Equations of motion for nonholonomic systems

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Slide 0

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Notes for Slide 0

I cannot think of a better short (not that it's all that short as is) title that really describes what I am going to say here, so this is what you get! The talk will have three parts, two of which are related, and the other being directly related to the other two only in that it talks about nonholonomic mechanics. The latter part of the talk will deal with the use of affine connections for representing the equations of motion for nonholonomically constrained systems. This is then necessarily restricted to systems with simple Lagrangians (i.e., kinetic energy minus potential energy), and constraints which are linear in the velocities. The two related parts of the talk, about the Gibbs-Appell equations and nonlinear variational principles, deal with a more general class of systems.

Affine Connections in Nonholonomic Mechanics

Slide 1

- Data:
  - configuration manifold Q;
  - Riemannian metric g on Q ($\overset{g}{\nabla}$ is the Levi-Civita connection);
  - distribution D on Q ($\mathscr{D}$ is sections of D, $P\colon TQ \to TQ$ is orthogonal projection onto $D^{\perp}$).
- Can also include potential energy easily, but let's not.

Notes for Slide 1

What we discuss in this section seems to have originated with Synge [1928]. Other authors have picked up on the idea [Bloch and Crouch 1995, Cattaneo 1963, Cattaneo-Gasparini 1963, Vershik 1984].

The addition of potential energy is rather simple, but for the purposes of this talk inhibits the discussion of the geometry which is the essentially interesting feature.

Derivation of equations

- The Lagrange-d'Alembert Principle gives equations as

$$\overset{g}{\nabla}_{\dot{c}(t)}\dot{c}(t) = \lambda(t) \in D^{\perp}_{c(t)} \tag{1a}$$
$$P(\dot{c}(t)) = 0 \tag{1b}$$

Slide 2

- Differentiate (1b) along c:

$$P(\overset{g}{\nabla}_{\dot{c}(t)}\dot{c}(t)) = -(\overset{g}{\nabla}_{\dot{c}(t)}P)(\dot{c}(t)).$$

- Apply P to (1a):

$$P(\overset{g}{\nabla}_{\dot{c}(t)}\dot{c}(t)) = \lambda(t).$$

- $\implies \lambda(t) = -(\overset{g}{\nabla}_{\dot{c}(t)}P)(\dot{c}(t)).$

Notes for Slide 2

Of course λ is simply the Lagrange multiplier.

Note that the constraint equation (1b) is not the only way to ask that the solution curves evolve in D. For example, we can replace P with AP for any invertible $(1,1)$ tensor field A. Furthermore, much of what we say will not change if such a modification is made. However, for the sake of concreteness, we stick to the description above.

Slide 3

- Solutions of (1) are in 1–1 correspondence with geodesics with initial velocities in D of the affine connection ∇ defined by

$$\nabla_X Y = \overset{g}{\nabla}_X Y + (\overset{g}{\nabla}_X P)(Y).$$

- We study the geometry of ∇ and some of its affine transformations.

Notes for Slide 3

As we shall see, it makes sense to restrict ∇ to D, that is, the geodesics of ∇ with initial velocities in D evolve so that all subsequent velocities are in D. Note that we are only concerned with those geodesics of ∇ whose initial velocities lie in D. This is the reason why we have the flexibility of replacing P with AP as mentioned above. This will only affect those geodesics whose initial velocities are not in D.

The Geometry of ∇

Slide 4

- $\nabla_X Y \in \mathscr{D}$ for $Y \in \mathscr{D}$. Thus ∇ restricts to a vector bundle connection in D (so have parallel translation, etc. in D).
- $(\nabla_X P)(Y) = \nabla_X Y - \overset{g}{\nabla}_X Y \in \mathscr{D}^\perp$ for $Y \in \mathscr{D}$.
- Hmm. . . second fundamental form. . .
  - Classically, a submanifold M of Q is *totally geodesic* if geodesics of $\overset{g}{\nabla}$ starting tangent to M remain on M $\iff$ the second fundamental form of M is zero.
  - Define $S_D \colon D \times D \to D^\perp$ by $S_D(X, Y) = (\nabla_X P)(Y)$. Call S_D the *second fundamental form* of D.

