Generalised splines via the maximum principle

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1. The approach

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- *Question:* What is a spline?
- *My answer:* A curve satisfying a differential equation arising from a minimisation problem.
- Typically, the necessary conditions arising from the minimisation problem are derived with a variational approach.
- Instead, I will use the maximum principle.
- This allows the solution of more general minimisation problems, including, for example, control constraints.
- The control systems I employ are well-suited to the generation of wide classes of curves on manifolds: *affine connection control systems*.

2. What are affine connection control systems?

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- Shortly, they are this:
  1. a configuration manifold Q;
  2. an affine connection ∇ on Q;
  3. a collection $\mathscr{Y} = \{Y_1, \dots, Y_m\}$ of vector fields on Q.
- The corresponding control system is

$$\nabla_{c'(t)} c'(t) = u^a(t) Y_a(c(t))$$

for a controlled trajectory (u, c).

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- Mechanical examples of affine connection control systems:
  1. Lagrangian systems with kinetic energy Lagrangians (∇ is the Levi-Civita connection for the kinetic energy Riemannian metric). For example (some of these need potential energy),
     - satellites,
     - robotic manipulators,
     - underwater vehicles, etc.
  2. Same as above with the addition of constraints linear in velocity. For example,
     - locomotion systems (wheeled vehicles),
     - grasping applications, etc.

3. Affine connection control systems as control affine systems

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- Convert

$$\nabla_{c'(t)}c'(t) = u^a(t)Y_a(c(t))$$

to control affine system on TQ:

$$\dot{v}(t) = f_0(v(t)) + u^a(t)f_a(v(t)),$$

$v \in TQ$.

- Turns out that
  1. the drift is the geodesic spray denoted $f_0 = Z$, and
  2. the control vector fields are the vertical lifts of the vectors fields from $\mathscr{Y}$: we write $f_a = Y_a^{\text{lift}}$.

4. The Maximum Principle for affine connection control systems

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- Noakes, Heinzinger, Paden, and Camarinha, Crouch, Silva Leite, and Sontag, Sussmann, and Fax, Murray, and Chyba, Leonard, Sontag.
- We shall investigate in a little detail *one* of the several consequences of the Maximum Principle as it applies to affine connection control systems.
- Start general—let's look at the Maximum Principle for

$$c'(t) = f_0(c(t)) + u^a(t)f_a(c(t)),$$

with $c(t) \in M$, u taking values in $U \subset \mathbb{R}^m$, and objective function $L(x, u)$.

- Have the **control Hamiltonian** on $U \times T^*M$:

$$H(\alpha_x, u) = \underbrace{\alpha_x(f_0(x))}_{H_1} + \underbrace{\alpha_x(u^a f_a(x))}_{H_2} - \underbrace{L(x, u)}_{H_3}.$$

- One of several consequences of the MP is that if (u, c) is a minimiser then there exists a one-form field λ along c with the property that $t \mapsto \lambda(t)$ is an integral curve for the time-dependent Hamiltonian $(\alpha_x, t) \mapsto H(\alpha_x, u(t))$.
- The Hamiltonian is a sum of three terms, and so too will be the Hamiltonian vector field. Let us look at the first term, that with (plain old) Hamiltonian $H_1(\alpha_x) = \alpha_x(f_0(x))$.

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- In local coordinates X_{H_1} is written as

$$\dot{x}^i = f_0^i(x)$$

$$\dot{p}_i = -\frac{\partial f_0^j}{\partial x^i} p_j \quad \longleftarrow \quad \textit{“adjoint equation”?}$$

- X_{H_1} is the **cotangent lift** of f_0 and we denote it $f_0^{T^*}$.

- *Objective:* Understand $f_0^{T^*}$ when $M = TQ$ and $f_0 = Z$.
- Begin with a change of subject: Let f_0 be a vector field on (general) M with f_0^T its **tangent lift** defined by

$$f_0^T(v_x) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} T_x F_t(v_x)$$

(F_t is the flow of f_0).

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- f_0^T is the “linearisation” of f_0 and in coordinates is given by

$$\dot{x}^i = f_0^i(x) \qquad \left(\text{compare } f_0^{T^*}: \quad \dot{x}^i = f_0^i(x)\right.$$

$$\dot{v}^i = \frac{\partial f_0^i}{\partial x^j} v^j \qquad \left. \dot{p}^i = -\frac{\partial f_0^j}{\partial x^i} p_j \right)$$

- The flow of f_0^T measures how the integral curves of f_0 change as we change the initial condition in the direction of v_x.

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- Perhaps we can understand Z^T—thus take $M = TQ$ and $f_0 = Z$ in the discussion of tangent lift.
- Note:
  - Projections of integral curves of Z to Q are geodesics of ∇.
  - Z^T measures variations of integral curves of Z.
  - Thus Z^T measures variations of geodesics.
  - But we know something else which measures variations of geodesics...
- Let $c(t)$ be a geodesic. By varying the initial condition for the geodesic we generate an "infinitesimal variation" ξ of the geodesic and it turns out to satisfy... the **Jacobi equation**:

$$\nabla^2_{c'(t)}\xi(t) + R(\xi(t), c'(t))c'(t) + \nabla_{c'(t)}\big(T(\xi(t), c'(t))\big) = 0.$$

- What is the *precise* relationship between Z^T and the Jacobi equation?

