Optimal control for a simplified hovercraft model

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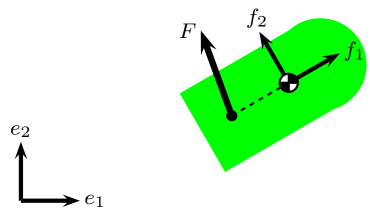
1. The system

- Okay... it is a *very* simplified hovercraft model:



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- The system is modelled by:
  1. configuration space $Q = SE(2)$ with coordinates (x, y, θ);
  2. kinetic energy Riemannian metric
     $g = m(\mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y) + J\mathrm{d}\theta \otimes \mathrm{d}\theta$;
  3. control vector fields

$$Y_1 = \frac{\cos\theta}{m}\frac{\partial}{\partial x} + \frac{\sin\theta}{m}\frac{\partial}{\partial y}, \qquad Y_2 = -\frac{\sin\theta}{m}\frac{\partial}{\partial x} + \frac{\cos\theta}{m}\frac{\partial}{\partial y} - \frac{h}{J}\frac{\partial}{\partial\theta}.$$

2. Some useful definitions

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- Let Y be the distribution spanned by the input vector fields.
- Let g_{Y} denote the restriction of g to Y.
- Let the orthogonal projection onto Y be denoted P_{Y}.
- Define the $(2,0)$ tensor h_{Y} by

$$h_{\mathsf{Y}}(\alpha_q, \beta_q) = g_{\mathsf{Y}}(g^{\sharp}(\alpha_q), g^{\sharp}(\beta_q)).$$

- Let $h_{\mathsf{Y}}^{\sharp}\colon T^*Q \to TQ$ be the associated bundle mapping.
- If Y is a vector field, ∇Y denotes the $(1,1)$ tensor defined by

$$\nabla Y(\alpha, X) = \langle \alpha; \nabla_X Y \rangle$$

for a one-form α and a vector field X.

3. Extremals

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- We look at force and time-optimal control for the system.
- The affine connection for the system is flat and torsionless. Thus the equations for extremals simplify from the equations for general affine connection control systems.

3.1. Time-optimal control

- The controls must be constrained. We use geometric constraints:

$$g(u^a(t)Y_a(c(t)), u^b(t)Y_b(c(t))) \leq 1.$$

- The necessary conditions of the maximum principle are given by

$$\begin{aligned}\nabla_{c'(t)}c'(t) &= u^a(t)Y_a(c(t))\\ \nabla^2_{c'(t)}\lambda(t) &= u^a(t)(\nabla Y_a)^*(\lambda(t)),\end{aligned}$$

where λ is the adjoint one-form field.

- For nonsingular extremals, the controls are determined by the maximum principle:

$$u^a Y_a(c(t)) = -\frac{P^*_{\mathrm{Y}}(\lambda(t))}{\|P^*_{\mathrm{Y}}(\lambda(t))\|_g}, \tag{1}$$

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- If an extremal has the property that $\lambda(t) \in \mathrm{ann}(\mathrm{Y}_{c(t)})$ for all t, then the extremal is **singular**, and (1) cannot be used to determine the controls.
- For the hovercraft system, it turns out to be possible to explicitly determine the form of all singular extremals.

3.2. Force-optimal control

- The cost function is

$$J_{\mathsf{force}}(\gamma) = \int_0^T \frac{1}{2} g(u^a(t)Y_a(c(t)), u^b(t)Y_b(c(t)))\,\mathrm{d}t.$$

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- Normal extremals satisfy

$$\begin{aligned} \nabla_{c'(t)} c'(t) &= -h^\sharp_{\mathrm{Y}}(\lambda(t)) \\ \nabla^2_{c'(t)} \lambda(t) &= \tfrac{1}{2} \nabla h_{\mathrm{Y}}(\lambda(t), \lambda(t)). \end{aligned}$$

- Abnormal extremals satisfy the same conditions as singular extremals for time-optimal control, but there are no control bounds.

