Control theory for mechanical systems

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1. Some sample systems

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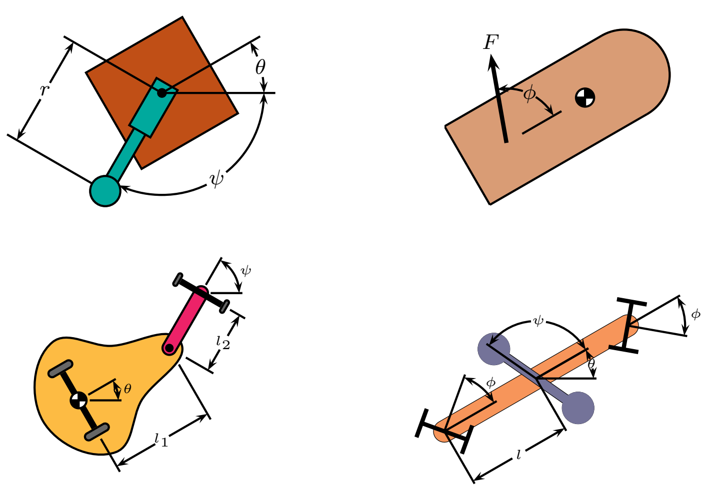

2. General system description

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- We consider "simple mechanical systems, possibly with constraints:"
  1. a configuration manifold Q;
  2. kinetic energy, defining a Riemannian metric on Q (essentially a q-dependent, positive-definite matrix);
  3. potential energy (a function on Q);
  4. possibly velocity constraints that allow rolling, but not slipping;
  5. a collection of forces whose direction and magnitude may be controlled (consider fewer forces than degrees of freedom).
- Control system properties (consider zero potential case):
  1. inherently nonlinear;
  2. linearisations are badly behaved (linear methods not applicable);
  3. none of the "standard" nonlinear methods generally apply.

System geometry

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- The mechanical systems described above are very "structured."
- The Euler-Lagrange equations (no external forces) for the Lagrangian $L(q,\dot{q}) = \frac{1}{2}g_{ij}(q)\dot{q}^i\dot{q}^j$:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\partial L}{\partial \dot{q}^i}\Big) - \frac{\partial L}{\partial q^i} &= g_{ij}\Big[\ddot{q}^j + g^{jk}\Big(\frac{\partial g_{k\ell}}{\partial q^m} - \frac{1}{2}\frac{\partial g_{\ell m}}{\partial q^k}\Big)\dot{q}^\ell\dot{q}^m\Big] \\ &= g_{ij}\Big[\underbrace{\ddot{q}^j + \overset{g}{\Gamma}{}^j_{\ell m}\dot{q}^\ell\dot{q}^m}_{\substack{\textit{geodesic equations}\\ \text{for the } \textit{Levi-Civita}\\ \text{affine connection}}}\Big].\end{aligned}$$

- This makes us think that the geometry of the affine connection may be important.
- It is in fact *extremely* important, and meshes beautifully with the control problems.

3. Problems to consider

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- Problems that have been considered:
  1. describe the states reachable from a given point (the controllability problem).
  2. Steer from point A at rest to point B at rest (the steering problem).
  3. Design the forces, perhaps as functions of configuration, velocity, and time, so that a desired operating point is rendered stable (point stabilisation problem).
  4. Follow a desired path in configuration space, possibly with a specific parameterisation (trajectory tracking problem).
  5. Perform one of the above tasks in a manner that minimises some cost function (optimal control).

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- Problems that have not yet received serious consideration:
  1. more detailed models (dissipative effects, for example);
  2. actuator dynamics;
  3. stability to disturbances and perturbations (robustness);
  4. implementation issues (!!)

4. Tools for analysis and design

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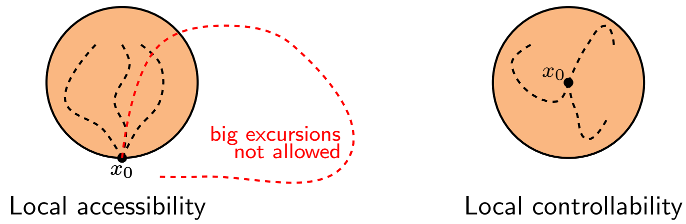
- For systems of the type we are considering, the controllability problem is fundamental. . .



$$\dot{x} = f_0(x) + \sum_{a=1}^{m} u_a(t) f_a(x)$$

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- Accessibility analysis is "standard:" Consider the following simple control system:

$$\dot{x} = u_1 f_1(x) + u_2 f_2(x).$$

- Apply the control

$$u(t) = \begin{cases} (1,0), & 0 \le t < \frac{T}{4} \\ (0,1), & \frac{T}{4} \le t < \frac{T}{2} \\ (-1,0), & \frac{T}{2} \le t < \frac{3T}{4} \\ (0,-1), & \frac{3T}{4} \le t \le T. \end{cases}$$

- Where does $x(T)$ end up?

