Sharp first-order controllability conditions for affine connection control systems

Ron M. Hirschorn* Andrew D. Lewis†

11/07/2001

Slide 0

*Department of Mathematics and Statistics, Queen's University
Email: ron@mast.queensu.ca
URL: http://www.mast.queensu.ca/~ron/
†Department of Mathematics and Statistics, Queen's University
Email: andrew.lewis@queensu.ca
URL: http://www.mast.queensu.ca/~andrew/

1. Introduction

Slide 1

- *Question:* Why talk about controllability?
- *Answer:* Because it is (1) hard, (2) interesting, and (3) possibly useful.
- The objective is *feedback-invariant* controllability conditions, just as controllability is a feedback-invariant notion.
- Many existing controllability tests are not stated in a feedback-invariant manner.

- An example of an intrinsically feedback-dependent condition is the good/bad bracket condition.
- Consider the control affine system

$$\dot{x}(t) = f_0(x(t)) + \sum_{a=1}^{m} u^a(t) f_a(x(t)).$$

Slide 2

- A **bad** bracket is one with an odd number of f_0's and an even number of each of the control vector fields. A **good** bracket is not bad.
- If at x_0, any bad bracket can be written as a linear combination of lower-order good brackets, then the system is locally controllable at x_0. (The "real" statement has a weaker hypothesis than we give here.)
- There are systems that do not satisfy the good/bad hypothesis (or the weaker "real" one), but can be made to satisfy it with a change of basis for the input vector fields.

2. Affine connection control systems

- An affine connection control system is
  1. a configuration manifold Q;
  2. an affine connection ∇ on Q;
  3. a collection $\mathscr{Y} = \{Y_1, \dots, Y_m\}$ of vector fields on Q.

Slide 3

- The corresponding control system is

$$\nabla_{c'(t)} c'(t) = u^a(t) Y_a(c(t))$$

for a controlled trajectory (u, c).

- As a control affine system we have

$$f_0 = Z \text{ (the geodesic spray)}, \quad f_a = Y_a^{\text{lift}} \text{ (the vertical lift)}.$$

3. Bracket structure for affine connection control systems

- For bounded inputs, local controllability is only feasible with a zero velocity initial condition, 0_q.
- When evaluated at 0_q, the only brackets that are nonzero are those for which the number of appearances of the inputs, minus the number of appearances of the drift, is either zero or one.

Slide 4

- For example, the brackets

$$f_a, \quad [f_a, [f_0, f_b]], \quad [[f_0, f_a], [f_0, f_b]]$$

are (possibly) nonzero when evaluated at 0_q, but the brackets

$$f_0, \quad [f_a, f_b], \quad [f_0, [f_0, f_a]]$$

are all zero when evaluated at 0_q.

- The nonzero brackets also have interesting geometric properties.
- Define the **symmetric product**:

$$\langle X : Y \rangle = \nabla_X Y + \nabla_Y X.$$

- Let $\overline{\mathrm{Sym}}(\mathscr{Y})$ be the distribution defined by the smallest $\mathbb{R}$-subspace of vector fields containing $\mathscr{Y}$ and closed under symmetric product.
- For a family of vector fields $\mathscr{F}$, let $\overline{\mathrm{Lie}}(\mathscr{F})$ be the distribution defined by the smallest $\mathbb{R}$-subspace of vector fields containing $\mathscr{F}$ and closed under Lie bracket.

Slide 5

- Using the canonical decomposition $T_{0_q}TQ \simeq T_qQ \oplus T_qQ$, if $\mathscr{F} = \{Z, Y_1^{\mathrm{lift}}, \dots, Y_m^{\mathrm{lift}}\}$, then

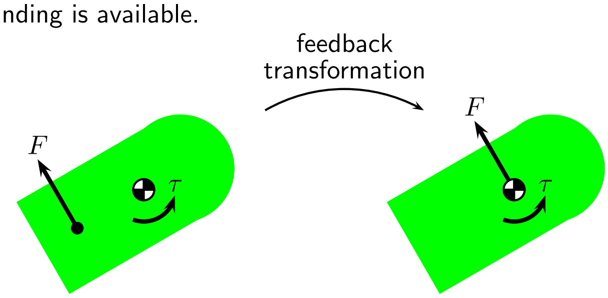
$$\overline{\mathrm{Lie}}(\mathscr{F})_{0_q} = \underbrace{\overline{\mathrm{Lie}}(\overline{\mathrm{Sym}}(\mathscr{Y}))_{0_q}}_{\text{horizontal}} \oplus \underbrace{\overline{\mathrm{Sym}}(\mathscr{Y})_{0_q}}_{\text{vertical}}.$$

- Furthermore, all bad brackets (obstructions to controllability) are in the vertical, symmetric product, component.

