Affine connection control systems
(probably just optimal control)

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1. What are affine connection control systems?

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- Shortly, they are this:
  1. a configuration manifold Q;
  2. an affine connection ∇ on Q;
  3. a collection $\mathscr{Y} = \{Y_1, \dots, Y_m\}$ of vector fields on Q.
- The corresponding control system is

$$\nabla_{c'(t)} c'(t) = u^a(t) Y_a(c(t))$$

for a controlled trajectory (u, c).

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- Examples of affine connection control systems:
  1. Lagrangian systems with kinetic energy Lagrangians (∇ is the Levi-Civita connection for the kinetic energy Riemannian metric). For example, (some of these need potential energy)
     - satellites,
     - robotic manipulators,
     - underwater vehicles, etc.
  2. Same as above with the addition of constraints linear in velocity. For example,
     - locomotion systems (wheeled vehicles),
     - grasping applications, etc.

2. Why are affine connection control systems interesting?

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- Lots of interesting applications, including some surprisingly subtle "simple" examples.
- The data for the systems, i.e., the affine connection ∇ and the input vector fields $\mathscr{Y}$, is a seemingly nice combination of structural simplicity and challenging geometry.
  - The systems are not at all amenable to linear methods (they are hard).
  - One can get complete answers to some fundamental questions (they are not *too* hard).
- Any area of (nonlinear, of course) control theory with a differential geometric foundation ought to have a specially structured counterpart for affine connection control systems.

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- In this talk we concentrate on two questions:
  1. optimal control;
  2. nonlinear controllability (time permitting).
- Other questions which have been successfully approached include:
  - trajectory generation when Q is a Lie group (Bullo and Leonard);
  - series expansions (Bullo, Ostrowski);
  - vibrational control (Baillieul, Bullo);
  - kinetic shaping using feedback (Bloch et al., Auckly et al., Hamberg)

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Affine connection control systems as control affine systems

- Convert

$$\nabla_{c'(t)}c'(t) = u^a(t)Y_a(c(t))$$

  to control affine system on TQ:

$$\dot{v}(t) = f_0(v(t)) + u^a(t)f_a(v(t)),$$

  $v \in TQ$.
- Turns out that
  1. the drift is the geodesic spray denoted $f_0 = Z$, and
  2. the control vector fields are the vertical lifts of the vectors fields from $\mathscr{Y}$: we write $f_a = Y_a^{\text{lift}}$.

3. The Maximum Principle for affine connection control systems

- Noakes, Heinzinger, Paden, and Crouch, Silva Leite, and Sontag, Sussmann, and Fax, Murray, and Chyba, Leonard, Sontag.
- We shall investigate in a little detail *one* of the several consequences of the Maximum Principle as it applies to affine connection control systems.

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- Start general—let's look at the Maximum Principle for

$$c'(t) = f_0(c(t)) + u^a(t) f_a(c(t)),$$

with $c(t) \in M$, u taking values in $U \subset \mathbb{R}^m$, and objective function $L(x, u)$.

- Have the **control Hamiltonian** on $U \times T^*M$:

$$H(\alpha_x, u) = \underbrace{\alpha_x(f_0(x))}_{H_1} + \underbrace{\alpha_x(u^a f_a(x))}_{H_2} - \underbrace{L(x, u)}_{H_3}.$$

- One of several consequences of the MP is that if (u, c) is a minimiser then there exists a one-form field λ along c with the property that $t \mapsto \lambda(t)$ is an integral curve for the time-dependent Hamiltonian $(\alpha_x, t) \mapsto H(\alpha_x, u(t))$.
- The Hamiltonian is a sum of three terms, and so too will be the Hamiltonian vector field. Let us look at the first term, that with (plain old) Hamiltonian $H_1(\alpha_x) = \alpha_x(f_0(x))$.

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- In local coordinates X_{H_1} is written as

$$\dot{x}^i = f_0^i(x)$$

$$\dot{p}_i = -\frac{\partial f_0^j}{\partial x^i} p_j \qquad \longleftarrow \qquad \textit{“adjoint equation”?}$$

- X_{H_1} is the **cotangent lift** of f_0 and we denote it $f_0^{T^*}$.

- *Objective:* Understand $f_0^{T^*}$ when $M = TQ$ and $f_0 = Z$.
- Begin with a change of subject: Let f_0 be a vector field on (general) M with f_0^T its **tangent lift** defined by

$$f_0^T(v_x) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} T_x F_t(v_x)$$

(F_t is the flow of f_0).

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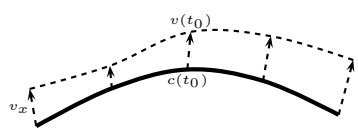
- f_0^T is the "linearisation" of f_0 and in coordinates is given by

$$\begin{aligned} \dot{x}^i &= f_0^i(x) \\ \dot{v}^i &= \frac{\partial f_0^i}{\partial x^j} v^j \end{aligned} \qquad \left(\text{compare } f_0^{T^*}: \quad \begin{aligned} \dot{x}^i &= f_0^i(x) \\ \dot{p}^i &= -\frac{\partial f_0^j}{\partial x^i} p_j \end{aligned} \right)$$

- The flow of f_0^T measure how the integral curves of f_0 change as we change the initial condition in the direction of v_x.

