Applications of differential geometry to mechanical control systems

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1. Some toys

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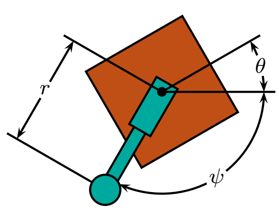

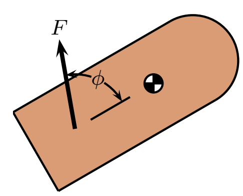

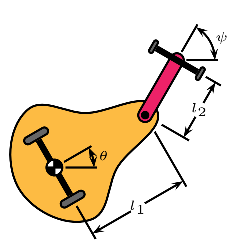

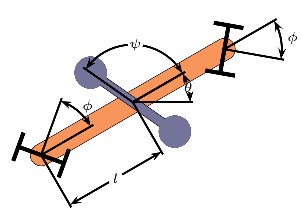

2. What is control theory?

- It is a *huge* field spanning the most applied engineering disciplines to essentially pure mathematics.

2.1. Differential equations

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- Consider a rather general differential equation:

$$\dot{x}(t) = f(t, x(t)), \qquad x(t_0) = x_0.$$

- This can only be "solved" in very special cases: e.g., in the linear case when $f(t, x) = Ax$.
- In the general case, one seeks *qualitative* descriptions: are certain known solutions stable? are there periodic solutions? etc.

2.2. Control systems

- A quite general control system may be written as

$$\dot{x}(t) = F(t, x(t), u(t)), \qquad x(t_0) = x_0.$$

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- One wishes to *design* a control $u(t)$ ("open loop") or $u(x)$ ("closed loop") so that the system behaves in a desired manner. For example
  - steer from a point x_0 to a point x_1;
  - render a point x_0 stable;
  - follow a desired trajectory $x_{\mathsf{des}}(t)$.
- Such design is normally far more difficult than solving differential equations ⟶ look for *qualitative* descriptions.

3. Controllability

- Controllability theory essentially deals with the question, "Where can you go from here?"
- It is an essential basic element in any theory of control, e.g., for "stabilisability."

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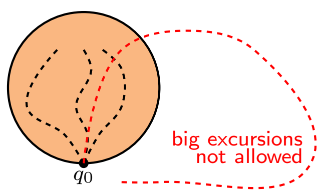
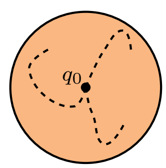
- Essentially there are two classes of problems:



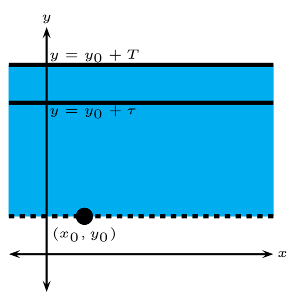


Local accessibility Local controllability

$\dot{x} = F(x, u)$

- Example:

$$\dot{x} = u$$
$$\dot{y} = 1$$



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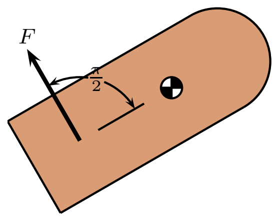
- This example is locally accessible, but not locally controllable.
- If y were a circular coordinate, the example would be locally accessible, but *globally* controllable.
- We stick to systems whose controllability can be described locally since one can essentially characterise local controllability using the knowledge of the system and its derivatives at the initial point.

- Here's a hard problem:



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- Is this system
  - locally accessible? (answer "standard")
  - locally controllable? (answer quite difficult)
  - globally controllable? (I do not know)

4. Controllability analysis

- Consider the following simple control system:

$$\dot{x}(t) = u_1(t) f_1(x(t)) + u_2(t) f_2(x(t)).$$

- Apply the control

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$$u(t) = \begin{cases} (1,0), & 0 \le t < \frac{T}{4} \\ (0,1), & \frac{T}{4} \le t < \frac{T}{2} \\ (-1,0), & \frac{T}{2} \le t < \frac{3T}{4} \\ (0,-1), & \frac{3T}{4} \le t \le T. \end{cases}$$

- Where does $x(T)$ end up?

- Then we determine that

$$x(T) = \sqrt{T}\,[f_1, f_2](x(0)) + \text{h.o.t.}, \qquad [f_1, f_2] = \frac{\partial f_2}{\partial x} f_1 - \frac{\partial f_1}{\partial x} f_2.$$

- $[f_1, f_2]$ is the **Lie bracket** of f_1 and f_2.
- More generally, we may consider a control system like

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$$\dot{x}(t) = f_0(x(t)) + u_1(t) f_1(x(t)) + \cdots + u_m(t) f_m(x(t)).$$

- By applying suitable controls, one may move in the directions

$$\begin{aligned}
&f_0,\ f_1, \dots, f_m,\\
&[f_a, f_b], \qquad a, b = 0, \dots, m,\\
&[f_a, [f_b, f_c]], \qquad a, b, c = 0, \dots, m,\\
&\text{etc..}
\end{aligned}$$

A simple exhibition of the Lie bracket

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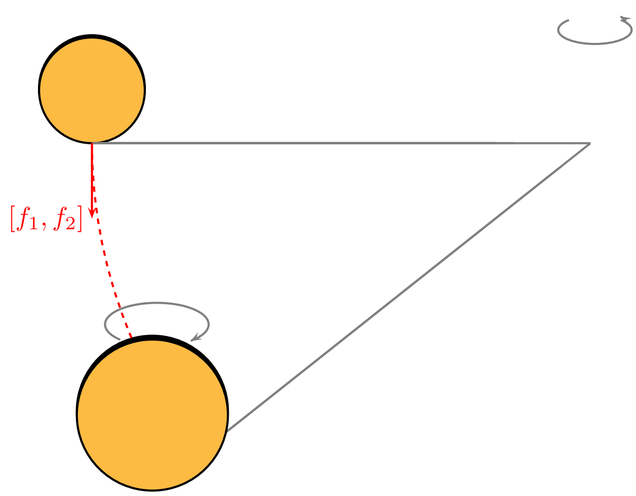

