The category of affine connection control systems

Andrew D. Lewis*

15/08/2002

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*Mathematics & Statistics, Queen's University
Email: andrew.lewis@queensu.ca
URL: http://www.mast.queensu.ca/~andrew/

The category CAS

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- We first consider a more standard class of system, following V. I. Elkin, *Reduction of Nonlinear Control Systems. A Differential Geometric Approach*, Kluwer, 1999.
- An ***object*** in CAS is a pair $\Sigma = (M, \mathscr{F} = \{f_0, f_1, \dots, f_m\})$ where $\mathscr{F}$ is a family of vector fields on the manifold M (think $\dot{x}(t) = f_0(x(t)) + u^a(t) f_a(x(t))$).
- A ***morphism*** sending $\Sigma = (M, \mathscr{F} = \{f_0, f_1, \dots, f_m\})$ to $\tilde{\Sigma} = (\tilde{M}, \tilde{\mathscr{F}} = \{\tilde{f}_0, \tilde{f}_1, \dots, \tilde{f}_{\tilde{m}})\}$ is a triple $(\psi, \lambda_0, \Lambda)$ where $\psi \colon M \to \tilde{M}$, $\lambda_0 \colon M \to \mathbb{R}^{\tilde{m}}$, and $\Lambda \colon M \to \mathsf{L}(\mathbb{R}^m; \mathbb{R}^{\tilde{m}})$ are smooth maps satisfying
  1. $T_x\psi(f_a(x)) = \Lambda^\alpha_a(x) \tilde{f}_\alpha(\psi(x))$, $a \in \{1, \dots, m\}$ and
  2. $T_x\psi(f_0(x)) = \tilde{f}_0(\psi(x)) + \lambda^\alpha_0 \tilde{f}_\alpha(\psi(x))$.

- This corresponds to a change of state-input by

$$(x, u) \mapsto (\psi(x), \lambda_0(x) + \Lambda(x)u).$$

- Elkin discusses equivalence, inclusion, and factorisation in the category CAS.
- He successfully considers local equivalence for various classes of system:
  1. single-input systems;
  2. systems with involutive input distributions;
  3. systems with three states and two inputs.
- The notions of factorisation are related to, but not the same as, the abstractions of Pappas, Lafferriere, and Sastry.
- *Punchline:* For control-affine systems, the category theoretic language is useful for organising a means of attack on various important control theoretic issues. We hope to do the same for affine connection control systems.

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What are affine connection control systems?

- They model a fairly general class of Lagrangian control systems:
  - a configuration manifold Q;
  - a Riemannian metric g on Q (kinetic energy);
  - a collection of input forces $\{F^1, \dots, F^m\}$;
  - possibly nonholonomic constraints, linear in velocity.
- The Lagrangian is kinetic energy: $L(v_q) = \frac{1}{2} g(v_q, v_q)$.
- Even with constraints, the equations of motion for these systems have the general form

$$\nabla_{c'(t)} c'(t) = u^a(t) Y_a(c(t)),$$

where ∇ is an affine connection on Q (the Levi-Civita connection when constraints are not present) and where the Y's are "related to" the F's.

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The category ACCS

- An ***object*** in the category of affine connection control systems (**ACCS**) is a triple $\Sigma_{\mathrm{aff}} = (Q, \nabla, \mathscr{Y} = \{Y_1, \dots, Y_m\})$.
- An **ACCS** ***morphism*** sending $\Sigma_{\mathrm{aff}} = (Q, \nabla, \mathscr{Y})$ to $\tilde{\Sigma}_{\mathrm{aff}} = (\tilde{Q}, \tilde{\nabla}, \tilde{\mathscr{Y}})$ is a triple (ϕ, S, Λ) where

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1. $\phi \colon Q \to \tilde{Q}$ is a smooth mapping,
2. S is a smooth section of $\mathbb{R}^{\tilde{m}}_Q \otimes \mathsf{TS}^2(TQ)$ and $\Lambda \colon Q \to \mathsf{L}(\mathbb{R}^m; \mathbb{R}^{\tilde{m}})$ is a smooth map, together satisfying
   (a) $T_q\phi(Y_a(q)) = \Lambda^\alpha_a(q)\big(\tilde{Y}_\alpha(\phi(q))\big)$ and
   (b) $T_q\phi(\nabla_X Y)_q = (\tilde{\nabla}_{\tilde{X}}\tilde{Y})_{\phi(q)} + S^\alpha_q(X(q), Y(q))\tilde{Y}_\alpha(\phi(q))$,
   where $\tilde{X}$ and $\tilde{Y}$ are ϕ-related to X and Y.

- What does an ACCS morphism *really* do?
- The map ϕ sends controlled trajectories for Σ_{aff} to controlled trajectories for $\tilde{\Sigma}_{\mathrm{aff}}$.
- If (γ, u) is a controlled trajectory for Σ_{aff}, then $(\phi \circ \gamma, \tilde{u})$ is a controlled trajectory for $\tilde{\Sigma}_{\mathrm{aff}}$ with

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$$\tilde{u}(t) = \Lambda(\gamma(t))u(t) - S^\alpha(\gamma'(t), \gamma'(t))\tilde{Y}_\alpha(\gamma(t)).$$

- Conversely, if ϕ sends every controlled trajectory for Σ_{aff} to a controlled trajectory for $\tilde{\Sigma}_{\mathrm{aff}}$, then there exists S and Λ so that (ϕ, S, Λ) is an ACCS morphism.
- Have notions of ***isomorphism*** (equivalence), ***epimorphism*** (projection or quotient), and ***monomorphism*** (subobjects).