4. Partial analysis of nonsingular extremals

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- A full analysis of the nonsingular extremals has not been undertaken, but is perhaps possible, at least qualitatively.
- We look at two types of nonsingular extremals, corresponding to the decoupling vector fields of Bullo and Lynch.
- Consider the two vector fields

$$X_1 = \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y},$$

$$X_2 = -\sin\theta \frac{\partial}{\partial x} + \cos\theta \frac{\partial}{\partial y} - \frac{mh}{J}\frac{\partial}{\partial \theta}.$$

4.1. X_1

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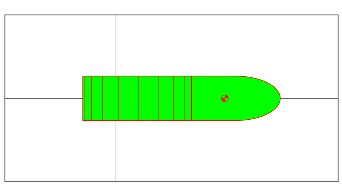
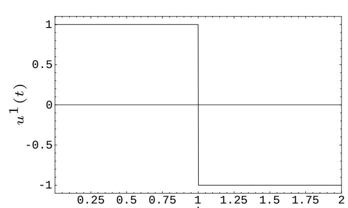
- This is a trivial problem as it boils down to optimal control of a mass moving on a line.
- A time-optimal extremal:





- The cost for the extremal is $J_{\text{time}} = 2$.

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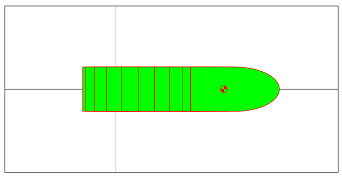
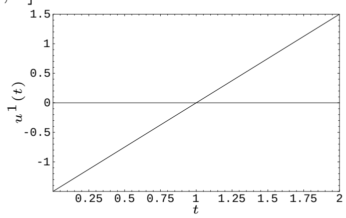
- A force-optimal extremal defined on $[0, 2]$:





- Comparison:

	J_{time}	J_{force}
time-optimal	2	2
force-optimal	2	$\frac{3}{2}$

4.2. X_2

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- Preliminary analysis suggests that there are no nonsingular extremals for the time or force-optimal problem that are reparameterised integral curves for X_2.
- However, one can restrict to such reparameterisations, and extremise over these.
- For a reparameterisation τ, the controls are given by

$$u^1(t) = \frac{m^2 h}{J}\tau'(t)^2, \quad u^2(t) = m\tau''(t).$$

- Since the reparameterisations are unrestricted, the problem is essentially fully actuated as a control problem.
- $\longrightarrow$ standard variational methods are applicable.

- For time-optimal control, the control bounds for a reparameterisation τ are given by
$$\frac{m^3h^2}{J^2}\tau'(t)^4 + \frac{mh^2+J}{J}\tau''(t)^2 \le 1$$
- Extremals satisfy a second-order variational problem with inequality constraints involving velocity and acceleration (standard problem).
- An example of a time-optimal extremal:

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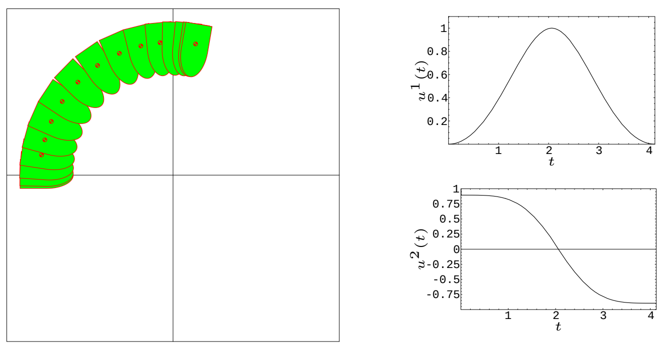

- *Note:* may not be an extremal for the full problem.

