda)$ satisfying the conditions of Lemma 5.1 is called ***C^κ-Picard data for X at (t_0, x_0, p_0)***. •

It is possible to partially order Picard data in an obvious way:

$$(\mathcal{U}_1, \mathcal{O}_1, \boldsymbol{\chi}_1, C_1, r_1, \alpha_1, \lambda_1) \preceq (\mathcal{U}_2, \mathcal{O}_2, \boldsymbol{\chi}_2, C_2, r_2, \alpha_2, \lambda_2) \tag{5.1}$$

if

1. $\mathcal{U}_1 \subseteq \mathcal{U}_2$,
2. $\mathcal{O}_1 \subseteq \mathcal{O}_2$,
3. $\boldsymbol{\chi}_1 | \mathcal{U}_1 = \boldsymbol{\chi}_2 | \mathcal{U}_2$,
4. $C_1 = C_2$,
5. $r_1 \leq r_2$,
6. $\alpha_1 \leq \alpha_2$, and
7. $\lambda_1 \leq \lambda_2$.

Picard data at (t_0, x_0, p_0) provides a backdrop for multiple representations for geometric constructions occurring near x_0, merely because $(\mathcal{U}, \boldsymbol{\chi}|\mathcal{U})$ is a chart for M. It will be convenient, depending on what we are doing, to variously use one of these representations in favour of the others. This is mere notation, in some sense, so let us give this notation here.

We denote by $\boldsymbol{\mathcal{U}} = \boldsymbol{\chi}(\mathcal{U}) \subseteq \mathbb{F}^n$ the image of the chart domain in $\mathbb{F}^n$. In like manner, we denote by $\boldsymbol{\mathcal{U}}(a, x) = \boldsymbol{\chi}(\mathcal{U}_{\boldsymbol{\chi}}(a, x))$ for any $a \in \mathbb{R}_{>0}$ and $x \in \mathcal{U}$ for which $\mathcal{U}_{\boldsymbol{\chi}}(a, x) \subseteq \mathcal{U}$. In our analysis to follow, we shall be interested in mappings $\Gamma \colon \mathcal{U}_{\boldsymbol{\chi}}(r, x_0) \to \mathcal{U}_{\boldsymbol{\chi}}(2r, x_0)$. Associated with such a mapping are two related mappings $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$, which are most easily defined by the following commutative diagrams:

$$\begin{array}{ccc} \mathcal{U}_{\boldsymbol{\chi}}(r, x_0) & \xrightarrow{\Gamma} & \mathcal{U}_{\boldsymbol{\chi}}(2r, x_0) \\ & \searrow^{\boldsymbol{\phi}} & \downarrow \boldsymbol{\chi} \\ & & \boldsymbol{\mathcal{U}}(2r, x_0) \subseteq \mathbb{F}^N \end{array} \qquad \begin{array}{ccc} \mathcal{U}_{\boldsymbol{\chi}}(r, x_0) & \xrightarrow{\Gamma} & \mathcal{U}_{\boldsymbol{\chi}}(2r, x_0) \\ \boldsymbol{\chi} \downarrow & & \downarrow \boldsymbol{\chi} \\ \boldsymbol{\mathcal{U}}(r, x_0) \subseteq \mathbb{F}^N & \xrightarrow[\boldsymbol{\psi}]{} & \boldsymbol{\mathcal{U}}(2r, x_0) \subseteq \mathbb{F}^N \end{array} \tag{5.2}$$

Any of the three mappings Γ, $\boldsymbol{\phi}$, and $\boldsymbol{\psi}$ are all clearly uniquely determined by the others. Thus, having chosen one, the other two are fixed. To reflect this, we will use notation like $\Gamma_{\boldsymbol{\phi}}$ and $\boldsymbol{\psi}_{\boldsymbol{\phi}}$ to reflect that, having chosen (in this case) $\boldsymbol{\phi}$, we have also specified (in this case) $\Gamma_{\boldsymbol{\phi}}$ and $\boldsymbol{\psi}_{\boldsymbol{\phi}}$. We will also sometimes use the symbols without the subscripts when the resulting brevity is convenient.

Also, we shall mainly use notation like the above in the setting where data depends additionally on times $\tau, \tau_0 \in \mathbb{T}_{t_0,\alpha}$ and parameter $p \in \mathcal{O} \subseteq \mathcal{P}$. In such cases, the mapping Γ above is $\mathrm{id}_{\mathbb{T}^2_{t_0,\alpha}} \times \Gamma \times \mathrm{id}_{\mathcal{O}}$, and all the notational conventions above still apply.

With these matters of notation out of the way, given Picard data, we introduce a variant of a standard operator used in the usual existence and uniqueness theory. Thus we let $m \in \mathbb{Z}_{\geq 0}$, let $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m+m', \infty, \omega, \mathrm{hol}\}$, and let $\kappa \in \{\infty, \omega, \mathrm{hol}\}$, as required. We let M be a C^κ-manifold, let $\mathbb{T} \subseteq \mathbb{R}$ be an interval, and let $\mathcal{P}$ be a topological space. We let $\mathscr{P} = (\mathcal{U}, \mathcal{O}, \boldsymbol{\chi}, C, r, \alpha, \lambda)$ be C^κ-Picard data for $X \in \Gamma^\nu_{\mathrm{PLI}}(\mathbb{T}; \mathsf{TM}; \mathcal{P})$ at $(t_0, x_0, p_0) \in \mathbb{T} \times \mathsf{M} \times \mathcal{P}$.

Let us define the spaces we use. There are really a small number of such spaces, but we make use of the various representations of them via the constructions in (5.2). Let $a \in (0, 2r]$. A space of intrinsic interest is the space

$$\mathrm{C}^0(\mathbb{T}^2_{t_0,\alpha} \times \mathcal{U}_{\boldsymbol{\chi}}(a, x_0) \times \mathcal{O}; \mathsf{M})$$

of continuous mappings. As per our above notational conventions, let us denote members of this space by Γ. We then have the related spaces

$$\mathrm{C}^0(\mathbb{T}^2_{t_0,\alpha} \times \mathcal{U}_{\boldsymbol{\chi}}(a, x_0) \times \mathcal{O}; \mathbb{F}^n), \;\; \mathrm{C}^0(\mathbb{T}^2_{t_0,\alpha} \times \boldsymbol{\mathcal{U}}(a, x_0) \times \mathcal{O}; \mathbb{F}^n)$$

whose members we denote by $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$. Note that we only generally have the relations induced by (5.2) if we place constraints on the codomain of the mappings.

In this class of merely continuous mappings, we shall obtain a coarse form of convergence by considering the subspace

$$\mathrm{C}^0_{\mathrm{bdd}}(\mathbb{T}^2_{t_0,\alpha} \times \mathcal{U}_{\boldsymbol{\chi}}(a, x_0) \times \mathcal{O}; \mathbb{F}^n) \subseteq \mathrm{C}^0(\mathbb{T}^2_{t_0,\alpha} \times \mathcal{U}_{\boldsymbol{\chi}}(a, x_0) \times \mathcal{O}; \mathbb{F}^n)$$

of bounded continuous functions with the topology defined by the norm

$$\|\boldsymbol{\phi}\|_\infty = \sup\{|\phi^j(\tau, \tau_0, x, p)| \mid (\tau, \tau_0, x, p) \in \mathbb{T}^2_{t_0,\alpha} \times \mathcal{U}_{\boldsymbol{\chi}}(r, x_0) \times \mathcal{O}, \; j \in \{1, \dots, n\}\}.$$

This normed space is complete.

In the more refined context of time- and parameter-dependence presented in Section 3, we also have the following spaces with their specific topologies.