- We determine that

$$x(T) = x(0) + \sqrt{T}\,[f_1, f_2](x(0)) + \text{h.o.t.}, \qquad [f_1, f_2] = \frac{\partial f_2}{\partial x} f_1 - \frac{\partial f_1}{\partial x} f_2.$$

- $[f_1, f_2]$ is the **Lie bracket** of f_1 and f_2.
- More generally, we may consider a control system like

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$$\dot{x}(t) = f_0(x(t)) + u_1(t) f_1(x(t)) + \cdots + u_m(t) f_m(x(t)).$$

- By applying suitable controls, one may move in the directions

$$\begin{aligned} & f_0,\ f_1, \dots, f_m, \\ & [f_a, f_b], \qquad a, b = 0, \dots, m, \\ & [f_a, [f_b, f_c]], \qquad a, b, c = 0, \dots, m, \\ & \text{etc.} \end{aligned}$$

A simple exhibition of the Lie bracket

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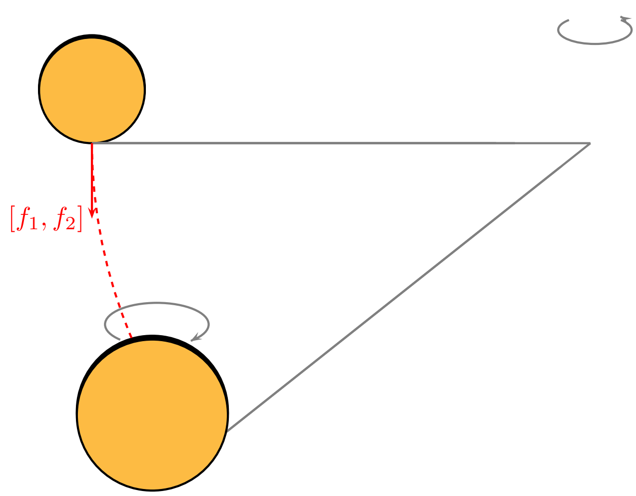

5. Some controllability results for mechanical systems

- For mechanical systems, the interaction of the Lie bracket and the system geometry (i.e., the affine connection) is very attractive. This gives nice accessibility results.[1,2,3]
- *Controllability* is a difficult problem.
- Let $\mathscr{U}$ be the set of measurable controls. For $u \in \mathscr{U}$ defined on the interval $[0, T]$ define $\|u\| = T + \|u\|_\infty$. For $x_0 \in \mathbb{R}^n$ consider the map sending u to the solution of the IVP

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$$\dot{x} = f_0(x) + \sum_{a=1}^{m} u_a(t) f_a(x), \qquad x(0) = x_0$$

at time T. Local controllability then becomes a nonlinear open mapping theorem from $\mathscr{U}$ to $\mathbb{R}^n$: Map a neighbourhood of the zero control to a neighbourhood of x_0.

[1] L/Murray, *SIAM Review*, **41**(3), 555–574, 1999
[2] L/Murray *Systems Control Lett.*, **31**(4), 199–205, 1997
[3] L, *Rep. Math. Phys.*, **42**(1/2), 135–164, 1998

From controllability to motion control

- L/Murray give sufficient conditions based on work of Sussmann.[4]
- These sufficient conditions lead to a class of control algorithms for certain systems that rely on specially constructed periodic inputs.[5]
- Problems treated include the steering problem, the point stabilisation problem, and the trajectory tracking problem.
- *Movies*

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[4] *SIAM J. Control Optim.*, **25**(1), 158–194, 1987
[5] Bullo/Leonard/L, *IEEE Trans. Automat. Control*, **45**(8), 1437–1454, 2000

- The steering problem for the class of systems treated by Bullo/Leonard/L turns out to more easily treatable.
- The key is the notion of a "decoupling vector field."[6]
- This relies on the affine differential geometric component of mechanical systems.

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- *Question:* What are the vector fields X on Q whose integral curves can be followed up to any reparameterisation?
- *Answer:* X and $\nabla_X X$ should lie in the span of the inputs.
- *Movies*

[6] Bullo/Lynch, To appear in *IEEE Trans. Robotics Automat.*, 2001, generalising L, *Proceedings of the 38th IEEE CDC*, 1162–1167, 1999

Improved controllability results

- The original sufficient conditions of L/Murray are not sharp.
- They suffer, as do many existing controllability results, from not being "feedback-invariant."
- Sharp conditions are known for single-input systems.[7]
- The first steps down the road to feedback-invariant conditions have been taken.[8]

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- For mechanical systems, we have sharp first-order conditions.
- All systems controllable at first-order turn out to admit a "full set" of decoupling vector fields.
- Sharp higher-order conditions may be attainable, and will hopefully provide useful insights into the controllability for systems that are more complicated than the simple examples used here.

[7] L, *Proceedings of the ECC*, 1997

[8] Basto-Gonçalves, *Systems Control Lett.*, **35**(5), 287–290, 1998 and Hirschorn/L, To appear in *Proceedings of the 40th IEEE CDC*, 2001

6. Lab toys

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