Notes for Slide 4

The properties $\nabla_X Y \in \mathscr{D}$ and $\nabla_X Y - \overset{g}{\nabla}_X Y \in \mathscr{D}^\perp$ for $Y \in \mathscr{D}$ are adequate to provide the affine connection we need. That is to say, any affine connection with these properties (and there are a lot of them) will give us the geodesics we want.

The reader will recall that the second fundamental form for a submanifold of a M Riemannian manifold (Q, g) is simply the normal component of the covariant derivative (with respect to the original Levi-Civita connection) of vector fields restricted to M. Clearly our definition generalises this.

Slide 5

- ∘ D is *geodesically invariant* if for every geodesic $c\colon [a,b] \to Q$ of $\overset{g}{\nabla}$, $\dot{c}(a) \in D_{c(a)}$ implies $\dot{c}(t) \in D_{c(t)}$ for $t \in\,]a,b]$
  - ∘ One can show that D is geodesically invariant if and only if S_D is skew-symmetric.
- ∇ is most interesting when it has torsion. Indeed, if ∇ has zero torsion, then D is integrable.

Notes for Slide 5

For more about geodesic invariance, we refer to [Lewis 1998].

The fact that D is integrable if ∇ is torsion-free follows easily. Since ∇ restricts to D, for $X, Y \in \mathscr{D}$ we have $[X,Y] = \nabla_X Y - \nabla_Y X \in \mathscr{D}$.

Transformations and Conservation Laws

- Transformations (i.e., diffeomorphisms or vector fields):
  - Especially interesting are those transformations of ∇ which also respect D as these will have "physical" meaning.
  - A subgroup (or subalgebra) of such transformations are those respecting both $\overset{g}{\nabla}$ and D. Others?

Slide 6

- Conservation Laws: For $X \in \mathscr{T}(Q)$ define $\boldsymbol{J}_X : TQ \to \mathbb{R}$ by $\boldsymbol{J}_X(v_q) = g_q(X(q), v_q)$.
  - If $X \in \mathscr{D}$ is a Killing field for g then $\boldsymbol{J}_X$ is constant along geodesics of ∇.

Notes for Slide 6

Recall that a diffeomorphism is a transformation for ∇ is it maps geodesics of ∇ to other geodesics. And, as usual, a vector field is an infinitesimal transformation of ∇ if its flow consists of a one-parameter of transformations of ∇.

When we say "physical meaning" for transformations, we mean that they map solutions to other solutions, at least in the case of diffeomorphisms.

The fact that transformations of $\overset{g}{\nabla}$ and D are also transformations for ∇ is something one has to prove, but is completely expected from the point of view of the physics. It is not clear how many more transformations are present.

Note that for a Killing vector field X of g to give rise to a conserved quantity, it need only be true that it be a section of D. It may not be that X is a transformation of D. Thus, vector fields may give rise to conserved quantities and yet not be transformations for all the problem data.

Slide 7

- If $X_1, \ldots, X_m$ are Killing fields for g and $\gamma^1, \ldots, \gamma^m$ are functions on Q so that

$$Y \triangleq \gamma^1 X_1 + \cdots + \gamma^m X_m \in \mathscr{D},$$

  then

$$\frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{J}_Y(\dot{c}(t)) = g(\dot{\gamma}^a(t) X_a(c(t)), \dot{c}(t)).$$

  This is the "momentum equation."

Notes for Slide 7

Note that in the momentum equation construction we do not ask for the vector fields $X_1, \ldots, X_m$ to be transformations for D. Thus both of our conservation laws arise from what are potentially only partial symmetries of the problem.

Things To Do

Slide 8

- Symmetries:
  - Understand the distinction between transformations for the problem and Killing fields which give rise to conserved quantities (should the idea of a transformation be enlarged?).
  - Reduction.
- There is freedom in the choice of affine connection which was not mentioned in the presentation. How may this be used? (Energy preserving affine connections.)