1. $\mathrm{C}^\nu_{\mathrm{PLAC}}(\mathbb{T}_{t_0,\alpha}; (\mathcal{U}_{\boldsymbol{\chi}}(a, x_0); \mathsf{M}); \mathcal{O})$: This is the space of continuous mappings from $\mathcal{O}$ into the topological space of locally absolutely continuous mappings with the topology prescribed by the semimetrics (3.6) and (3.7).
2. $\mathrm{C}^\nu_{\mathrm{PLAC}}(\mathbb{T}_{t_0,\alpha}; (\mathcal{U}_{\boldsymbol{\chi}}(a, x_0); \mathbb{F}^n); \mathcal{O})$: Because the codomain $\mathbb{F}^n$ has a vector space structure, this is the space of continuous mappings from $\mathcal{O}$ into the locally convex topological vector space of absolutely continuous functions with the seminorms given in Lemma 3.5.
3. $\mathrm{C}^\nu_{\mathrm{PLAC}}(\mathbb{T}_{t_0,\alpha}; (\boldsymbol{\mathcal{U}}(a, x_0); \mathbb{F}^n); \mathcal{O})$: As in the previous item, this is the space of continuous mappings from $\mathcal{O}$ into the locally convex topological vector space of absolutely continuous functions with the seminorms given in Lemma 3.5.

If we place restrictions on the codomains of the mappings as in (5.2), then there are natural identifications of these spaces since $\boldsymbol{\chi}$ is a C^κ-diffeomorphism between the various domains and codomains. These time- and parameter-dependent mappings with regularity ν will arise from mappings as in the preceding paragraph, i.e., from mappings with two time arguments but with one fixed. The notation we use to specify this fixed time is as follows. If

$$\Gamma \in \mathrm{C}^0(\mathbb{T}^2_{t_0,\alpha} \times \mathcal{U}_{\boldsymbol{\chi}}(r, x_0) \times \mathcal{O}; \mathsf{M}),$$

then we denote

$$\Gamma_{\tau_0}(\tau, x, p) = \Gamma^p_{\tau_0}(\tau, x) = \Gamma^p_{\tau,\tau_0}(x) = \Gamma(\tau, \tau_0, x, p).$$

We shall not have occasion to fix τ only, thus we will not make use of $\Gamma_\tau(\tau_0, x, p)$. This ensures that there can be no ambiguity as to which of the time arguments is being fixed.

Now, having introduced all spaces required, we define the operator. For our purposes, we elect to represent this operator in the space

$$\mathrm{C}^0(\mathbb{T}^2_{t_0,\alpha} \times \mathcal{U}_{\boldsymbol{\chi}}(r, x_0) \times \mathcal{O}; \mathbb{F}^n),$$

where the mappings are denoted by $\boldsymbol{\phi}$ in the diagrams (5.2). For each $j \in \{1, \dots, n\}$, we denote by $\phi_0^j \in \mathrm{C}^0_{\mathrm{bdd}}(\mathbb{T}^2_{t_0,\alpha} \times \mathcal{U}_{\boldsymbol{\chi}}(r, x_0) \times \mathcal{O}; \mathbb{F})$ the mapping $\phi_0^j(\tau, \tau_0, x, p) = \chi^j(x)$, and denote by $\overline{\mathsf{B}}(r, \boldsymbol{\phi}_0)$ the closed ball of radius r about $\boldsymbol{\phi}_0$ in $\mathrm{C}^0_{\mathrm{bdd}}(\mathbb{T}^2_{t_0,\alpha} \times \mathcal{U}_{\boldsymbol{\chi}}(r, x_0) \times \mathcal{O}; \mathbb{F}^n)$. We have the mapping

$$F_{\mathscr{P}} \colon \overline{\mathsf{B}}(r, \boldsymbol{\phi}_0) \to \mathrm{C}^0(\mathbb{T}^2_{t_0,\alpha} \times \mathcal{U}_{\boldsymbol{\chi}}(r, x_0) \times \mathcal{O}; \mathbb{F}^n)$$

defined by

$$F_{\mathscr{P}}(\phi^1, \dots, \phi^n)^p_{\tau,\tau_0} = \left(\chi^1 + \int_{\tau_0}^{\tau} (X^p_s \chi^1) \circ \Gamma^p_{\boldsymbol{\phi},s,\tau_0} \, \mathrm{d}s, \dots, \chi^n + \int_{\tau_0}^{\tau} (X^p_s \chi^n) \circ \Gamma^p_{\boldsymbol{\phi},s,\tau_0} \, \mathrm{d}s\right).$$

We call $F_{\mathscr{P}}$ the ***Picard operator***.

Associated with this operator are representations where the domain is $\boldsymbol{\mathcal{U}}(r, x_0)$ rather than $\mathcal{U}_{\boldsymbol{\chi}}(r, x_0)$. Precisely, we denote by $\boldsymbol{\psi}_0 \in \mathrm{C}^0_{\mathrm{bdd}}(\mathbb{T}^2_{t_0,\alpha} \times \boldsymbol{\mathcal{U}}(r, x_0) \times \mathcal{O}; \mathbb{F}^n)$ the mapping $\boldsymbol{\psi}_0(\tau, \tau_0, \boldsymbol{x}, p) = \boldsymbol{x}$, so that we have a mapping

$$\boldsymbol{F}_{\mathscr{P}} \colon \overline{\mathsf{B}}(r, \boldsymbol{\psi}_0) \to \mathrm{C}^0(\mathbb{T}^2_{t_0,\alpha} \times \boldsymbol{\mathcal{U}}(r, x_0) \times \mathcal{O}; \mathbb{F}^n)$$

defined so that the diagram

$$\begin{array}{ccc} \overline{\mathsf{B}}(r, \boldsymbol{\phi}_0) & \xrightarrow{F_{\mathscr{P}}} & \mathrm{C}^0(\mathbb{T}^2_{t_0,\alpha} \times \mathcal{U}_{\boldsymbol{\chi}}(r, x_0) \times \mathcal{O}; \mathbb{F}^n) \\ \downarrow & & \downarrow \\ \overline{\mathsf{B}}(r, \boldsymbol{\psi}_0) & \xrightarrow[\boldsymbol{F}_{\mathscr{P}}]{} & \mathrm{C}^0(\mathbb{T}^2_{t_0,\alpha} \times \boldsymbol{\mathcal{U}}(r, x_0) \times \mathcal{O}; \mathbb{F}^n) \end{array}$$

commutes, where the vertical arrows are defined by the chart map $\boldsymbol{\chi}$.

The following result records the essential properties of the Picard operator. The first two properties are standard, while the third is not.

5.3 Lemma: (Properties of Picard operators) *Let $m \in \mathbb{Z}_{\geq 0}$, let $\nu \in \{m, \infty, \omega, \mathrm{hol}\}$, and let $\kappa \in \{\infty, \omega, \mathrm{hol}\}$, as required. Denote*

$$\nu' = \begin{cases} \mathrm{lip}, & \nu = 0, \\ \nu, & \textit{otherwise}. \end{cases}$$

Let M be a C^κ-manifold, let $\mathbb{T} \subseteq \mathbb{R}$ be an interval, and let $\mathcal{P}$ be a topological space. Then the following statement hold:

(i) if $X \in \Gamma^0_{\mathrm{PLI}}(\mathbb{T}; \mathsf{TM}; \mathcal{P})$ and if $\mathscr{P} = (\mathcal{U}, \mathcal{O}, \boldsymbol{\chi}, C, r, \alpha, \lambda)$ is C^κ-Picard data for X at (t_0, x_0, p_0), then

$$F_{\mathscr{P}}(\overline{\mathsf{B}}(r, \boldsymbol{\phi}_0)) \subseteq \overline{\mathsf{B}}(r, \boldsymbol{\phi}_0);$$

(ii) *if* $X \in \Gamma^{\mathrm{lip}}_{\mathrm{PLI}}(\mathbb{T};\mathsf{TM};\mathcal{P})$ *and if* $\mathscr{P} = (\mathcal{U},\mathcal{O},\boldsymbol{\chi},C,r,\alpha,\lambda)$ *is* C^κ*-Picard data for* X *at* (t_0,x_0,p_0)*, then* $F_{\mathscr{P}}$ *is a contraction mapping in the complete metric space* $\overline{\mathsf{B}}(r,\boldsymbol{\phi}_0)$*;*