4. A motivating example

- Here's an example where the good/bad business indicates that a better understanding is available.

Slide 6



- The system on the left fails the good/bad test.
- The system on the right is feedback equivalent, but now passes the good/bad test (and is obviously configuration controllable).

5. The key geometric object

Slide 7

- The "right" controllability result for affine connection control systems should take account of how the affine connection ∇ "interacts" with the input distribution Y, and should involve the symmetric product.
- Let $\Sigma_{\mathsf{aff}} = (Q, \nabla, \mathscr{Y}, U)$ be an affine connection control system.
- Let Y be the distribution (possibly with nonconstant rank) spanned by the vector fields $\mathscr{Y}$.
- Define

$$\mathrm{Sym}^{(1)}(\mathscr{Y})_q = \mathrm{span}_{\mathbb{R}}(\langle Y_a : Y_b \rangle(q)|\ a, b = 1, \dots, m) + \mathsf{Y}_q$$

- Define a T_qQ/Y_q-valued symmetric bilinear map on Y_q by

$$B_{\mathsf{Y}_q}(u, v) = \pi_{\mathsf{Y}_q}(\langle U : V \rangle(q)),$$

  where U and V are vector fields extending $u, v \in \mathsf{Y}_q$, and where $\pi_{\mathsf{Y}_q} : \mathsf{Y}_q \to T_qQ/\mathsf{Y}_q$ is the canonical projection.

Slide 8

- Thus we make use of a vector-valued symmetric bilinear map.
- Some terminology for a generic one of these, $B\colon U \times U \to V$:
  - for $\lambda \in V^*$ denote B_λ to be the symmetric $(0,2)$-tensor $B_\lambda(u_1, u_2) = \langle \lambda; B(u_1, u_2) \rangle$;
  - B is **definite** (resp. **semidefinite**) if there exists $\lambda \in V^*$ so that B_λ is positive-definite (resp. positive-semidefinite);
  - B is **indefinite** if it is not semidefinite.

6. Statement of result

Slide 9

- Denote by $i_{\mathsf{Y}_q}\colon \mathsf{Sym}^{(1)}(\mathscr{Y})_q/\mathsf{Y}_q \to T_qQ/\mathsf{Y}_q$ the inclusion.
- Define $i^*_{\mathsf{Y}_q} B_{\mathsf{Y}_q}$ to be the restriction to $\mathsf{Sym}^{(1)}(\mathscr{Y})_q/\mathsf{Y}_q$ of B_{Y_q}.

Theorem *Let $\Sigma_{\mathrm{aff}} = (Q, \nabla, \mathscr{Y}, U)$ be an affine connection control system and let $q_0 \in Q$. Let $\mathsf{S}(\mathscr{Y}, q_0) \subset TQ$ be the integral manifold for the control system through 0_{q_0}. The following statements hold:*

(i) if $\mathsf{Sym}^{(1)}(\mathscr{Y})_{q_0} = \overline{\mathsf{Sym}}(\mathscr{Y})_{q_0}$ and if $i^*_{\mathsf{Y}_{q_0}} B_{\mathsf{Y}_{q_0}}$ is indefinite, then the restriction of Σ_{aff} to $\mathsf{S}(\mathscr{Y}, q_0)$ is STLC from 0_{q_0}.

(ii) if q_0 is a regular point for the distribution Y and if $B_{\mathsf{Y}_{q_0}}$ is definite, then Σ_{aff} is not STLCC from q_0.

7. Outline of proof

7.1. Sufficiency

Slide 10

- It turns out that the sufficient condition ensures that there is a choice for the input vector fields with the property that the "real" good/bad condition is satisfied.
- This was essentially noticed (unknown by us, *a priori*) for control affine systems by Basto-Gonçalves.[1]

7.2. Necessity

- Use the series expansion for affine connection control systems of Bullo.[2]
- Show that a linear function which is zero at q_0 attains only positive values for small times.

[1] *Systems Control Lett.*, **35**(5), 287–290, 1998
[2] To appear in *SIAM J. Control Optim.*

8. From here...

Slide 11

- Our first-order conditions can be improved.
- As they are, they may be the best possible for first-order brackets, but by allowing first-order derivatives, one should be able to get rid of the hypothesis of the regularity of the distribution in the necessary condition.
- Similarly, there are probably further directions that can be incorporated into the sufficient condition, involving higher-order brackets, but still first-order derivatives.
- *Higher-order conditions:* One should understand the "gap" between the sufficient and necessary conditions. Should be possible...
- Adapt for general control affine systems.