Notes for Slide 8

As a simple example of where conservation laws are not understood (at least by me), consider the Heisenberg system which has as conserved quantities

$$J_{X_2}, \qquad J_{X_1+X_4}, \qquad J_{X_1-X_6}, \qquad J_{X_4+X_6}.$$

where $X_1 = \frac{\partial}{\partial x}$, $X_2 = \frac{\partial}{\partial y}$, $X_3 = \frac{\partial}{\partial z}$, $X_4 = -z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}$, $X_5 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}$, and $X_6 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$. The first conserved quantity comes by virtue of one of the general results we stated, but the other three do not fit into a general scheme of which I am aware.

It is interesting to consider whether the affine connection structure can be used to assist in understanding reduction for these systems. This is future work.

When doing computations, I often make use of one of the affine connections defined by using AP instead of P as it makes the computations easier. However, there is no sound theoretical reason behind these choices, but rather they are made after some intermediate computations are made.

One can show is that it is possible to define an affine connection with the following properties [Lewis 1997a]: (I) the geodesics whose initial velocities are in D are solutions of the constrained system (1), and (II) the kinetic energy is preserved along the integral curves of the corresponding geodesic spray. Further, this connection may be shown to have the property that transformations of both g and D are full affine transformations (i.e., on all of TQ).

The Gibbs-Appell Equations

The Classical Construction

Slide 9

- For a point mass in $\mathbb{R}^3$ define $G = \frac{1}{2}m\|\ddot{\boldsymbol{x}}\|^2$. The unconstrained equations of motion are
$$\frac{\partial G}{\partial \ddot{\boldsymbol{x}}} = 0.$$
- Add a constraint as a one-form ω on $\mathbb{R}^3$. Let X_1 and X_2 be generators for $\ker(\omega)$.

Notes for Slide 9

To illustrate the classical version of the Gibbs-Appell equations, we use a simple example. The classical "theory" is more of a methodology in any case — one of those instances where a few examples makes the method clear in principle. The Gibbs-Appell equations are discussed by Pars [1965].

Of course the Gibbs-Appell method applied to a point mass is quite ridiculous. It is its applicability with constraints which makes it interesting. Also, it is interesting, as we shall see, to simply try to adapt the unconstrained method to general Lagrangians. The classical method we present only works for point masses and rigid bodies. The rigid body "Gibbs function" is

$$G = \frac{1}{2}\|\ddot{\boldsymbol{x}}\|^2 + \frac{1}{2}\big(I_1\dot{\omega}_1^2 + I_2\dot{\omega}_2^2 + I_3\dot{\omega}_3^2 + 2(I_3 - I_2)\omega_2\omega_3\dot{\omega}_1 + \\ 2(I_1 - I_3)\omega_3\omega_1\dot{\omega}_2 + 2(I_2 - I_1)\omega_1\omega_2\dot{\omega}_3\big)$$

where $\boldsymbol{x}$ is the position of the centre of mass and $\boldsymbol{\omega}$ is the body angular velocity.

Slide 10

- Let $(\nu^1, \nu^2) \mapsto \nu^1 X_1 + \nu^2 X_2$ be the inclusion (gives $(\dot{x}, \dot{y}, \dot{z})$ as functions of (ν^1, ν^2)).
- Obtain expressions for $(\ddot{x}, \ddot{y}, \ddot{z})$ as functions of (ν^1, ν^2) and $(\dot{\nu}^1, \dot{\nu}^2)$.
- Substitute the accelerations into $G = \frac{1}{2}\|\ddot{\boldsymbol{x}}\|^2$.
- Punchline: the Lagrange-d'Alembert principle is equivalent to the *Gibbs-Appell equations*:

$$\frac{\partial G}{\partial \dot{\boldsymbol{\nu}}} = 0.$$

- Now geometrise this.

Notes for Slide 10

The addition of external forces into the classical Gibbs-Appell methodology is readily accomplished, but we will present it neither here nor for our geometrical construction which follows. It is a simple encumberment.

The geometrisation of the classical Gibbs-Appell equations has several facets. First one must determine a suitable candidate "Gibbs function" for general Lagrangians. One must then make sense of differentiation with respect to acceleration. Finally, constraints must be added in to the formulation in the proper manner. This is accomplished in a jet bundle setting for Lagrangian mechanics [Giachetta 1992, Lewis 1996].

