When is a mechanical control system kinematic?

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What's the problem?

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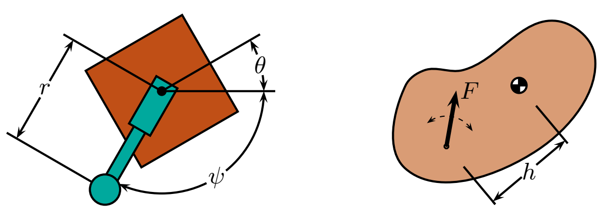
- Consider two mechanical control systems:



- Think about starting at rest and then applying controls.
- Do the systems behave somehow "differently"?
- Are either, in any sense, "nonholonomic (driftless) control systems"?

1. Are all mechanical control systems "nonholonomic" (i.e., driftless) systems?

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- Short answer: *No!* Longer answer requires defining the words...
- "Nonholonomic" control systems are control affine systems with no drift:

$$\dot{q}(t) = u_1(t)X_1(q(t)) + \ldots + u_m(t)X_m(q(t))$$

for vector fields $X_1, \ldots, X_m$.

- What is a mechanical control system? The data for such as we will consider is:
  1. a configuration manifold Q;
  2. a Riemannian metric g (kinetic energy);
  3. (possibly) a distribution D on Q describing nonholonomic velocity constraints.

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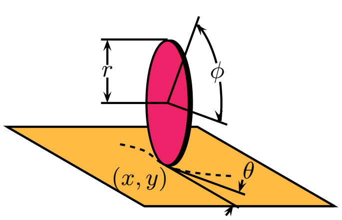
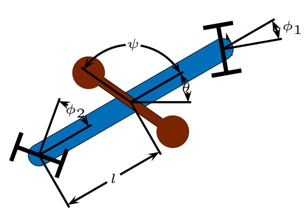
- The two earlier examples are unconstrained systems.
- Two nonholonomically constrained systems are:





- Do *these* control systems behave somehow "differently" from one another?
- Are either, in any sense, "nonholonomic (driftless) control systems"?

2. Affine connection control systems

- The four mechanical examples we have seen (leg, planar body, rolling disk, snakeboard) are examples of a special and interesting class of control systems.
- Consider systems with no constraints—write their equations of motion.

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- Use coordinates $(q^1, \dots, q^n)$ for Q and write the Euler-Lagrange equations for the Lagrangian $L(q, \dot{q}) = \frac{1}{2} g_{ij}(q) \dot{q}^i \dot{q}^j$:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\partial L}{\partial \dot{q}^i}\Big) - \frac{\partial L}{\partial q^i} &= g_{ij}\Big[\ddot{q}^j + g^{jk}\Big(\frac{\partial g_{k\ell}}{\partial q^m} - \frac{1}{2}\frac{\partial g_{\ell m}}{\partial q^k}\Big)\dot{q}^\ell \dot{q}^m\Big] \\ &= g_{ij}\Big[\ddot{q}^j + \overset{g}{\Gamma}{}^j_{\ell m}\dot{q}^\ell \dot{q}^m\Big]. \end{aligned}$$

- The n^3 functions

$$\overset{g}{\Gamma}{}^i_{jk} = \frac{1}{2} g^{i\ell}\Big(\frac{\partial g_{\ell j}}{\partial q^k} + \frac{\partial g_{\ell k}}{\partial q^j} - \frac{\partial g_{jk}}{\partial q^\ell}\Big)$$

  are the *Christoffel symbols* for the *Levi-Civita affine connection* associated with the metric g.

- Denote this affine connection by $\overset{g}{\nabla}$—we then have

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$$\begin{aligned} &\ddot{q}^i + \overset{g}{\Gamma}{}^i_{jk}\dot{q}^j \dot{q}^k = 0, \quad i = 1, \dots, n \\ \iff \quad &\overset{g}{\nabla}_{\dot{q}(t)}\dot{q}(t) = 0 \quad \sim \quad \text{mass}\times\text{acceleration} = 0 \end{aligned}$$

  These are the unforced equations (i.e., *geodesic equations*).

- *Not completely trivial fact:* For systems with nonholonomic constraints, the unforced equations are still geodesic equations, but with respect to a more complicated affine connection (Synge [1928], Bloch and Crouch [1995]).

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- Punchline: we study *affine connection control systems:*

$$\nabla_{\dot{q}(t)}\dot{q}(t) = u_1(t)Y_1(q(t)) + \ldots + u_m(t)Y_m(t) \qquad \text{(DYN)}$$

  for general affine connections ∇ and input vector fields $Y_1, \ldots, Y_m$.

- We are interested in when such a system is "equivalent" to one like

$$\dot{q}(t) = \tilde{u}_1(t)X_1(q(t)) + \ldots + \tilde{u}_{\tilde{m}}(t)X_{\tilde{m}}(t). \qquad \text{(KIN)}$$

- One must define "equivalent" properly—for example, the systems (DYN) and (KIN) are *never* equivalent with the same class of inputs.