(iii) *if* $X \in \Gamma^{\nu'}_{\mathrm{PLI}}(\mathbb{T};\mathsf{TM};\mathcal{P})$*, if* $(\mathcal{U},\mathcal{O},\boldsymbol{\chi},C,r,\alpha,\lambda)$ *is* C^κ*-Picard data for* X *at* (t_0,x_0,p_0)*, if* $\tau_0 \in \mathbb{T}_{t_0,\alpha}$*, and if*

$$\boldsymbol{\phi} \in \mathrm{C}^0(\mathbb{T}^2_{t_0,\alpha}\times\mathcal{U}_{\boldsymbol{\chi}}(r,x_0)\times\mathcal{O};\mathbb{F}^n)$$

satisfies

$$\boldsymbol{\phi}_{\tau_0} \in \mathrm{C}^{\nu}_{\mathrm{PLAC}}(\mathbb{T}_{t_0,\alpha};(\mathcal{U}_{\boldsymbol{\chi}}(r,x_0);\mathbb{F}^n);\mathcal{O})\cap\overline{\mathsf{B}}(r,\boldsymbol{\phi}_0),$$

then

$$F_{\mathscr{P}}(\boldsymbol{\phi})_{\tau_0} \in \mathrm{C}^{\nu}_{\mathrm{PLAC}}(\mathbb{T}_{t_0,\alpha};(\mathcal{U}_{\boldsymbol{\chi}}(r,x_0);\mathbb{F}^n);\mathcal{O})\cap\overline{\mathsf{B}}(r,\boldsymbol{\phi}_0);$$

Proof: (i) Let $\boldsymbol{\phi} \in \overline{\mathsf{B}}(r,\boldsymbol{\phi}_0)$. Then

$$\begin{aligned}|\phi^j(\tau,\tau_0,x,p)| &= |\phi^j(\tau,\tau_0,x,p)-\phi^j_0(\tau,\tau_0,x_0,p)|\\ &\le |\phi^j(\tau,\tau_0,x,p)-\phi^j_0(\tau,\tau_0,x,p)|+|\phi^j_0(\tau,\tau_0,x,p)-\phi^j_0(\tau,\tau_0,x_0,p)| \le 2r,\\ &\qquad (\tau,\tau_0,x,p)\in\mathbb{T}^2_{t_0,\alpha}\times\mathcal{U}_{\boldsymbol{\chi}}(r,x_0)\times\mathcal{O},\end{aligned}$$

and so $\Gamma_{\boldsymbol{\phi}}(\mathcal{U}_{\boldsymbol{\chi}}(2r,x_0)) \subseteq \mathcal{U}$ by properties of Picard data from Lemma 5.1. Therefore, for $f \in \mathrm{C}^\kappa(\mathsf{M})$,

$$f\circ\Gamma_{\boldsymbol{\phi}} \in \mathrm{C}^0(\mathbb{T}^2_{t_0,\alpha}\times\mathcal{U}_{\boldsymbol{\chi}}(r,x_0)\times\mathcal{O}).$$

Thus, for fixed $\tau_0 \in \mathbb{T}_{t_0,\alpha}$, $p\in\mathcal{O}$, and $f\in\mathrm{C}^\kappa(\mathsf{M})$,

$$(s\mapsto f\circ\Gamma^p_{\boldsymbol{\phi},s,\tau_0}) \in \mathrm{C}^0(\mathbb{T}_{t_0,\alpha};\mathrm{C}^0(\mathcal{U}_{\boldsymbol{\chi}}(r,x_0)))$$

by [Jafarpour and Lewis 2016, Proposition 3]. Thus

$$(s\mapsto\Gamma^p_{\boldsymbol{\phi},s,\tau_0}) \in \mathrm{C}^0(\mathbb{T}_{t_0,\alpha};\mathrm{C}^0(\mathcal{U}_{\boldsymbol{\chi}}(r,x_0);\mathsf{M})), \qquad \tau_0\in\mathbb{T}_{t_0,\alpha},\ p\in\mathcal{O},$$

by definition of the topology for the space of mappings. Therefore,

$$(s\mapsto (X^p_s\chi^j)\circ\Gamma^p_{\boldsymbol{\phi},s,\tau_0}) \in \mathrm{C}^0_{\mathrm{LI}}(\mathbb{T}_{t_0,\alpha};\mathcal{U}_{\boldsymbol{\chi}}(r,x_0)), \qquad \tau_0\in\mathbb{T}_{t_0,\alpha},\ p\in\mathcal{O},$$

by Lemma 4.6. We then calculate

$$\begin{aligned}|F_{\mathscr{P}}(\boldsymbol{\phi})^j(\tau,\tau_0,x,p)-\phi^j_0(\tau,\tau_0,x,p)| &\le \int_{\mathbb{T}_{t_0,\alpha}}|(X^p_s\chi^j)\circ\Gamma^p_{\boldsymbol{\phi},s,\tau_0}(x)|\,\mathrm{d}s < r,\\ &(\tau,\tau_0,x,p)\in\mathbb{T}^2_{t_0,\alpha}\times\mathcal{U}_{\boldsymbol{\chi}}(r,x_0)\times\mathcal{O},\ j\in\{1,\dots,n\},\end{aligned}$$

since $\Gamma_{\boldsymbol{\phi}}(\tau,\tau_0,x,p) \in \mathcal{U}$ and by the properties of Picard data from Lemma 5.1. Thus $F_{\mathscr{P}}(\boldsymbol{\phi}) \in \overline{\mathsf{B}}(r,\boldsymbol{\phi}_0)$, as claimed.

(ii) Let $\boldsymbol{\phi}_1,\boldsymbol{\phi}_2 \in \overline{\mathsf{B}}(r,\boldsymbol{\phi}_0)$. Let $\Gamma_1,\Gamma_2 \in \mathrm{C}^0(\mathbb{T}_{t_0,\alpha}\times\mathcal{U}_{\boldsymbol{\chi}}(r,x_0)\times\mathcal{O};\mathsf{M})$ be the corresponding mappings satisfying

$$\chi^j\circ\Gamma_a(\tau,\tau_0,x,p) = \phi^j_a(\tau,\tau_0,x,p), \qquad j\in\{1,\dots,n\},\ a\in\{1,2\},$$

using (5.2). Then we have, for each $j \in \{1, \dots, n\}$,

$$\begin{aligned}
&|F_{\mathscr{P}}(\boldsymbol{\phi}_1)^j(\tau, \tau_0, x, p) - F_{\mathscr{P}}(\boldsymbol{\phi}_2)^j(\tau, \tau_0, x, p)| \\
&\quad \leq \int_{\mathbb{T}_{t_0,\alpha}} |(X^p_s \chi^j) \circ \Gamma^p_{1,s,\tau_0}(x) - (X^p_s \chi^j) \circ \Gamma^p_{2,s,\tau_0}(x)| \, \mathrm{d}s \\
&\quad \leq \frac{\lambda}{C} \sup\{\mathrm{d}_{\mathbb{G}}(\Gamma_1(\tau', \tau_0', x', p'), \Gamma_2(\tau', \tau_0', x', p')) \mid (\tau', \tau_0', x', p') \in \mathbb{T}^2_{t_0,\alpha} \times \mathcal{U}_{\boldsymbol{\chi}}(r, x_0) \times \mathcal{O}\} \\
&\quad \leq \lambda \sup\{|\phi^k_1(\tau', \tau_0', x', p') - \phi^k_2(\tau', \tau_0', x', p')|| \\
&\qquad\qquad k \in \{1, \dots, n\},\ (\tau', \tau_0', x', p') \in \mathbb{T}^2_{t_0,\alpha} \times \mathcal{U}_{\boldsymbol{\chi}}(r, x_0) \times \mathcal{O}\},
\end{aligned}$$

using again the properties of Picard data from Lemma 5.1. The desired conclusion follows.