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- The previous "arguments" form the basis for deciding whether a system is locally *accessible*. For real analytic systems the issue has been decided since 1972.[1]
- Local *controllability* is darn hard!
- For analysts... Let $\mathscr{U}$ be the set of measurable controls. For $u \in \mathscr{U}$ defined on the interval $[0, T]$ define $\|u\| = T + \|u\|_\infty$. For $x_0 \in \mathbb{R}^n$ consider the map sending u to the solution of the IVP

$$\dot{x}(t) = F(x(t), u(t)), \qquad x(0) = x_0$$

at time T. Local controllability then becomes a nonlinear open mapping theorem from $\mathscr{U}$ to $\mathbb{R}^n$: Map a neighbourhood of the zero control to a neighbourhood of x_0.

[1] H. J. Sussmann and V. Jurdjevic, Controllability of nonlinear systems, *Journal of Differential Equations*, **12**, 95–116, 1972.

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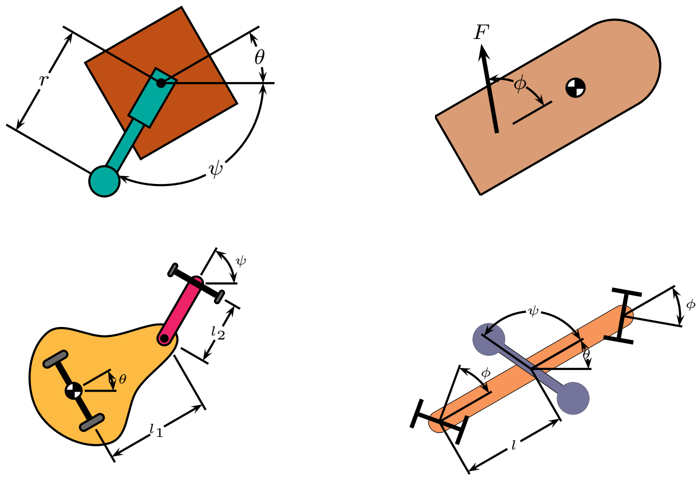
- What is known about local controllability?
  - For special systems (e.g., linear), computable necessary and sufficient conditions exist.
  - For quite general systems, computable sufficient *or* necessary conditions exist.
  - The state of the art is a morass of seemingly related results with no as yet understood unity.

5. Controllability of toys



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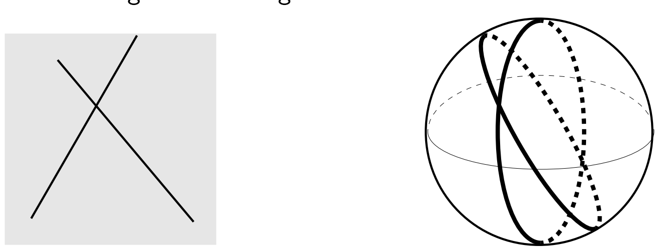
6. Mechanical control systems

- The toys are very special—they are "mechanical."
- Differential geometric methods are important in control for general nonlinear systems. . . they are indispensable for mechanical systems!
- **Geodesics** are length minimising curves.



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- What does length minimisation mean?
  For planar curves the length of a curve is

$$\ell = \int_0^1 \sqrt{\dot{x}(t) + \dot{y}(t)}\,\mathrm{d}t.$$

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- More generally, we may define a version of length by

$$\ell = \int_0^1 \sqrt{\dot{x}^T(t) M(x(t)) \dot{x}(t)}\,\mathrm{d}t$$

  for a positive-definite matrix function M.
- Geodesics will now depend on M, and they can be pretty wild.

- Now consider a particle of mass m in the plane. Its kinetic energy is

$$K = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2).$$

- We may define a more general kinetic energy by

$$K = \frac{1}{2} \dot{x}^T(t) M(x(t)) \dot{x}(t)$$

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  for a positive-definite matrix function M.
- Fact: length minimising curves also minimise the corresponding kinetic energy.
- Principal which governs the mechanical world: *A system left to its own devices will move in such a way that it (locally) minimises its kinetic energy.*

- The study of the free motions of a mechanical system is "the same" as the study of length minimising curves.
- The correct object is the "Levi-Civita connection corresponding to the Riemannian metric defining the kinetic energy of the mechanical system."
- Now start adding stuff.

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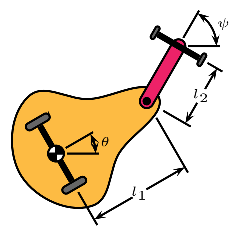
- Constraints:



- Nontrivial and important fact: the "form" of the equations describing the free motion remain unchanged when constraints are added.

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- *Summary of mechanics:* The study of unforced mechanical systems reduces to studying geodesic equations ⟶ **affine differential geometry**.
- To do control, one needs forces; one force for each direction in which we have authority.
- *"Fact:"* The geometry of the mechanical structure reacts well with the geometry of the control problem, e.g.,
  ⟶ Lie brackets look nice
  ⟶ One can answer questions which are as yet untouchable in general.

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7. Controllability analysis (sometimes) leads to control design