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Some tangent bundle geometry using Z

- To make the "connection" between Z^T and the Jacobi equation, we perform constructions on the tangent bundle using the spray Z.
- ∇ comes from a linear connection on Q which induces an Ehresmann connection on $\pi_{TQ}\colon TQ \to Q$.
- Thus we may write $T_{v_q}TQ \simeq T_qQ \oplus T_qQ$.
- Z^T is not a spray, but... if $I_Q\colon TTQ \to TTQ$ is the canonical involution then $I_Q^* Z^T$ *is* a spray (it is the spray for the so-called *complete lift of* ∇).
- Use $I_Q^* Z^T$ to induce an Ehresmann connection on $\pi_{TTQ}\colon TTQ \to TQ$.
- Thus

$$\begin{aligned} T_{X_{v_q}}TTQ &\simeq T_{v_q}TQ \oplus T_{v_q}TQ \\ &\simeq \underbrace{T_qQ \oplus T_qQ}_{\text{geodesic equations}} \oplus \underbrace{T_qQ \oplus T_qQ}_{\text{variation equations}} \end{aligned}$$

- One represents Z^T in this splitting and determines that the Jacobi equation sits "inside" one of the four components.
- Now one applies similar constructions to T^*TQ and Z^{T^*} to derive (all going to plan) a one-form version of the Jacobi equation.
- Need a little notation:

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$$\langle R^*(\alpha, u)v; w\rangle = \langle \alpha; R(w, u)v\rangle, \quad \langle T^*(\alpha, u); w\rangle = \langle \alpha; T(w, u)\rangle.$$

- After the dust settles, we get what we are after which is the **adjoint Jacobi equation**:

$$\nabla^2_{c'(t)}\lambda(t) + R^*(\lambda(t), c'(t))c'(t) - T^*(\nabla_{c'(t)}\lambda(t), c'(t)) = 0.$$

- Why did I do this?
  - The adjoint Jacobi equation captures the interesting part of the Hamiltonian vector field Z^{T^*}, which comes from the MP, and words it in terms of affine differential geometry, i.e.,

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$$Z^{T^*}$$

$$\Updownarrow$$

$$\nabla_{c'(t)}c'(t) = 0$$
$$\nabla^2_{c'(t)}\lambda(t) + R^*(\lambda(t), c'(t))c'(t) - T^*(\nabla_{c'(t)}\lambda(t), c'(t)) = 0.$$

  - The geometry of Z on TQ provides a way of **globally** pulling out the "adjoint equation" from the MP in an intrinsic manner—this is not generally possible in the MP.

- The adjoint Jacobi equation forms the backbone of a general statement of the MP for affine connection control systems.
  - The contribution of the inputs needs to be added (easy).
  - The contribution of the objective function needs to be added (difficulty depends on the nature of the function).

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- Take the case when objective function is $L(u, v_q) = \frac{1}{2}g(v_q, v_q)$ for a Riemannian metric, and the affine connection is not necessarily the Levi-Civita connection. (In the case when ∇ is the Levi-Civita connection, a result is obtained by Silva Leite, Camarinha, and Crouch.)
- In this case, it is possible for there to be abnormal extremals (and probably abnormal minimisers).

- The normal extremals satisfy

$$\begin{aligned}&\nabla_{c'(t)}c'(t) = -h^\sharp_{\mathsf{Y}}(\lambda(t))\\&\nabla^2_{c'(t)}\lambda(t) + R^*(\lambda(t), c'(t))c'(t) - T^*(\nabla_{c'(t)}\lambda(t), c'(t)) =\\&\qquad\qquad \tfrac{1}{2}\nabla h_{\mathsf{Y}}(\lambda(t), \lambda(t)) - T^*(\lambda(t), h^\sharp_{\mathsf{Y}}(\lambda(t))),\end{aligned}$$

and abnormal extremals satisfy three conditions:

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1. $\nabla_{c'(t)}c'(t) = u^a(t)Y_a(c(t))$,
2. $\lambda(t) \in \operatorname{ann}(\mathsf{Y}_{c(t)})$ for $t \in [a, b]$ and
3. λ satisfies the equation along c given by:

$$\nabla^2_{c'(t)}\lambda(t) + R^*(\lambda(t), c'(t))c'(t) - T^*(\nabla_{c'(t)}\lambda(t), c'(t)) = B_{\mathsf{Y}}(\lambda(t), u^a(t)Y_a(t)).$$

- When ∇ is the Levi-Civita connection for g, and when the system is fully actuated, then we recover the equation of Noakes, Heinzinger, and Paden and Crouch and Silva Leite:

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$$\nabla^3_{c'(t)} c'(t) + R(\nabla_{c'(t)} c'(t), c'(t)) = 0.$$

- Where to go from here?
  - Other cost functions (time, length, etc.)
  - Constructible examples.