Geometric first-order controllability conditions for affine connection control systems

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1. Introduction

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- *Question:* Why talk about controllability?
- *Answer:* Because it is (1) hard, (2) interesting, and (3) possibly useful.
- The objective is *feedback-invariant* controllability conditions, just as controllability is a feedback-invariant notion.
- Many existing controllability tests are not stated in a feedback-invariant manner.

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- An example of an intrinsically feedback-dependent condition is the good/bad bracket condition.
- Consider the control affine system

$$\dot{x}(t) = f_0(x(t)) + \sum_{a=1}^{m} u^a(t) f_a(x(t)).$$

- A **bad** bracket is one with an odd number of f_0's and an even number of each of the control vector fields. A **good** bracket is not bad.
- If at x_0, any bad bracket can be written as a linear combination of lower-order good brackets, then the system is locally controllable at x_0. (The "real" statement has a weaker hypothesis than we give here.)
- There are systems that do not satisfy the good/bad hypothesis (or the weaker "real" one), but can be made to satisfy it with a change of basis for the input vector fields.

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2. Affine connection control systems

- An affine connection control system is
  1. a configuration manifold Q;
  2. an affine connection ∇ on Q;
  3. a collection $\mathscr{Y} = \{Y_1, \dots, Y_m\}$ of vector fields on Q.
- The corresponding control system is

$$\nabla_{c'(t)} c'(t) = u^a(t) Y_a(c(t))$$

for a controlled trajectory (u, c).

- As a control affine system we have

$$f_0 = Z \text{ (the geodesic spray)}, \quad f_a = Y_a^{\text{lift}} \text{ (the vertical lift)}.$$

3. Bracket structure for affine connection control systems

- For bounded inputs, local controllability is only feasible with a zero velocity initial condition, 0_q.
- When evaluated at 0_q, the only brackets that are nonzero are those for which the number of appearances of the inputs, minus the number of appearances of the drift, is either zero or one.

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- For example, the brackets

$$f_a, \quad [f_a, [f_0, f_b]], \quad [[f_0, f_a], [f_0, f_b]]$$

are (possibly) nonzero when evaluated at 0_q, but the brackets

$$f_0, \quad [f_a, f_b], \quad [f_0, [f_0, f_a]]$$

are all zero when evaluated at 0_q.

- The nonzero brackets also have interesting geometric properties.
- Define the **symmetric product**:

$$\langle X : Y \rangle = \nabla_X Y + \nabla_Y X.$$

- Let $\overline{\mathrm{Sym}}(\mathscr{Y})$ be the distribution defined by the smallest $\mathbb{R}$-subspace of vector fields containing $\mathscr{Y}$ and closed under symmetric product.
- For a family of vector fields $\mathscr{F}$, let $\overline{\mathrm{Lie}}(\mathscr{F})$ be the distribution defined by the smallest $\mathbb{R}$-subspace of vector fields containing $\mathscr{F}$ and closed under Lie bracket.

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- Using the canonical decomposition $T_{0_q}TQ \simeq T_qQ \oplus T_qQ$, if $\mathscr{F} = \{Z, Y_1^{\mathrm{lift}}, \dots, Y_m^{\mathrm{lift}}\}$, then

$$\overline{\mathrm{Lie}}(\mathscr{F})_{0_q} = \underbrace{\overline{\mathrm{Lie}}(\overline{\mathrm{Sym}}(\mathscr{Y}))_{0_q}}_{\text{horizontal}} \oplus \underbrace{\overline{\mathrm{Sym}}(\mathscr{Y})_{0_q}}_{\text{vertical}}.$$

- Furthermore, all bad brackets (obstructions to controllability) are in the vertical, symmetric product, component.

4. A motivating example

- Here's an example where the good/bad business indicates that a better understanding is available.

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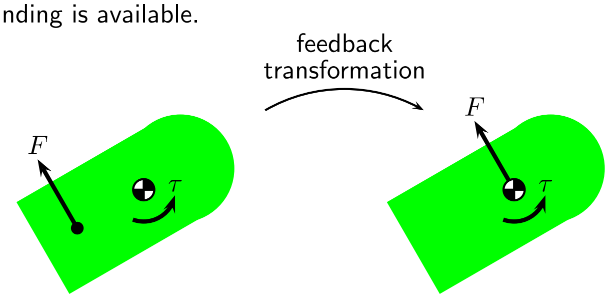

- The system on the left fails the good/bad test.
- The system on the right is feedback equivalent, but now passes the good/bad test (and is obviously configuration controllable).