- For force-optimal control, the cost function is
$$J_{\text{force}} = \int_0^T \left(\frac{m^3h^2}{J^2}\tau'(t)^4 + \frac{mh^2+J}{J}\tau''(t)^2\right) \mathrm{d}t.$$
- $\longrightarrow$ straight calculus of variations problem.
- An example of a force-optimal extremal:

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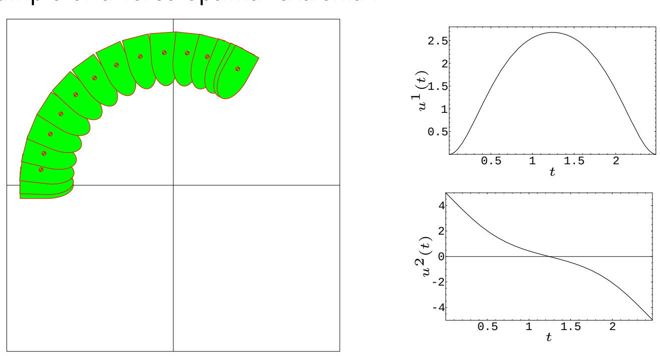

- *Note:* may not be an extremal for the full problem.

4.3. Punchline

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- It is very easy to design open-loop controls to follow whatever reparameterisation of integral curves of X_1 and X_2 one wants.
- The restriction of the optimal control problem to X_2 integral curves is not something one can do "by hand."
- $\Longrightarrow$ in practice, one would likely go for some sort of suboptimal controls for computational efficiency.

5. Complete analysis of singular extremals

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- Let us first introduce a simple class of singular extremals.
- Consider a trajectory—parameterised in a very specific, but not here specified, manner—of the hovercraft as follows:
  - it is defined on $]-\infty,\infty[$;
  - $x^2(t)+y^2(t)=\left(\frac{J}{mh}\right)^2$ for all $t\in\mathbb{R}$;
  - $\theta(t)=\pi+\arctan\left(\frac{y(t)}{x(t)}\right)$ for all $t\in\mathbb{R}$;
  - $\lim_{t\to\infty}(x(t),y(t))=-\lim_{t\to-\infty}(x(t),y(t))$;
  - $\lim_{t\to\infty}\theta(t)=\pi+\lim_{t\to-\infty}\theta(t)$.

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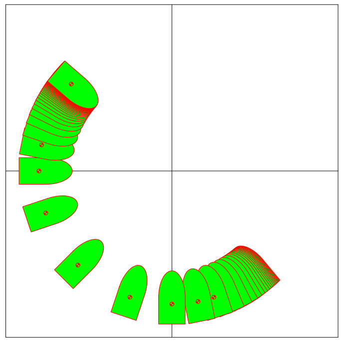
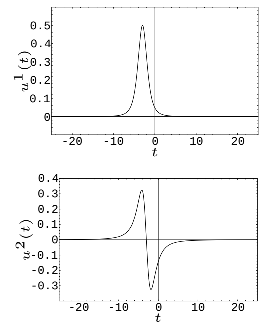
- A picture tells a thousand words...





- Any subarc of such a trajectory is a singular extremal, and we call these **stationary** singular extremals.

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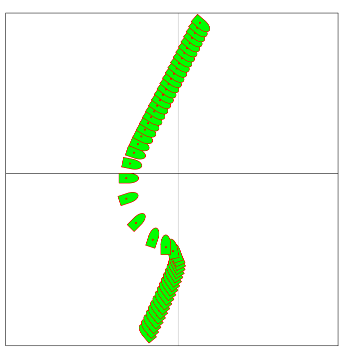
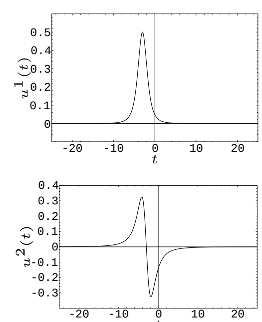
- A general singular extremal is the superposition of a stationary one and a uniform linear motion:





- Note that the uniform linear motion is accomplished without the addition of any input.

6. And then. . .

- Are any of the extremals we have found optimal?
- General theorems corresponding to some of the observations.
- Come up with path planning strategies based on extremals (if they can be sufficiently well understood).
- Higher-order necessary conditions.

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- A hardware hovercraft is in the works (thanks to Dave Tyner and Mark Levkoe).