(iii) By Lemma 3.11,

$$(p \mapsto (t \mapsto X^p_t \chi^j)) \in \mathrm{C}^{\nu'}_{\mathrm{PLAC}}(\mathbb{T}_{t_0,\alpha}; \mathcal{U}_{\boldsymbol{\chi}}(r, x_0); \mathcal{O}), \qquad j \in \{1, \dots, n\}.$$

By hypothesis and by the diagrams (5.2),

$$(p \mapsto (t \mapsto \Gamma^p_{\boldsymbol{\phi},t,\tau_0})) \in \mathrm{C}^{\nu}_{\mathrm{PLAC}}(\mathbb{T}_{t_0,\alpha}; (\mathcal{U}_{\boldsymbol{\chi}}; \mathsf{M}); \mathcal{O}).$$

By Lemma 4.9,

$$(p \mapsto (t \mapsto (X^p_t \chi^j) \circ \Gamma^p_{\boldsymbol{\phi},t,\tau_0})) \in \mathrm{C}^{\nu}_{\mathrm{PLAC}}(\mathbb{T}_{t_0,\alpha}; \mathcal{U}_{\boldsymbol{\chi}}(r, x_0); \mathcal{O}), \qquad j \in \{1, \dots, n\}.$$

This part of the result follows from the preceding observation and part (i). ■

5.2. Local flows. Now we state a result regarding local flows. Let us first define what we mean by a local flow.

5.4 Definition: (Time- and parameter-dependent local flow) Let $m \in \mathbb{Z}_{\geq 0}$, let $\nu \in \{m, \infty, \omega, \mathrm{hol}\}$, and let $\kappa \in \{\infty, \omega, \mathrm{hol}\}$, as required. Denote

$$\nu' = \begin{cases} \mathrm{lip}, & \nu = 0, \\ \nu, & \text{otherwise.} \end{cases}$$

Let $\kappa \in \{\infty, \omega, \mathrm{hol}\}$ as required, let M be a C^{κ}-manifold, let $\mathbb{T} \subseteq \mathbb{R}$ be an interval, and let $\mathcal{P}$ be a topological space. Let $X \in \Gamma^{\nu'}_{\mathrm{PLI}}(\mathbb{T}; \mathsf{TM}; \mathcal{P})$ and let $(t_0, x_0, p_0) \in \mathbb{T} \times \mathsf{M} \times \mathcal{P}$. A $\mathbf{C}^{\boldsymbol{\nu}}$***-local flow*** for X about (t_0, x_0, p_0) is a quadruple $(\Phi^X, \mathbb{S}, \mathcal{U}, \mathcal{O})$ with the following properties:

(i) $\mathbb{S} \subseteq \mathbb{T}$ is an interval for which $t_0 \in \mathbb{S}$;
(ii) $\mathcal{U} \subseteq \mathsf{M}$ is a neighbourhood of x_0;
(iii) $\mathcal{O} \subseteq \mathcal{P}$ is a neighbourhood of p_0;
(iv) $\Phi^X \in \mathrm{C}^0(\mathbb{S}^2 \times \mathcal{U} \times \mathcal{O}; \mathsf{M})$;
(v) $\Phi^X_{\tau_0} \in \mathrm{C}^{\nu}_{\mathrm{PLAC}}(\mathbb{S}; (\mathcal{U}; \mathsf{M}); \mathcal{O})$ for each $\tau_0 \in \mathbb{S}$;
(vi) $\tau \mapsto \Phi^X(\tau, \tau_0, x, p)$ is the integral curve for X^p through x at time τ_0 for each $(\tau_0, x, p) \in \mathbb{S} \times \mathcal{U} \times \mathcal{O}$, and defined for $\tau \in \mathbb{S}$. •

In making this definition, we are making use of the fact that locally absolutely continuous integral curves exist and are unique for vector fields $X \in \Gamma^{\mathrm{lip}}_{\mathrm{LI}}(\mathbb{T}; \mathsf{TM})$. This is not, on its face, exactly the standard result for existence and uniqueness of solutions to ordinary differential equations, simply because our definition of $\Gamma^{\mathrm{lip}}_{\mathrm{LI}}(\mathbb{T}; \mathsf{TM})$ does not obviously imply the standard conditions. However, the standard conditions do indeed follow, essentially by virtue of Lemma 3.14. We shall see in the proof of the next theorem that this follows as a consequence of more general constructions.

5.5 Theorem: (Existence of C^0-local flow) *Let M be a C^∞-manifold, let $\mathbb{T} \subseteq \mathbb{R}$ be an interval, let $\mathcal{P}$ be a topological space, and let $X \in \Gamma^{\mathrm{lip}}_{\mathrm{PLI}}(\mathbb{T}; \mathsf{TM}; \mathcal{P})$. Let $(t_0, x_0, p_0) \in \mathbb{T} \times \mathsf{M} \times \mathcal{P}$. Let $\mathscr{P} = (\mathcal{U}, \mathcal{O}, \boldsymbol{\chi}, C, r, \alpha, \lambda)$ be C^∞-Picard data for X at (t_0, x_0, p_0). Then there exists a C^0-local flow $(\Phi^X, \mathbb{T}_{t_0,\alpha}, \mathcal{U}_{\boldsymbol{\chi}}(r, x_0), \mathcal{O})$ for X about (t_0, x_0, p_0). Additionally, for fixed $\tau, \tau_0 \in \mathbb{T}_{t_0,\alpha}$ and $p_0 \in \mathcal{O}$, the mapping $x \mapsto \Phi^X(\tau, \tau_0, x, p_0)$ is a bi-Lipschitz homeomorphism onto its image.*

Proof: Let $\mathscr{P} = (\mathcal{U}, \mathcal{O}, \boldsymbol{\chi}, C, r, \alpha, \lambda)$ be C^∞-Picard data for X at (t_0, x_0, p_0). By Lemma 5.3(ii) and the Contraction Mapping Theorem [Abraham, Marsden, and Ratiu 1988, Theorem 1.2.6], there is a unique fixed point of $\boldsymbol{F}_{\mathscr{P}}$ in

$$\overline{\mathsf{B}}(r, \boldsymbol{\psi}_0) \subseteq \mathrm{C}^0(\mathbb{T}^2_{t_0,\alpha} \times \boldsymbol{\mathcal{U}}(r, x_0), \mathcal{O}; \mathbb{R}^n).$$

We denote this unique fixed point by $\boldsymbol{\psi}_\infty$, noting that there are then corresponding mappings

$$\boldsymbol{\phi}_\infty \in \mathrm{C}^0(\mathbb{T}^2_{t_0,\alpha} \times \mathcal{U}_{\boldsymbol{\chi}}(r, x_0) \times \mathcal{O}; \mathbb{R}^n), \qquad \Gamma^+_\infty \in \mathrm{C}^0(\mathbb{T}^2_{t_0,\alpha} \times \mathcal{U}_{\boldsymbol{\chi}}(r, x_0) \times \mathcal{O}; \mathsf{M}).$$

We define

$$\Phi^X(\tau, \tau_0, x, p) = \Gamma_\infty(\tau, \tau_0, x, p), \qquad (\tau, \tau_0, x, p) \in \mathbb{T}^2_{t_0,\alpha} \times \mathcal{U}_{\boldsymbol{\chi}}(r, x_0) \times \mathcal{O},$$

noting that Φ^X is continuous. Note that

$$\boldsymbol{\psi}_\infty(\tau, \tau_0, \boldsymbol{x}, p) = \boldsymbol{x} + \int_{\tau_0}^{\tau} \boldsymbol{X}^p_s \circ \boldsymbol{\psi}^p_{\infty,s,\tau_0}(\boldsymbol{x})\, \mathrm{d}s,$$

because $\boldsymbol{\psi}_\infty$ is a fixed point of $\boldsymbol{F}_{\mathscr{P}}$. Therefore, according to (5.2), the curve $\tau \mapsto \Phi^X(\tau, \tau_0, x, p)$ satisfies

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\tau}\Phi^X(\tau, \tau_0, x, p) = X(\tau, \Phi^X(\tau, \tau_0, x, p)), \quad \text{a.e. } \tau \in \mathbb{T}_{t_0,\alpha},\\
&\Phi^X(\tau_0, \tau_0, x, p) = x,
\end{aligned}$$

i.e., the curve is the (necessarily unique) integral curve of X through x at time τ_0 defined for $\tau \in \mathbb{T}_{t_0,\alpha}$.