- The general picture you might have in mind for integral curves of f_0^T is this:



- If the integral curve of f_0 is stable to perturbations in the direction of v_x:

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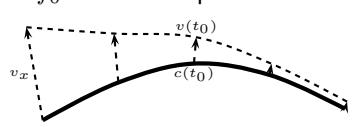

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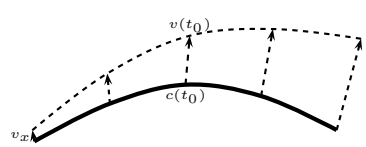

- Perhaps we can understand Z^T—thus take $M = TQ$ and $f_0 = Z$ in the discussion of tangent lift.
- Note:
  - Projections of integral curves of Z to Q are geodesics of ∇.
  - Z^T measures variations of integral curves of Z.
  - Thus Z^T measures variations of geodesics.
  - But we know something else which measures variations of geodesics...

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- Let $c(t)$ be a geodesic. By varying the initial condition for the geodesic we generate an "infinitesimal variation" ξ of the geodesic and it turns out to satisfy... the **Jacobi equation**:

$$\nabla^2_{c'(t)}\xi(t) + R(\xi(t), c'(t))c'(t) + \nabla_{c'(t)}\big(T(\xi(t), c'(t))\big) = 0.$$

- What is the *precise* relationship between Z^T and the Jacobi equation?

Some tangent bundle geometry using Z

- To make the "connection" between Z^T and the Jacobi equation, we perform constructions on the tangent bundle using the spray Z.
- ∇ comes from a linear connection on Q which induces an Ehresmann connection on $\pi_{TQ}\colon TQ \to Q$.
- Thus we may write $T_{v_q}TQ \simeq T_qQ \oplus T_qQ$.

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- Z^T is not a spray, but... if $I_Q\colon TTQ \to TTQ$ is the canonical involution then $I_Q^*Z^T$ *is* a spray.
- Use $I_Q^*Z^T$ to induce an Ehresmann connection on $\pi_{TTQ}\colon TTQ \to TQ$.
- Thus

$$\begin{aligned} T_{X_{v_q}}TTQ &\simeq T_{v_q}TQ \oplus T_{v_q}TQ \\ &\simeq \underbrace{T_qQ \oplus T_qQ}_{\text{geodesic equations}} \oplus \underbrace{T_qQ \oplus T_qQ}_{\text{variation equations}} \end{aligned}$$

- One represents Z^T in this splitting and determines that the Jacobi equation sits "inside" one of the four components.
- Now one applies similar constructions to T^*TQ and Z^{T^*} to derive (all going to plan) a one-form version of the Jacobi equation.

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- Need a little notation:

$$\langle R^*(\alpha, u)v; w\rangle = \langle \alpha; R(w, u)v\rangle, \quad \langle T^*(\alpha, u); w\rangle = \langle \alpha; T(w, u)\rangle.$$

- After the dust settles, we get what we are after which is the **adjoint Jacobi equation**:

$$\nabla^2_{c'(t)}\lambda(t) + R^*(\lambda(t), c'(t))c'(t) - T^*(\nabla_{c'(t)}\lambda(t), c'(t)) = 0.$$

- Why did I do this?
  - The adjoint Jacobi equation captures the interesting part of the Hamiltonian vector field Z^{T^*}, which comes from the MP, and words it in terms of affine differential geometry, i.e.,

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$$Z^{T^*}$$

$$\Updownarrow$$

$$\boxed{\begin{aligned} &\nabla_{c'(t)}c'(t) = 0 \\ &\nabla^2_{c'(t)}\lambda(t) + R^*(\lambda(t), c'(t))c'(t) - T^*(\nabla_{c'(t)}\lambda(t), c'(t)) = 0. \end{aligned}}$$

  - The geometry of Z on TQ provides a way of **globally** pulling out the "adjoint equation" from the MP in an intrinsic manner—this is not generally possible in the MP.

- The adjoint Jacobi equation forms the backbone of a general statement of the MP for affine connection control systems.
  - The contribution of the inputs needs to be added (easy).
  - The contribution of the objective function needs to be added (difficulty depends on the nature of the function).
- When objective function is $L(u, v_q) = \frac{1}{2}g(v_q, v_q)$, when ∇ is the Levi-Civita connection for g, and when the system is fully actuated, then we recover the equation of Noakes, Heinzinger, and Paden and Crouch and Silva Leite:

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$$\nabla^3_{c'(t)}c'(t) + R(\nabla_{c'(t)}c'(t), c'(t)) = 0.$$

- Where to go from here?
  - Work out some examples!
  - Examine conditions for extremals to be nonsingular.
  - Time-optimal control and controllability.
  - Infinite-horizon stabilising controllers.