The Gibbs-Appell Equations for Unconstrained Systems

Slide 11

- Let $\pi\colon \mathscr{Q} \to \mathbb{R}$ be a locally trivial fibre bundle with $J^k\mathscr{Q}$ the bundle of k-jets of sections. Use coordinates (t, q^i, v^j, a^k) for $J^2\mathscr{Q}$.
- Let E be the pull-back of $V\mathscr{Q} \triangleq \ker(T\pi) \to \mathscr{Q}$ to $J^1\mathscr{Q} \to \mathscr{Q}$.
- Recall that $J^k\mathscr{Q} \to J^{k-1}\mathscr{Q}$ is an affine bundle with the fibres being affine spaces modelled on the fibres of $V\mathscr{Q}$. Also recall $J^k\mathscr{Q} \subset T(J^{k-1}\mathscr{Q})$.
- Let L be a function on $J^1\mathscr{Q}$ so that the matrix
$$\frac{\partial^2 L}{\partial v^i \partial v^j}$$
is nondegenerate (i.e., L is regular).

Notes for Slide 11

We do not work with trivial bundles as the nontrivial setting enables one to better come to grips with the geometry which is useful in the general formulation of the Gibbs-Appell equations. In this setting time and configuration are interwoven and their is no natural notion of time-independent. In order to talk about time-independence, one must work with a trivial bundle $\mathscr{Q} = \mathbb{R} \times Q$.

Note that E is a vector bundle over $J^1\mathscr{Q}$. It is the pull-back described by the following diagram:

$$\begin{tikzcd} E \arrow[r] \arrow[d, "\mathrm{pr}_1"'] & J^1\mathscr{Q} \arrow[d] \\ V\mathscr{Q} \arrow[r] & \mathscr{Q} \end{tikzcd}$$

Further, it is isomorphic to the kernel of the derivative of the projection $J^1\mathscr{Q} \to \mathscr{Q}$ and so is a subbundle of $T(J^1\mathscr{Q})$.

One may verify that the object
$$\frac{\partial^2 L}{\partial v^i \partial v^j}$$
is intrinsic and may be thought of as a symmetric bilinear form on the fibres of E.

- The Lagrangian vector field associated with L is a vector field on $T(J^1\mathscr{Q})$, but may in fact be thought of as a map $\xi_L \colon J^1\mathscr{Q} \to J^2\mathscr{Q} \subset T(J^1\mathscr{Q})$.

- The *Gibbs function* is the function on $J^2\mathscr{Q}$ given by

$$G_L(t,q,v,a) = \frac{1}{2}\frac{\partial^2 L}{\partial v^i \partial v^j}(\xi_L^i - a^i)(\xi_L^j - a^j).$$

Slide 12

- It is fairly obvious that the Euler-Lagrange equations are equivalent to the *unconstrained Gibbs-Appell equations*:

$$\boldsymbol{d}_2 G_L \triangleq \frac{\partial G_L}{\partial \boldsymbol{a}} = 0.$$

- The definition of G_L and the Gibbs-Appell equations are intrinsic because of the affine structure of $J^2\mathscr{Q}$.

Notes for Slide 12

We are being a bit sly here and skipping some details which make the presentation intrinsic. But everything does, in fact, work. When we write $\xi_L - a$ we mean subtraction as done in an affine space so the result is naturally a point in the model vector space. It thus makes sense to apply to this the quadratic form $\frac{\partial^2 L}{\partial v^i \partial v^j}$. In a similar manner, one uses the affine structure to define $\boldsymbol{d}_2 G_L$. It may then be regarded in several ways, one being as a morphism from $J^2\mathscr{Q}$ to E^* so that the following diagram commutes:

$$\begin{tikzcd}
J^2\mathscr{Q} \arrow[rr, "\boldsymbol{d}_2 G_L"] \arrow[dr] & & E^* \arrow[dl] \\
& J^1\mathscr{Q} &
\end{tikzcd}$$