Next we claim that the map

$$\mathcal{U}(r, x_0) \ni x \mapsto \Phi^X(\tau, \tau_0, x, p) \in \mathsf{M}$$

is Lipschitz for every $\tau, \tau_0 \in \mathbb{T}_{t_0,\alpha}$ and $p \in \mathcal{O}$. Thus we fix $\tau, \tau_0 \in \mathbb{T}_{t_0,\alpha}$ and $p \in \mathcal{O}$. By parts (ix) and (xi) of Lemma 5.1, we have

$$\int_{\tau_0}^{\tau} \left|X^p_s\chi^j(x_1) - X^p_s\chi^j(x_2)\right| \mathrm{d}s \\ \leq \lambda \max\left\{\left|\chi^l(x_1) - \chi^l(x_2)\right| \;\middle|\; l \in \{1,\dots,k\}\right\}, \qquad x_1, x_2 \in \mathcal{U}.$$

Let $x_1, x_2 \in \mathcal{U}(r, x_0)$. We have

$$f \circ \Phi^X(\tau, \tau_0, x_a, p) = f(x_a) + \int_{\tau_0}^{\tau} (X^p_s f) \circ \Phi^{X^p}_{s,\tau_0}(x_a)\, \mathrm{d}s, \qquad a \in \{1, 2\},\ f \in \mathrm{C}^\infty(\mathsf{M}).$$

Therefore, for $j \in \{1, \dots, n\}$,

$$\begin{aligned}
&\left|\chi^j \circ \Phi^{X^p}_{\tau,\tau_0}(x_1) - \chi^j \circ \Phi^{X^p}_{\tau,\tau_0}(x_2)\right| \\
&\quad\leq \left|\chi^j(x_1) - \chi^j(x_2)\right| + \int_{\tau_0}^{\tau} \left|(X^p_s\chi^j) \circ \Phi^{X^p}_{s,\tau_0}(x_1) - (X^p_s\chi^j) \circ \Phi^{X^p}_{s,\tau_0}(x_2)\right| \mathrm{d}s \\
&\quad\leq \max\left\{\left|\chi^l(x_1) - \chi^l(x_2)\right| \;\middle|\; l \in \{1,\dots,n\}\right\} \\
&\qquad + \lambda \sup\left\{\left|\chi^l \circ \Phi^{X^p}_{s,\tau_0}(x_1) - \chi^l \circ \Phi^{X^p}_{s,\tau_0}(x_2)\right| \;\middle|\; s \in |\tau_0, \tau|,\ l \in \{1,\dots,n\}\right\}.
\end{aligned}$$

Abbreviate

$$\xi^p(\tau, \tau_0) = \max\left\{\left|\chi^l \circ \Phi^{X^p}_{\tau,\tau_0}(x_1) - \chi^l \circ \Phi^{X^p}_{\tau,\tau_0}(x_2)\right| \;\middle|\; l \in \{1,\dots,n\}\right\}$$

and

$$\sigma^p = \sup\left\{\xi^p(\tau, \tau_0) \mid \tau, \tau_0 \in \mathbb{T}_{\alpha,t_0}\right\}.$$

Our computation just preceding is then abbreviated

$$\left|\chi^j \circ \Phi^{X^p}_{\tau,\tau_0}(x_1) - \chi^j \circ \Phi^{X^p}_{\tau,\tau_0}(x_2)\right| \leq \xi^p(\tau_0, \tau_0) + \lambda\sigma^p, \quad \tau, \tau_0 \in \mathbb{T}_{t_0,\alpha},\ p \in \mathcal{O},\ j \in \{1,\dots,n\},$$

and taking the sup over $j \in \{1, \dots, n\}$ and $\tau, \tau_0 \in \mathbb{T}_{t_0,\alpha}$ on the left gives

$$\sigma^p \leq \xi^p(\tau_0, \tau_0) + \lambda\sigma^p \implies \sigma^p \leq (1 - \lambda)^{-1}\xi^p(\tau_0, \tau_0).$$

Since

$$\xi^p(\tau_0, \tau_0) = \max\{\left|\chi^j(x_1) - \chi^j(x_2)\right| \mid j \in \{1,\dots,n\}\},$$

we can use part (ix) of Lemma 5.1 to give

$$\mathrm{d}_{\mathbb{G}}(\Phi^X(\tau, \tau_0, x_1, p), \Phi^X(\tau, \tau_0, x_2, p)) \leq C\xi^p(\tau_0, \tau_0) \leq C\sigma^p \leq C^2(1 - \lambda)^{-1}\mathrm{d}_{\mathbb{G}}(x_1, x_2),$$

which shows that $x \mapsto \Phi^X(\tau, \tau_0, x, p)$ is Lipschitz. Incidentally, the Lipschitz constant is independent of $\tau, \tau_0 \in \mathbb{T}_{t_0,\alpha}$ and $p \in \mathcal{O}$.

Now we show that we can choose the Picard data such that, for each τ, τ_0, and p, the mapping $x \mapsto \Phi^X(\tau, \tau_0, x, p)$ is a bi-Lipschitz homeomorphism onto its image. To do this, we retain the Picard data $\mathscr{P}=(\mathcal{U},\mathcal{O},\boldsymbol{\chi},C,r,\alpha,\lambda)$ as above and note that, as we saw in the proof of Lemma 5.3(i),

$$\Phi^X(\tau,\tau_0,x,p)\in\operatorname{cl}(\mathcal{U}_{\boldsymbol{\chi}}(2r,x_0)),\qquad \tau,\tau_0\in\mathbb{T}_{t_0,\alpha},\ x\in\operatorname{cl}(\mathcal{U}_{\boldsymbol{\chi}}(r,x_0)),\ p\in\mathcal{O}.$$

We then take $\alpha'\in(0,\alpha]$ and a neighbourhood $\mathcal{O}'$ of p_0 such that $\mathscr{P}'=(\mathcal{U},\mathcal{O}',\boldsymbol{\chi},C,r'=\frac{r}{2},\alpha',\lambda)$ is Picard data for X at (t_0,x_0,p_0). That this is possible follows immediately from the constructions of α and $\mathcal{O}$ from the proof of Lemma 5.1. Since $\mathscr{P}'\preceq\mathscr{P}$ (according to the partial ordering of Picard data as following (5.1)), we have that Φ^X restricted to $\mathbb{T}^2_{t_0,\alpha'}\times\mathcal{U}_{\boldsymbol{\chi}}(r',x_0)\times\mathcal{O}'$ is continuous and that $x\mapsto\Phi^X(\tau,\tau_0,x,p)$ is Lipschitz with Lipschitz constant $C^2(1-\lambda)^{-1}$ for $\tau,\tau_0\in\mathbb{T}_{t_0,\alpha'}$ and $p\in\mathcal{O}'$. Moreover,

$$\Phi^X(\tau,\tau_0,x,p)\in\operatorname{cl}(\mathcal{U}_{\boldsymbol{\chi}}(r,x_0)),\qquad \tau,\tau_0\in\mathbb{T}_{t_0,\alpha'},\ x\in\operatorname{cl}(\mathcal{U}_{\boldsymbol{\chi}}(r',x_0)),\ p\in\mathcal{O}',$$

again from the proof of Lemma 5.3(i). Now, for $\tau,\tau_0\in\mathbb{T}_{t_0,\alpha'}$ and $p\in\mathcal{O}'$, we have