4. Some controllability results

- The controllability results are for "configuration controllability"—determine the character of the set of configurations reachable from an initial state with zero velocity.
- Results are local $\longrightarrow$ we use the control Lie algebra structure. This structure is very rich!
- Convert to control affine system on TQ:

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$$\dot{v}(t) = f_0(v(t)) + u^a(t)f_a(v(t)),$$

$v \in TQ$.

- Recall that
  1. the drift is the geodesic spray denoted $f_0 = Z$, and
  2. the control vector fields are the vertical lifts of the vectors fields from $\mathscr{Y}$: we write $f_a = Y_a^{\text{lift}}$.

- To evaluate brackets at $0_q \in T_qQ$, note that $T_{0_q}TQ \simeq T_qQ \oplus T_qQ$.
- Given Z, we have seen that for *any* $v_q \in TQ$ we have a decomposition $T_{v_q}TQ \simeq T_qQ \oplus T_qQ$, but that at 0_q is natural.

Some sample brackets

- All brackets $[Y_a^{\mathrm{lift}}, Y_b^{\mathrm{lift}}]$ vanish identically.

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- $[Z, Y_a^{\mathrm{lift}}](0_q) = (-Y_a(q), 0)$.
- Globally we have $[Y_a^{\mathrm{lift}}, [Z, Y_b^{\mathrm{lift}}]] = (0, \langle Y_a : Y_b \rangle)$ where

$$\langle Y_a : Y_b \rangle = \nabla_{Y_a} Y_b + \nabla_{Y_b} Y_a \qquad (\textbf{symmetric product}).$$

- $[[Z, Y_a^{\mathrm{lift}}], [Z, Y_b^{\mathrm{lift}}]](0_q) = ([Y_a, Y_b](q), 0)$.
- *Punchline:* When evaluating brackets at 0_q we get symmetric products (in the vertical direction) and Lie brackets of symmetric products (in the horizontal direction) of vector fields from $\mathscr{Y}$.

- This can be turned into a theorem. Let $\overline{\mathrm{Sym}}(\mathscr{Y})$ be the distribution generated by symmetric products from $\mathscr{Y}$ and let $\overline{\mathrm{Lie}}(\overline{\mathrm{Sym}}(\mathscr{Y}))$ be the involutive closure of $\overline{\mathrm{Sym}}(\mathscr{Y})$.

Theorem 1 *Let $q \in Q$ and let Λ_q be the integral manifold through q of the distribution $\overline{\mathrm{Lie}}(\overline{\mathrm{Sym}}(\mathscr{Y}))$. For an analytic affine connection control system, the set of configurations reachable from $q \in Q$ is contained in Λ_q and contains a nonempty open subset of Λ_q.*

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- If $\dim(\Lambda_q) = \dim(Q)$ then the system is **locally configuration accessible** at q.
- **Local configuration controllability** (i.e., the ability to reach a *neighbourhood* of the initial configuration) is a more subtle question.
  - We have sufficient conditions.
  - When $m = 1$: Local configuration controllability $\iff \dim(Q) = 1$.

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- *Note:* Necessary and sufficient conditions are not known for general single-input systems. . . affine connection control systems have a *very* structured control Lie algebra.
- Perhaps necessary and sufficient conditions for local controllability are possible for multi-input affine connection control systems.
- The sufficient conditions for configuration controllability suggest motion control algorithms which may be implemented, e.g., on Lie groups.
- Controllability away from zero velocity? Involves curvature, i.e., the holonomy of the affine connection.

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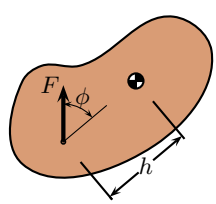
Controllability for a few examples

- *Planar rigid body*



1. ϕ fixed and not $0, \pi$: Locally configuration accessible, but not locally configuration controllable (it is single-input).
2. ϕ fixed at 0 or π: Not locally configuration accessible ($\dim(\Lambda_q) = 1$ for every $q \in Q$).
3. ϕ free to vary: Locally configuration controllable.

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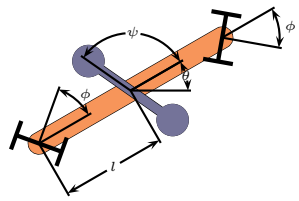
- *Snakeboard*



1. With either single input: Not locally configuration accessible ($\dim(\Lambda_q) = 1$ for almost every $q \in Q$).
2. With both inputs: Locally configuration controllable.

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5. Other things concerning affine connection control systems

- Can affine connection control systems be simplified or be put into a form desirable for certain ends (equivalence and feedback).
- Linear stabilisation methods fail $\Longrightarrow$ can we find nice stabilisation algorithms. Homogeneity useful here?
- Trajectory generation.
- Systematic investigation of effects of symmetry.
- etc. etc.