Control theory for mechanical systems

Andrew D. Lewis*

Collaborators: Francesco Bullo, Theo Coombs, Jorge Cortés,
Ron Hirschorn, Kevin Lynch, Sonia Martínez, David Tyner

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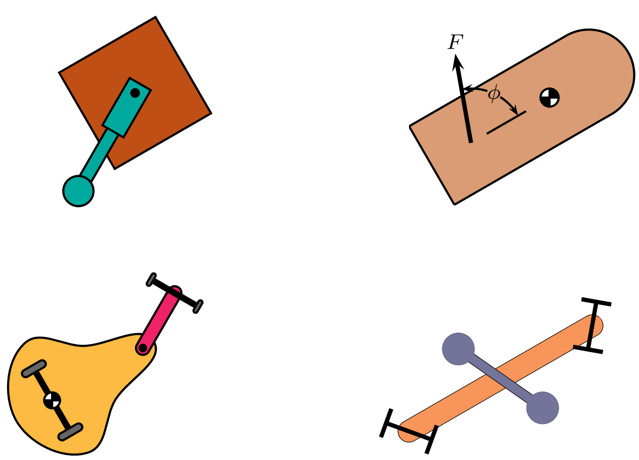
*Department of Mathematics and Statistics, Queen's University
Email: andrew.lewis@queensu.ca
URL: http://www.mast.queensu.ca/~andrew/

Simple examples



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What are we interested in?

- Broadly, a general methodology that encompasses modelling, analysis, and design.
- More specifically, for one of the example systems, or any system "like" them,

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  - can we model it in a unified manner that is conducive to the further objectives of analysis and design?
  - can one describe its reachable set?
  - if given a suitable cost function, can one analyse the corresponding extremals of the optimal control problem?
  - are there simple collections of trajectories that are sufficiently rich to do motion planning?

Modelling

- For us, a **simple mechanical control system** consists of a 6-tuple $(\mathsf{Q}, \mathbb{g}, V, F, \mathcal{D}, \mathscr{F} = \{F^1, \dots, F^m\})$ where

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  1. Q is a finite-dimensional configuration manifold,
  2. $\mathbb{g}$ is a Riemannian metric on Q,
  3. V is a potential function on Q,
  4. F represents all non-potential forces that are not controlled (e.g., dissipative forces),
  5. $\mathcal{D}$ is a distribution on Q modelling linear velocity constraints,
  6. $\mathscr{F}$ is a collection of one-forms on Q, each representing a force over which we have control.

- We generally simplify to the situation where $V = 0$ and $F = 0$, although potential forces have received some attention,[1] as have dissipative forces.[2]
- With these simplifications, the problem is reduced to an **affine connection control system** which is described by a 4-tuple $\Sigma_{\mathsf{aff}} = (\mathsf{Q}, \nabla, \mathcal{D}, \mathscr{Y} = \{Y_1, \dots, Y_m\})$ with

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1. Q as before,
2. ∇ an affine connection (which is not generally Levi-Civita),
3. $\mathcal{D}$ a distribution to which ∇ restricts,
4. $\mathscr{Y}$ a collection of vector fields on Q (these are related to the one-forms $\mathscr{F}$).

[1] L/Murray, *SIAM J. Control Optim.*, **35**(3), 766-790, 1997.
[2] Cortés/Martínez/Bullo, *IEEE Trans. Automat. Control*, submitted, July 2001.

- When $\mathcal{D} = \mathsf{TQ}$ then ∇ is the Levi-Civita affine connection $\overset{\mathbb{g}}{\nabla}$ associated with $\mathbb{g}$.
- When $\mathcal{D} \subsetneq \mathsf{TQ}$ then ∇ is defined by

$$\nabla_X Y = \overset{\mathbb{g}}{\nabla}_X Y - (\overset{\mathbb{g}}{\nabla}_X P)(Y),$$

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where P is the orthogonal projection onto $\mathcal{D}^{\perp}$.

- The equations of motion for such systems are

$$\nabla_{\dot{\gamma}(t)} \dot{\gamma}(t) = \sum_{a=1}^{m} u_a(t) Y_a(\gamma(t))$$

for a controlled trajectory (γ, u) satisfying $\dot{\gamma}(t) \in \mathcal{D}_{\gamma(t)}$ for some (and hence all) t.

Controllability analysis

- Suppose that the controls $u\colon [0,T] \to U \subset \mathbb{R}^m$ are measurable and take their values in a compact set U for which $0 \in \operatorname{int}(\operatorname{conv}(U))$.
- For $q_0 \in \mathsf{Q}$ and $T > 0$ let $\mathcal{R}_{\mathsf{TQ}}(q_0, \leq T) \subset \mathsf{TQ}$ be the set of states reachable from 0_{q_0} in time at most T and let $\mathcal{R}_{\mathsf{Q}}(q_0, \leq T) = \tau_{\mathsf{Q}}(\mathcal{R}_{\mathsf{TQ}}(q_0, \leq T))$.