$$\begin{aligned}\Phi^{X^p}_{\tau_0,\tau}\circ\Phi^{X^p}_{\tau,\tau_0}(x)&=x,\qquad x\in\mathcal{U}_{\boldsymbol{\chi}}(r',x_0),\\ \Phi^{X^p}_{\tau,\tau_0}\circ\Phi^{X^p}_{\tau_0,\tau}(x)&=x,\qquad x\in\Phi^{X^p}_{\tau,\tau_0}(\mathcal{U}_{\boldsymbol{\chi}}(r',x_0)),\end{aligned}$$

by elementary properties of flows that follow from uniqueness of integral curves (we do not prove these elementary properties here). Thus $\Phi^{X^p}_{\tau,\tau_0}$ is a bijection from $\mathcal{U}_{\boldsymbol{\chi}}(r',x_0)$ onto its image with inverse $\Phi^{X^p}_{\tau_0,\tau}$. That $\Phi^{X^p}_{\tau,\tau_0}$ is a bi-Lipschitz homeomorphism from $\mathcal{U}_{\boldsymbol{\chi}}(r',x_0)$ onto its image follows from the preceding part of the proof since there we showed that $\Phi^{X^p}_{\tau_0,\tau}$ is Lipschitz with Lipschitz constant $C^2(1-\lambda)^{-1}$.

Given the preceding paragraph, we reassign to the symbol $\mathscr{P}$ the Picard data $\mathscr{P}'$ from the preceding paragraph. It remains to show that, for each $\tau_0\in\mathbb{T}_{t_0,\alpha}$,

$$\Phi^X_{\tau_0}\in\mathrm{C}^0_{\mathrm{PLAC}}(\mathbb{T}_{t_0,\alpha};(\mathcal{U}_{\boldsymbol{\chi}}(r,x_0);\mathsf{M});\mathcal{O}).$$

We prove this by constructing a sequence in

$$\mathrm{C}^0(\mathcal{P};\mathrm{C}^0_{\mathrm{LAC}}(\mathbb{T};(\mathcal{U}_{\boldsymbol{\chi}}(r,x_0);\mathsf{M})))$$

that converges uniformly to the mapping

$$p\mapsto(\tau\mapsto\Phi^{X^p}_{\tau,\tau_0})$$

The continuity of this mapping then follows since uniform limits of continuous functions are continuous [Willard 1970, Theorem 42.10]. In the various representations of mappings from (5.2), we will work with mappings from $\boldsymbol{\mathcal{U}}(r,x_0)$ to $\mathbb{R}^n$, i.e., those denoted with $\boldsymbol{\psi}$. We take $\boldsymbol{\psi}_0$ as above, i.e.,

$$\boldsymbol{\psi}_0(\tau,\tau_0,\boldsymbol{x},p)=\boldsymbol{x},\qquad \tau,\tau_0\in\mathbb{T}_{t_0,\alpha},\ \boldsymbol{x}\in\boldsymbol{\mathcal{U}}(r,x_0),\ p\in\mathcal{O}.$$

We then recursively define a sequence $(\boldsymbol{\psi}_k)_{k\in\mathbb{Z}_{\geq0}}$ of mappings by $\boldsymbol{\psi}_{k+1}=\boldsymbol{F}_{\mathscr{P}}(\boldsymbol{\psi}_k)$, noting that $\boldsymbol{\psi}_{k,\tau_0}\in\mathrm{C}^0_{\mathrm{PLAC}}(\mathbb{T};(\boldsymbol{\mathcal{U}}(r,x_0);\mathbb{R}^n);\mathcal{O})$, $k\in\mathbb{Z}_{\geq0}$, by Lemma 5.3(iii). We will show that this sequence converges in $\mathrm{C}^0_{\mathrm{PLAC}}(\mathbb{T};(\boldsymbol{\mathcal{U}}(r,x_0);\mathbb{R}^n);\mathcal{O})$ by showing that it is Cauchy.

We define seminorms $p^0_{K,\infty}$ for $\mathrm{C}^0(\boldsymbol{\mathcal{U}}(r,x_0);\mathbb{R}^n)$ by

$$p^0_{K,\infty}(\boldsymbol{\psi})=\sup\left\{\|\boldsymbol{\psi}(\boldsymbol{x})\|_\infty \;\middle|\; \boldsymbol{x}\in\boldsymbol{\mathcal{U}}(r,x_0)\right\},\qquad K\in\mathscr{K}(\boldsymbol{\mathcal{U}}(r,x_0))$$

noting that this differs from the usual definition of seminorms for $\mathrm{C}^0(\mathcal{U}(r,x_0);\mathbb{R}^n)$ in the unimportant way that the usual definition uses the Euclidean norm for $\mathbb{R}^n$.

Note that

$$\boldsymbol{\psi}^p_{k+1,\tau,\tau_0}(\boldsymbol{x}) = \boldsymbol{x} + \int_{\tau_0}^{\tau} \boldsymbol{X}^p_\tau \circ \boldsymbol{\psi}^p_{k,\tau,\tau_0}(\boldsymbol{x})\,\mathrm{d}\tau.$$

Thus we should show that, for $\epsilon \in \mathbb{R}_{>0}$, for each compact $K \subseteq \mathcal{U}(r,x_0)$, for each compact interval $\mathbb{K} \subseteq \mathbb{T}_{t_0,\alpha}$, there exists $N \in \mathbb{Z}_{>0}$ such that

$$\begin{aligned} &\sup\{p^0_{K,\infty}(\boldsymbol{\psi}^p_{k,\tau,\tau_0} - \boldsymbol{\psi}^p_{l,\tau,\tau_0}) \mid \tau \in \mathbb{K}\} < \epsilon, \\ &\int_{\mathbb{K}} p^0_{K,\infty}(\boldsymbol{X}^p_\tau \circ \boldsymbol{\psi}^p_{k,\tau,\tau_0} - \boldsymbol{X}^p_\tau \circ \boldsymbol{\psi}^p_{l,\tau,\tau_0})\,\mathrm{d}\tau < \epsilon \end{aligned} \tag{5.3}$$

for all $p \in \mathcal{O}$ and all $k,l \geq N$, according to our characterisation in Lemma 3.5 of the topology of the space of locally absolutely continuous functions with values in $\mathrm{C}^0(\mathcal{U}(r,x_0);\mathbb{R}^n)$. For $k,l \in \mathbb{Z}_{>0}$, and $\tau \in \mathbb{K}$, $\boldsymbol{x} \in K$, and $p \in \mathcal{O}$, we have

$$\begin{aligned} \|\boldsymbol{\psi}^p_{k,\tau,\tau_0}(\boldsymbol{x}) - \boldsymbol{\psi}^p_{l,\tau,\tau_0}(\boldsymbol{x})\|_\infty &= \|\boldsymbol{F}_{\mathscr{P}}(\boldsymbol{\psi}_{k-1})(\tau,\tau_0,\boldsymbol{x},p) - \boldsymbol{F}_{\mathscr{P}}(\boldsymbol{\psi}_{l-1})(\tau,\tau_0,\boldsymbol{x},p)\|_\infty \\ &\leq \int_{|\tau_0,\tau|} \|\boldsymbol{X}^p_\tau \circ \boldsymbol{\psi}^p_{k-1,\tau,\tau_0}(\boldsymbol{x}) - \boldsymbol{X}^p_s \circ \boldsymbol{\psi}^p_{l-1,\tau,\tau_0}(\boldsymbol{x})\|_\infty\,\mathrm{d}s \\ &\leq \int_{|\tau_0,\tau|} p^0_{K,\infty}(\boldsymbol{X}^p_s \circ \boldsymbol{\psi}^p_{k-1,s,\tau_0} - \boldsymbol{X}^p_s \circ \boldsymbol{\psi}^p_{l-1,s,\tau_0}))\,\mathrm{d}s, \end{aligned}$$

similarly to our computation above done to show that $\Phi^{X^p}_{\tau,\tau_0}$ is Lipschitz. Therefore, similarly to Lemma 4.8, it suffices to show only that the second of the conditions from (5.3) is satisfied. This we now turn to.