The Gibbs-Appell Equations for Constrained Systems

Slide 13

- Define a $(1,1)$ tensor field on $J^1\mathscr{Q}$ by $S = \frac{\partial}{\partial v^i} \otimes (\mathrm{d}q^i - v^i \mathrm{d}t)$.
- Let Λ be a codistribution on $J^1\mathscr{Q}$ so that $\dim(\Lambda) = \dim(S^*\Lambda)$.
- Let $C = \mathrm{coann}(\Lambda) \cap J^2\mathscr{Q}$. C is an affine subbundle of $J^2\mathscr{Q}$. Let $\bar{C}$ be the vector subbundle of E upon which C is modelled.
- Let $i\colon \bar{C} \to E$ be the inclusion (a bundle map over $J^1\mathscr{Q}$).
- The *constrained Gibbs-Appell equations*,

$$i^*(\boldsymbol{d}_2 G_L) = 0,$$

agree with the Lagrange-d'Alembert principle where both apply.

Notes for Slide 13

The $(1,1)$ tensor field S is the generalisation to $J^1\mathscr{Q}$ of the almost tangent structure on the tangent bundle of a manifold.

Note that we allow a rather general class of constraints. In fact, it is most natural in this context to allow this general type of constraint. To fit standard linear constraints (i.e., specified by a codistribution on $\mathscr{Q}$) into this framework, one needs to differentiate them once to arrive in the setting we describe here. It is also true that linear constraints, after differentiation, satisfy the condition $\dim(\Lambda) = \dim(S^*\Lambda)$.

The fact that C is an affine subbundle of $J^2\mathscr{Q}$ is a consequence of our asking that $\dim(\Lambda) = \dim(S^*\Lambda)$. This condition is a natural one, in the same way that asking a linear constraint distribution to have constant rank is natural.

The Gibbs Function for Riemannian Manifolds

Slide 14

- Let $\mathscr{Q} = \mathbb{R} \times Q$ and let g be a Riemannian metric on Q.
- TQ inherits a natural Riemannian metric g^{TQ} from Q (Sasaki).
- Let I be the natural involution of TTQ.
  $J^2\mathscr{Q} \simeq \mathbb{R} \times \mathrm{Fix}(I) \subset \mathbb{R} \times TTQ$.
- Let $L(t, v) = \frac{1}{2}g(v, v)$ (kinetic energy).
- Let $L^{TQ}(t, w) = \frac{1}{2}g^{TQ}(w, w)$ (kinetic energy wrt Sasaki restricted to $J^2\mathscr{Q}$).
- $G_L(t, w) = L^{TQ}(t, w) - L(t, \pi_{TTQ}(w))$ (second term doesn't depend on acceleration).

Notes for Slide 14

For more detail on the Sasaki metric, we refer to [Sasaki 1958].

Note that we work in the time-dependent trivial setting here as this is natural. In doing so we are able to make an identification of $J^2\mathscr{Q}$ with a standard tangent bundle like object that we are perhaps more familiar with. This cannot be done in the general nontrivial setting.

Note that the Gibbs function G_L in this case differs from the Sasaki kinetic energy L^{TQ} only by a term which goes away upon application of $\boldsymbol{d}_2$. Therefore, for practical purposes, we may use L^{TQ} as a Gibbs function for this system.

Things To Do

Slide 15

- Examples with nonlinear constraints.
- Symmetry and reduction.

Notes for Slide 15

One of the advantages of the Gibbs-Appell formulation is that it naturally handles nonlinear constraints. It would be interesting to come up with, and analyse, problems which utilise this generality. Jerry has mentioned systems of particles which preserve kinetic energy.

It might be interesting to see if the jet bundle structure appears in a reduction methodology for these systems. The interaction of all the various bundles (affine and otherwise) with the symmetry is something unexplored.

Variational Principles for Nonlinear Constraints

Gauss's Principle of Least Constraint

- Same setting as Gibbs-Appell equations: $\pi\colon \mathscr{Q} \to \mathbb{R}$ and $\Lambda \subset T^*(J^1\mathscr{Q})$ such that $\dim(\Lambda) = \dim(S^*\Lambda)$.
- Equations of motion determined by specifying the acceleration.