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Definition $\Sigma_{\text{aff}} = (\mathsf{Q}, \nabla, \mathcal{D}, \mathscr{Y})$ is

(i) ***accessible*** from 0_{q_0} if $\operatorname{int}_{\mathcal{D}}(\mathcal{R}_{\mathsf{TQ}}(q_0, \leq T)) \neq \varnothing$ for all sufficiently small T, is

(ii) ***configuration accessible*** from q_0 if $\operatorname{int}(\mathcal{R}_{\mathsf{Q}}(q_0, \leq T)) \neq \varnothing$ for all sufficiently small T, is

(iii) ***small-time locally controllable*** (***STLC***) from 0_{q_0} if $0_{q_0} \in \operatorname{int}_{\mathcal{D}}(\mathcal{R}_{\mathsf{TQ}}(q_0, \leq T))$ for all sufficiently small T, and is

(iv) ***small-time locally configuration controllable*** (***STLCC***) from q_0 if $q_0 \in \operatorname{int}(\mathcal{R}_{\mathsf{Q}}(q_0, \leq T))$ for all sufficiently small T.

Some old results[1]

- Define the **symmetric product** by $\langle X : Y \rangle = \nabla_X Y + \nabla_Y X$.
- Let $\mathcal{Y}$ be the distribution generated by $\mathscr{Y}$.
- Let $\overline{\operatorname{Sym}}(\mathcal{Y})$ be the distribution generated by $\mathscr{Y}$ under symmetric product.
- Let $\overline{\operatorname{Lie}}(\mathscr{V})$ be the distribution generated by a family of vector fields $\mathscr{V}$ under Lie bracket.

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Theorem $\Sigma_{\text{aff}} = (\mathsf{Q}, \nabla, \mathcal{D}, \mathscr{Y})$ *is*

(i) accessible from 0_{q_0} *if* $\overline{\operatorname{Sym}}(\mathcal{Y})_{q_0} = \mathsf{T}_{q_0}\mathsf{Q}$ *and is*

(ii) configuration accessible from q_0 *if* $\overline{\operatorname{Lie}}(\overline{\operatorname{Sym}}(\mathcal{Y}))_{q_0} = \mathsf{T}_{q_0}\mathsf{Q}$.

[1] L/Murray, *SIAM J. Control Optim.*, **35**(3), 766-790, 1997.

Low-order controllability results

- These revolve around vector-valued quadratic forms.
- For $\mathbb{R}$-vector spaces V and W, let $\mathsf{TS}^2(V;W)$ be the collection of symmetric bilinear maps $B\colon V \times V \to W$.
- For $B \in \mathsf{TS}^2(V;W)$ and $\lambda \in W^*$ define $\lambda B(w_1, w_2) = \langle \lambda; B(w_1, w_2)\rangle \in \mathbb{R}$.

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Definition $B \in \mathrm{TS}^2(V;W)$ is

(i) ***indefinite*** if for each $\lambda \in W^* \setminus \mathrm{ann}(\mathrm{image}(B))$, λB is neither positive nor negative-semidefinite and is

(ii) ***definite*** if there exists $\lambda \in W^*$ so that λB is positive-definite.

Some newer results[1]

- For $q \in \mathsf{Q}$ define $B_{\mathcal{Y}}(q) \in \mathsf{TS}^2(\mathcal{Y}_q; \mathsf{T}_q\mathsf{Q}/\mathcal{Y}_q)$ by

$$B_{\mathcal{Y}}(q)(v_1, v_2) = \pi_{\mathcal{Y}_q}(\langle V_1 : V_2\rangle(q)),$$

where V_1 and V_2 are vector fields extending $v_1, v_2 \in \mathcal{Y}_q$.

Theorem *Let $\Sigma_{\mathrm{aff}} = (\mathsf{Q}, \nabla, \mathcal{D}, \mathscr{Y})$. If $q_0 \in \mathsf{Q}$ is a regular point of $\mathcal{Y}$ then Σ_{aff} is*

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(i) not STLCC from q_0 if $B_{\mathcal{Y}}(q_0)$ is definite.

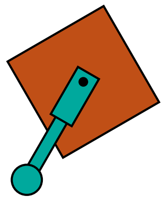
Assume that $\overline{\mathrm{Sym}}(\mathcal{Y})_{q_0}$ is generated by symmetric products of degree at most two. Then Σ_{aff} is

(ii) STLC from 0_{q_0} if $\overline{\mathrm{Sym}}(\mathcal{Y})_{q_0} = \mathsf{T}_{q_0}\mathsf{Q}$ and if $B_{\mathcal{Y}}(q_0)$ is indefinite, and is

(iii) STLCC from q_0 if $\overline{\mathrm{Lie}}(\overline{\mathrm{Sym}}(\mathcal{Y}))_{q_0} = \mathsf{T}_{q_0}\mathsf{Q}$ and if $B_{\mathcal{Y}}(q_0)$ is indefinite.

[1] Hirschorn/L, *Proceedings of 40th IEEE CDC*, 4216-4221, Dec. 2001.