For $k,l \in \mathbb{Z}_{>0}$, and $\tau \in \mathbb{K}$, $\boldsymbol{x} \in K$, and $p \in \mathcal{O}$, we calculate

$$\begin{aligned} \int_{\mathbb{K}} \Big\| \boldsymbol{X}^p_\tau \circ \boldsymbol{\psi}^p_{k,\tau,\tau_0}(\boldsymbol{x}) - &\ \boldsymbol{X}^p_\tau \circ \boldsymbol{\psi}^p_{l,\tau,\tau_0}(\boldsymbol{x}) \Big\|_\infty\,\mathrm{d}\tau \\ &\leq \lambda \sup\Big\{ \|\boldsymbol{\psi}^p_{k,\tau,\tau_0}(\boldsymbol{x}) - \boldsymbol{\psi}^p_{l,\tau,\tau_0}(\boldsymbol{x})\|_\infty \;\Big|\; \tau \in \mathbb{K} \Big\} \\ &\leq \lambda \int_{\mathbb{K}} \|\boldsymbol{X}^p_\tau \circ \boldsymbol{\psi}^p_{k-1,\tau,\tau_0}(\boldsymbol{x}) - \boldsymbol{X}^p_\tau \circ \boldsymbol{\psi}^p_{l-1,\tau,\tau_0}(\boldsymbol{x})\|_\infty\,\mathrm{d}\tau \\ &\leq \lambda \int_{\mathbb{K}} p^0_{K,\infty}(\boldsymbol{X}^p_\tau \circ \boldsymbol{\psi}^p_{k-1,\tau,\tau_0} - \boldsymbol{X}^p_\tau \circ \boldsymbol{\psi}^p_{l-1,\tau,\tau_0})\,\mathrm{d}\tau. \end{aligned}$$

Thus we have

$$p^0_{K,\infty,\mathbb{K}}(\boldsymbol{\psi}^p_{k,\tau_0} - \boldsymbol{\psi}^p_{l,\tau_0}) \leq \lambda p^0_{K,\infty,\mathbb{K}}(\boldsymbol{\psi}^p_{k-1,\tau_0} - \boldsymbol{\psi}^p_{l-1,\tau_0}), \qquad p \in \mathcal{O}.$$

In particular,

$$p^0_{K,\infty,\mathbb{K}}(\boldsymbol{\psi}^p_{k+1,\tau_0} - \boldsymbol{\psi}^p_{k,\tau_0}) \leq \lambda p^0_{K,\infty,\mathbb{K}}(\boldsymbol{\psi}^p_{k,\tau_0} - \boldsymbol{\psi}^p_{k-1,\tau_0}), \qquad p \in \mathcal{O},$$

and so an elementary induction argument gives

$$p^0_{K,\infty,\mathbb{K}}(\boldsymbol{\psi}^p_{k+1,\tau_0} - \boldsymbol{\psi}^p_{k,\tau_0}) \leq \lambda^k p^0_{K,\infty,\mathbb{K}}(\boldsymbol{\psi}^p_{1,\tau_0} - \boldsymbol{\psi}^p_{0,\tau_0}), \qquad p \in \mathcal{O}.$$

Therefore, supposing that $k > l$,

$$\begin{aligned} p^0_{K,\infty,\mathbb{K}}(\boldsymbol{\psi}^p_{k,\tau_0} - \boldsymbol{\psi}^p_{l,\tau_0}) &\le p^0_{K,\infty}\left(\sum_{j=l}^{k-1} \boldsymbol{\psi}^p_{j+1,\tau_0} - \boldsymbol{\psi}^p_{j,\tau_0}\right) \\ &\le \sum_{j=l}^{k-1} \lambda^j p^0_{K,\infty}(\boldsymbol{\psi}^p_{1,\tau_0} - \boldsymbol{\psi}^p_{0,\tau_0}) \\ &= \sum_{j=0}^{k-l-1} \lambda^{j+l} p^0_{K,\infty}(\boldsymbol{\psi}^p_{1,\tau_0} - \boldsymbol{\psi}^p_{0,\tau_0}) \\ &\le \lambda^l p^0_{K,\infty}(\boldsymbol{\psi}^p_{1,\tau_0} - \boldsymbol{\psi}^p_{0,\tau_0}) \sum_{j=0}^{\infty} \lambda^j = \frac{\lambda^l}{1-\lambda} p^0_{K,\infty}(\boldsymbol{\psi}^p_{1,\tau_0} - \boldsymbol{\psi}^p_{0,\tau_0}) \end{aligned}$$

for all $p \in \mathcal{O}$. Thus, by choosing N sufficiently large, we deduce the second condition of (5.3). ■

5.6 Remarks: (On the preceding theorem)

1. We *do not* show that a time- and parameter-dependent vector field in $\Gamma^{\mathrm{lip}}_{\mathrm{PLI}}(\mathbb{T}; \mathsf{TM}; \mathcal{P})$ gives rise to a $\mathrm{C}^{\mathrm{lip}}$-local flow, but only to a C^0-local flow. This is a consequence of the lack of continuity of the locally Lipschitz superposition operator, explained in Example 4.3. The continuity of the joint composition operator for the regularity classes $\nu \in \mathbb{Z}_{>0} \cup \{\infty, \omega, \mathrm{hol}\}$ can be expected to give the existence of a C^ν-local flow for $X \in \Gamma^\nu_{\mathrm{PLI}}(\mathbb{T}; \mathsf{TM}; \mathcal{P})$ in these cases. This will be fleshed out a little in the next section.
2. Note that, the preceding remark notwithstanding, we prove that, for fixed values of the parameter, the local flow is Lipschitz and satisfies a Lipschitz bound like that given in Lemma 3.20. Note that this does not actually follow from Lemma 3.20 since, as we observed above, we do not show that there is a $\mathrm{C}^{\mathrm{lip}}$-local flow.
3. Note that, in the last part of the proof of the theorem, we show that the local flow is indeed obtained as the limit of a sequence within the green blob in Figure 2.
4. In the last part of the proof one can see that we are essentially proving the continuous-dependence on parameters of the fixed point defined by iterating a contraction mapping in a space of parameter-dependent functions. This is carried out in a general and elegant way in the draft book of Glöckner and Neeb [2023], and utilised in [Glöckner 2015, Glöckner 2023].
5. One might observe that the proof of the theorem uses more or less standard tools. One could conclude from this that there is nothing new contributed by the proof. An alternative view, and the one we prefer, is that the proof demonstrates that the standard hypotheses and methods give stronger conclusions than are normally asserted. In the theorem, we are considering the most basic of hypotheses—essentially those of the standard existence and uniqueness theorem—and arriving at the weakest conclusions—that there is a C^0-local flow. But one can expect that the results with richer hypotheses will give richer conclusions. It is to a discussion of this that we now turn. •

5.3. Further developments. The part of Theorem 5.5 that is not classical is showing that the local flow has the property that

$$(p \mapsto (t \mapsto \Phi^{X^p}_{\tau_0})) \in \mathrm{C}^0_{\mathrm{PLAC}}(\mathbb{T}_{t_0,\alpha}; (\mathcal{U}(r, x_0); \mathsf{M}); \mathcal{O}).$$

This itself follows from the nonclassical property of the Picard operator given as Lemma 5.3(iii), which, in turn, relies on the machinery developed in the previous parts of the paper, and specifically the results from Section 4 on composition operators. It is all of this that enables the possibility of a Picard iteration scheme within the green blob in Figure 2. It remains to prove convergence in the green blob, as was done in Theorem 5.5 for vector fields with local Lipschitz regularity. This will be a bit of a project, and in this section we outline how it can be done. The full development of this will be the subject of future papers.