3. $(\mathcal{U}_{\mathrm{dyn}}, \mathcal{U}_{\mathrm{kin}})$-reducibility

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- We define a pair of suitable classes of inputs:
  1. $\mathcal{U}_{\mathrm{dyn}}$ are bounded, measurable inputs;
  2. $\mathcal{U}_{\mathrm{kin}}$ are absolutely continuous inputs.
- Inputs from $\mathcal{U}_{\mathrm{kin}}$ are "one level smoother" than those from $\mathcal{U}_{\mathrm{dyn}}$.
- Let us define a pair of distributions corresponding to systems (DYN) and (KIN):
  1. $D_{\mathrm{dyn}} = \mathrm{span}\{Y_1, \ldots, Y_m\}$;
  2. $D_{\mathrm{kin}} = \mathrm{span}\{X_1, \ldots, X_{\tilde{m}}\}$.

- **Definition 1** An affine connection control system (DYN) is $(\mathcal{U}_{\rm dyn},\mathcal{U}_{\rm kin})$*-reducible* to a driftless system (KIN) if the following two conditions hold:

  (i) for each controlled trajectory (σ, u) of (DYN) with $u \in \mathcal{U}_{\rm dyn}$ and with initial condition $\sigma(0)$ in the distribution $D_{\rm kin}$, there exists a controlled trajectory $(c, \tilde{u})$ of (KIN) with $\tilde{u} \in \mathcal{U}_{\rm kin}$ and with the property that $c = \tau_Q \circ \sigma$;

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  (ii) for each controlled trajectory $(c, \tilde{u})$ of (KIN) with $\tilde{u} \in \mathcal{U}_{\rm kin}$, there exists a controlled trajectory (σ, u) of (DYN) with $u \in \mathcal{U}_{\rm dyn}$ and with the property that $\sigma(t) = c'(t)$ for a.e. $t \in [0, T]$.

- This is as good as one might expect—it is impossible that *all* trajectories of (DYN) be lifts of trajectories of (KIN) since the latter always has velocities in $D_{\rm kin}$.

4. Conditions for $(\mathcal{U}_{\rm dyn},\mathcal{U}_{\rm kin})$-reducibility

- We need a simple operation associated with ∇—the *symmetric product*:

$$\langle X : Y\rangle = \nabla_X Y + \nabla_Y X.$$

- In coordinates

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$$\big(\nabla_X Y\big)^i = \frac{\partial Y^i}{\partial q^j} X^j + \Gamma^i_{jk} X^j Y^k, \quad i = 1, \dots, n.$$

- **Theorem 1** *The affine connection control system (DYN) is* $(\mathcal{U}_{\rm dyn},\mathcal{U}_{\rm kin})$*-reducible to a system of the form (KIN) if and only if the following two conditions hold:*

  (i) $D_{\rm dyn} = D_{\rm kin}$*;*

  (ii) $\langle X : Y\rangle \in \Gamma^\infty(D_{\rm dyn})$ *for every* $X, Y \in \Gamma^\infty(D_{\rm dyn})$*.*

- Condition (ii) of the theorem has a geometric interpretation.
- ADL [1998] shows that condition (ii) holds if and only if $D_{\mathrm{dyn}} \subset TQ$ is invariant under the unforced dynamics of the affine connection control system (DYN).

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- The symmetric product has a rôle sort of like that of the Lie bracket:
  - a distribution D is integrable if and only if it is closed under Lie bracket (Frobenius's theorem);
  - a distribution D is "invariant under the geodesic dynamics" if and only if it is closed under symmetric product.

5. Examples again

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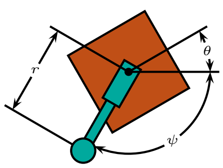
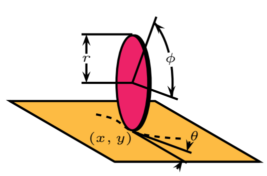
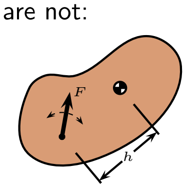
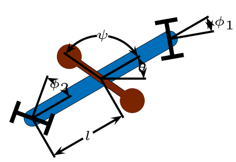
- Of the four systems we have looked at, these two are $(\mathcal{U}_{\mathrm{dyn}}, \mathcal{U}_{\mathrm{kin}})$-reducible:





- These two are not:





6. What to conclude?

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- Not very many mechanical systems are $(\mathcal{U}_{\mathrm{dyn}}, \mathcal{U}_{\mathrm{kin}})$-reducible (since a generic distribution is not closed under $\langle \cdot : \cdot \rangle$).
- Those which *are* $(\mathcal{U}_{\mathrm{dyn}}, \mathcal{U}_{\mathrm{kin}})$-reducible are very likely amenable to more simplified control techniques.
- *Our theorem is another indication of the utility of the affine connection formalism in investigating the types of mechanical control systems we have discussed today.**

*See ADL/Murray, *SIAM Review*, **41**(3), 555-574.