Examples



- not accessible but configuration accessible;
- STLCC.

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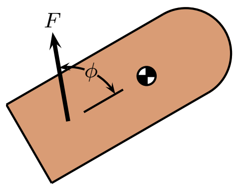
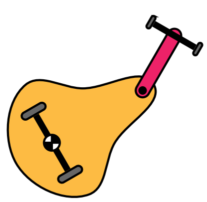


- accessible with ϕ fixed or variable (but not fixed at 0 or π),
- STLCC if ϕ variable;
- not STLCC if ϕ fixed.



- accessible;
- not STLCC.

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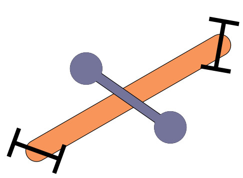

- accessible;
- STLCC.

Application to motion planning

- Think of $B_{\mathcal{Y}}$ as a bundle mapping on all of Q.
- A vector field X is a **decoupling vector field** for Σ_{aff} if one can follow any reparameterisation of any integral curve of X with a controlled trajectory of Σ_{aff}.

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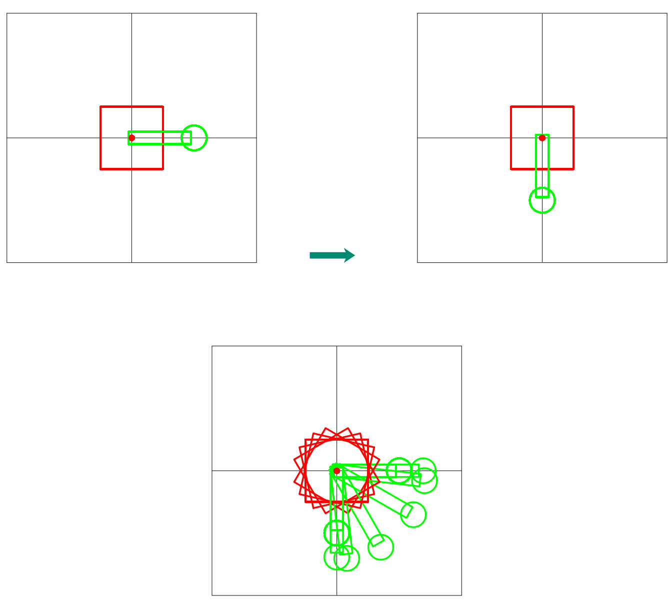
Theorem[1] X is a decoupling vector field if and only if $X \in \Gamma^\infty(\mathcal{Y})$ and $B_{\mathcal{Y}}(X, X) = 0$.

- If there exists generators $\mathscr{Y} = \{Y_1, \dots, Y_m\}$ for $\mathcal{Y}$ that are all decoupling vector fields, then $B_{\mathcal{Y}}(q)$ is indefinite for each $q \in$ Q. (If $\operatorname{codim}(\mathcal{Y}) = 1$ then the converse is also true.)
- For each of the example systems that is STLCC, $\mathcal{Y}$ possesses a collection of decoupling generators!

[1] Bullo/Lynch, *IEEE Trans. Robotics and Autom.*, **17**(4), 402–412, 2001.

Some demos



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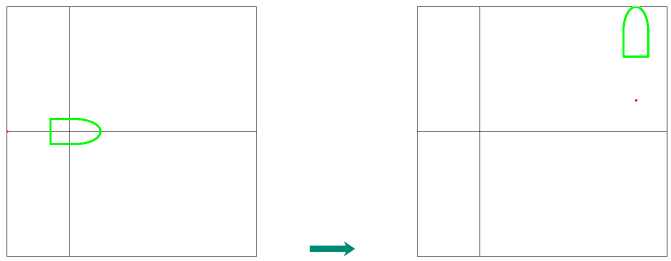

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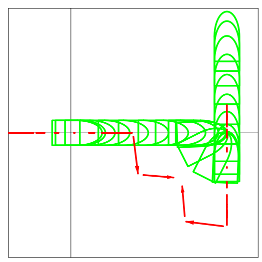

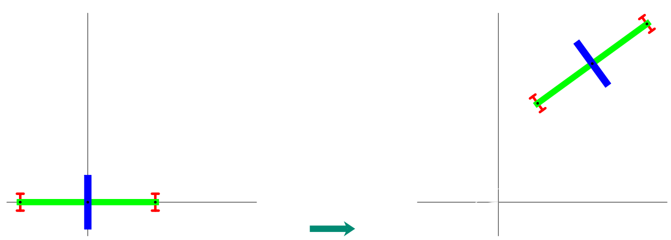

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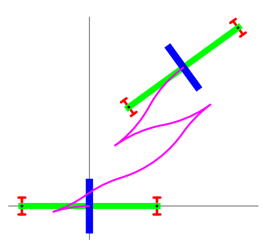

Work to be done

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- Higher-order controllability.
- Connections between decoupling vector fields and extremals for optimal control problems?
- Effects of potential and dissipative forces.