The finitely differentiable and smooth cases. The fact that, if, for $\nu \in \mathbb{Z}_{>0} \cup \{\infty\}$, $X \in \Gamma^{\nu}_{\mathrm{LI}}(\mathbb{T}; \mathsf{TM})$, then its local flow is in the space $\mathrm{C}^{\nu}_{\mathrm{PLAC}}(\mathbb{T}; (\mathcal{U}(r, x_0); \mathsf{M}); \mathcal{O})$ is almost classical. In the smooth case, this follows from the results of Agrachev and Gamkrelidze [1978]. However, the methods in that paper do not apply to the finitely differentiable case. Let us, therefore, outline how the methods we give here can be used to give the result in this finitely differentiable case, and consequently in the smooth case.

First of all, one wishes to use induction to reduce to the case of $m = 1$. The way one can approach this is to prove the existence of a C^0-local flow for the first prolongation $\nu_1 X$ defined in Section 2.5. There is a wrinkle in this, however, and this is that $\nu_1 X^p \notin \Gamma^{\mathrm{lip}}_{\mathrm{LI}}(\mathbb{T}; \mathsf{TM})$. Thus Theorem 5.5 does not apply. However, the vector field $\nu_1 X$ is *linear*. One can prove a general result as follows. Let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a smooth vector bundle. Then a linear vector field $X \in \Gamma^0_{\mathrm{PLI}}(\mathbb{T}; \mathsf{TE}; \mathcal{P})$ on a vector bundle over a vector field $X_0 \in \Gamma^{\mathrm{lip}}_{\mathrm{PLI}}(\mathbb{T}; \mathsf{TM}; \mathcal{P})$ possesses a unique C^0-local flow, defined on a local vector bundle rather than on an open subset of M and for which $\Phi^{X^p}_{\tau,\tau_0}$ is a C^0-vector bundle mapping. In this way, one proves the result in the case $m = 1$. Using Lemma 2.2, one proves the finitely differentiable case by induction. The smooth case follows since it is the intersection of all finitely differentiable cases; said more sophisticatedly, the smooth topology is the inverse limit of the finitely differentiable topologies.

The holomorphic case. The holomorphic case follows immediately from the smooth case. This is a consequence of the fact that a uniform limit of holomorphic mappings is holomorphic [Gunning and Rossi 1965, Lemma I.A.11].

The real analytic case. The real analytic case is the most challenging. Here one needs to have a better understanding of the real analytic topology than is afforded by merely knowing its seminorms. Indeed, it seems necessary that one has to use the descriptions from the original work of Martineau [1966] and described by Lewis [2023]. A key ingredient in this description is that, given a compact subset $K \subseteq \mathsf{M}$, one has a mapping from $\Gamma^{\omega}(\mathsf{TM})$ to the set $\mathscr{G}^{\mathrm{hol}}_{K,\mathsf{T}\overline{\mathsf{M}}}$ of germs of holomorphic vector fields about K in a complexification $\overline{\mathsf{M}}$ of M. One can use then use the results concerning local flows from the holomorphic case to give results in the real analytic case. In the parameter-independent case, this is [Jafarpour and Lewis 2014, Theorem 6.26]. Ideas of Glöckner [2023] are useful for establishing continuous dependence on parameters in the real analytic case.

Extension from local flows to global flows. If M is not compact, then generally the flow of $X \in \Gamma^{\nu}_{\mathrm{LI}}(\mathbb{T};\mathsf{TM})$ is not complete, meaning that, for some $(t_0,x_0) \in \mathbb{T}\times\mathsf{M}$, the integral curve though x_0 at initial time t_0 is not defined for all $t \in \mathbb{T}$. To this end, for $X \in \Gamma^{\nu}_{\mathrm{PLI}}(\mathbb{T};\mathsf{TM};\mathcal{P})$, $\nu \in \mathbb{Z}_{>0} \cup \{\infty,\omega,\mathrm{hol}\}$, we define the ***domain*** of X to be

$$\mathrm{D}(X) = \{(t,t_0,x,p) \in \mathbb{T}\times\mathbb{T}\times\mathsf{M}\times\mathcal{P} \mid \text{the maximal integral curve } \xi \text{ for } X^p \text{ with } \xi(t_0)=x_0 \text{ is defined on } |t,t_0|\}.$$

One then can define the ***flow*** of X to be the mapping $\Phi^X\colon \mathrm{D}(X) \to \mathsf{M}$ by requiring that

$$t \mapsto \Phi^X(t,t_0,x,p)$$

be the maximal integral curves for X^p with initial condition x at time t_0. One then wants to know what are the regularity properties of Φ^X. One can show that these regularity properties are as follows:

1. Φ^X is continuous;
2. $t \mapsto \Phi^X(t,t_0,x,p)$ is locally absolutely continuous;
3. $t_0 \mapsto \Phi^X(t,t_0,x,p)$ is locally absolutely continuous;
4. $x \mapsto \Phi^X(t,t_0,x,p)$ is a C^{ν}-diffeomorphism onto its image.

We do this as follows. Let $(t_0,x_0,p_0) \in \mathbb{T}\times\mathsf{M}\times\mathcal{P}$. Denote by $J(t_0,x_0,p_0) \subseteq \mathbb{T}$ the set of $t \in \mathbb{T}$ such that, for each $t' \in |t_0,t|$, there exists a relatively open interval $J \subseteq \mathbb{T}$, a neighbourhood $\mathcal{U}$ of x_0, and a neighbourhood $\mathcal{O}$ of p_0 such that

1. $t' \in J$,
2. $J \times \{t_0\} \times \mathcal{U} \times \mathcal{O} \subseteq \mathrm{D}(X)$,
3. $J \times \mathcal{U} \times \mathcal{O} \ni (t,x,p) \mapsto \Phi^X(t,t_0,x,p) \in \mathsf{M}$ is continuous, and
4. for each $t \in J$ and $p \in \mathcal{O}$, $\mathcal{U} \ni x \mapsto \Phi^X(t,t_0,x,p)$ is a C^{ν}-diffeomorphism onto its image.

One wishes to show that

$$\sup J = \sup\{t \in \mathbb{T} \mid (t,t_0,x_0,p_0) \in \mathrm{D}(X)\}, \quad \inf J = \inf\{t \in \mathbb{T} \mid (t,t_0,x_0,p_0) \in \mathrm{D}(X)\}.$$

This is achieved by contradiction, supposing that $t_1 \triangleq \sup J$ satisfies

$$t_1 < \sup\{t \in \mathbb{T} \mid (t,t_0,x_0,p_0) \in \mathrm{D}(X)\}$$

(a similar argument applies for $\inf J$). One then chooses C^{ν}-Picard data for X at $(t_1,\Phi^X(t_1,t_0,x_0,p_0),p_0)$ to arrive at a contradiction by virtue of the properties of local flows.

Smoothness of local flows in spaces of vector fields and mappings. A different variant of the properties of global flows concerns the smoothness of the map sending $X \in \Gamma^{\nu}_{\mathrm{LI}}(\mathbb{T};\mathsf{TM})$ to the mapping

$$((t,x) \mapsto \Phi^X(t,t_0,x)) \in \mathrm{C}^{\nu}_{\mathrm{LAC}}(\mathbb{T};(\mathsf{M};\mathsf{M})).$$

Of course, as we have already pointed out, this map is not well-defined when M is not compact. In the compact case, this mapping is known to be smooth for the smooth and real analytic regularity [Glöckner 2015, Glöckner 2023]. In this compact case, the problems described are closely connected to the notion of regularity of a Lie group introduced by Milnor [1984]. This notion of regularity refers to the existence of a smooth evolution map from a curve in the Lie algebra to a curve in the group. Thus this is related to the existence of an exponential map for the group.

In the noncompact situation, a few difficulties arise. First of all, as we have already observed, vector fields are generally not complete so they will have a restricted domain. Second and relatedly, the group of diffeomorphisms is problematic for noncompact manifolds, cf. [Neeb 2006, page 296]. Therefore, one must modify the form of the results, and one way to do this is to consider the restriction of the flow to a compact subset of initial conditions. The development of this is ongoing work of the author and H. Glöckner.

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