ACCS $\subset$ CAS

- To render an object $\Sigma_{\rm aff} = (Q, \nabla, \mathscr{Y})$ in ACCS an object $\Sigma = (M, \mathscr{F})$ in CAS, take
  1. $M = TQ$,
  2. f_0 is the geodesic spray for ∇ (a second-order vector field on TQ), and
  3. f_a is the "vertical lift" of Y_a.

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- To an ACCS morphism (ϕ, Λ) one associates a CAS morphism $(\psi, \lambda_0, \Lambda)$ with $\psi = T\phi$ and $\lambda_0(v_q) = S(v_q, v_q)$.
- *Question:* Is the collection of morphisms for ACCS simply the collection of morphisms for CAS restricted to systems in ACCS?
- *Answer:* No, there are CAS morphisms of objects in ACCS that are not ACCS morphisms $\Longrightarrow$ the extra structure of ACCS objects needs special attention.

Decompositions of ACCS morphisms

- An ACCS morphism (ϕ, S, Λ) is a ***morphism over controls*** (a **CACCS** ***morphism***) if $Q \subset \tilde{Q}$ and $\phi\colon Q \to \tilde{Q}$ is the inclusion.
- CACCS morphisms are simply algebraic Q-dependent transformations of the control (if $\tilde{Q} = Q$, think "partial feedback linearisation").
- An ACCS morphism (ϕ, S, Λ) is a ***morphism over configurations*** (a **QACCS** ***morphism***) if $S = 0$ and $\Lambda(q) = \mathrm{id}_{\mathbb{R}^m}$.

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- QACCS morphisms leave alone the controls, and are simply coordinate mappings. One can show that for a QACCS morphism, $\phi\colon Q \to \tilde{Q}$ must satisfy
  1. ϕ maps geodesics of ∇ to geodesics of $\tilde{\nabla}$ (ϕ is ***totally geodesic***) and
  2. the control vector field $\tilde{Y}_a$ must be ϕ-related to the control vector field Y_a, $a = 1, \ldots, m$.

Proposition: *An isomorphism in ACCS is the composition of a QACCS morphism with a CACCS morphism.*

- One should be able to put together tools from the theory of distributions and from affine differential geometry to obtain some equivalence results for affine connection control systems.
- *This has not even been started.* Of all the equivalence classes is CAS determined by Elkin, *none* of the representatives are affine connection control systems!
- $\longrightarrow$ Lots of stuff to do for the equivalence problem.

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Factor systems

- $\tilde{\Sigma}_{\text{aff}} = (\tilde{Q}, \tilde{\nabla}, \tilde{\mathscr{Y}})$ is a ***factor system*** for $\Sigma_{\text{aff}} = (Q, \nabla, \mathscr{Y})$ if there is an ACCS morphism (ϕ, S, Λ) so that $\phi\colon Q \to \tilde{Q}$ is a surjective submersion.
- Factor systems are interesting for several reasons, including
  1. reduction in the presence of symmetry can often be thought of as factorisation in ACCS,
  2. Factor systems that are fully actuated seem to come up fairly often (i.e., fully actuated base space, in the case of reduction).

Proposition: *Under some assumptions, if $\tilde{\Sigma}_{\text{aff}} = (\tilde{Q}, \tilde{\nabla}, \tilde{\mathscr{Y}})$ is a factor system for $\Sigma_{\text{aff}} = (Q, \nabla, \mathscr{Y})$ then for every controlled trajectory $(\tilde{\gamma}, \tilde{u})$ for $\tilde{\Sigma}_{\text{aff}}$ there is a controlled trajectory (γ, u) for Σ_{aff} so that $\tilde{\gamma} = \phi \circ \gamma$.*

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- QACCS factor systems are "simple."

Proposition: *$\tilde{\Sigma}_{\mathrm{aff}} = (\tilde{Q}, \tilde{\nabla}, \tilde{\mathscr{Y}})$ is a factor system for $\Sigma_{\mathrm{aff}} = (Q, \nabla, \mathscr{Y})$ via a QACCS morphism if and only if*

(i) $\nabla_X X$ is ϕ-projectable for all ϕ-projectable vector fields X and

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(ii) the vector fields $\mathscr{Y}$ are ϕ-projectable.

- As with isomorphisms, we may decompose factorisations.

Proposition: *(Roughly), if $\Sigma_{\mathrm{aff}} = (Q, \nabla, \mathscr{Y})$ factors to $\tilde{\Sigma}_{\mathrm{aff}} = (\tilde{Q}, \tilde{\nabla}, \tilde{\mathscr{Y}})$ by an ACCS morphism, then one can render $\tilde{\Sigma}_{\mathrm{aff}}$ a QACCS factor system by the pre-application of a CACCS morphism.*

Wrap up

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- This has been a rambling, barely coherent presentation of a loosely organised collection of ideas.
- With some significant effort, it is possible that this will one day come together to produce a collection of significant results.
- It is also possible that with some significant effort, nothing will happen because problems such as system equivalence are intrinsically extremely difficult.