5. The key geometric object

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- The "right" controllability result for affine connection control systems should take account of how the affine connection ∇ "interacts" with the input distribution Y, and should involve the symmetric product.
- Let $\Sigma_{\mathsf{aff}} = (Q, \nabla, \mathscr{Y}, U)$ be an affine connection control system.
- Let Y be the distribution (possibly with nonconstant rank) spanned by the vector fields $\mathscr{Y}$.
- Define

$$\mathrm{Sym}^{(1)}(\mathscr{Y})_q = \mathrm{span}_{\mathbb{R}}(\langle Y_a : Y_b \rangle(q) |\ a, b = 1, \dots, m) + \mathsf{Y}_q$$

- Define a T_qQ/Y_q-valued symmetric bilinear map on Y_q by

$$B_{\mathsf{Y}_q}(u, v) = \pi_{\mathsf{Y}_q}(\langle U : V \rangle(q)),$$

where U and V are vector fields extending $u, v \in \mathsf{Y}_q$, and where $\pi_{\mathsf{Y}_q} \colon \mathsf{Y}_q \to T_qQ/\mathsf{Y}_q$ is the canonical projection.

- Thus we make use of a vector-valued symmetric bilinear map.
- Some terminology for a generic one of these, $B\colon U \times U \to V$:
  - for $\lambda \in V^*$ denote B_λ to be the symmetric $(0,2)$-tensor $B_\lambda(u_1, u_2) = \langle \lambda; B(u_1, u_2)\rangle$;
  - B is **definite** (resp. **semidefinite**) if there exists $\lambda \in V^*$ so that B_λ is positive-definite (resp. positive-semidefinite);
  - B is **indefinite** if it is not semidefinite.

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6. Statement of result

- Denote by $i_{\mathsf{Y}_q}\colon \mathsf{Sym}^{(1)}(\mathscr{Y})_q/\mathsf{Y}_q \to T_qQ/\mathsf{Y}_q$ the inclusion.
- Define $i^*_{\mathsf{Y}_q} B_{\mathsf{Y}_q}$ to be the restriction to $\mathsf{Sym}^{(1)}(\mathscr{Y})_q/\mathsf{Y}_q$ of B_{Y_q}.

Theorem *Let $\Sigma_{\mathrm{aff}} = (Q, \nabla, \mathscr{Y}, U)$ be an affine connection control system and let $q_0 \in Q$. Let $\mathsf{S}(\mathscr{Y}, q_0) \subset TQ$ be the integral manifold for the control system through 0_{q_0}. The following statements hold:*

(i) if $\mathsf{Sym}^{(1)}(\mathscr{Y})_{q_0} = \overline{\mathsf{Sym}}(\mathscr{Y})_{q_0}$ and if $i^*_{\mathsf{Y}_{q_0}} B_{\mathsf{Y}_{q_0}}$ is indefinite, then the restriction of Σ_{aff} to $\mathsf{S}(\mathscr{Y}, q_0)$ is STLC from 0_{q_0}.

(ii) if q_0 is a regular point for the distribution Y and if $B_{\mathsf{Y}_{q_0}}$ is definite, then Σ_{aff} is not STLCC from q_0.

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7. Outline of proof

7.1. Sufficiency

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- It turns out that the sufficient condition ensures that there is a choice for the input vector fields with the property that the "real" good/bad condition is satisfied.
- This was essentially noticed (unknown by us, *a priori*) for control affine systems by Basto-Gonçalves.[1]

7.2. Necessity

- Use the series expansion for affine connection control systems of Bullo.[2]
- Show that a linear function which is zero at q_0 attains only positive values for small times.

[1] *Systems Control Lett.*, **35**(5), 287–290, 1998
[2] To appear in *SIAM J. Control Optim.*

8. From here...

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- Our first-order conditions can be improved.
- As they are, they may be the best possible for first-order brackets, but by allowing first-order derivatives, one should be able to get rid of the hypothesis of the regularity of the distribution in the necessary condition.
- Similarly, there are probably further directions that can be incorporated into the sufficient condition, involving higher-order brackets, but still first-order derivatives.
- *Higher-order conditions:* One should understand the "gap" between the sufficient and necessary conditions. Should be possible...
- Adapt for general control affine systems.