Slide 16

- Fix $j^1c(t) \in J^1\mathscr{Q}$ and consider the quadratic function
$$G_L(j^2c(t)) = \frac{1}{2}\frac{\partial^2 L}{\partial v^i \partial v^j}(\xi_L^i - a^i)(\xi_L^j - a^j)$$
on the fibre of $C = \mathrm{coann}(\Lambda) \cap J^2\mathscr{Q}$ over $j^1c(t)$.
- The physical acceleration at $j^1c(t)$ is the unique critical point of this quadratic function.

Notes for Slide 16

In Gauss's Principle one fixes $(t, q, v) \in J^1\mathscr{Q}$ and asks which value of a in the fibre over this point gives the correct equations of motion. By correct we mean "agreeing with the Gibbs-Appell equations."

Note that since the fibres of $J^2\mathscr{Q}$ are affine spaces, and since C is an affine subbundle, it makes sense to say that a function defined there is quadratic. Clearly such a function will have a unique critical point. If we make some definiteness assumptions on the Lagrangian, the critical point will also be a minima. Note that this reduces finding the physical motions to a finite-dimensional constrained problem.

A Generalisation of the Lagrange-d'Alembert Principle

Slide 17

- Recall the Lagrange-d'Alembert principle. If the constraints are specified by a distribution D on Q, the equations of motions are defined by
$$\delta \int_{t_1}^{t_2} (\dot{c})^* L \, \mathrm{d}t = 0$$
for admissible variations δ (i.e., those taking values in D).
- The specification consists of two parts:
  - saying what the constraints are;
  - saying what an admissible variation looks like.
- Adapt this philosophy to a general situation.

Notes for Slide 17

That we must specify the nature of the constraints is clear. It is perhaps somewhat less clear that the Lagrange-d'Alembert principle, and the Principle of Virtual Work also, are simply ways of prescribing what an admissible variation should look like. This is perhaps best seen by equating these principles with the multiplier method. After all, a Lagrange multiplier is nothing other that a force added to the equations which annihilates the admissible variations.

Slide 18

- Again work with $\pi\colon \mathscr{Q} \to \mathbb{R}$ and constraints modelled by a codistribution Λ with $\dim(\Lambda) = \dim(S^*\Lambda)$.
- Again define $C = \operatorname{coann}(\Lambda) \cap J^2\mathscr{Q}$ which is modelled on $\bar{C} \subset E$.
- Define admissible variations to be sections of $(j^1c)^*\bar{C}$ (this makes sense!).
- A local section $c\colon [t_1, t_2] \to \mathscr{Q}$ of π is a solution of the constrained problem (e.g., a solution of the Gibbs-Appell equations) if and only if

$$\delta \int_{t_1}^{t_2} (j^1c)^* L \, \mathrm{d}t = 0$$

for admissible variations.

Notes for Slide 18

At no point have we been precise by what we mean by a solution of the constrained problem in the case where the constraints are general. Indeed, there are a few equivalent ways of doing this, two being the Gibbs-Appell equations and the variational principle we describe here.

It needs a second of thought to see that admissible variations are in fact variations of a section c. That they are is explained by Lewis [1997b].

Slide 19

Symmetries

- There is a Noether Theorem: Lift a $V\mathcal{Q}$-valued vector field X on $\mathcal{Q}$ to X^{lift} on $J^1\mathcal{Q}$. If X^{lift} is a section of $\bar{C}$ then the momentum $\frac{\partial L}{\partial v^i}X^i$ is conserved.
- There is a momentum equation if the infinitesimal symmetry X does not give rise to a section of $\bar{C}$.

Notes for Slide 19

The Noether Theorem we state is also proved by Lewis [1997b]. It really looks a lot like the normal Noether Theorem, but one has to take into account the constraints and the jet bundle geometry. In particular, one should take a moment to see that the definition of momentum as given makes sense.

The momentum equation will look a lot like that of Bloch, Krishnaprasad, Marsden, and Murray [1996], but I haven't gone through it formally. Perhaps the jet bundle structure will help in understanding some of the geometry involved.

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